Classical Physics
Prof. V. Balakrishnan
Department of Physics
Indian Institution of Technology, Madras

Lecture No. # 13

Now, let me formalize the idea of symmetry, what I mean by symmetry, what we mean by the constant of the motion and so on. And this is now going to take us into a very general relation between the concepts of symmetry on the one hand, in variance on the other and conservation loss. All these three are sort of jumbled up in our minds and they are closely linked to each other, and we will try to formalize this relationship.

The first step in this regard was taken long ago by the mathematician M. E. Noether, the beginning of the 20 th century is one of the greatest algebraist of the last century. And she discovered this theorem in a much more general context, than the one I am going to give it. And the one I am going to state here applies to the simple case of Lagrangian mechanics, but the theorem itself is applicable in much greater generality. So, let me write this down.

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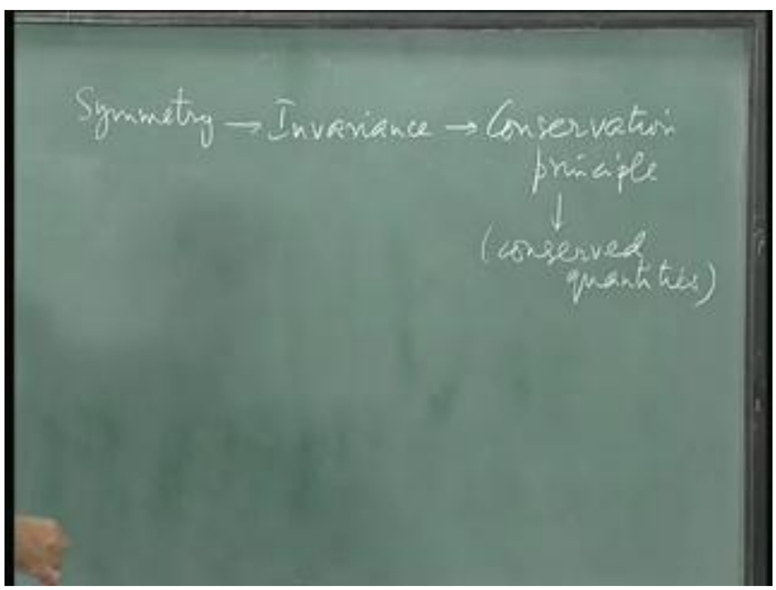

Noether's theorem and it is going to formalize the connection between symmetry invariance and conservation, conservation principle. And of course, reading from conservation principle is going to be is going to have conserved quantities; such as total angle of momentum or total momentum, total energy and so on. We have all follow from special cases of this general relationship, which leads from a symmetry right up to a cons conservation law. On the other side, Noether's theorem going to colors, how this happens, the idea is the following.

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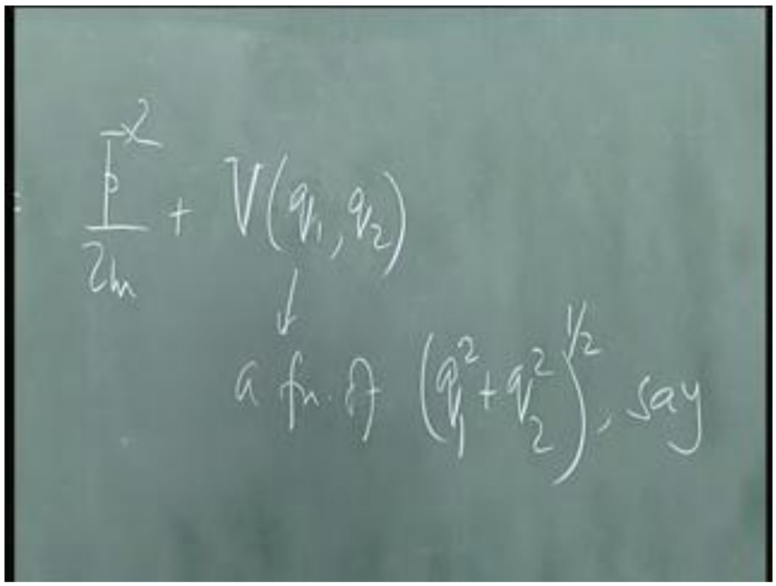

