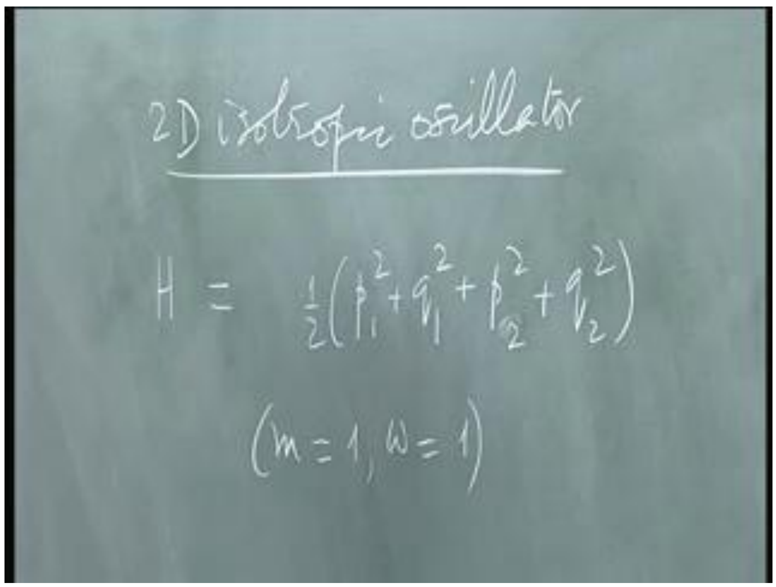
Classical Physics
Prof. V. Balakrishnan
Department of Physics
Indian Institute of Technology, Madras

Lecture No. # 14

Let us begin, where we left off last time, and I would like to do a couple of examples today to put things in proper frame work.

(Refer Slide Time: 01:18)



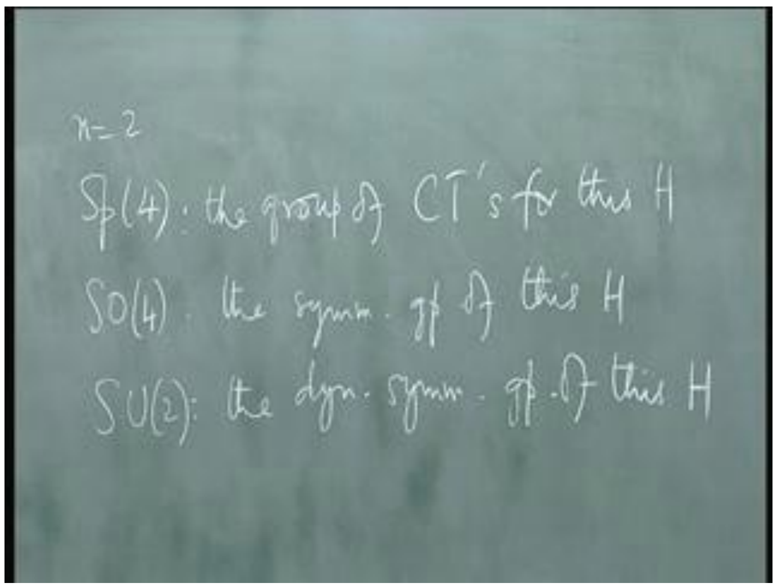
Recall that you looked at the 2D isotropic oscillator, let us quickly recover; what we capitulate what we had here, this H was equal to one-half p1 square plus q 1 square plus p 2 square plus q 2 square. In units in which the mass is equal to 1 the frequency equal to 1 choosing those units and this was our Hamiltonian.

Now, this problem is integrable the reason being that a constant f 1 which is essentially p 1 squared plus q 2 squared and another constant of the motion f 2, p 2 squared plus q 2 squared. I mean involution with each other it is a 2 degree of freedom system and therefore, the problem is completely integrable. We could go to action angle variables and so on. And we also discovered that since the frequencies are the same the motion is periodic.

Period one this case or whatever, and we know that the motion lies on a torus and it is not a geodic on the torus because, the trajectory is in fact, just a close curve simple close curve on this torus. Now, the symmetry of this Hamiltonian system the dynamical symmetry I pointed out would be that set of transformations of the four phases space variables which would live the equations of motions unchanged. And which would therefore, say take the solution set to the solution set it.

That symmetry involved a certain set of generators it is a symmetry group of transformations.

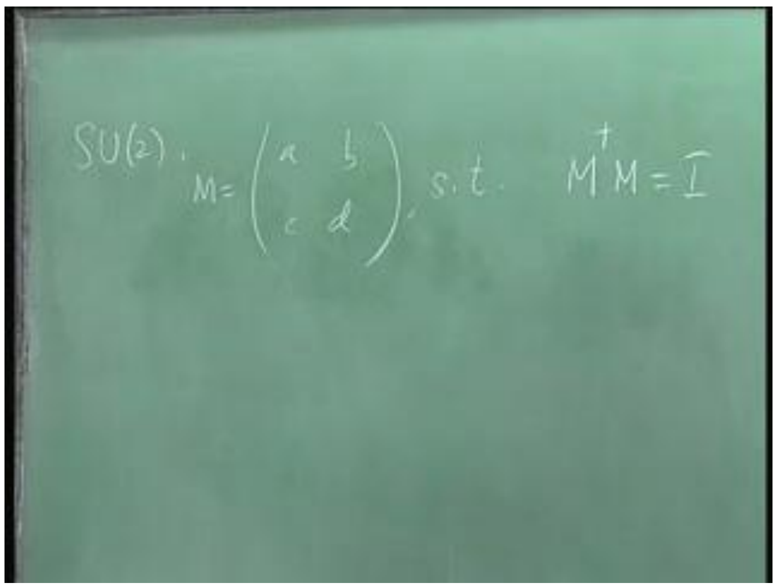
(Refer Slide Time: 03:07)



And the statement we made was that since this is a two degree of freedom system n equal to 2. The symplectic group Sp of 4 the symplectic group of 4 by 4 matrices is isomorphic; to the group of canonical transformations of this Hamiltonian. And the intersection of this group or rather the sub group of this group which leaves the equation the Hamiltonian unchanged is. In fact, the sub group the group that leaves this form unchanged and that is the set of rotations in four dimensional spaces.

Thus there are four variables. So, while Sp of 4 is the group of CT's for this Hamiltonian, SO of 4 the symmetry group of this Hamiltonian. And the dynamical symmetry group of this problem is a sub group of the symplectic group which leaves which which canon a group of the (( )) canonical transformations which also leave the Hamiltonian unchanged. And that group I said was SU of 2 the dynamical symmetry of this Hamiltonian. And this symmetry group the abstract group is a group of 2 by 2 matrices which are unitary with complex entries, and which have unit modules.

(Refer Slide Time: 04:56)



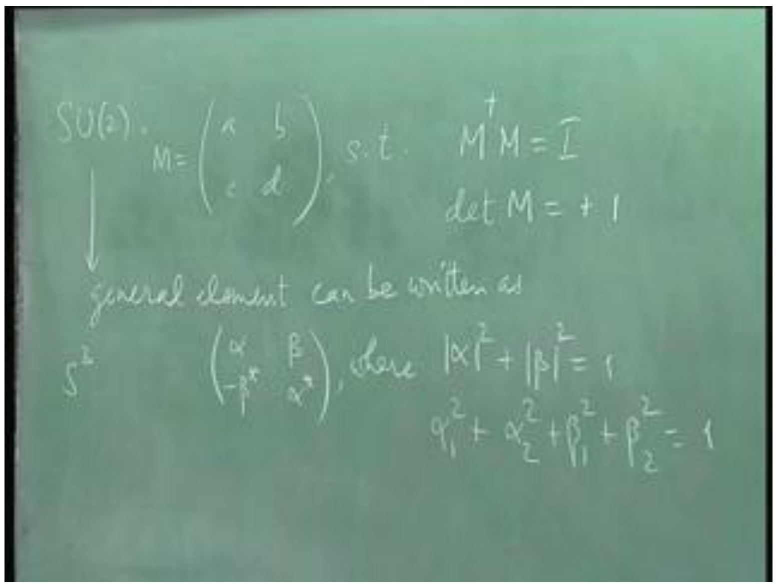
Now, let me say few words about SU 2 separately here. SU 2 it consists of all 2 by 2 matrices a b c d, such that. So, this is a metrics M such that M dagger M equal to I that is, what is meant by unitary all matrices which satisfied M dagger M equal to 1.

And the S for the SU 2 stands for the fact the determinant M equal to plus 1. If you, put in these conditions then you get the general group belong general metrics belonging to this group SU 2. Now, what we have to do is to do some counting of parameters how many parameters are there real parameters are there in a general 2 by 2 metrics eight of them because each entry can be a complex number. So, there are eight parameters.

Now, we have to reduce that once you say the metrics is unitary it is a certain set of conditions and you have to impose all those conditions and what ,how many conditions would there be, by the way this thing says M dagger equal to M inverse. So, you have to find the inverse and then you have to put the determinant equal to 1 and then you start imposing these conditions.

I mean you do that it turns out that 4 conditions obtain the moment you say M dagger M equal to I that gives four conditions. One for each element out here and therefore, the number of parameters is reduced from eight to four and then you say the determinant must be plus 1.

