Classical Physics

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Lecture No. # 16

Yesterday, I mentioned that exponential divergences of space trajectories can be characterize by a Lyapunov exponent. And the question I ask to us why I needed to take the long time limit of the Lyapunov exponent, my answer was not quite satisfactory. You can find finite time Lyapunov exponents but, it is not very relevant for the purposes, we have in mind the reason you need a very long time average is the following. If a system is kaotic and let us look at the kind of maps we been talking about here.

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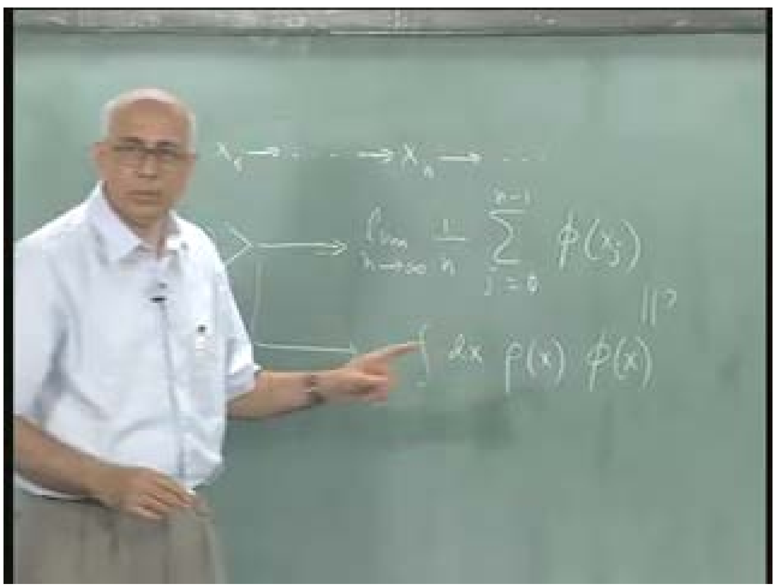

Then what happens is you start with the variable x naught, after 1 times step. It goes to a value x 1 and then after n times steps goes to x n and so on, so forth. If there is exponential divergence of neighboring projectories, then the rough idea is that, any initial error in x naught is amplified exponentially. And therefore x n after sometime could be nonsense for instance if you write two numbers in binary. And they differ from each other in the 100 decimal place and at each step let us say the error double. In the Lyapunov exponent is log 2, because the error becomes 2 to the n after n steps, which you can write us e to the n log 2, in the lambda is log 2 in that case, which is positive.

Now, what will happen is that as we will see today with the simple model, after hundred iterations, the error has transmitted itself to the first decimal place. And then of course it becomes meaningless completely. So, the whole idea is that x n is not computable, on the other hand I might have a physical variable, x might represent a physical quantity and then maybe some function of x say phi of x, which I would like to calculate whose average value are like to calculate on the average.

As the dynamics runs or like to find phi of x at each incident of time, calculate arithmetic average of all these things that would be the average of phi of x. So, what would this be this would be equal to 1 over n summation j equal to 1 to n or n minus 1 phi of x j j equal to 0 to n minus 1. This should be the average value the time average n is time now in desecrates steps of sometimes step.

And of course if x j is not computable in this sense, then this quantity is meaningless but, I would like to know what it does over a long period of time. In principle I would like to know what this quantity is this is the long time average of a dynamical variable or a function of a dynamical variable. And the difficulty is what you do if x n is meaningless, has got so much error in it that its completely not even computable it does not even make sense.

So, what one you like to do in that is, that is stage is to replace a time average by an average over a probability distribution called sample average. This is the whole idea of statistical mechanics which you going to come to very shortly.

So, I would like to replace this instead of this quantity, I would like to replace it by and integral over all possible values of x, some probability density of x multiplied by phi of x. And the question is these two equal, is this equal to that the statement is if the system is ergodic. Ergodicity implies that time averages are equal to unsample averages, in other words you can get read of this quantity.

Provided you have knowledge of this probability density and then, you can convert a long time average into one sample average or a statistical average. For this you must be sure, that this rho of x is an invariant density. In other words it does not change under evolution itself and to attain this in where intensity the dynamics has to run for a long, long time. And that is why we need here to find out what this n to infinitive limit is, we need the long time average of quantities. So, this is the reason why I am interested in defining a Lyapunov exponent, which is in fact infinitive time limit.

First, there is an epsilon going to zero limit and then there is a key go to infinitive limit, we will see this in with the help of the example, we started off yesterday. Now, yesterday we looked at simple maps one dimensional maps in particular we looked at linear maps. And then we discovered that if you have a fix point of the origin depending on the slope of this fix point at the fix point of the map, you could either have a stable fix point or a non stable fix point.

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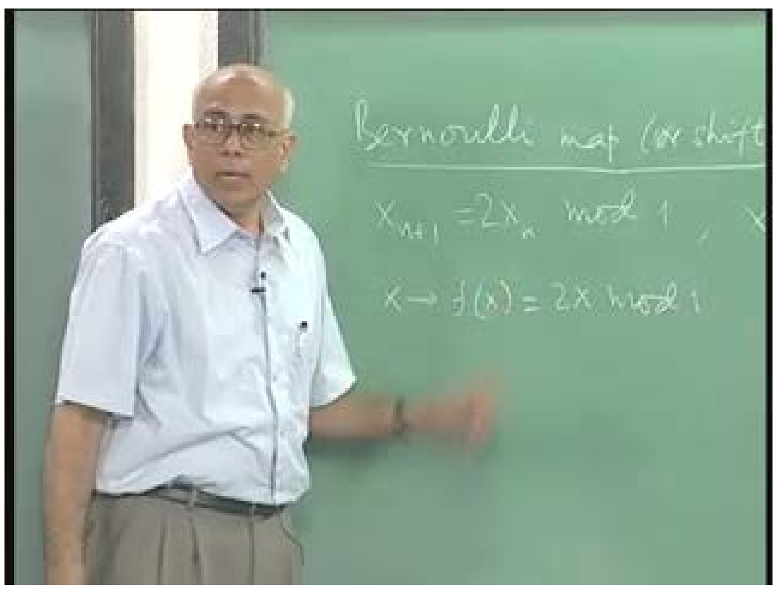

Now, let us look at what happens if you have a non-linear map, in a non-linear map can be extremely complicated the simplest of these models is so called Bernoulli map; it is also called the Bernoulli shift. I will use this term kind of interchangeably, there are very precise mathematical definitions of each terms. And the whole of dynamical systems theory in mathematics is sometimes goes under the name of ergodic theory. Because, we talking about transformation of variables and these transformations are suppose to displace certain ergodicity properties.

Now, let me define what the Bernoulli map is, it is a map of the unit interval to itself, so we start with a number x naught between 0 and 1. And you give me a, I give you a rule by which you find the number x 1, which is a function of x naught, which also lies between 0 and 1. In the rule is a following, x n plus 1 equal to x n 2 x n modulo 1, when I say modulo 1, I mean it is a fraction. And then x naught goes you start of in the interval 0 to 1, this is my map function. Now, let us plot, what this map function is, so the map actually says, you take x to f of x and f of x is 2 x modulo 1, what is that function look like.

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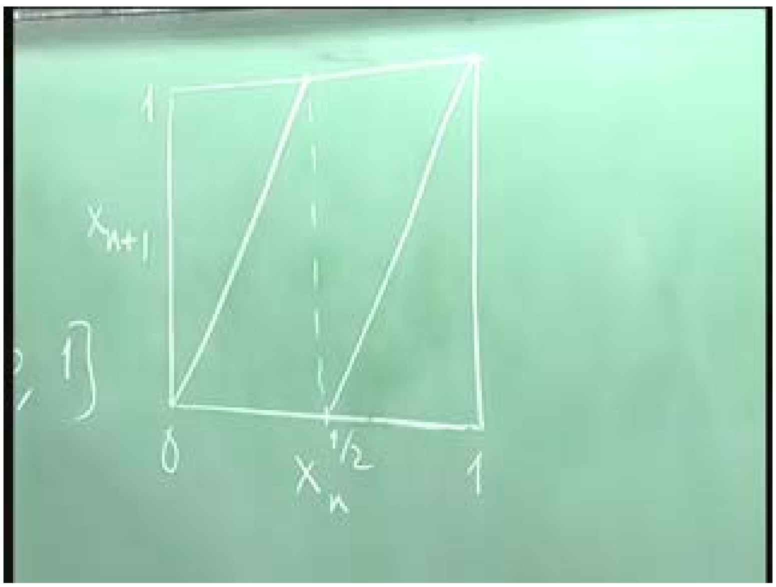

So, here 0, here 1, if this is x n and I plot x n plus 1, which is a function of x sub n by this rule, it is says simply double take the number between 0 and 1 and double it. And if you do that of course you realize this is 1 by the time you get the point of half the graph goes out of scale. But, then you are suppose to subtract the 1, because it is modulo 1 and bring it back that is equivalent to taking this straight line cutting of the second piece and bringing it back here and attaching it to this. And this is a half that is the map, this is the rule.

So, very simple rule you capacitance do this in the pocket calculator, take a number between 0 and 1 double it, if you remains with 0 to 1 you leave it, if it is more than 1, then you subtract 1 and you get the fraction back and you keep you keep doing this. Now what are the fix point of this map, well the fix point should arise, whenever the map is equal to the function x itself;

whenever f of x is equal to x. So, where are the fix points, 0 is a fix point and 1 is a fix point clear, wherever the 45 degree line intersect the map function, these are the fixed points what is the slope of this map at the fixed points.

1 is equal to 0 there equivalent points, so the movement you come to one is equivalent to this immediately yeah 1 is the same as 0. 1 is the identify at 0 what the slope at these fixed. Slope is 2 right the slope at this point 2 x slope is 2 is that greater than 1 yes. So, are they stable fixed point or unstable? Unstable, badly unstable right.

Now, the thing that could happen is if the fixed point becomes unstable, then you should ask also is there any iterate of this map, which as a fixed point. In other words the second iterate of this map f of f of x might have a fixed point and that could mean a period 2 point for the original map, because you see a fixed point means a following.