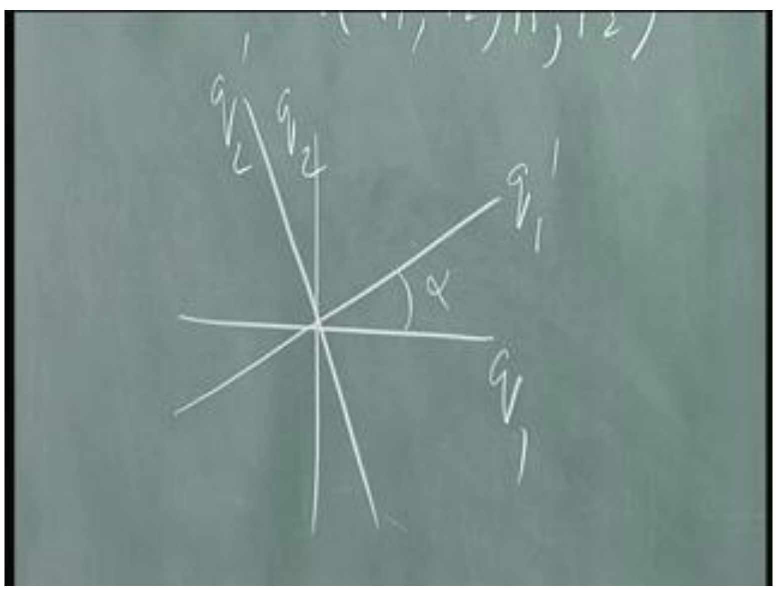

Suppose you have to give an example a two dimension motion a particle moving in two dimensions and let us suppose this particle moves under the influence of a force. So, you have two dimensional motion on a plane and the Hamiltonian, in this case is q 1 q 2 p 1 p 2. And let suppose these are Cartesian components for instance, let suppose that this is equal to as usual plus a potential, which would in general depend on q 1 and q 2 and p squared it stands for p 1 squared plus p 2 squared.

Now of course, if this potential has circular symmetry, which is a two-dimensional analog of rotational symmetry. Then, we know that the system is invariant these Hamiltonian is invariant under rotations in the plane. So, if this is a function of q 1 square plus q 2 square to the power half say, the function only of the distance from the origin when if I make a rotation of the coordinate axis in the q 1 q 2 planes; the new Hamiltonian is not going to look different from the old Hamiltonian; it simply not going to change at all.

You would expect, that the physics of the problem will therefore, not change, what will remain invariant. If I take for example, the Euler Lagrange equation and solve them, what would be invariant under this set of transformation that is instead of choosing; these coordinates q 1 and q 2. I choose some coordinates q 1 prime and q 2 prime, which is tilted which are tilted, these axis are tilted with respect to the original axis by some angle alpha I make this transformation and choose q 1 prime q 2 prime; and cor correspondingly p 1 prime p 2 prime as my coordinate as my generalized coordinates and momenta, what is going to remain invariant?

Hamiltonian does not change, so what would remain invariant the differential equations would not change in form, they would look exactly the same in the new coordinates in the new variables as they were in the old variables.

And what is consequence of that, the equations of motions do not change, in the old coordinates or in the new coordinates they look exactly the same. Therefore, what happens can I assert that the solution does not change, looks exactly the same as before; can I add that well apart from adding to constants and so on. Can I assert that a given solution does not change, can I do that, can I can I do that? Well, we also must be careful about initial condition and so on.

So, the numerical values in the unprime coordinates and in the prime coordinate variables are different of course. So, can you assert that the thing does not change at all what does not change, what does not change? Is it not possible for example, suppose take a very, very simple example.