(Refer Slide Time: 06:50)



So, one more parameter is gone and the number of independent parameters is just 3. I leave it as an exercise to you to show that the general element general element can be written as some number alpha beta, minus beta star alpha star.

So, you can write a general number of this group of 2 by 2 matrices that SU 2 SU 2 groups in the form of some complex number alpha some complex number beta, and then minus beta star and alpha star such that mode alpha squared plus mode beta squared equal to one you can always do that. Now, what sort of parameters space do you have you can ask what is the parameter space of this group. The space of this group is all numbers alpha and beta such that mode alpha squared plus mode beta squared is 1 and of course, you could also say that if alpha is equal to alpha 1 plus. I alpha 2 and beta as beta 1 plus I beta 2 then this can be written as alpha 1 squared plus alpha 2 squared equal to 1.

So, you have 4 real numbers alpha 1 to beta 2 such that the some of the squares of these 4 real numbers equal to 1. What kind of object is that in this four dimensional space it is a surface of a units sphere in the 4 dimensional spaces. So, one says that the parameter space of SU 2 the parameter space is a surface of a sphere and 4 dimensional space it is denoted by S 3, a sphere of 3 dimensional sphere S 3, S 1 is just a circle S 2 is what you have used to as a surface of a balloon in 3 dimensions a 2 dimensional sphere a 3 dimensional sphere is a surface of a unit sphere embedded in four Euclidean dimensions and so on.

So, this provides a great deal of simplification you would not get into that right now, but I want to point out that this is a symmetry group of this Hamiltonian, it is non trivial now you could ask every group is generated by some generators generator of transformations.

What generates this group here it turns out that the constants of the motion; I wrote down earlier namely J 1 is 1 quarter. Q 1 squared plus p 1 squared minus q 2 squared minus p 2 squared J 2 equal to one- half q 1 q 2 plus p 1 p 2 and j 3 is one-half q 1 p 2, these 3 quantities and these 3 functions of the face space variables are constants of the motion, but they are not in involution with each other and they turn out to be the generator, of the group SU 2, the of the dynamical symmetry group, you need 3 parameters for that group and there are 3 generators. So, these turn out to be the generators. Later if time permits in this course, I will talk a little bit about generators of Li groups and. So, on but right now I want to go ahead and incidentally verify that J i J j is equal to epsilon ijk so, they are not in the involution.

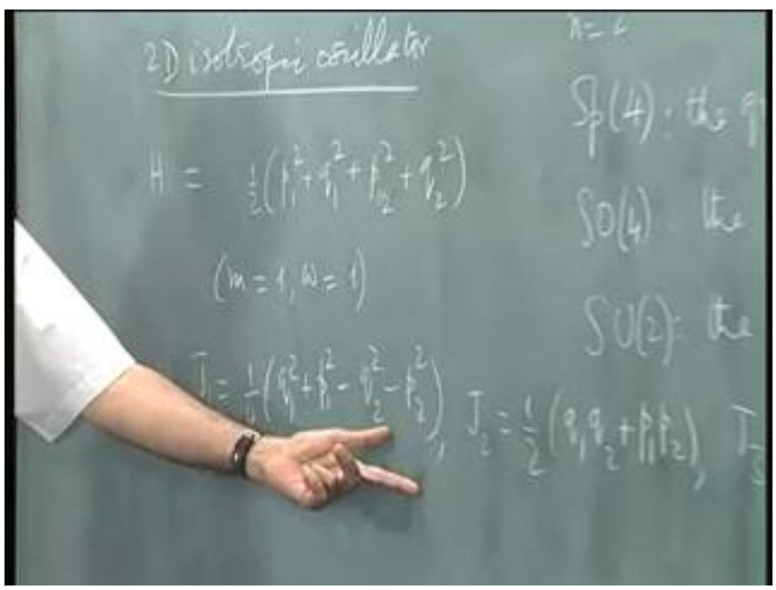
Of course, this means the pasaon bracket of J 1 with J 2 with J 3 and then cyclic permutation. and they are not all independent because, in this problem how many independent constants of the motion can you have which are time independent constants of the motions, well the face space is four dimensional so, you could add best have three at best you could have 3, but it is integrable therefore you really have 2 constants of the motion independent which are in involution with each other.

But you can have more constants of the motion you can have a large number you have f one f 2 this is f 1 minus f 2. So, it is not independent then you have f 3 and then you have 4. So, you really have a fairly large number 4 constants of the motion all of them are time independent, but they can not all be independent of each other.

So, discovering this is not a trivial task it is trivial in this problem, but it is not true in in general finding what the symmetry group of a Hamiltonian system is finding how to integrated how how to find the symmetry group is non-trivial.

So, let us go from here to a real problem this was a 2 dimensional loss ladder which would be a model in many physical situation, but let us go to a real problem the problem of a planet moving around the sun the Kepler problem, and let us see what happens there. I should mention by the way just as a matter of mention the three dimensional isotropic oscillator in three D would mean this.

(Refer Slide Time: 09:40)



What would be this replaced by Sp of 6 what would this be replace by so, 6 because 3 or 2 of guys and you could ask what is this going to be replaced by it turns out it is replaced by 3. How many generators would SU 3 have?

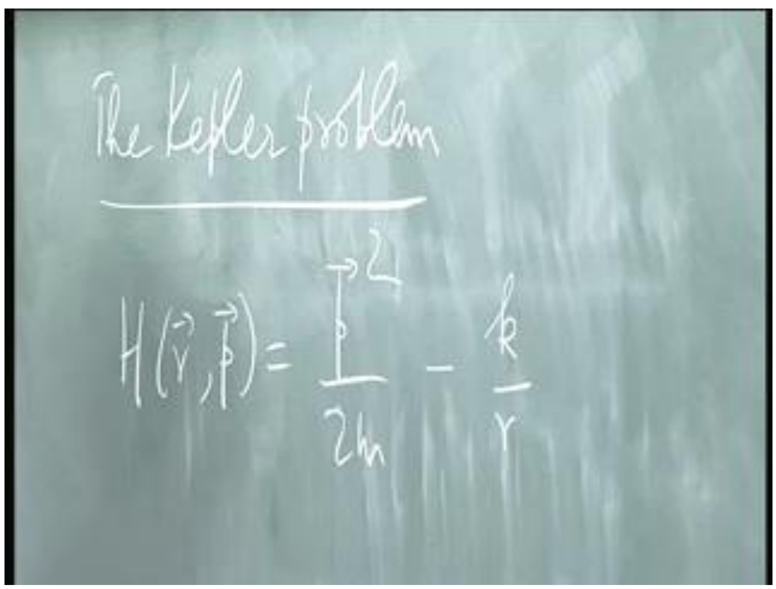
(( ))

No. So, what you have to ask is how many parameters independent parameters does not n by n unitary metrics have and then you must subtract 1 from it for the determinant equal to 1 and it is n squared. Un has n squared generators the number of elements is 2 n squared because , there are n squared elements each of them is complex for there are 2 n squared, but the number of independent parameters is only n squared for un the unitary matrices in n by n matrices.

And then when you put the S condition is the determinant equal to 1 it becomes n squared minus 1. So, 3 squared minus 1 are 8 therefore, SU of 3 has 8 generators. So, there are eight of these quantities you have to discover eight combinations not these, but there are eight of these combinations which would the invariant, which would be constants of the motion satisfying a certain algebra namely the SU 3 algebra, so that is not our immediate concern.

Yes.

(Refer Slide Time: 14:40)



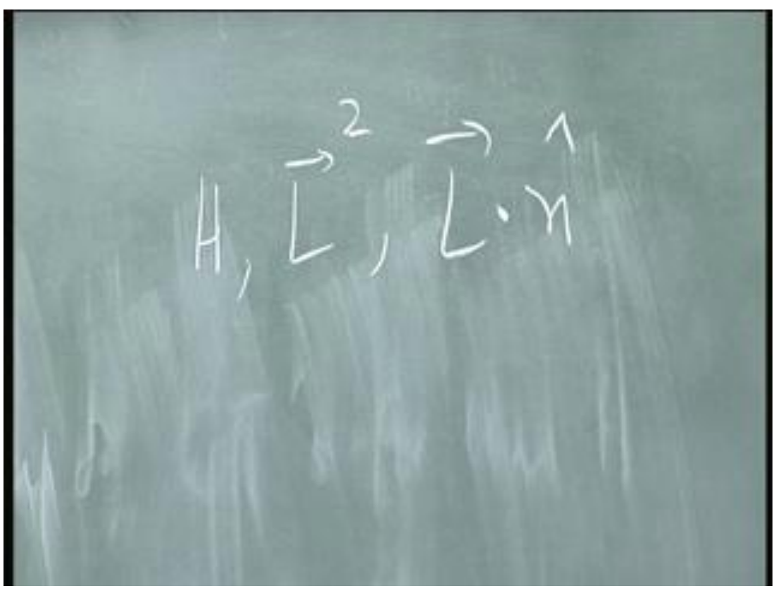
Those is not our immediate concern, but let us look at the Kepler problem or the **(( ))** potential once again the Hamiltonian as a function of r and p is just p squared over 2 m that is the square of the kinetic n. That is the kinetic energy the square of the momentum over 2 m minus let us assume that it is an attractive potential although it does not matter could be attractive or repulsive minus some constant with appropriate physical dimensions over r in spherical polar coordinates a 1.