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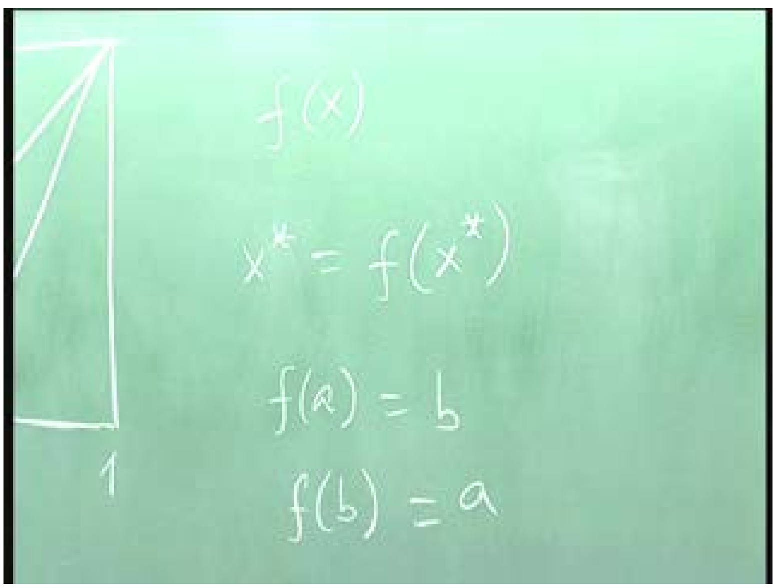

In the map f of x fixed point satisfies x star equal to f of x star. That is what a fixed point as but, it is also conceivable that you might have the following situation, you might have f of a equal to b and f of b equal to a it is could happen right. So, you start with the number a and you do the map, it goes to a number b and you do a map again its comes back to a. This is obviously a period to cycle it is going to be a flip back between a and b keeps going right.

Now, it could happen that this map has a period 2 cycle which is stable in which case everything will fall into that. Now, what is the period 2, what is the first iterate of this map look like,

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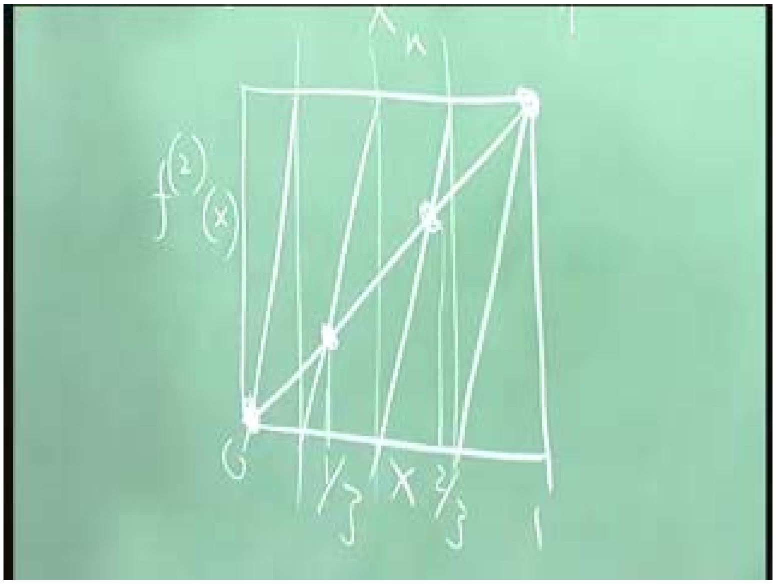

I have to do this map twice. So, here x and here f 2 of x by superscript 2, I mean f of f of x and f 2 of x is obviously equal to 4 x modulo 1. Who is already have a 2 x and then you have to take each of those x and make it a 2 x. So, it is 4 x but, everything is modulo 1, what is that function looks like, it is going to have a lot of steps right, it is going to have a step at a half, it is going to step pre quarters and one forth switch going to be slope 4.

And where are the fixed point of this map, well all these lines this is a fixed point that is a fixed point these are the original fixed point. But, now you got two more fixed points, these two points stable or unstable. Everything is unstable because the slope is now 4, which is bigger than 1, so its even more unstable, 1 is the same as 0 it is equivalent to 0; you put x equal to 1 here, we are 2 modulo 1, which is 1 right.

There are no unstable, there are no stable points are off, if stable points exist, then you would expect there alternative in the (()) sense that we had earlier that maxima and minima would follow each other. But this map as you can see fixed points or unstable the period 2 points are unstable also. So, looks like there are no stable points at all, what about period, what is the next step f 3 of x that is going to have slope 8 and all those points are unstable as well.

So, you can see here what is happening is a this is 0, what is this point by the way 1 third it is 1 third, this point is 1 third, this point is 2 thirds and that is 1. So, I start with 0, I remain at 0, I started at 1 I remain at 1 but, I start with 1 third I double it, it becomes 2 thirds, I double that it become 4 thirds. But, then subtract 1, so it is comes back to 1 third and 2 third and so on. So, it goes the clock between these two and its unstable there are more in an example of the next period, period four, what will be the next.

1 fifth I start with 1 fifth it goes to 2 fifths, it goes to 4 fifths that goes to 8 fifths, which is the same as 3 fifths and that goes to 6 fifths, which is same as 1 fifths and show it fifth back. And so it is clear that you have an infinite number of unstable periodic orbits as well and infinite number of time with all unstable completely. What do you think is a set of periodic orbits, they are unstable it is clear that if you took any rational number, it is going to end up as part of a periodic orbit. The right way to look at this problem is actually instead of missing around with this x here to write these numbers in binary.

So, we can write any number between 0 and 1 in binary then what happens. So, let us do that and then you will understand, why it is called the Bernoulli shift.

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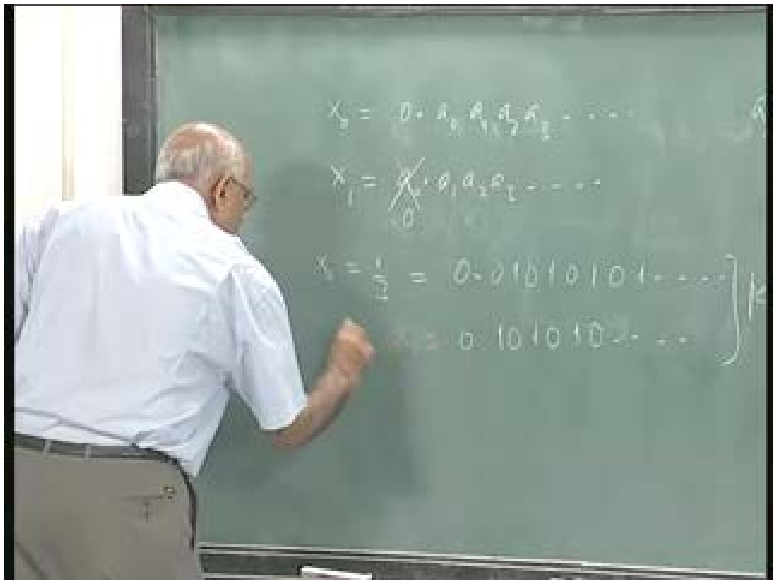

So, I start with x naught and this is equal to 0 point it is a number between 0 and 1, I could write this number as a naught, a 1, a 2, a 3 and dot, dot. Now, decimal place like this where each of the a i is, is a 0 or 1. Now, what is x 1, what is x 1 equal to, what happens here is that, this stands for x naught is numerically equal to a naught divided by 2 to the power 1 plus a 1 divided by 2 square that is, what this binary decimal if you like stands for right.

Now, I multiply by 2, then all that happens is everything shifts and then this is equal to a naught dot a 1 a 2, a 3. But, then if a naught is 0, there is no problem, this means x naught was less than half but, if a naught is 1, then the numbers between half and 1 to start with and when I double it becomes bigger than 1 and this a naught is 1 and that thrown out. So, it is again 0 point whatever now you see what is happening here. Each time you are simply shifting the decimal point and throwing away the integer part are you losing information yes indeed.

So, is this map invertible no, because if you give me an x 1, I cannot tell you what x naught was uniquely here, another way of seeing it is right here. If you give me and x 1 or x n plus 1, there are 2 possible x n from which it arose and you cannot tell, which one it was because I got thrown away.

So, in this map is loss of information in the forward direction you can see that, you cannot invert the map and the reason you cannot invert the map is because the map function is non linear. And it may linear you could invert the whole thing and it is this non linearity that lead into k going to lead to k off, going to lead the problems that we have now. But you also see that every number that rational is in fact part of a periodic orbit. What is a rational number? How do I represent the rational number here, either it is terminates after sometime and then you get only 0's or it repeats and as a pattern which repeats.

And every number for which the pattern repeats is going to keep become part of a periodic orbit but, we are guaranteed that the periodic orbits are all unstable complete. For instance 1 fifth for instance 1 third, what is that in this binary thing, what is well it is suddenly less than half, so there is a 0, here is greater than a quarter, so there is a 1 here. And then the difference between the 2 is 1 12 right and that is less than 1 8, so this is zero here, it is greater than 1 16, so there is a 1 here and so on. And on the whole you see that, x 1 equal to 0. 1 0 1 0 1 0 and this forms a period 2 cycle and a flip flop between each other. I leave you to figure out what is 1 5th it is again a part of a period recycle therefore you see that every rational number is part of a periodic orbit.

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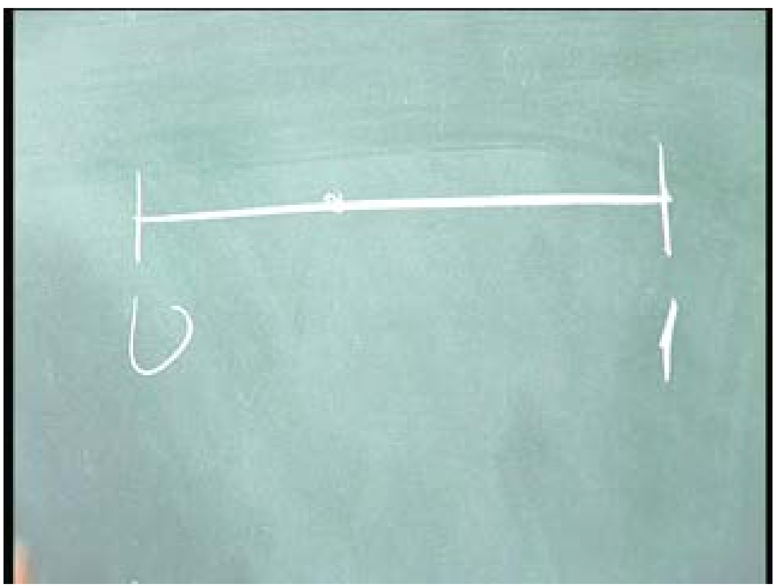

And it is unstable, so what is evidently happening is that, because it is unstable here 0 to 1 every rational number including a rational number here for example its unstable.