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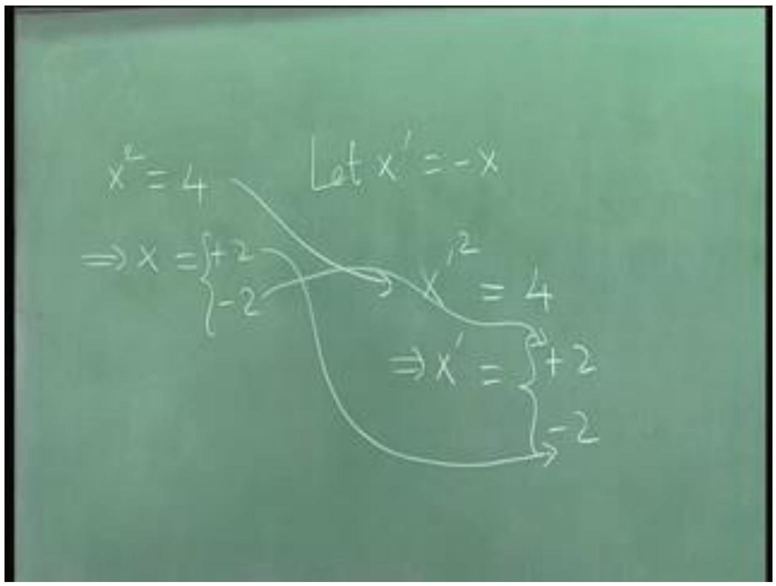

Suppose, I have an equation x square is equal to 4 look at this as 1 example and I want you to impress upon you what the meaning of it I have an equation x squared equal to 4. This equation is invariant under a change of coordinates, which takes x to x prime, but x prime equal to minus x. So, what has remained changed this implies x equal to plus 2 minus 2 you have 2 solutions, so what has remain unchanged. Now, I look I work, I look at this equation in the primed coordinates what does it look like. So, say let x prime equal to minus x, then this equation gets transform to x prime squared equal to 4, what are the solutions here.

Suppose, I look at this solution x equal plus 2, does it remain unchanged, what happens to it, it becomes minus x prime equal to minus 2. So, this solution corresponds to that and that solution corresponds to this. So, what is remained unchanged, the set of solutions is unchanged. Once you understood that you understood a great deal of the secrets of the universe. And then in invariance it is the full set of solution; that is unchanged, the solution space is unchanged individual solutions would go from one solution to another, but, within the same set.

This is the meaning of invariance dynamical invariance obviously, so it is the full set of equations that is going to be unchanged. A given solution will look like some other solution after you transform but, the full set is unchanged. So, therefore, we now began to understand what we mean by dynamical symmetry? The dynamical symmetry of a given system is a set of transformations of the face space variables, such that the equations of motions do not change and as a consequence the set of solutions do not change, does not change. So, that is what I would mean by symmetry of the system.

Now, let us formalize it let us write this down the following way, I will do it first in the Lagrangian framework and then in the Hamiltonian framework, where it will become much clear.

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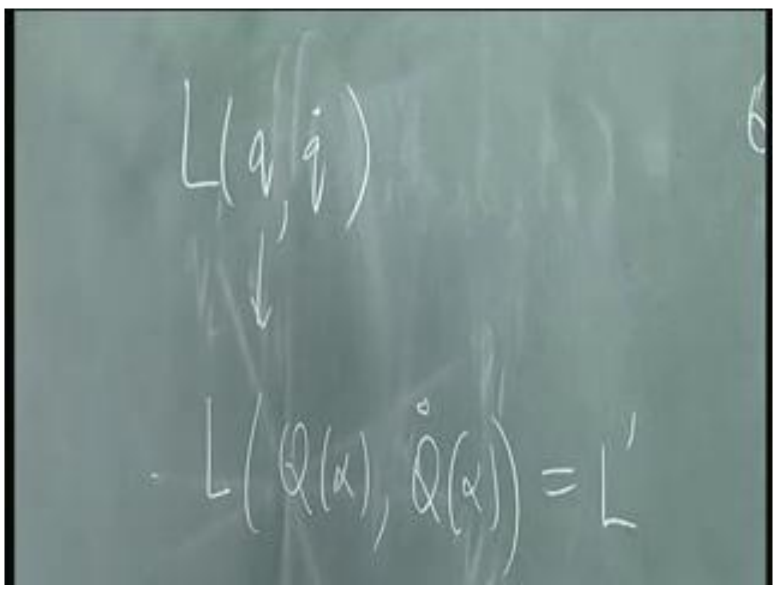

So, let us suppose you have a Lagrangian L of q q dot ignore the t for a movement it is not allowing; and I make change of variables from q to something else, I would like to parameterize this, I would like to make a set of continuous change of variable.