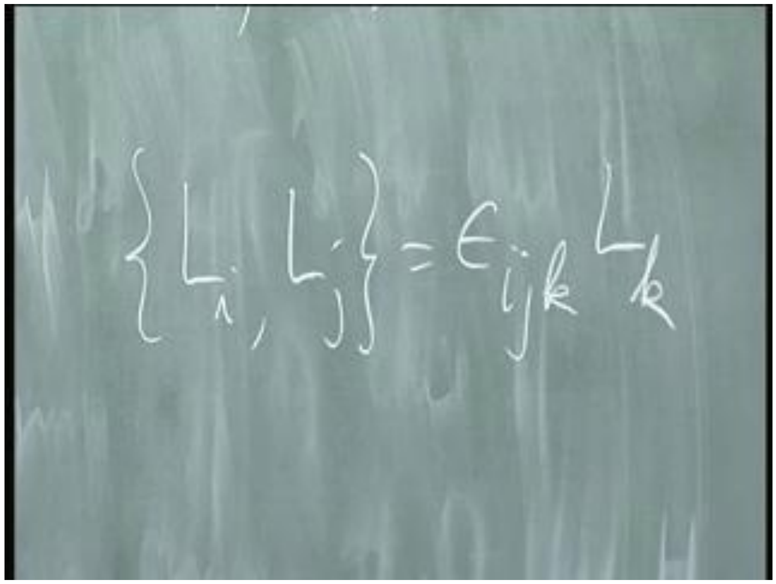
Over, r potential leads to a 1 over r squared force once you differentiate it. So, this is the central force here now, we know in practice that the orbits as Newton proved are actually ellipsis. But, there are other orbits to this potential you could also have a hyperbolic orbit you could have a parabolic orbit. In fact, they are conic sections orbit. And the motion in an elliptic all all bounded orbits are actually closed curves to their periodic motion; that is very special to this potential.

The moment I change this one over r and make it is a one over r squared or one over r cubed or something like that this property is gone it is immediately lost the only central potentials for which all the bounded orbits are closed curves are periodic orbits. The only central potentials are, in fact, the 1 over r potential and the r squared potential corresponding to the isotropic 3 dimensional oscillators these are the only 2 potentials.

The oscillator has a symmetry group SU of 3. We just saw that the question is what is the symmetry group of this Hamiltonian? Because a 2 is integrable completely, integrable.

(Refer Slide Time: 16:32)





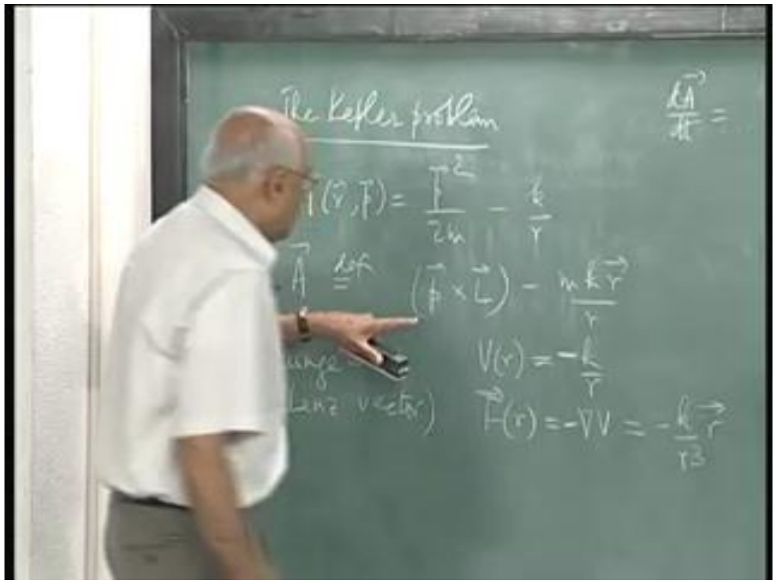
What are the constants of the motions that, one can write the Hamiltonian itself what else? The angular momentum about the centre of attraction about centre of force, because this is the central potential, so certainly this; of course, the moment i say L is a constant of the motion it means any all the 3 components of it in any frame of reference are constants of the motion L 1 L 2 L 3.

Now, is this problem integrable; Yes it is a integrable what are the, what are the three independent constants of the motion in involution with each other which tell you that it is integrable.

So, the 3 independent once are H L squared L dot n any 1 component you choose whatever you like the component of the angel of moment i mean any arbitrary direction the total angel of momentum on squared and H. These are three independent constants of the motion and they are in involution with each other, so it is integrable, but we are actually finding more of them there is L 1 L 2 L 3 here they are not in involution with each other.

And remember the angular momentum algebra L i L j is epsilon ijk are there any other constants of the motion these would s be the constants of the motion even if the potential were not one over r any central potential be of r would have this set of constant of the motion, but there are something very special about the Kepler about this problem. The one over r here and that is the existence of further constants of the motion

(Refer Slide Time: 18:33)



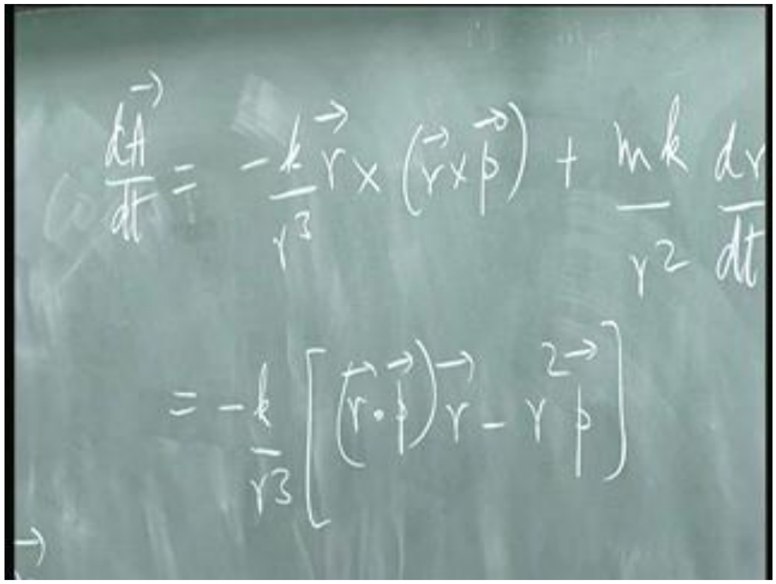
There is another vector quantity A and it is called the Laplace Runge Lenz vector. And it is defined as p cross L minus mk r over r this is the definition.

And the claim is that a is the constant of the motion. You have to verify whether this is true or not and then we interpret what this thing does and where it stands in relation to H and L. It is quite clear that it involves L it involves p the momentum p it involves the coordinate r and so on. But it is a very strange combination of things which is suppose to be constant.

Now, let us verify if this is true, so let us, do d A over d T. I need some space dA over d T Equal to I start differentiating this incidentally if this is the potential V of r is minus minus K over r with a minus sign what is the force on this particle what is that equal to minus the gradient minus del V and that is equal to what ?

Well this minus kills that minus, but then if I differentiate one over r i get another minus sine. So, it is minus K over r squared times the unit vector on the r if i differentiate with respect to r use the formula for the gradient in spherical polar coordinates. Since there is no theta or free dependents the other two thing derivatives do not act and this is it. It is more convenient to write this as minus K over r cubed r vector. So, I do not want to fool around these unit vector; because, I change this from point to point and I forget to differentiate it, so if this r, so what is dA over dT.

(Refer Slide Time: 21:10)



The first term is dp over d t, but d p over d t is equal to the force immediately. So, this is equal to minus k over r cubed r cross P L, but what is L in terms of let us get read of L and write everything in terms of r and t. So, what is L r cross p I have to keep track of this bracket. So, let us do that, and then the next term is p cross d L over d t, but that is 0 because d L over d t is 0 L is a constant of the motion.

And then the next term is minus mk and i differentiate this. So, you get an r squared that becomes a plus d r over d t r and then minus mk over r d r vector over d t. So, this is the full and derivative of a were we have used the fact that the forces even by a one over r squared force we have used the fact that angular momentum this concern and we have put in Newton's equation d p over d t has been written equal to the force.