And it is acts like a separate tricks, so it throws out numbers on either side of it but, the rational are dense on the unit interval. Arbitrary closed every point, you can find rational and it is unstable. So, in this magical system you have a dense set of unstable orbits buried in this unit interval but, most numbers are not rational. The set of rational is a set of measure 0 you put, you close your eyes and put a put a pencil on this point on the unit interval, the probability that you are going to hit a rational number is actually 0.

You are going to definitely hit an irrational number for which the decimal expansion does not terminate goes on forever and this is unstable it is kaotic it is going to behave completely random fashion. So, what happens as time goes on and we will explicitly see, this is that the iterates of every rational number, rational initial condition or on a periodic arbitrary just unstable but, the moment you have an initial condition, which is irrational this does not settle down anywhere. It moves it wanders all over the unit interval and it is iterates finally densely fill up the entire unit integral.

So, any give prove this we prove, we will show that there is uniform density of the iterates of any irrational number, no matter where you start. And is system ergodic it is suddenly is ergodic is it conservative or dissipate that is a much trickier question to answer, because you must know what meant by dissipative in this problem. We do not know the gate we will see that it is a dissipative system this is actually dissipate. We will see why, it is a little tricky but we will see why.

But, before I do that lets define the Lyapunov exponent for this, what do you think is a Lyapunov exponent for this problem. First of all it is clearly kaotic, it must have positive Lyapunov exponent what do you think it is. It is log 2 easy to see, because you know you have the power 2 here. So, I start with any x naught I start with a x naught plus epsilon the 2 of them would get separated by precisely log 2 e to the power log 2 n log 2 is how the error could propagate you will prove that.

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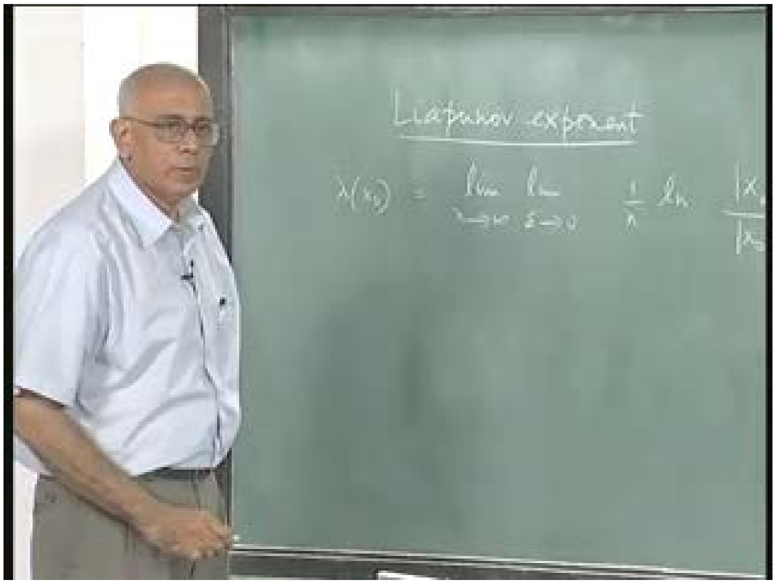

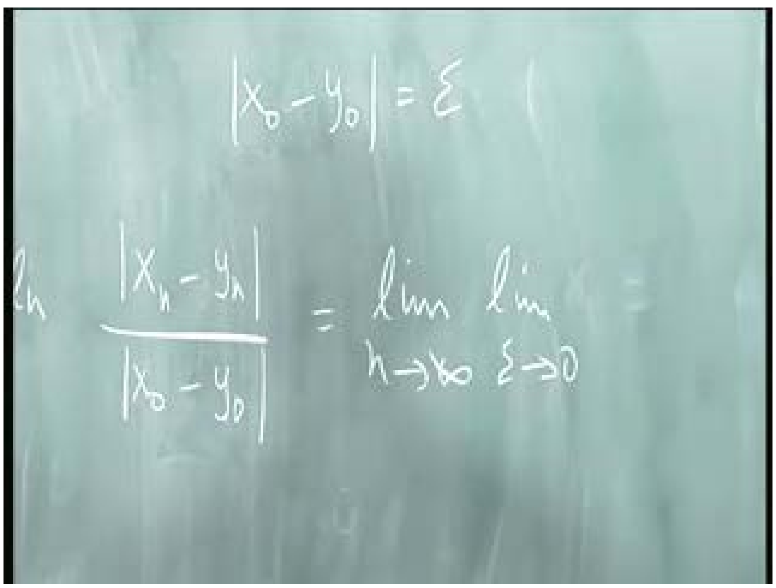

So, let us write down the formula for the Lyapunov exponent for a 1 dimensional map, I define a lambda, which is the function of the initial value x naught is equal to limit n tend into infinity limit epsilon goes to 0 of 1 over n log x modulus x n minus x naught x n minus this is how we find it. But, then you could also write this as equal to limit n tends to infinity limit epsilon goes to 0.

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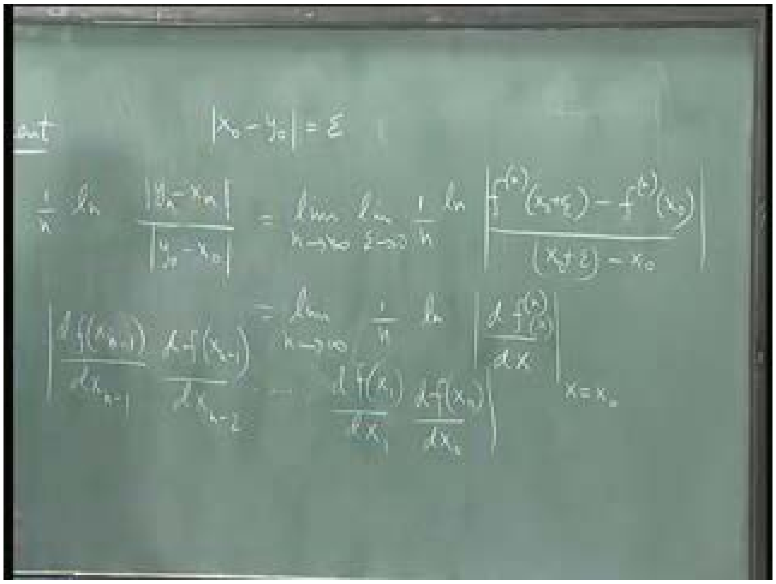

Log f n of x naught plus epsilon minus f n just confused y n minus x n, because I start with y naught, I iterate n times that is f superscript n of y naught and similarly for x naught but, y naught is x naught plus epsilon, so it is equal to the log of the nth power of this. So, what is this equal to and I have to take limit epsilon goes to 0, it is that is a that is a 1 over end very important it is a 1 over n.

So, this is equal to limit n tends to infinity 1 over n log d f n over d x at x equal to x it is the nth derivative of this function f superscript n, with respect to x at the point x equal to x naught and the modulus of this. But you see you could also write this as equal to limit n tends to infinitive 1 over n log d f x n, d f x x n over d f x n minus 1 over d x n minus 1 d f x n minus 1 over d x n minus 2. All the way up to d f x 1 over d x 1 d f modulus f x, all you have to use is the fact at x j plus 1 equal to f of x j wide definition.

Therefore I take this nth iterate and that is equal to f n of x is f n minus 1 of f of x, which is equal to f n minus 2 of f f of x and so on. And since I have called the iterate of x naught f of x naught I called it x one this cancels this all the way down till you end up with this function. So, all I have done is to write x n minus 1 sorry this is n minus 2 f of f x n minus 2 is the same as x n minus 1 and so on.

So, I just inserted this one and now you see that this Lyapunov exponent is a property not just of x naught but the entire orbit of x naught. Because, x naught goes to x 1 it goes to x 2 and the derivative of this map function at the point x 1, x 2, x 3 there all involved. So, it is really a property not just of the initial point but of the orbit on, which the initial point runs and the whole thing will be a function of that initial points so let me keep this (()) can I simplify this.

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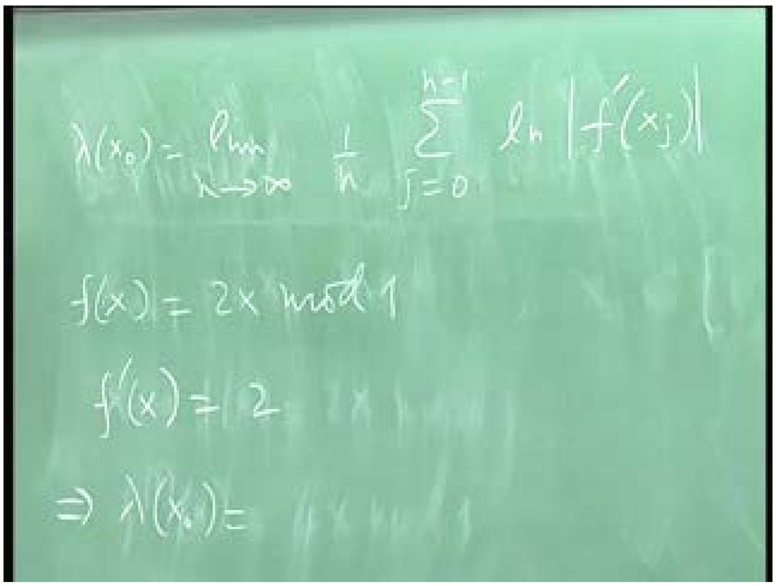

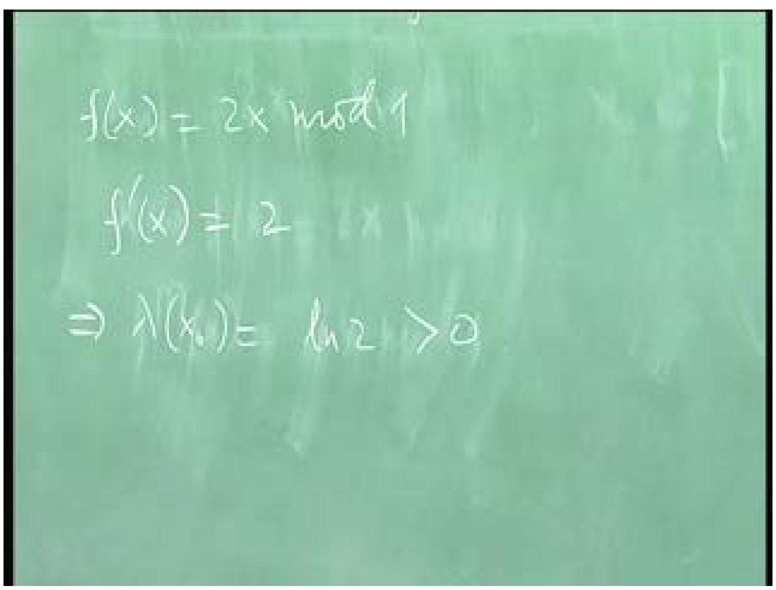

Yes indeed because its obvious that it once this could be written as equal to lambda of x naught equal to limit n tends to infinity of 1 over n log of a b is log of a plus log of b. So, you could write this as a summation from j equal to 0 to n minus 1 log f prime of x j modulus f prime of x at every point the modulus of that is like a stretch factor, the local contraction expansion factor We will take suppose take the log of this guy and some it over the entire orbit and then divide by n and therefore it is a time average. It is a long time average of the stretch factor, that is apply to our case.