For example, when I rotated the coordinate system I could have chosen a whole lot of alpha’s, infinity of alpha’s, any number of angles right all between 0 on 2 pi. So, I would like to parameterize the transformation by a certain parameter or a set of parameters. And let us call that set of parameters alpha and therefore, let me call Q of alpha the coordinate in a frame, which is rotated by an angle alpha as alpha changes this Q of 0 alpha becomes something else.

And Q of 0 is my original Q start with an original core set of coordinates and that is why there is no rotational angle at all. Then I start going to sets of new coordinates each time parameterize by certain alpha and let me call that Q of alpha. The Lagrangian here goes into the Lagrangian Q of alpha Q dot of alpha and let me call this Lagrangian L prime L prime is equal to L, when alpha is equal to 0 by definition. Now I ask alright how would be equations of motions remain unchanged, if L prime is equal to L, they would not change right.

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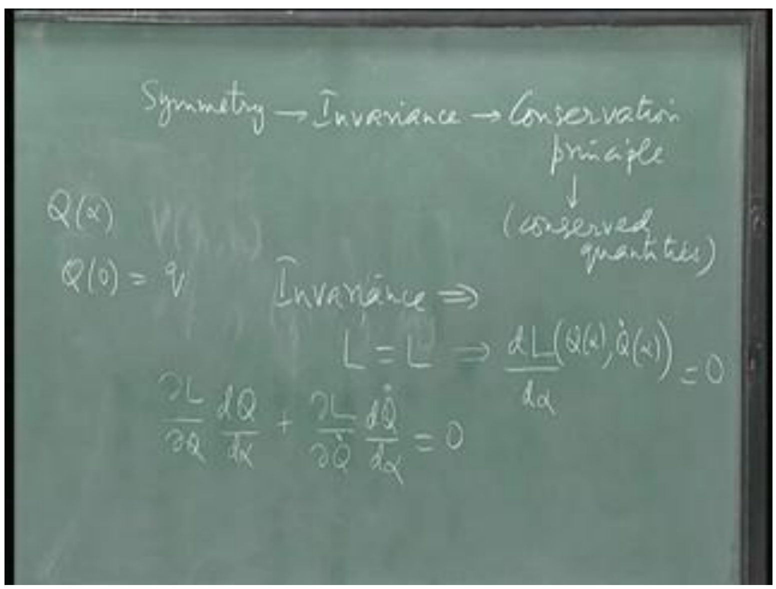

So, let us say invariance implies L prime equal to L which in turn implies that d L of Q of alpha Q dot of alpha over d alpha must be 0. It means this Lagrangian cannot depend on alpha, even though you made a change of coordinates. And think of this potential problem think of the potential is a function of Q 1 squared plus Q 2 squared. If I now make a rotation Q 1 prime is Q 1 cos alpha plus Q 2 sine alpha and similarly for Q 2 etcetera.

And now you can easily see that Q 1 prime squared plus Q 2 prime squared is the same as Q 1 square plus Q 2 square and therefore, the Lagrangian does not depend on this alpha at all. But, the consequence of this immediately say's delta L over delta Q d Q over d alpha, because L depends on alpha through Q of alpha plus delta L over delta Q dot dQ dot over d alpha must be 0.

Because L is not directly a function of alpha it depends on alpha through Q of alpha and Q dot of alpha this must be 0. But, that immediately tells us that d over d t delta L over delta Q do; that is what the Euler Lagrangian equation is for this d over d t delta L over delta Q dot, d Q over d alpha plus delta L over delta Q dot times d over d alpha of d Q over d t.