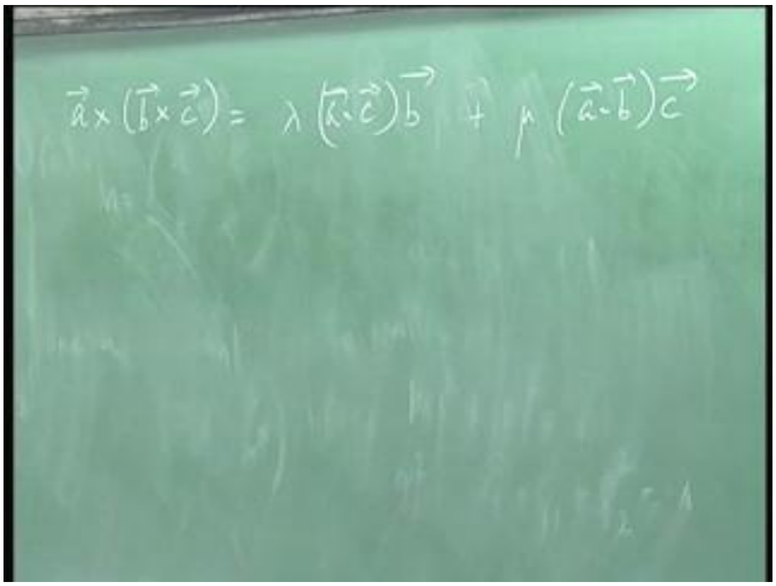
So, on the solution set were all these things are valid this is what the d A over d T is and now we have to start. Simplifying of this term I can simplify right away n dr over dt is just the momentum. So, permit me to remove this m and write this as p in this fashion. And what is this equal to this is minus k over r cubed and let us use the formula for the triple product a cross b cross c what is a cross b cross c?

A dot c a dot c, so that is r dot p a dot c p minus a dot b that is r squared p.

By the way how do i know that a cross b cross c is a dot c b minus?

How do I know this we are on our desert island no information, no information. So, you have to start up fresh. Well, we can always write it in components and do this,

(Refer Slide Time: 23:37)



But let me give you a simple argument here a cross of b cross c equal to now what can it possibly be a b c are three arbitrary vectors in space generally not collinear.

It is got to be in the plane of b and c because, you could use these as oblique coordinates you got to be in the plane of b and c very nice. So, it is equal to some alpha times b plus beta times c. Anything in the plane of b and c can be expanded in terms of b and c linearly. Now, what should alpha and beta be what sort of object should alpha and beta be vector scalars.

Scalars

Scalars formed out of what?

a b and c.

a b and c, but the left hand side is linear in a b c each of the vectors therefore, the right hand side must also be linear therefore, this guy can not involve b. It must be formed from a and c what are the possible vectors you can have scalars you can have from a and c? You can have a dot c you can have a squared you can have c squared, but a squared and c squared are not linear. So, you got have a dot c.

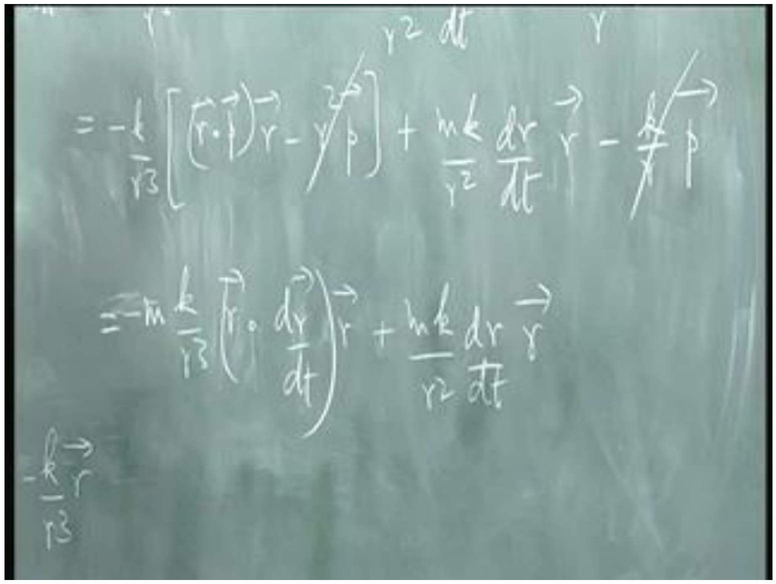
So, this is a dot c a dot c b, but nobody told you it is 1 times a dot c it could be squared of 3 pie times. So, this thing you know lambda times a dot c plus some other constant mu times again a dot b it is got to be this where lambda and mu are numbers real numbers and they cannot depend on a b c. So, they are universal numbers for every a b c there must be the same numbers now what happens.

If you interchange b and c on the left hand side you get a minus sign there for that must be true here too and how can that happen? One of them has to be minus the other. So, this is the only way it can happen is minus that is it.

Now, what you do next you used linearity you used homogeneity and you said the whole thing they are linearly independent in three dimensions or etcetera. So, the only thing you can have is.

Put a special case now put a special case put ijk and then you discover lambda as minus 1, so the job is done. I leave you to find a cross b cross c cross d in the same way doing this, but any ways. So, we have successfully found this formula and we came back here.

Refer Slide Time: 26:28)

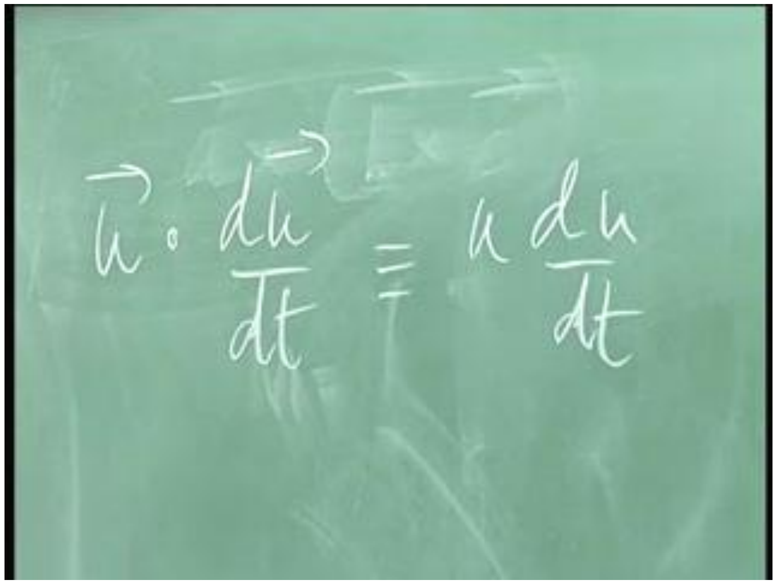


So, you got this plus m k over r squared d r over d t r minus k over r p.

Now, is there any cancellation the second term this becomes plus k over, r p and that is minus k over r p. So, permit me to cancel this term against that term. And then you have this term here is equal to minus k over r cubed r dot p, but that is d r over d t and then there is an m and then r plus m k over r squared d r over d t. What do I do with this?

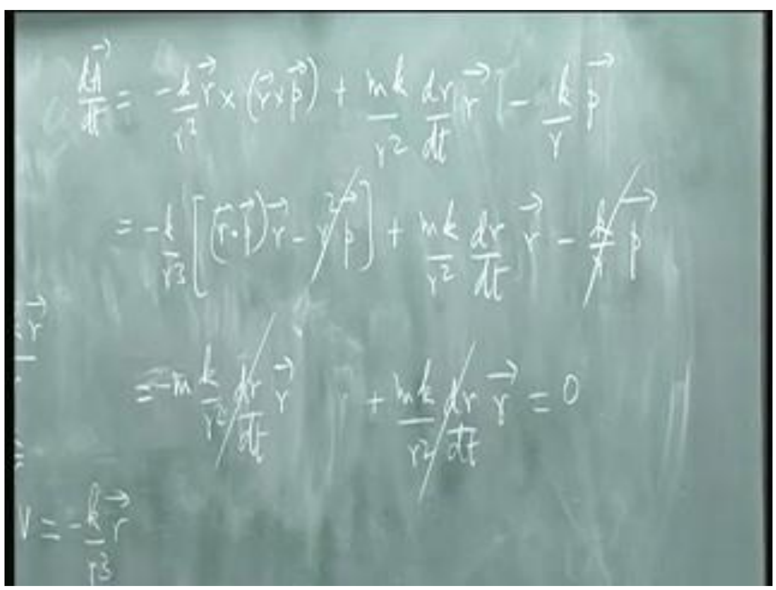
r dot d r over d t that is d over dt of half d over d t of r squared, but r squared you write as little r squared and then it becomes r dr by dt right.

(Refer Slide Time: 27:34)



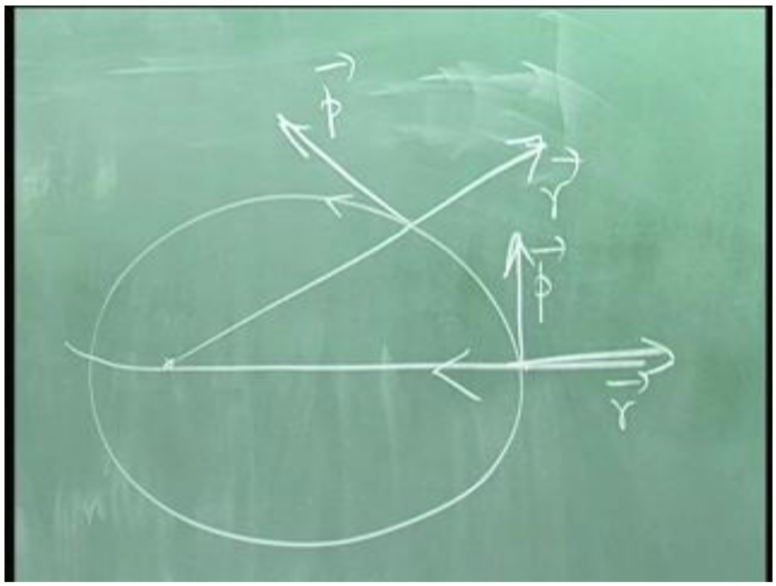
So, you use this well known vector identity u dot du over d t is always equal to u du over d t. Because you can write each side as the half the derivative of either u squared or u dot u very, very useful identity. And then immediately you realize this is equal to r d r over d t and you remove one of the r you get this cancel that cancel equal to 0.