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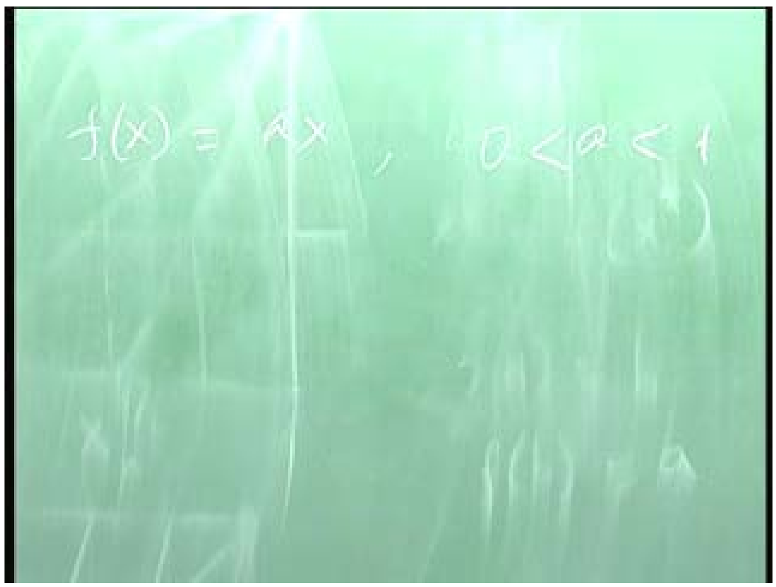

So what is this equal to for our map f of x equal to 2 x modulo 1, so f prime of x equal to 2 no matter where you are the slope is always 2 the prime of x is 2 right. And therefore what does this give you, what is lambda of x not equal to it is 1 over n times log 2 sum over n times in that n cancels out and you end up with a log 2, which is greater than 0.

As I said every time you have a Lyapunov exponent, which is positive we have kayas we are Kaotic behavior. See, your guaranteed that in this problem you actually lose the error initial error epsilon amplifies at this rate, you lose one bit of information every iteration iterative.

(())

Yeah you sets of measure 0, may for for instance you have incidentally it is true, not true if you have a periodic orbit. Then it is not true if you start with an x naught, which is on a periodic orbit, then it does not expand anything like that part of a periodic orbit its comes back to itself. What could be the Lyapunov exponent on a periodic orbit what it be? Let us look at the map let us ask our self, what it could be, it is a good question, what is the Lyapunov exponent on a periodic orbit.

Let us look at the stable periodic orbit, so let me look at this map f of x equal to a x and a is less than 1 a is a positive number between 2 and 1 what is going to be the Lyapunov exponent in this case, it just becomes a to the power n x. So, any initial error epsilon will become a to the n epsilon and the epsilon divides out goes away and then, you have log a to the power n 1 over n outside and so its log a. And what is log a is positive, negative or zero? It is negative.

So, you see if you have stable fixed point the Lyapunov exponent becomes negative everything falls into that place, everything falls into 0, what about, what about the situation where you have a period two cycle, what could the Lyapunov exponent be.

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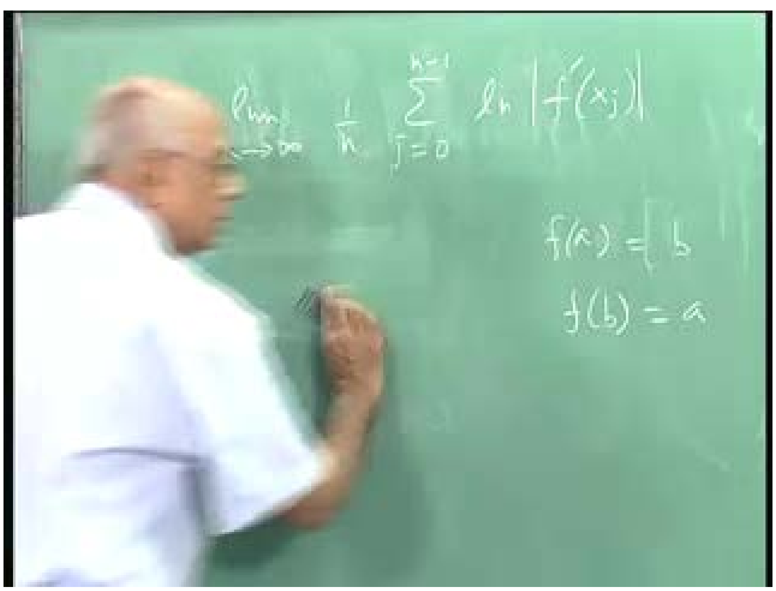

If it is stable if the period two cycle is stable, so let us suppose f of a equal to b f of b equal to a. For stability you require that the slope be positive that the slope be modulus of the slope is less than 1. So, this could imply this could be the condition for the stability of a periodic orbit, the slope of the each of these points or product of these two must be less than 1. What could the Lyapunov exponent end up positive, negative definitely negative, definitely negative.

So, you look at the examples, so it could be zero and still not kaotic but the moment is positive you have this exponential amplification of errors. Now, what I would like to show you is that we have to by the way, we computes the Lyapunov exponent very easily in this problem, because I knew that this slope was actually constant. So, there is really nothing to compute but, if the map is a not a simple piece wise linear map, then this is no longer so trivial.

And you need the better algorithm to calculate this Lyapunov exponent and this you do so numerically but, you actually needs some formula for it which you a resistance going to produce in a while. So, that requires a little more information about this map but, before I do that I would like to point out that this map is random in the following sense. You agree that the rule is completely deterministic you give me a x naught and I give you a definite x 1, x 2, x naught, naught 1.

So, in the forward direction the map is completely unique it specify for every x naught I give you a definite x 1 and it is deterministic there is no randomness here at all. And yet and yet it is as random as a coin toss in the following sense in the following precise sense. Suppose and this is what happens in real life in a dynamical system.

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If this is your phase space in a dynamical system it is very hot to keep track of points to infinite accuracy you always has some resolution, so what you end up doing is to form histograms. So, you partition in the phase space into cells you partition in the entire phase space into cells into little boxes or bin and on a computer, when you do numeric’s. You keep track of how many times the point representative point comes into a particular bin. For example if you doing on the unit interval and your accuracy is 1 over 1000, when your partition it into a 1000 bins and you keep track up to 3 decimal places where the point is and build up histograms right.

Now a representative point could starts here and the next time it is goes here and then it is goes here its goes here and so on. And you keep track of this whole thing and you build up a histogram over a long period of time. And eventually the histogram will stabilizes will be your invariant measure it will be the one, that steady state, in the steady state.

That is what we are trying to discover and this is called partitioning phase space or course training phase space. Now, let us do that without Bernoulli map and now I am lazy, I do the simplest possible course training, which is the following.

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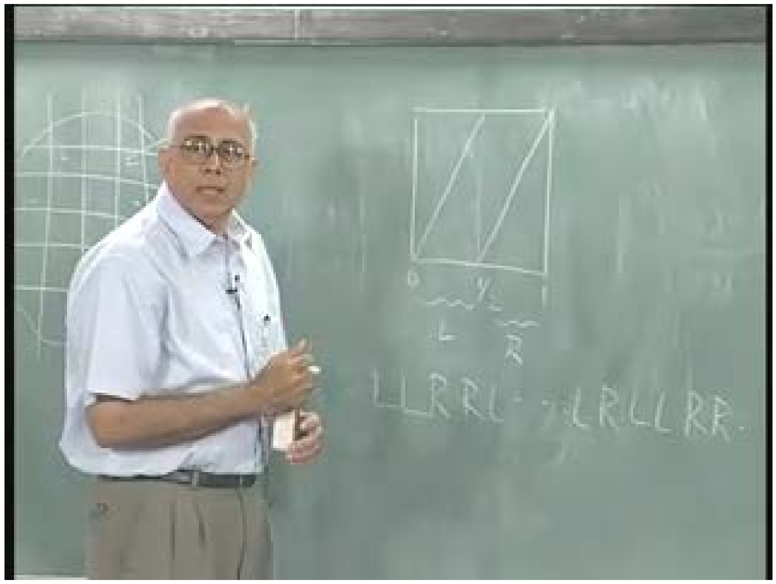

So, here is the map 0 to half and half to 1, this I call 1 bin and that I call 1 bin and this is say the left bin and the right bin. So, I am not even interested in whether this where this number is, I am just going to keep track of is this less than half or bigger than half that is it. And as you can see this corresponds to asking whether each of those a i is 0 or 1. If it is less than half it is 0, if it is more than half it is a 1. And I do this over a long period of time and I keep track of where the iterates of random point irrational point and if are get a long point string L L R R L L R R L L R R.

Typical segment of the iterates of some arbitrary irrational number is going to look like this, string of this kind. Now the statement of randomness in this deterministic dynamics is that under that Bernoulli shift it turns out that, given any coin toss experiment which you take a coin and you toss it. And if it is tails you put on L if it is heads put on R you going to get a long strings of L’s and R’s and if it is completely random sequence you going to get all possible sequences. The statement is that for every result of such a coin toss experiment string of some arbitrary length form from a coin toss experiment there exist an initial condition x naught in the Bernoulli map, such that it is iterate could be exactly equal to this string.