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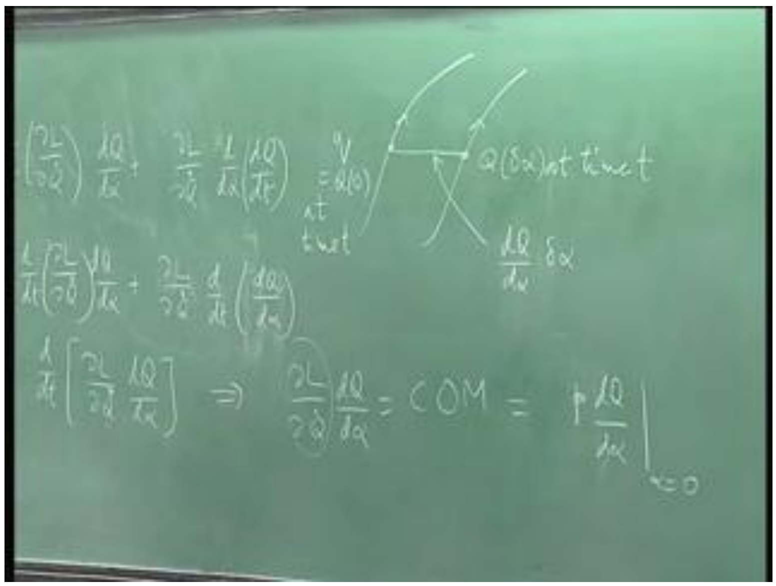

But alpha has nothing to do with t, this is just a set of coordinates which I am using a transformation right. And if you look at the trajectory for example, at some instant of time if this point here is q, which is equal to Q of 0. If I go to a set of changed coordinates the numerical values of these quantities would change and the new trajectory would perhaps look like this. And this point is map to this point, if I change the alpha and let suppose this is an infinite decimal change.

So, this is Q of delta alpha a very small alpha rotation for example, then what is this equal to this is Q of delta alpha at time t, at some instant of time t this is at time t. And of course, this Q changes in this fashion of a function of time at the particular instant of time. I look at this coordinate or set of coordinates in the original frame of reference in the original coordinates and in the transform coordinates. And if it is in infinite decimal transformation of this kind, it is quit evident that this is equal to delta Q over delta alpha d alpha for delta alpha. So, let us write its better d Q over d alpha delta alpha.

So, this alpha has nothing to do with t, it just changes of frames of reference or coordinates. Therefore, you can write 0 equal to this equal to d over d t delta L over delta Q dot d Q over d alpha plus delta L over delta Q dot d over d t of d Q over d alpha, which of course. Immediately is d over d t of delta L over delta Q dot d Q over d alpha and that is 0. So, what does that imply, the quantity in the bracket is a constant of the motion, this implies immediately that delta L over delta Q dot d Q over d alpha equal to constant.

It is conserve and therefore, I can evaluate it at any instant of time including t equal to 0 in the original coordinates but, in the original coordinates, what does this become what is delta L. So, this becomes delta L over delta Q dot and what is that equal to this quantity here, it is the original p. So, this is equal to p d Q over d alpha at alpha equal to 0. I can evaluate it at any instant of time including t equal to 0 and therefore, even the slope d Q over d alpha and a Q at alpha equal to, at t equal to 0.

So, that gives me a prescription for finding the constant of motion, corresponding to invariance I can generalize this to many variables. Now not just the single alpha, but a whole set of alphas if I rotate in three dimension for instance I would have 3 rotation angles. And then I would have three constants of the motions corresponding to three angular momentum; we will work this out in terms of some simple exercises but, this is the idea of Noether's theorem, is this the most general way you can write this theorem is there something else you can do which immediately generalizes this theorem.

Well, remember that I said that a Lagrangian is not unique, you can add the total derivative of function of the generalize coordinates and time to it. And nothing is going to happen total time derivative could be added to it. Therefore, it is not necessary that invariance implies that L prime equal to L. So, how could L and L prime differ and still get you have get away with the fact that the equations do not change, you can add a d f over d t to it right.

So, you work through the whole thing again by saying L prime is not equal to this is not equal to 0 but, the difference L prime minus L to first order in delta alpha equal d f over d t you plug that whole thing back again. And then you discover there is an extra contribution here, which would depend on f. And that is the most general invariance you could have this is the content of Noether's theorem, what is absolutely crucial for it is that the set of transformation must be continuous.

If differentiated with respect to the parameters and they must be connected to the identity. In other words you must have a situation, where you have the original coordinates and then you make a transformation, which takes of continuously from no transformation at all. So, this is a connected set of transformation parity is not one is not such a connected set of connected transformation. It is not a continuous transformation because, we go from a right handed coordinated system to a left handed coordinate system you cannot do.