(Refer Slide Time: 27:57)



So, indeed this a is a constant of the motion therefore, it remains unchanged in both direction and magnitude, what is the direction of L. L is perpendicular to the plane of the orbit because r and p define a plane and this plane is the same at all times that is Kepler's second law and then L is perpendicular to it and what is the direction of a it is also unchanged.

(Refer Slide Time: 28:49)



So, what is the direction of a well you can see that what is going to happen is that if this is your orbit going in this fashion the plane of the black board let us say is the plane of the orbit then at any point this is the direction of r this is the direction of p. So, r cross p comes out of the board in fashion and that is the direction of a angular momentum perpendicular to this plane.

Now, this guy here is p cross L there is a further p cross L and then there is an r. So, in which direction is it well you could find this at any point in the orbit it is a same directions at all point places right. So, for instance if you found it here at this point this is r and that is p.

So, r cross p the angular momentum comes out and then you got to do an L cross p and that goes in here and. So, does r with a minus sign that guy also goes in here. So, it is in the direction of the major axis it is along the major axis of this ellipse and now it says it does not change. So, what does that imply for the ellipse what does that imply for this orbit it does not process otherwise the ellipse could process very slowly.

So, you could have a motion of this kind and this happens if you put a little perturbation it would do this it would no longer strictly be an ellipse, but it would process slowly. That does not happen for the Kepler problem with the strict one over r potential, because the lense vector prevents it from doing. So, the constancy of the lense vector and it is in the direction of the semi major axis.

Now, I ask.

p is m dr over dt, so there was an m here you check the dimensionality of these things I have observed all a lots of dimensions in this k here k over r is dimensions of energy.

What is meant by actual motion, how does it behave in space, which I mean the one over r problem or the actual an actual real planet real life planet.

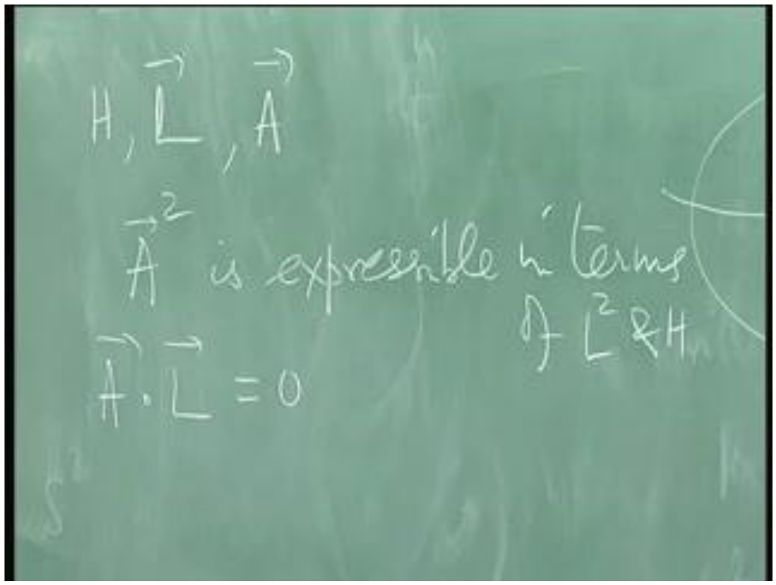
Real planet.

What do you think what do you think a real planet does. Certainly process it certainly process there are perturbation due to other planets there is Jupiter sitting their. Then there is tidal friction there is tidal forces there is a solar wind there are lots and lots of other effects many, many other perturbations and then there is a general relativistic correction to the universe square law of force itself that too will cause a precesation; a very, very tiny one in the case of the earth extremely tiny. In fact, in the solar system the biggest perception is caused in mercury just closest to the sun and that is 43 seconds of arc per century 43 seconds of arc not even a minute not even a degree.

Per century, but mercury it self has an orbit that process of the order of 500 odd seconds of arc per century and then of that 43 is the general relativistic correction very, very tiny of course, there are other systems were this is extremely pronounced. There is a binary pulsar which is like processing 4 degrees per orbit per rotate per revolution. So, this is because of masses involved are ignoramus much much greater.

So, this is the idea behind the lense vector now tell me how many constants of the motion are there, it looks like we have an embarrassment of riches there is an H there is an L and there is an A.

(Refer Slide Time: 32:45)



So, there are seven of these constants of the motion the face space itself is only six dimensional. They are not independent of each other yeah.

The actual direction is along the apilin or something like that it is either out or in, but it is on this axis it is on this axis incidentally, I could have change the sign of k it is true this constant of the motion is nothing to do with the k is positive or negative. So, even if you had a repulsive potential even if you have a hyperbolic orbit there is still a renail vector it is still a lense vector of this kind.

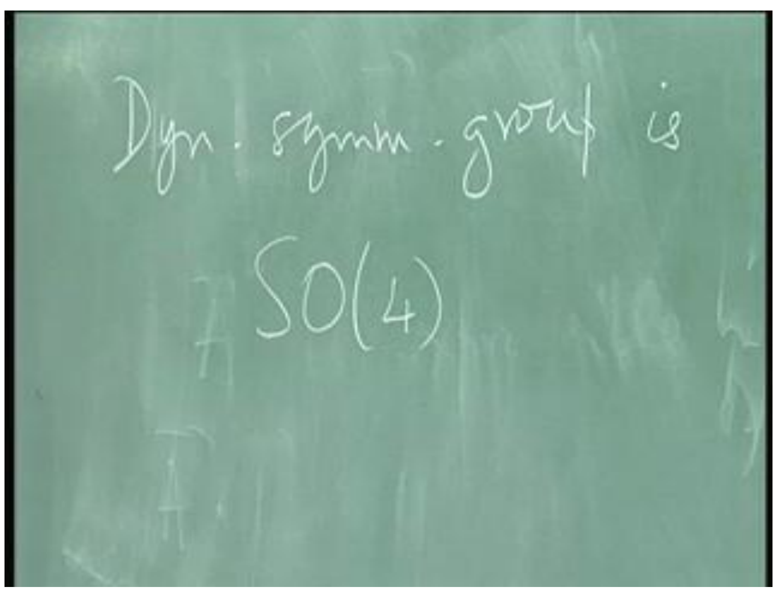
It is always in the on that axis at all times because if it. So, at the 2 extremes of the orbit it is got to be, so at all times because it is constant. So, you can find the direction of A at any convenient point in the orbit and it is guaranty to do the same at all times just like the angular of momentum it is always normal to the orbit normal to the plane of the orbit.

So, you have the seven guys, but they are not independent of each other you can quickly see that A squared can be written in terms of H and L squared. So, once you tell me H and L squared the numerical value of the total angular momentum squared and the energy then I also tell you the magnitude of A moreover you immediately see that A dot L equal to 0 is expressible also A dot L is 0.

So, one component of A is gone anyway the one that is along the angular momentum A lies in the plane of the orbit just as L always lies normal to the plane of the orbit . So, you really do not have that many independent constants of a motion in the next question you can ask is what is the symmetry group of this Hamiltonian. That is harder to discover because unlike the oscillator case were I could identify j 1 j 2 j 3 easily this is harder because this A has a 1 over r term in it and that is not an easy thing to handle this guy is sitting here these are not polynomials or anything like that.

So, the next problem said I am going to give you A and I am also going to tell you how to form combinations of A and L such that you have a simple algebra. It turns out that if you take suitable units and you take linear vector. Six components for A plus or minus L 3 for A plus L 3 for A minus L and combinations of A and L A plus or minus L roughly apart from constants then you have 6 these combinations would obey a certain Lie algebra you can convert it to angular momentum algebras, but not in A and L directly, but A plus L and A minus L in that algebra is isomorphic to the algebra of rotations and four dimensions.

(Refer Slide Time: 36:23)



So, the symmetry group of this Hamiltonian dynamical symmetry group is SO of 4 this extra symmetry here translates in quantum mechanics to degeneracy. So, the moment you have symmetry it implies degeneracy in quantum mechanics it means it is the set of transformations under which the energy does not change, so whole lot of states. Defined by, different quantum numbers would have the same energy. So, symmetry in classical mechanics goes over into degeneracy in quantum mechanics.