Let me repeat back given in arbitrary random sequence, random string with an alphabet consisting of just these two letters L and R. And I look at that string and say oh where it is come from, this came from iterating, this came from a coin toss experiment completely and correlated coin toss experiment is called a Bernoulli trail right; you keep doing this and your records this L's and R's

On the other hand somebody produces for you a string of L's and R's which came by iterating a deterministic rule, the Bernoulli shift you cannot tell the difference. The statement is for every such string which you get by coin toss experiment, where exist an initial condition x naught which will produce the same string, if you did the deterministic evolution by the Bernoulli shift. In that sense randomness and determinism a very closed link to each other this is ask random as a coin toss. This is the modulus think about kaotic dynamics that under suitable conditions, the kaoyas could be so extreme, this is really an extreme form of kaoyas that there is no correlation at all.

You could ask what is the correlation after all if it is a coin toss there is no correlation between whether one the first 25 th toss of the head or tail and the 26 toss for the head or tail there is no correlation it is a delta function. That is exactly the correlation even here all possible strings could occur, it is completely uncorrelated, even though that map is deterministic completely, job then is to find out what is the density with which fills things. Already this suggest to you since L's and R's are appearing with equal frequency, it is suggesting to you that okay.

Suppose I did a course training like this 0 to 1 4 th, 1 4th to half after this is what would now happen the same thing once again. Now you have an alphabet with four letters call them A B C D and again all possible such sequences could appear. And could be indistinguishable from Bernoulli tracks, it could be indistinguishable from a suitable (( )) completely.

So, that is the level of this is randomness appearing, we need to know, how to prove that everything is filled up uniform, so let us do that. That is interesting need to be able to handle things like that, I need to able to find this quantity I need a formula for it.

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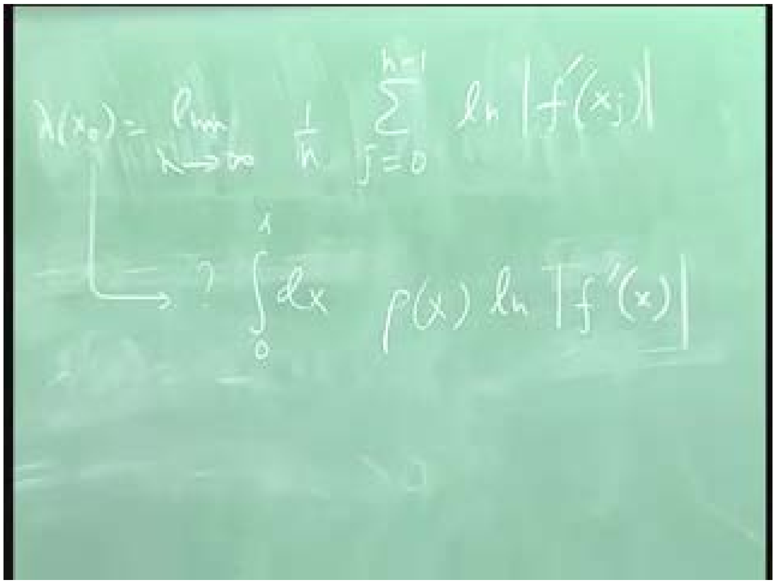

In fact I am extremely happy if this can be replace by ergodicity to an integral over the interval say zero to one that's the interval of the map.

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Very deep question is now trying to point out that even the toss of a coin is itself kaotic machine, because whether it is false or head or tails depends on extremely small changes in the initial conditions. I toss it a little higher, the wind blows a little harder or the coin is a little oiler or it lands on it is a edge and falls this way or that little tilde, that will tell you finally what the (()). So, in that sense yes the very paradigm of random machine the coin toss or die is itself kaoyas machine, yes that is true. Now, we going to get deeper into this I am lying a little bit we do not know how to define a random number.

So, we will come back to this over and over again, there is really no satisfactory definition of random number about. On the other hand there is no satisfactory definition what regularity means either, if you give me a piece of music and I ask you this is random or not, what are you going to tell me. If I play a piece of music up to certain point is no way that you can tell me, what the next note is going to be. Of course if you know what the piece of music is you know what it is but, if I give you the sample this little strip I could argue that this entire piece of music up to this point was a realization of a random process.

And now its going to differ from what the actual original piece of music was in the next instant But, if you give me a finite string there is no way of telling, so in that sense everything is random also. So, we going to assume that we have a paradigm of randomness called the Bernoulli trial, how I produce it is a different question, because produce with the coin and I am extremely accurate with I know everything about the coin, then of course I can tell you whether it is going to be heads or tails. And of course you would be able to must have a enormous computing power to able to do this that is the whole point.

You have to tell me with arbitrarily high precision, how I toss the coin then I can predict have you write this book all the Newtonian casino, how many people have read this. When the early days of Kaoyas a bunch of people in the US try to break the loss wages casino system is suppose run by the mobs it is dangerous try to do this by smuggling by sort of in the early days. They had wired themselves and they had a computer program which could actually calculate, where the (()) is going to stop. They try to break the casino by doing this by putting in the initial conditions and solving this equations numerically to see where its going to begin, and so on.

Now illegal to do this, because it is very dangerous and illegal to do this, because it is possible to some extend to make predictions. But, to the extent that these machines are more more sophisticated theoretic devices its harder and harder to do this. In principle you need infinite position to do this right okay that’s a good conceptual. So, I would like to be able to write this guy as equal to an integral d x some density times log mod f prime of x and we have to do this. Aand like to be able to find this rho of x if I can do that then the job is done I can calculate the Lyapunov exponent in this problem in terms of this in variant density rho of x.

In the question is how do I write an equation for rho of x and this is going to take as to this very basic question, where does dynamic stop and where does statistic begin, where does a statistics mechanics begin. Because we really getting into that right now; unlike statistical mechanics which leads to thermo dynamics and so on; as we will see a little later. There is no extra input here the rule of the game is that dynamical equation itself I give you a map function and that is the end of it everything has to be produce from that how do I do that?

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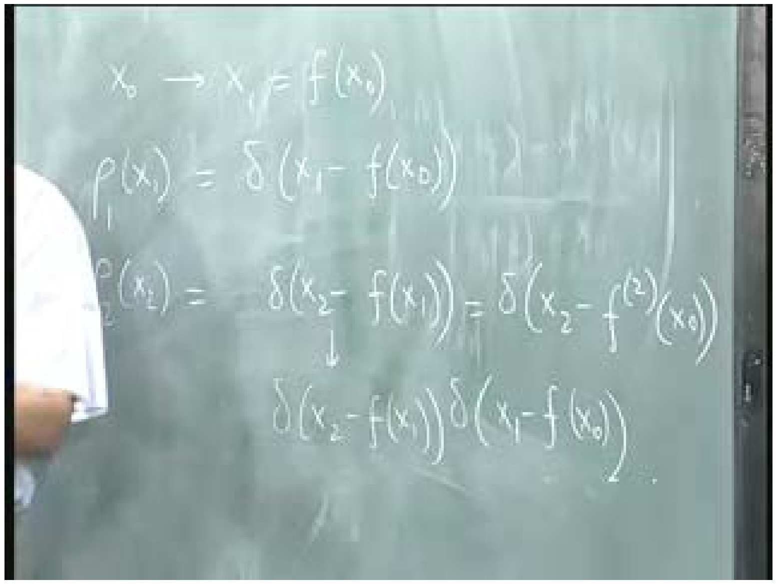

Well I argue as follows, suppose I start with a point x not then under the map this goes to an x one equal to f of x naught. Therefore I can ask what is the probability density of the point x 1 after 1 iteration give me x naught, I give you a x 1. But now ask what is the probability density of x 1; of course since that's a definite sharp x 1 the answer is that it is a delta function at which point, in terms of x naught yes x naught. So, the probability density at the end of 1 times step of reaching a point x 1 is equal to delta of x 1 minus f of x naught. Do you agree?

I could have reached any x 1, but I reached only that x 1, which correspond f of x naught, because I give you x naught, what is rho of x 2 I put this here to show that it is after n 2 times that is. What is this equal to this is equal to a delta of x 2 minus f of x 1 of course provided x 1 is given but if x 1 itself is uncertain and it was x naught that was given. So, this is equal to delta of x 2 minus f 2 x naught you could also write it us delta of x 2 minus you could also write this as delta of x 2 minus f of x 2 delta of x 1 minus f x naught. Would this be correct, should I write it like this, is this correct? I have to integrate, I have to integrate over what over some interval, over what variable x 1.

Because, it says starting with an x naught i go to an x 2 through in intermediate point x 1 and I have to integrate overall possible x. Once but the system this delta function will pick out that x 1, which is equal to f of x naught. So, I have to integrate over the intermediate tie, so I could write it like this or like that or like this, everything will depend on x naught where end up and I am asking for a density in x 2. So, there should not be any x 1, in there, it is just gets in integrated over, so what is the moral of the story.

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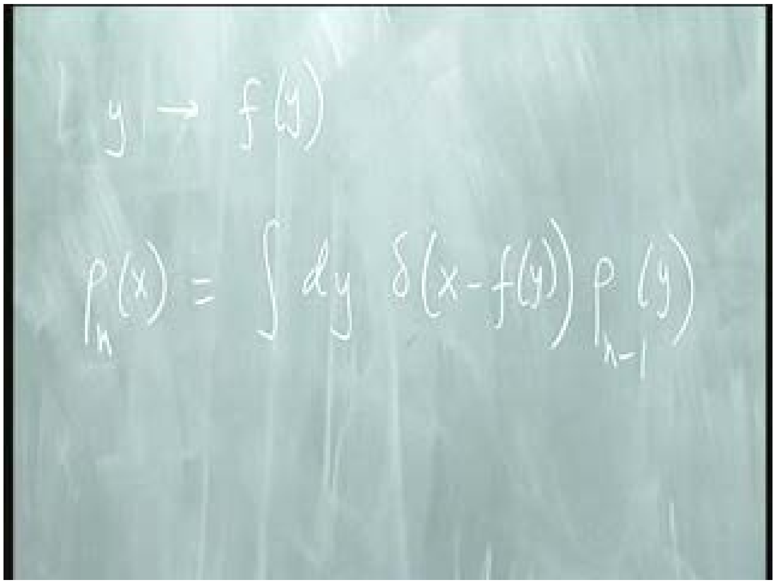

If you give me a point y under the iteration it goes to f of y, therefore if I start if I ask after n times steps. What is the probability density in x this is equal to an integral over all y's delta function of x minus f of y multiply by rho n minus 1 of y. So, it is say I start with some initial density x not itself is arbitrarily define some density it goes to a whole spread of points x 1, every x naught goes to f of x naught a spread of x naught goes to a spread of x, once by this rule.