So, continuously it is a discontinuous transformation but, rotation is a continuous transformation, a shift of the origin is a continuous transformation, a shift of the original time is a continuous transformation. Is there any other continuous transformation you are familiar with you can scale things out. I can always multiply the coordinates by constants, I can share the coordinates I can do all sorts of linear transformations; is there anything else, any other physical example of a continuous transformation, which is nothing to do with the coordinates.

Well, they are all on the coordinates, these are all transformations induced on the coordinates but, there are other transformations. For example, if you go back to the example of a charge particle in an electromagnetic field, you can make a continuous set of gauge transformations. It could be continuous right you could start with an infinite decimal and then keep going and that would again be a transformation, which would lead to a conservation principle; in this case it leads to the conservation of charge.

So, at last to we begin to understand where the conservation of charge comes from it comes from the gauge invariance of the electromagnetic field equation after all equation electromagnetic. So, this is really the principle that underlies what the conservation law is trying to tell you. Now, having done this we work out an example with regard to this application of this Noether's theorem yes.

You can evaluate it at any point, the point is at any point the whole idea is I would like to write it in terms of a quantity which I know which is p. But of course, this p I need not have started with my original coordinate system like this. I could have started like this at some finite value of alpha, I call that my initial p and q. So, it does not matter, the whole point is both terms in this bracket have to be evaluated at the same value of alpha, you cannot do this at different values.

So, if I did at alpha equal to 0, the other term also had better be done at alpha equal to zero that is the only point. Now, you could ask alright, so much for the Lagrangian situation to summarize, what is happened is we found if the Lagrangian is invariant under a continuous set of transformations. When Noether's theorem tells you there exist a corresponding conserve quantity, there are as many conserve quantities.

As there are parameters for the transformations in three-dimensional rotation, there are three Euler angles because of parameters. And therefore, you have three conserve quantities, if you shift the origin in three dimensions, once again there are three constants by which you can shift the origin, 1 for the x, 1 for the y, 1 for the z direction.

Therefore there are three constants once again and they would turn out to be the components of the linear momentum, for rotations it is the angular momentum; you can even change the origin of time. And if the equation of remain motion remain unchanged then it means that there is a constant of the motion, in a radian constant of the motion and that is the Hamiltonian itself.

So, each time you see that conserve quantities arise as a consequence of this principle, there is a certain symmetry in the system, which leads to an invariance of the equations of motions. Therefore the set of solutions which leads to a conservation principle, now let us go and ask in the Hamiltonian framework what does this look like, what would this mean, what would the constant of the motion imply. And that is even more interesting let me do that, little bit of preliminary work is needed.

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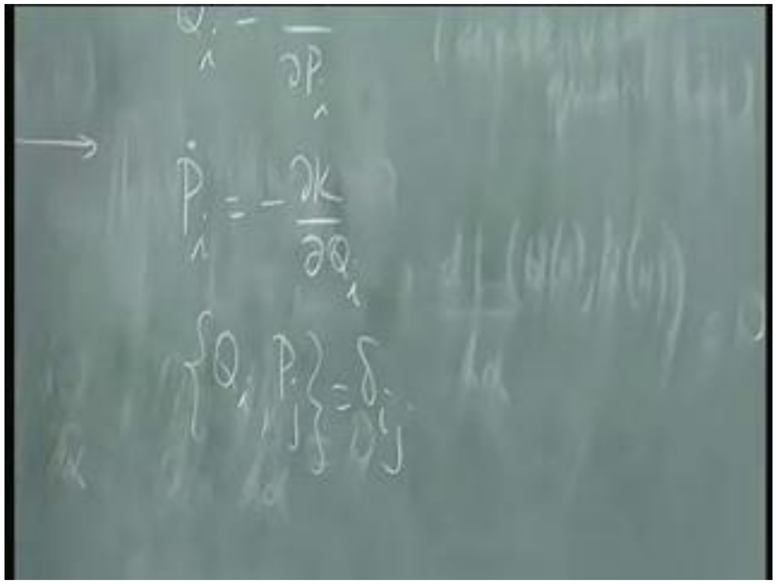