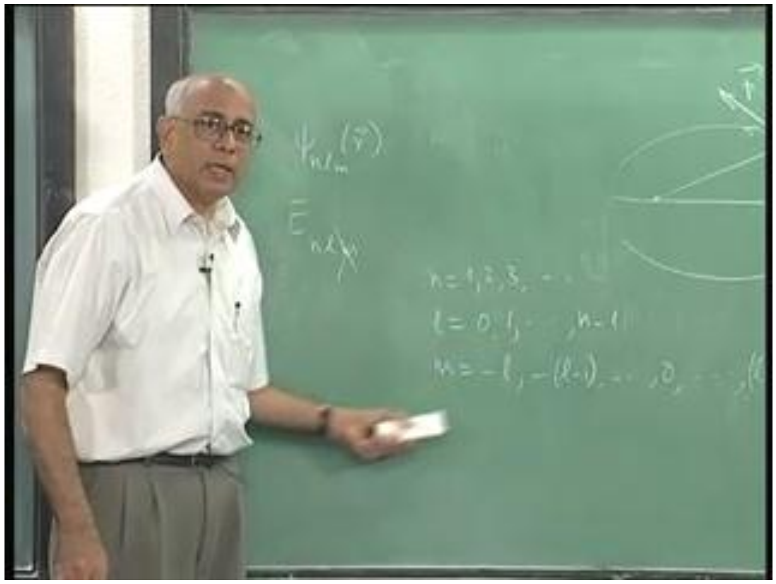
And what is the degeneracy of the hydrogen atom the principle quantum number is n and what is the degeneracy of a state of a level specified by n.

It is actually 2 n squared it is 2 n squared and the 2 comes from the fact that the electron has a spin and there are 2 internal degrees of freedom.

So, never mind the 2 there is an squared still and you have to explain that and it comes about in the following way it comes about because normally. The state could be defined by 3 quantum numbers n l m apart the wave function would be defined by 3 quantum numbers principle quantum number orbital angular momentum quantum number and what is called the magnetic quantum number.

This magnetic quantum number is the Eigen value of 1 component of the angular momentum say Lz or something like that. The energy levels E would also be the functions of n l and m, but because it is a central potential no single direction can be single doubt is special and there for the energy levels do not depend on m, but they would depend on n and l.

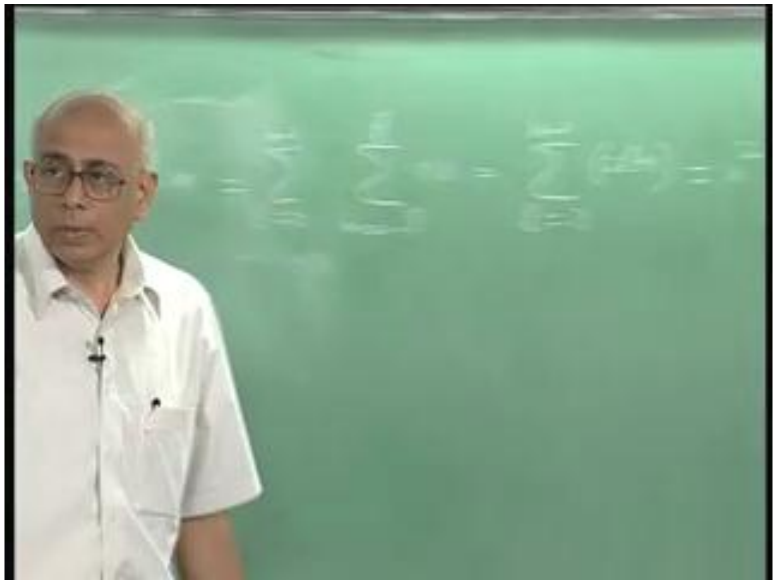
(Refer Slide Time: 38:26)



Now, what kind of degeneracy would you have in that situation remember that in the Kepler problem for example, n equal to 1 2 3 dot dot dot. L equal to 0, 1 up to n minus 1 and what does m do m runs from minus l minus of l minus 1 0; l minus 1 runs from minus l to plus l how many of these states are there for a given l 1 2 plus 1.

So, the normal degeneracy of the hydrogen atom would just be 12 plus 1 of a central potential any central potential the energy levels would be specified by n and l, and for given n and l it would be 2 12 plus 1 fold degeneracy, but because of the Runge Lenz vector the hydrogen atom has an extra degeneracy which is called accidental degeneracy. It is not accidental it is it comes about because of the nature of the 1 over r potential and this 2 goes.

(Refer Slide Time: 39:40)



So, what is the degeneracy? Now, summation m equal to minus l to plus l time 1 that many states that is 12 plus 1, but now you also have the sum from l equal to 0 to n minus 1 of 1. So, this is equal to summation l equal to 0 to n minus 1 of 12 plus 1 that is equal to n square.

That comes out to the n squared and when you multiply that by 2 for the spin then you get the 2 n squared degeneracy, and of course, the moment you depart from the one over r potential due to any correction what. So, ever this accidental degeneracy disappears completely it is lifted and you have a splitting of the levels this is the reason why in the hydrogen atom the 2 s level and the 2 p level are not the same; whereas the strictly had just pure 1 over r potential 2 s and 2 p you should not be able to distinguish between no l dependants, but you actually have a deference due to what is call the fine structure.

So, you learned this in quantum mechanics, but let us get back to classical mechanics, but I want to point out to you that the origin of this whole business is in this extra symmetry this dynamical symmetry for a Kepler problem. So, as you can see it is not easy to find the dynamical symmetry group of a problem of a particular integrable problem.

I would like to switch stream a little bit and go on to several other things, I wanted to talk about, but first any questions what we have done, so far.

What I am going to do is to give you a problem sheet in which, I will right down the specific generators and ask you to check out the algebras. So, this involves putting in all the constants and so on which I will do carefully and give this; both for the isotropic oscillator in two and three dimensions as well as the Kepler problem. So, I will, I will write this out in the form of notes and give it to you.

Yes

You need three of them which are independent of each other. So, it is clear that you do not have that many that are independent of each other they have to be functionally independent and they must be involution with each these guys are not in involution with each other.

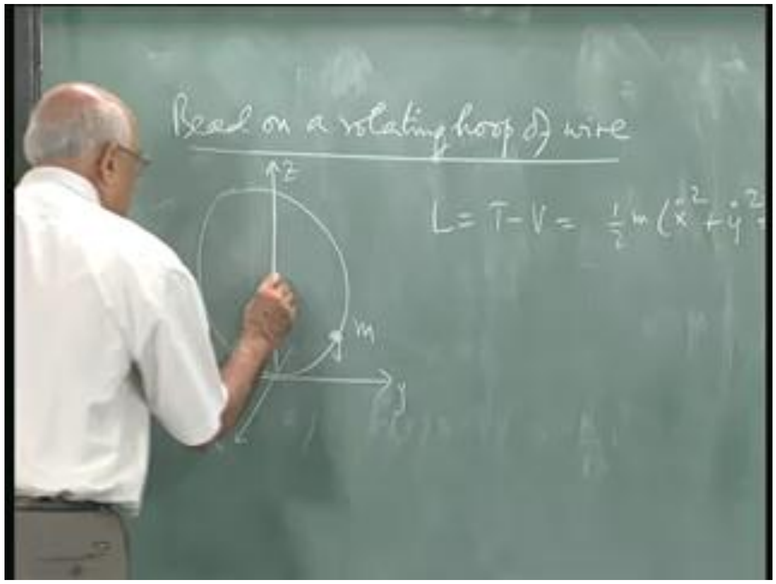
Which 3 H L squared and L 3 for example, that we will do that is it. The person commentator of the person bracket of L I with A J is not 0, that is exactly what the algebra is going to be that is not going to be true. So, they they do not commute with each other.

What you do with that is this really takes me into group theory I really at some stage we should come to terms with this and ,I should define for you what a Lie group is, what the role of the generators is and, how you generate the elements of the group from the infinite as mel generators. So, probably this would be the best thing to do.

Let me devote an hour to this separately, I start with the rotation group and do this because, it is very, very useful how many people here have had a little bit of exposure to group theory? Then it is important, then, let us do then it is then it is let us do that let us do that since the majority has not had it. So, let me do that, we have fifteen minutes today. So, let me do the following not west this time I will talk about generators and the symmetry group a little more.

Because I need to we need to slowly move into statistical physics and the way we are going to do that is by, asking what happens when integrability is lost how does kiosk set in and then what happens after things becomes irregular and. So, on but before, I do that let me fruitfully spend the remaining time today by going back a little bit and showing you how the Lagrangian framework as well as the Hamiltonian framework and a framework of a stability analyses is useful in analyzing a problem with constrains.

(Refer Slide Time: 44:43)



So, let us switch stream and do this and let us look at bead on a rotating hoop of wire, I promised that do this long ago, but never got down to it. So, let us do that problem right away. And it will help us revise all that we know about the Lagrangian and Hamiltonian framework. So, let us do this quickly. Let us suppose have a vertical hoop of wire a circular hoop of wire let us draw the circle first, and then put the axis a vertical circle there is a gravity acting downwards let us call that the z axis and let us call the this is the y axis x comes out of the plane of the board.

And I have right handed coordinate system to start with the the circle the hoop is in the y z plane and I put a little bead on it of mass m with gravity acting downwards I am going to assume it is frictionless. I am going to assume that this bead moves smoothly and frictionless or, this wire what is the Lagrangian and I will assume the potential energy to be zero here and m g z as you move upward.