And says you give me the density at the n minus 1 times steps if that is rho n minus 1 of y each of those y's goes to a particular x by this singular kernel here. And if I integrate over all y's I end up with this that density at the nth time step in x that's my dynamical equation. It is now an evolution equation for probability densities not for the variable themselves. So, we gone over to a statistical picture already okay, is this equation clear; its equivalent to saying rho 1 of x 1 is delta of x 1 minus f of x naught provided I have a sharp x naught.

But, if I have a density in x naught rho naught of x naught, then I must integrate over all possible x naught with that weight factor to produce the density in x 1. And similarly for 2 with from 1 and 3, 4, 2 and so on. So this is what the equations that is the evolution equation for the density. We do not know, we do not know rho not, you give me any rho not I tell you what rho n is by this prescription. So, what is happen is the dynamics now, has gone into statistics some sense by saying if you do not know your initial condition but, you know what it is probability distribution is I will tell you the probability distribution of the nth iterate.

Now, what happens if we take the limit n goes to infinitive here well rho n minus 1 is not distinguishable from rho n, when n goes to infinitive whether its n minus 1 or not does not matter.

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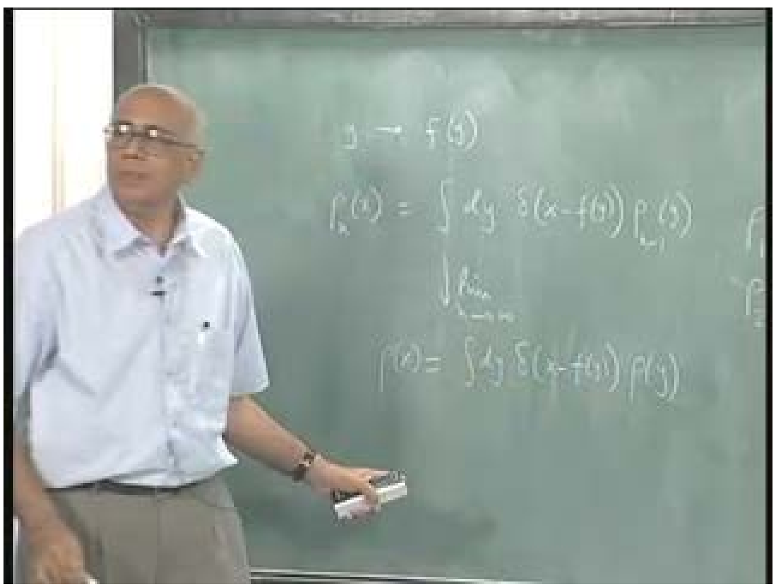

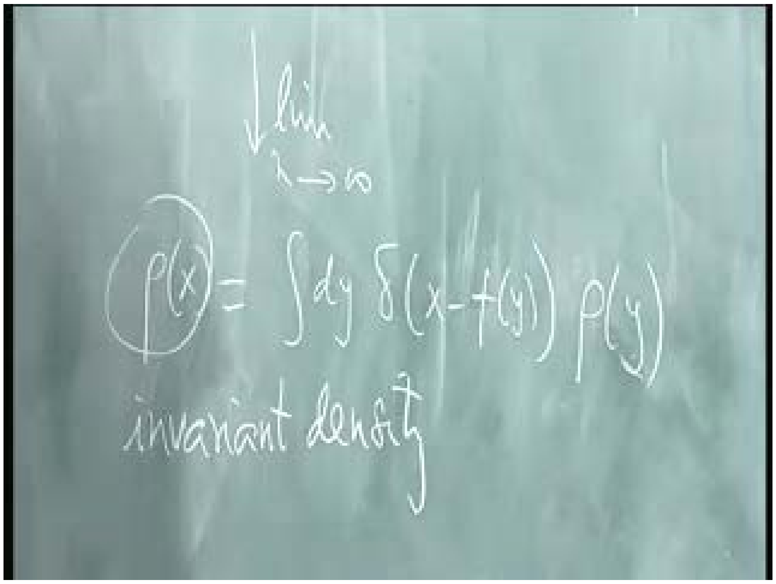

So, it is clear that if you take the limit n goes to infinity here when if there is a limit if there is a limit if this function has a limit that limit could satisfy rho of x equal to d y delta of x minus f of y rho y this rho and that rho are the same. If the limit exist, it is in this sense that the max value distribution of velocities in this room is in invariant density, each of these particle is colliding. But, if the distributed according to the maximal in distribution even after the collision is still I am maximal in distribution.

So, even after the dynamics taken place which is buried here, you still have the same distribution that is the meaning of equilibrium. And since does not this distribution does not change under iteration it is called the invariant measure, this is called the invariant density. What sort of equation is this is it differential equation for this rho of phi, is it an integral equation. Yes it is an integral equation it is not a differential equation, it is an integral equation. It says you have to find rho phi this is an unknown quantity you are give an f of 1 given this kernel.

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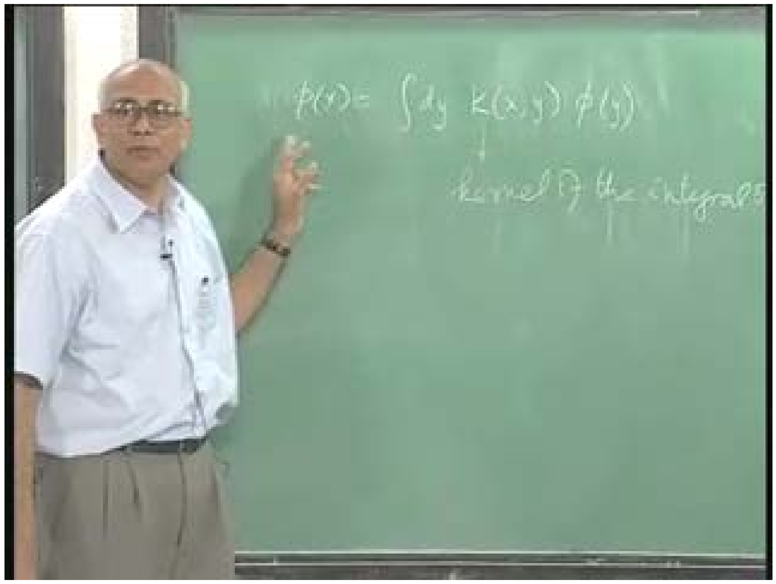

And this is a an example of an integral equation, which looks like this, if you have a phi of x given by some integral d y kernel k of x y phi of y this is called the integral equation an example of an integral equation. And this quantity k is a function of x and y is called the kernel of the integral operator is this a homogenous equation or n homogenous equation? Homogeneous.

If you multiply rho of y by constant it cancels out on both sides, so it is certainly a homogeneous equation is it an Eigen value equation, it is an Eigen value equation? Yes this is some operator acting on phi of y and it produces the same phi, what is the Eigen value 1. So, it is actually an integral equation homogeneous integral equation with Eigen value one of this kernel k of x comma y is this of that type, is this equation of that type? Yes indeed it is. But, is there something special about the kernel it is a bad kernel, it is a delta function it is a singular term.

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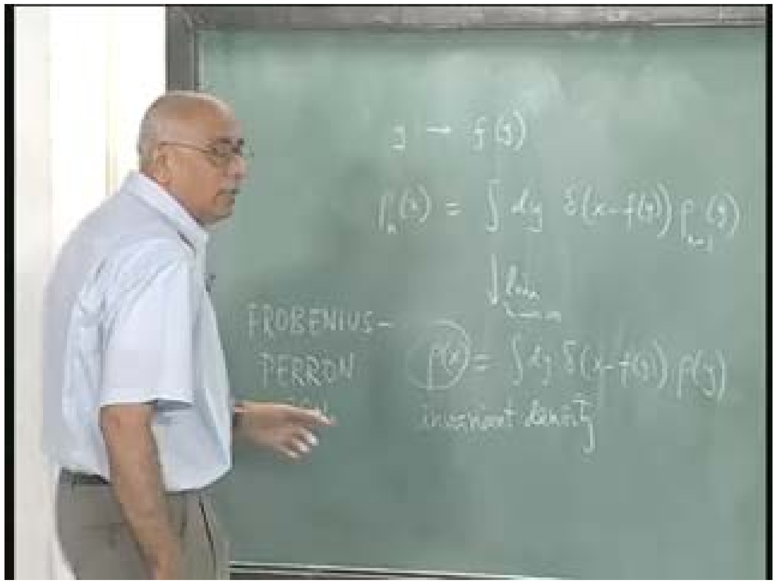

So, you see you have to this equation but, it is a singular integral equation the kernel is singular this equation as a name it is called the Frobenius. It is called the Frobenius Perron equation and it is a singular integral equation.

So, the matter is no so trivial is you will have to deal with some mathematical complexities in you have to solve these equations. And of course if the x variable is in many dimensions then you have to solve a multidimensional equation it is not a trivial problem at all. But, it is a homogeneous integral equation and you would like to find that Eigen function, which as Eigen value plus 1, is there any other requirement on rho. However you going to find this because after all I can multiply this rho by constant and I do not change anything.