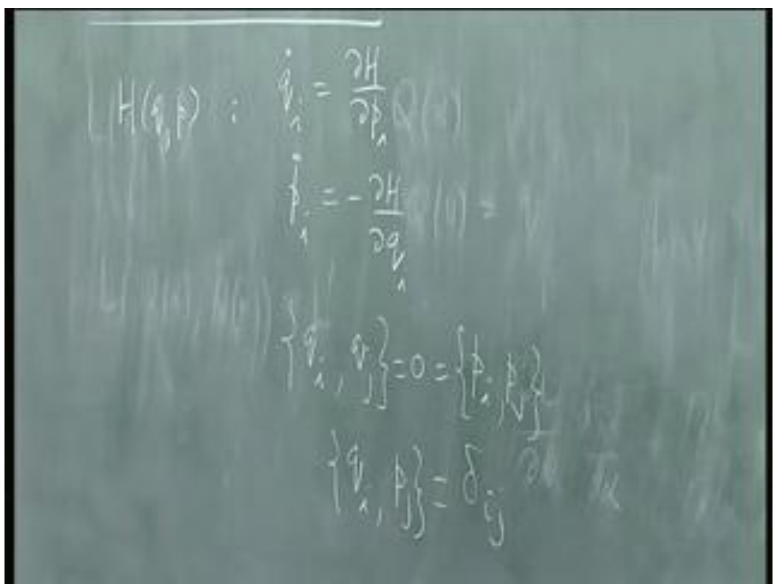

So, let us look at it in the Hamiltonian, this framework I have a Hamiltonian function, function of generalized coordinates and generalized momenta. I am only going at look at autonomous systems for the time being. And I know that the equations of motion are such that q dot is delta H over delta p i p i dot is minus delta H over delta q i. And I also know that there exists canonical Poisson bracket relation, this ensures main standard structure Hamiltonian structure.

Now, I make a transformation, which leaves Hamilton's equations of motions an invariant. Otherwise, you do not have a symmetry at all to start with what set of transformations leaves Hamiltonian equations unchanged canonical transformations. So, it is quite clear, that whatever symmetry transformations. We are talking about must to start with the canonical transformations that may not be enough you have to start with canonical transformations.

So, this thing here must go over into something which looks like Q i dot equal to delta k over delta Pi delta Pi. And moreover you must have Q i P j equal to delta i j this ensures that it is canonical but, for Hamilton's equations remain unchanged the Hamiltonian must remain unchanged, so that the set of solutions look exactly the same. So, you need something more, you need those canonical transformations for which k is the same as H. Otherwise it is not going to be a symmetry of course, into some other Hamiltonian and that is not a symmetry actually.

So, dynamical symmetry transformation is one, where Hamilton's equations not only remain unchanged in form. And the new Q's are canonical coordinates but, the new Q's and P's are canonical variables but Hamilt the Hamiltonian itself does not change in form. The functional form k must be the same as the functional form H. But, even without that this is what a canonical transformation is now can we say something more about it what kind of transformations a canonical transformations.

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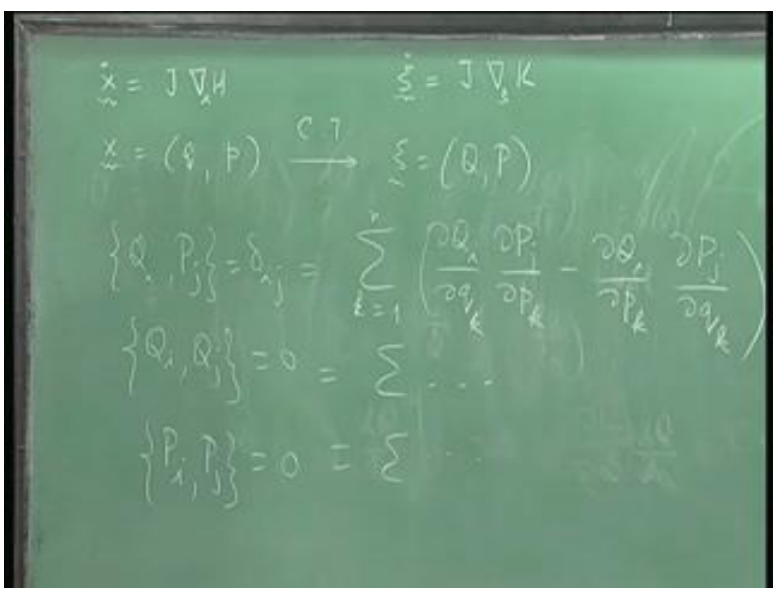

Let us look at that we need to understand that, how to we do this well let suppose x equal to the q's and p's q 1 to q n and p 1 to p n. And let us suppose you make a canonical transformation to a new set a variables let us call it see which is capital Q capital P. Once again I work in this 2 n-dimensional face space and I have these Poisson bracket relations. Now, what do these bracket relations imply for example, this relation what does it mean says Q i P j equal to delta i j implies this is equal to a summation from k equal to 1 to n delta Q i over delta q k delta P j over delta P k. It implies this relation, because that is what is a meaning of Poisson bracket?