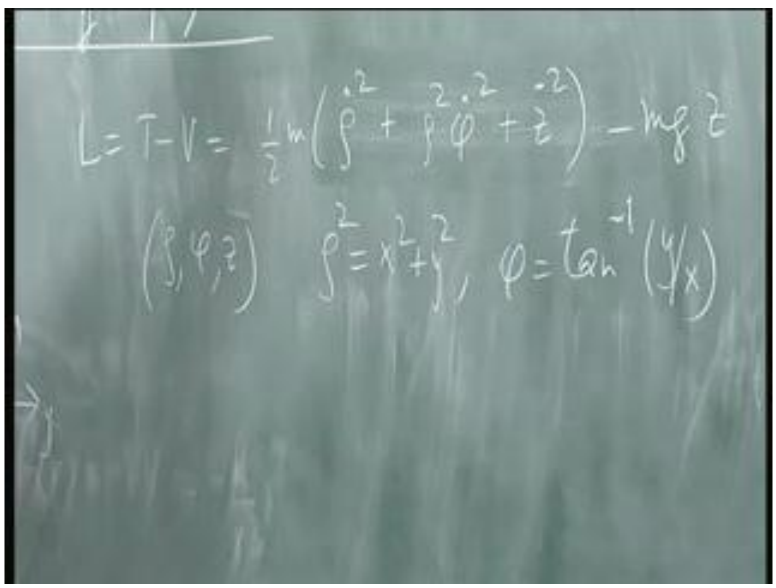
So, this Lagrangian is T minus V and that is equal to one-half m x dot squared plus y dot squared plus z dot squared minus m v z and there is constraint in this problem. And the constraint is that the bead is sitting on this hoop of wire at all times. So, we need to know the equation of this wire and this wire has this, this hoop has radius R.

And it is here and, what is the equation to this, hoop it is a circle in the y z plane and therefore, in the circle centre is that y equal to 0, z equal to r. So, it is y squared plus z minus

R whole squared equal to R squared. So in principle, I can eliminate z in favor of y now I would like to make matters interesting there is an elementary problem.

One thing is to find out what is the period of oscillations of this would that the simple harmonic in general, no except for very small oscillations near the bottom of this hoop, it is not going to be simple harmonic. But let us make matters interesting by saying that this is actually rotating with some uniform angular speed omega the moment I said this rotating I have now cylindrical symmetry about the axis or axial symmetry. So, what coordinate system should I choose? Cylindrical polar coordinates, so let us chose cylindrical polar coordinates

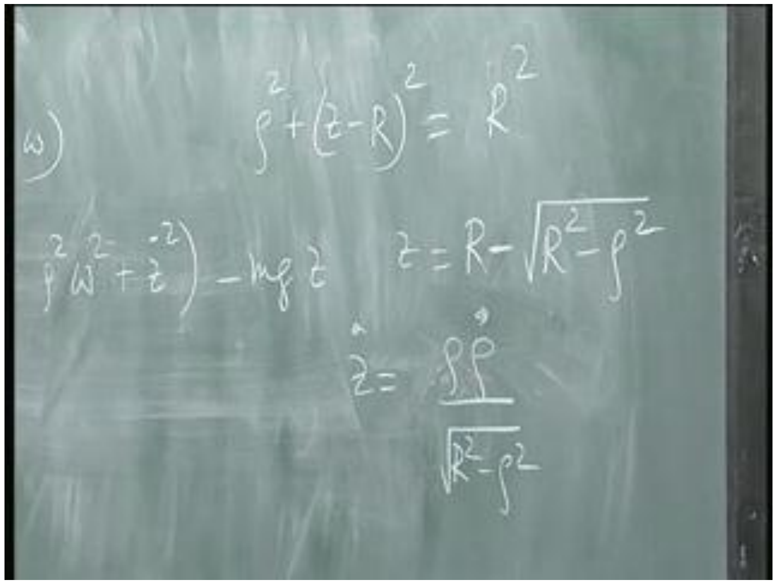
(Refer Slide Time: 48:03)



And let us use rho p and z rho squared is x squared plus y squared t is equal to tan inverse y over x and z remains as it is what the kinetic energy in these coordinates is. So, I should erase this right what is the kinetic energy in this coordinates, well rho dot squared certainly plus rho squared pie dot squared plus z dot squared minus mgz in this fashion.

So, it is certainly true, but then z is related, not to y now but remember, it is rotating. So, it is actually related to this actual distance, which is x squared plus y squared or rho squared.

(Refer Slide Time: 49:10)



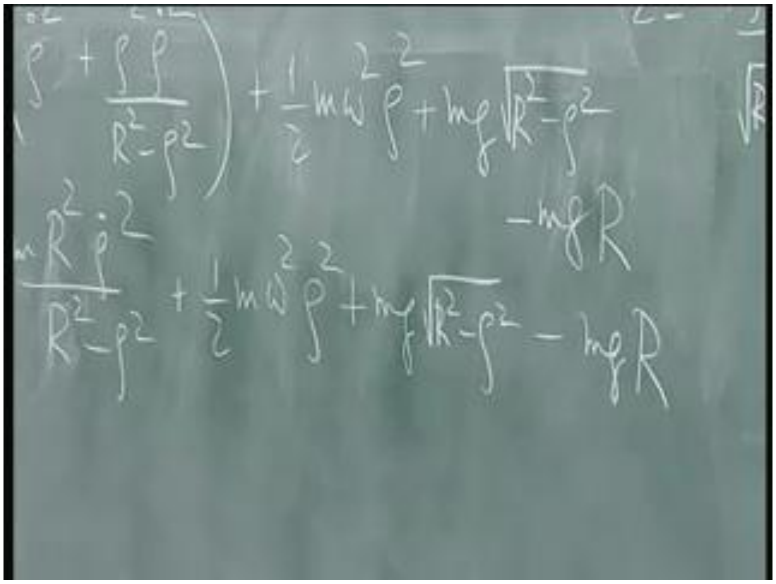
So, the constrain is really z minus r squared is rho squared which is x squared plus y squared because that is the distance from of this point from the z x square of the distance, radial distance actual distance from the z axis, but pie dot is omega because radial azimuthal angle changes is fixed by you, you are rotating this hope at constant angular speed.

So, we can, in fact, get rid of this two write this as omega squared because pie dot is always omega. Now, let us remove the whole point is there is a lot of rho depends here let us remove this z in favor of row. So, let us solve for it and you get z minus r whole squared is equal to r squared or z equal to r plus or minus square root of r squared minus rho squared there are 2 roots and what are they correspond to let us say there are two possible values of z for a given rho. So, if you here at this value of rho well, this root and this root.

One above this half-half way point and the other below, we are interested in motion here. So, which root should, I choose for the plus or the minus, I should choose the minus, I should be very careful to choose the minus. You are shore the minus is what I should chose because if rho 0, I better have z equal to 0 which corresponds to this point.

Otherwise it will go there, so I should choose the minus sign therefore, what z dot it is equal to minus 1 over twice square root of r squared minus rho squared then I have to differentiate minus rho squared which is equal to minus 2 rho rho dot their and the minus signs go away the two goes away and I have rho row dot over r square.

(Refer Slide Time: 51:35)



So, let us put this in and I have L equal to one-half m rho dot squared plus z dot squared which is rho squared rho dot squared over r squared minus rho squared plus one-half m omega squared row. Squared minus m g and z which is plus m g square root of R squared minus rho squared minus mgR and, I can simplify this right R squared. So, it becomes mR squared over R min R squared minus rho squared because this rho squared cancels.

What is this L of function of rho and rho dot that is it, so how many independent degrees of freedom are their now left why do you say two what are they is it one or two I would say just row, I just say rho everything, else is gone I know how pie changes pie is equal to omega t plus a constant and, I know z in terms of row. So there is only one degree of freedom rho and a corresponding velocity rho dot. So, it is a problem which is really one dimensional. So, this L is a function of rho one rho dot that is it just a one dimensional problem, but a fairly difficult kinetic energy this is a kinetic energy part.

But there is another part which came out of the kinetic energy which does not involve a generalize velocity except that a velocity has gone it is disappeared it is gone here. In omega what does this looks like to you it looks like a potential it looks like an oscillator potential rho squared in this redial coordinates, but remember l is t minus v. So, it should have look like minus half rho squared except. It looks like a plus it looks like plus. So, it is like an oscillator potential, but an inverted oscillator potential it is trying to push you away now what do you call this tendency when you rotate something it pushes you away it is the centrifugal term.

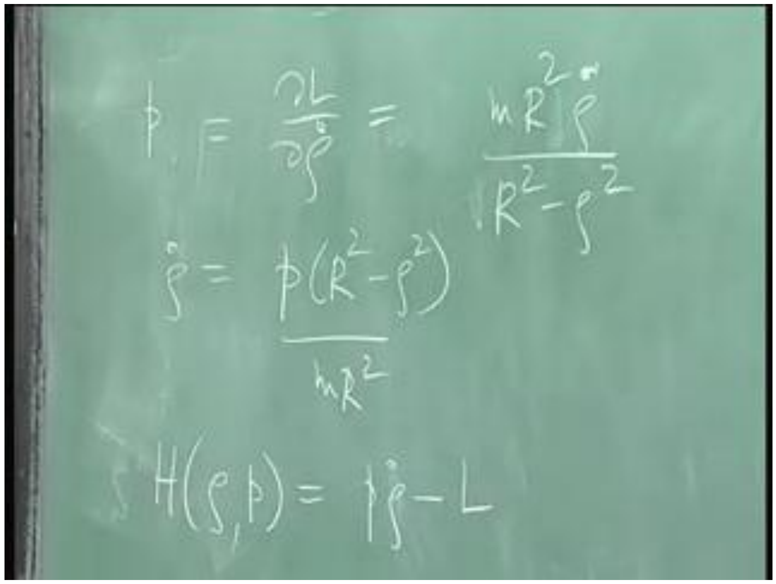
This is the centrifugal term. So, you see what is happening is not only that constrains have been taken into account automatically, I have not putting any constrained force I used that equation and got rid of z, but a non inertial force is also put into is also included automatically.