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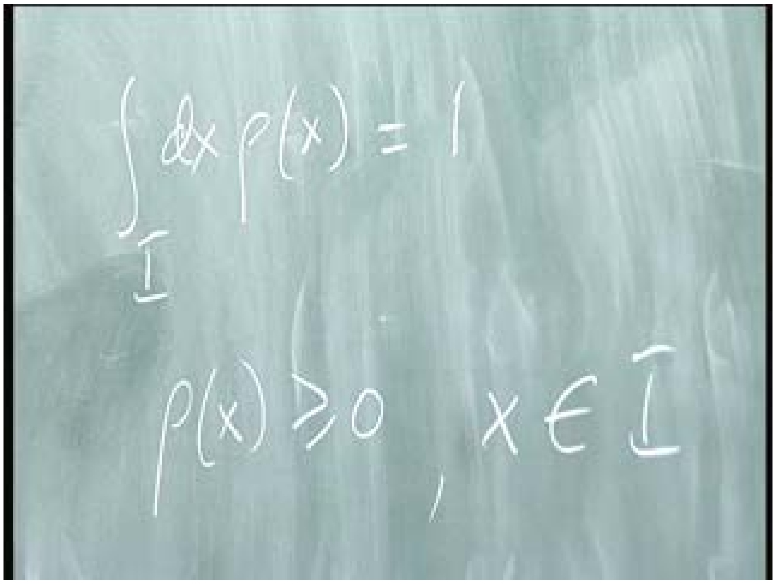

I normalize I need to normalize of course what is the normalization condition on this rho, over the interval I. This could be the unit interval or whatever interval you are 0 to infinity minus infinity to infinity we do not care. Whatever is a range of this map over that interval if you integrate this variant density you should get 1; that is the normalization condition.

So, is there any other requirement, it cannot be negative rho of x is a probability density. So its cannot be negative it could be 0 or some points, we do not care can it be infinite, yes why not why not, but it has to be an integrable infinity. You still must satisfy that condition the total area under the curve must be 1 the function maybe unbounded.

So, you still have to put the condition rho of x greater than equal to 0 per all x in this integral could be unbound could well be unbounded you are familiar with unbounded densities this no reason why take a pendulum clock. And it is swinging back and forth in this fashion and I ask what is the measure in variant density of the angle of the angle and amplitude, what to do you say.

I go to a clock shop in the old variety not the modern day clocks also take a photograph of all the clocks they have on display. Let assume all fellows got an display all are pendulum clock you invariably notice that most of the pendulum’s are either like this or like that very few of them are like this, why is that? Because out here the velocity is 0, it is about to turn it about to turn here it is spends more time there, then it done exist through here and then it as (()).

So, you take a random snapshot most of them are going to be like this or like that, so it is immediately clear that if you are going to take snapshot at random intervals of time uniformly over a time period.

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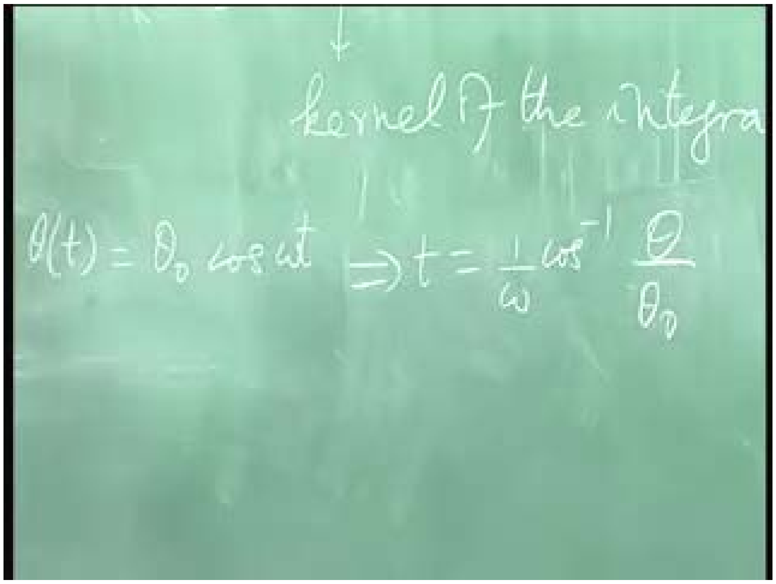

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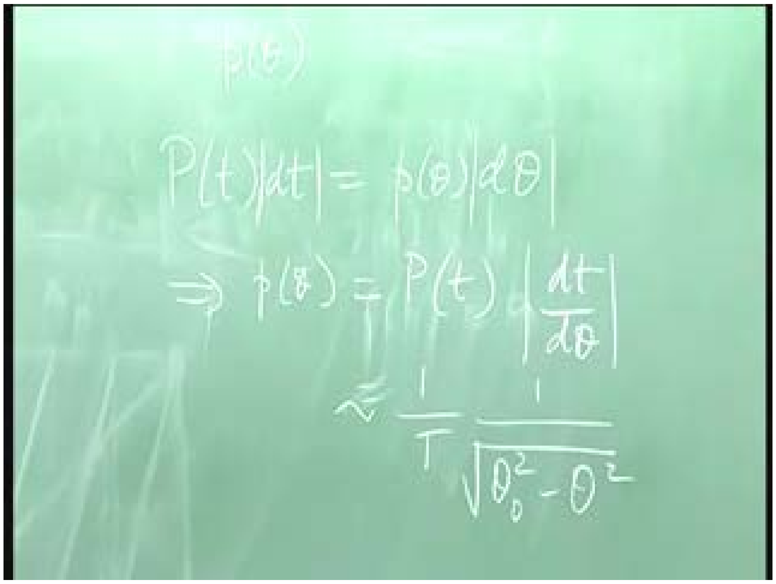

Then the angle of distribution is no longer going to be uniform it is going to be predominantly like this or like that at easy to find out, because if the solution is something like theta of t equal to theta naught cos omega t, if that what oscillator looks like the simple harmonic motion for example.

this could imply that omega t equal to theta of t then just call it theta, theta over theta not um implies t equal to 1 over omega cos inverse theta over theta naught. Then of course you know that if the probability density in theta is p of theta.

And the probability density in time is just uniform, then you know the p of t d t must be equal to p of theta d theta. If the time between time t and t plus d t the angular amplitude is between theta and d theta probabilities must be conserved on both sides. So, this is suddenly two need to have these types positives, so you must have at sitting there. And therefore this implies that p of theta equal to p of t modulus d t over d theta modulus.

But, p of t is equal to one over capital T the time period its uniform in a time period and keeps repeating. Now, just some constant and then d t over d theta its not to find, because you know that d theta over d t equal to theta naught omega sin omega t and I take the modulus of this type therefore this no assign and I inverted and I write it in terms of theta.

So, what is this give you d theta over d t it is just sign, so what does that give you d t over d theta is your sign in the denominator but, I am write it as a function of theta. So, what it give you goes like this some constant we do not care, we normalizing 1 over square root theta naught square minus theta square. So, the probability distribution in the angle angle is not uniform its in fact predominantly towards theta equal to theta naught, because its becomes infinitive those points.

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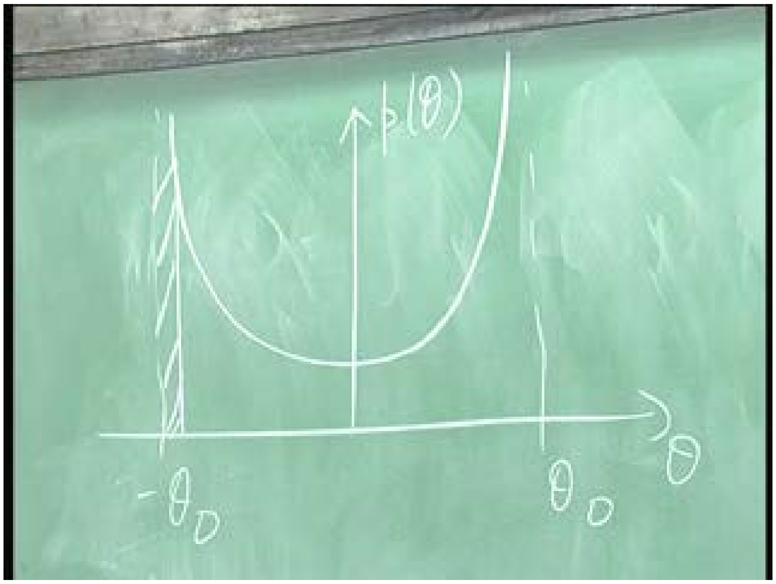

What this graph look like, it is infinite those points the symmetric function you blose up at the two end points its actually infinite but, the area under the curve is guaranty to be unity. So, probability densities can become infinite become unbounded we do not care but, it is a square root inverse square root similarity can be integrated give you 1.

And now you see of course immediately that this is the probability that you are in a small range the theta in the left end under the amplitude and similarly for the right end. And of course that much, much bigger than this, so that is why you do not see, it ever in the middle but, you see it here predominantly. So, remember if you have two variables let to be each other and one of them is here got a uniform distribution, there is no guaranty the other one as a uniform distribution at all; you could have a bias and that exactly what happens.

So, all you need is that this type be positive, now let us apply this immediately it our problem of the Bernoulli shift, you can write it down. So, this is equal to for the Bernoulli shift, this is equal to integral 0 to 1 d y delta function of x minus f of y but, what is f of y it is 2 y modulo 1 right but, that so that not going to be so trivial.

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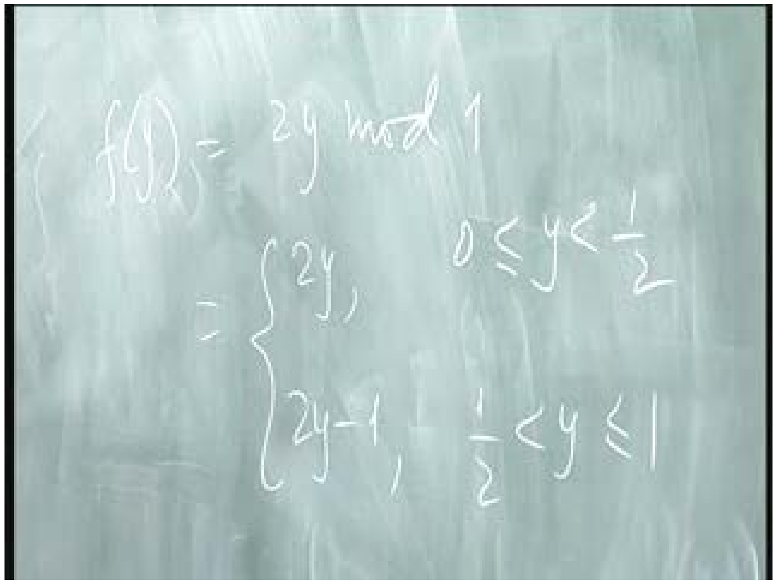

This is f of y equal to 2 y modulo 1 that is equal to 2 y for 0 less than y less than half and it is equal to 2 y minus 1 half less than y less than equal to 1, because two different functions all together.