Yes, certainly there are…

No, I am saying rho is the degree of freedom rho and rho dot are the pair of dynamic variables a degree of freedom is just a coordinate always, but I need a degree of freedom a generalize coordinate as well as a generalize velocity or a generalized moment. So, I am very carefully distinction between degrees of freedom and dynamical variables dynamical variables come in pairs they are the degrees of freedom or the coordinates together with the corresponding velocity or momentum depending on whether you are looking at the Lagrangian or the Hamiltonian.

So, all I am pointing out is that we have got rid of all the other you have got rid of pie you got rid of z we are left with rho pie dot has been put equal to omega constant and z dot has been expressed in terms of rho and pie dot and rho dot and here we are. So, now I could write the Euler Lagrangian equations down there is just one equation to right and that is got for the delta L over delta rho equal to etcetera. But let us go and do the Hamiltonian and what would you do you would first start by defining a piece of rho the momentum conjugate the rho the redial momentum as equal to delta L over delta rho dot. And what is that equal to that is highly non-trivial it has this term this whole mess here. So, this is equal to m r squared and there is n 2 rho dots. So, the two cancels rho dot over R squared minus rho squared.

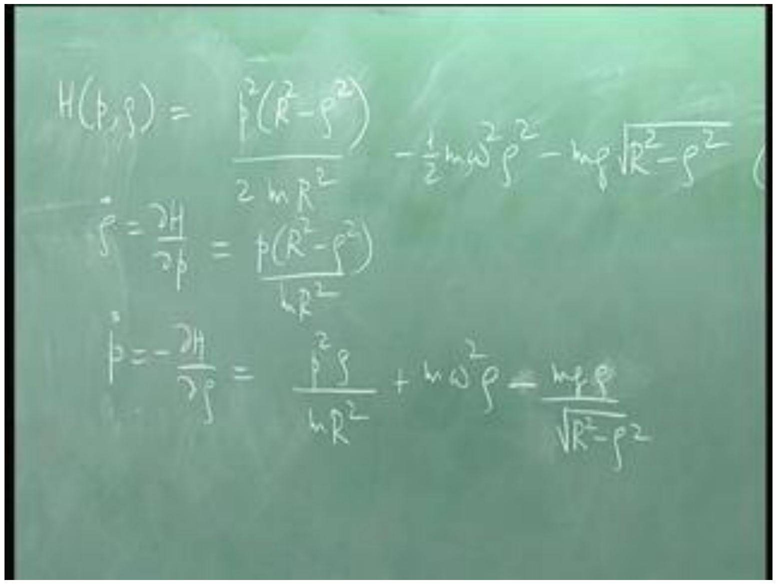
(Refer Slide Time: 56:17)



Since there is just one degree of freedom rho let me not right this subscript here let me just call it P for ease of notation, it is the momentum conjugate to rho and the next step is of course, you must right this as rho dot equal to p times R squared minus rho squared over mR squared and then you must find the Hamiltonian which is a function of rho and p equal to p rho dot minus L.

But you must be careful to remove all rho dots on the right hand side and express everything in terms of p and rho and that would be a Hamilton Hamiltonian.

(Refer Slide Time: 57:45)



So, what happens next as we have H of p and rho equal to and we could write this down immediately if I do this rho dot squared here. It is going to be p squared. So, let us let us do that it is it is a painful thing, but p rho dot is p squared R squared minus rho squared over mR squared that takes care of the p rho dot term minus this term in which rho dot is replaced by that formula there.

And as always happens in this case just as half m v squared becomes p squared over 2 m we are going to end up with just a factor of two just half that minus the rest of the Lagrangian. So, minus one-half m omega squared rho squared minus mg square root of R squared minus rho squared plus a constant, I am not interested in the constant m g r is just a constant that is the Hamiltonian.

Now, let us write down now let us write down the Hamilton equations and then, I am going to do leave the rest of if for you have to do. So, rho dot equal to delta H over delta p and that is guaranteed to be this p times R squared minus rho squared nothing new, but what is p dot that is minus delta H over delta rho.

And that is equal to now things get interesting you have to first differentiate this term and put a minus sign. So, it is equal to p squared rho divided by m R squared if, I differentiate I get the minus 2 rho the 2 cancels. The minus goes away with this term here and then, I differentiate this with a minus sign. So, plus m omega squared rho and this plus mg over twice root R squared minus rho squared minus twice rho; so mg rho with a minus sign.

These are non-trivial equations, you have to solve these two equations and they are fairly non-linear as you can see where are the critical points where would the critical points be first we have to ask physically does it make sense if you started at this point and whether you rotated or not nothing is going to happen.

So, p equal to 0 rho equal to 0 must certainly be a critical point and that is certainly true if said p equal to 0 this goes away if I said rho equal to 0 this way whole thing goes away notice that rho cannot be equal to R that is not a critical point because in this blows up. So, the only possibility for this right thing the only way can vanish is if you if p is 0 now what I want you to do is to look at the critical point at p equal to 0 rho equal to 0 linearise about this point and ask.

What happens linearise and find out whether it is what sort of critical point this is stable unstable etcetera, is there any other possibility actually there is one more possibility and the possibility is if p is 0 this side is 0 this term goes away. Then, this could vanish independently and it may have a root other than rho equal to 0, it may have a finite root provided it is real now, I leave that as an exercise to you to complete and we will write down the solution tomorrow.

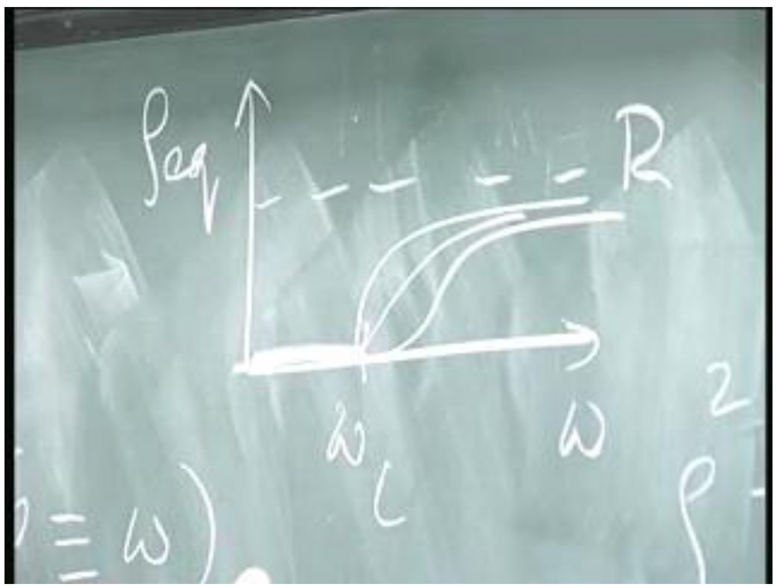
If you look at that term it will turn out that if rho if omega is sufficiently high then there is a possibility that you have a non-trivial row; that means, there is some distance here depending on the speed of omega which is stable and then at that point this guy will become unstable and the bead will move out and remain at that point.

So, the osculation's would be about that point and what is called the bifurcation will happen once a critical value of this angular speed is reached in this problem it will be something like square root of g over R or something like that then you have to find the stability changes the stability the origins becomes unstable and the new stable point becomes the new point critical point becomes stable. And we will look at this tomorrow and complete the rest of it. So, stability analysis is going to help us to interpret the solution and this problem displaces an example of what is called a bifurcation a small change in the angular frequency omega.

In the vicinity of a critical value either this side or that side of it is going to cause a large qualitative change in the behavior of the dynamical system. So, this is a problem also serves to give you an example of bifurcation in a dynamical system once you do that then of course, you can change the shape here and ask what is going to happen etcetera. Tell me physically what is going to happen if I start rotating this with infinite angular velocity arbitrarily high angular velocity what you think will happen?

It should go in stick here that should be the solution. So, already you expect that this rho which is the stable value of rho once you exceed a critical point should go and stick there.

(Refer Slide Time: 01:03:46)



So, in fact, we can anticipate ourselves and write what is rho equilibrium or steady state as a function of angular speed and if you do this you discover it is the origin here upto a certain angular speed and then at a critical value it is going to take of and then saturate to the value R. So, what would this curve look like? We do not know, we do not know yet it might do this it might do this it might start with the finite slope and go there it is of great interest and importance find out exactly what does it do with that point. We will discover that this threshold behavior is like a square root singularity.

So, this perpendicular slope here and then goes and saturates in this fashion that is characteristic of what is called a pitch fork bifurcation. So, as you can see even this very, very simple looking problem, you do not really have to solve the entire problem, but we can understand completely what the dynamical system is going to do by this combination of techniques the Lagrangian followed by the stability analysis it is going to take care of this, so let me stop today.