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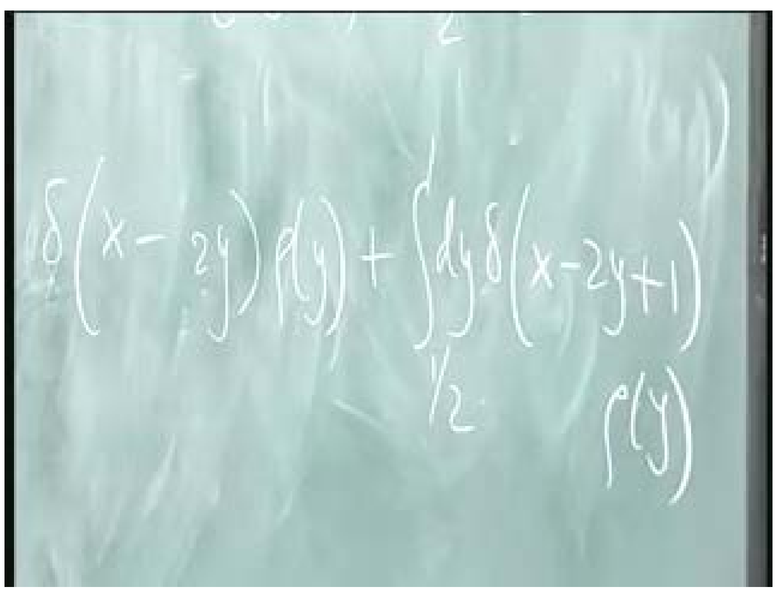

And you have to use that fact here and therefore I have no choice but to write it us 0 to half delta x minus 2 y rho of y plus integral half to 1 d y delta of x minus 2 y plus 1 rho of y choice but, do this but I can I do this integral using the delta function. Yes I can but I have to use the fact that I am it write it as a delta function of y minus something only then can I do it.

So, I need a formula which says what should I do if I got a delta function in a function of y I have to convert it to a function of y minus whatever and what formula would you use you need to use the fact that delta of a x is 1 over modulus a delta of x and it is a symmetric function. So, I could write this as delta of y minus x over 2 but, I have to pull out the 2 and write it in the denominator and it is see do this. And I have do the same thing here, so this is equal to y minus x plus 1 over 2 rho y and a half, then the next step is I can do this integral provided I have provided the delta functions phi what you see.

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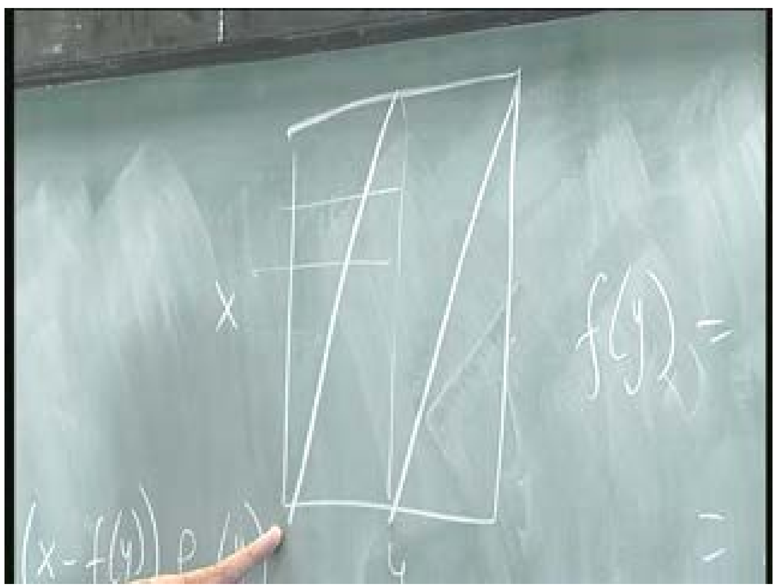

If I go back to the map function this was the first branch the left branch and this is the right branch and it this function that I have written here. And this function that I written here, this place and the idea is that for every x between 0 and 1 any x, I put there is a certain y in the delta function contributes as y runs from zero to y. Therefore I can straight away do this in detail and write this as equal to 1 half rho of x over 2, wherever y appears is write x over q. Similarly for every y between 0 and 1 half and 1 there exist and x the map function fires, so I am guarantee that I can use the delta function and write it as rho of x plus 1 over 2.

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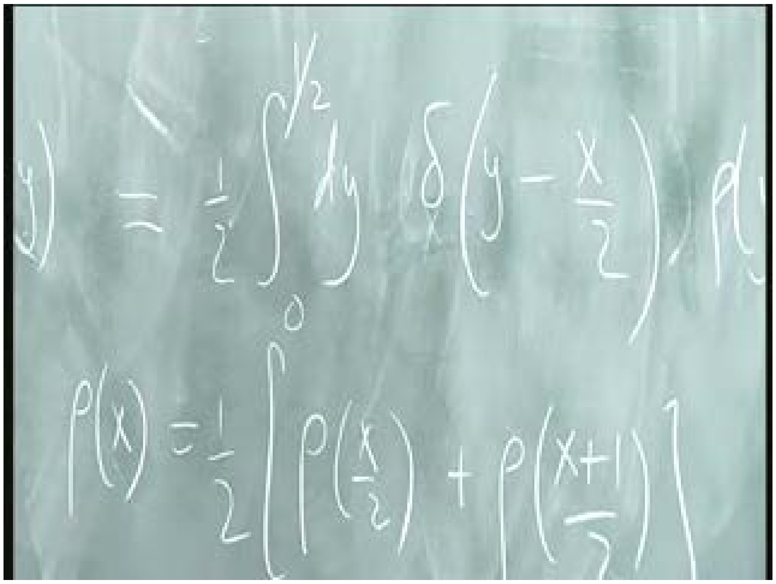

That is my Frobenius Perron equation rho of x is equal to this is this an integral equation, it is no integral, is it a differential equation. It is a functional equation it says rho at any argument x is the average is the arithmetic mean of rho at half that argument plus 1 plus that argument over 2. It is not easy to solve there are no algorithm to solve functional equations at all.

Now, you have to take it from me as an article of faith that this is an extreme this is an (()) problem, that if you have a Frobenius Perron equation of this kind, this is now become a functional equation. And you impose the condition of non negativity and the fact that is integrable, normalizable then, there are theorem which tell you and suitable conditions which apply here that a solution if it is exist is unique you (()). And therefore once you know a solution if it is exist is unique you can make guesses.

What will be the simplest guess is a constant and this things has to be normalize to unity between 0 and 1. So, what should the constant b 1 yeah 1. Suppose I put rho of x equal to 1, what happens and therefore guarantee that is the solution and moreover it’s a unique solution not this not always going to happen but, in this case it happens. But, you need that very powerful theorem, which is not easy to establish and then it turns out that the solution is in fact a constant density.

This is why a certain that the iterates of irrational points on the Bernoulli shift could uniformly and density fill up the unity integral. Of course with infinite number of exceptions all the rational points but, they are a set of measures 0. But, the dense you need them you need those unstable orbits to skew this trajectories on either side of each of these unstable orbits. And produce this kaoyas but, having done that the job of those points is over the rest of the point should uniformly and density fill up this entire unit interval.

The unit interval is called the attract or not the whole interval is become the attract completely not always true but, in this particular problem terms out true. So, indeed we can now, now that we know this they can compute the Lyapunov exponent.

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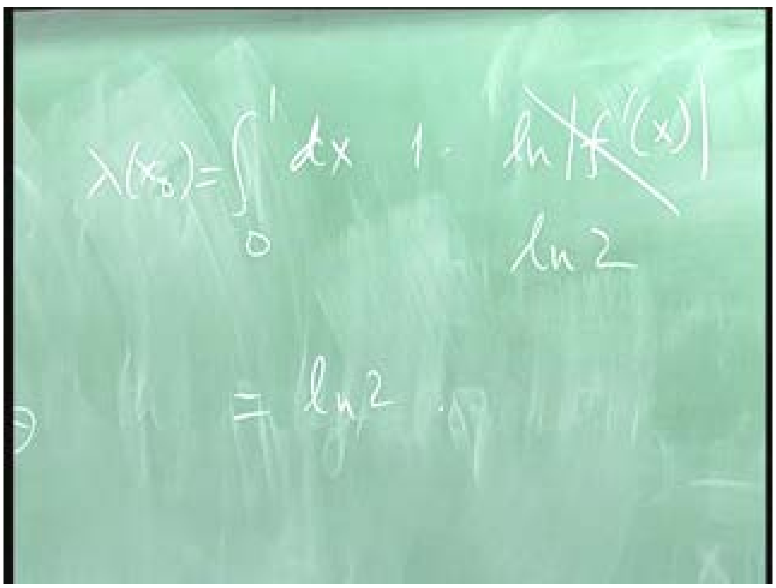

Lambda is equal to of any x not which is irrational 0 to 1 d x rho of x, which is 1 times log mod f prime of x but, that is log 2 therefore its equal to log 2. So, do not have to go through this long time series, we can apply a formula is in this rho of x axis, what's the value of log 2 point. Whatever it is, 693 693 log 2 0.693 okay. So, it is possible that is it what you need. Suppose, I not done the Bernoulli shift but I had done f of x equal to 3 x modulo 1, what could happen we have log 3 it is Lyapunov exponent right.

Again all points which are rational could be parts of periodic orbits, if you write it us ternary decimals in an alphabet containing 0 1 and 2. Then all decimal points, which could all numbers whose decimal expansion who either the terminate or recure would lie on periodic orbits. And all irrational points which are really a set of measure one would become kaotic, completely kaotic.

So, these Bernoulli shifts are very, very simple examples which you can see they already have fairly complicated amount of mathematics in them. Because if you ask what are all the possible Eigen values of this operator you can put a lambda here do not call it rho of x, call it phi of x then it is very intricate, completely very intricate. If you relax those conditions of positive it non negativity and normalizability, then you can get extremely complicated. You may not even a continuous functions you may have factor functions and so on.

So, the full set of Eigen values are highly non trivial but we are looking for a very, very simple Eigen function, which is unit Eigen value, largest Eigen value and it is non-negative normalizable the answer is its just the constant it is easy in this case. Now, what are do tomorrow morning is to show you that this map is actually dissipated but, we would like to mimic Hamiltonian dynamics, which is conservative measure preserving.

So, we will change this little bit, I give you a similar kind of shift slightly more complicated two dimensional shift, which would in fact de-measure preserve we look at that then this is lead as in another couple of lectures to statistical physics.