So, it says that this set of relations must be true, when you calculate this Poisson bracket of the new Q's and P's you going to have to get delta i j, but that is a definition of the Poisson bracket. And it says this quantity must be equal to delta I j, it further says Q i Q j equal to 0. So, you also going to have Q i Q j equal to 0 and P i P j must be equal to 0. So, that equal to a similar relation with Q j here and Q j there and similarly P i and P j there.

So, you have all these three sets of relations must be valid; otherwise it is not a canonical transformation. That looks very messy and you could ask what does it mean, what does it finally, mean is there a simple way of understanding it.

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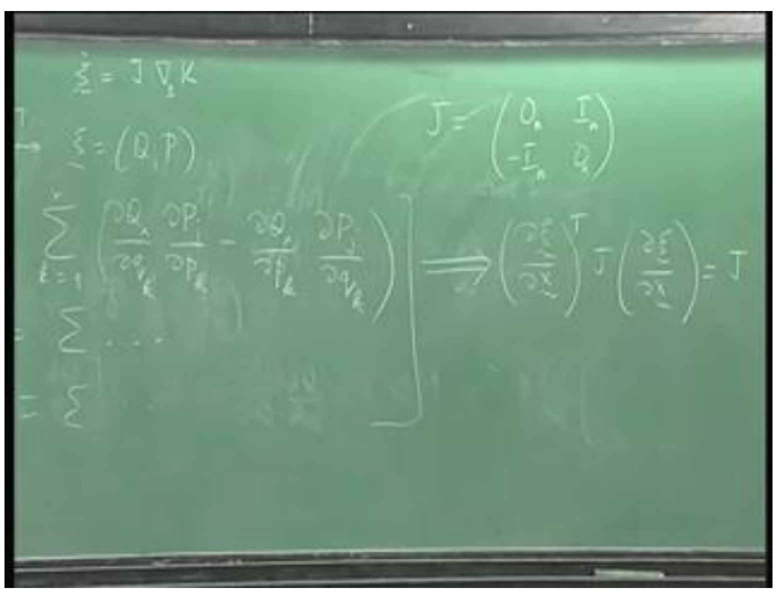

The answer is yes there is recall that we have defined this matrix J, which was equal to 0 unit matrix minus unit matrix 0. And I put an n here to show these are n by n matrices we define this matrix J and then I said Hamilton's equations take on a very simple form. And they form that this set of equations took on here was x dot equal to J gradient of H an extremely simple form that we took on. So, this full set of equations can be summarized in just that single line and of course, when you make a canonical transformation.

And go to Q and P, then once again this thing here would says that c dot equal to j gradient of k, but the gradient in the new variables; so gradient with respect to the components of c and with respect to the components of x. So, these are very simple structures, now you can ask we also saw that the Poisson bracket of a with b could be written as a gradient of a transpose j the gradient of b.

So, you can ask can this set of equations be written an a simple form and the answer is the yes you can take the Jacobian matrix delta c over delta x. This stands for 2 n by 2 n matrix in which each component of c is differentiated with the corresponding component with the components of x here and you have a 2 n by 2 n matrix. Let us see Jacobean matrix for the transformation, that matrix transpose J and that matrix once again must be equal to J.

And that is what the content of this set of equations is little bit of work and you can write down the entire set of canonical transformations any canonical transformations is canonical if and only if this Jacobean matrix J, the Jacobean transpose J, the matrix equal to J itself.

Suppose, you did not have this suppose you did not have this, then it says matrix transpose matrix equal to the unit matrix what you call such a matrix.

Transpose r, transpose r equal to I.

Well you do not call it unitary you call it orthogonal if it is real then of course, a unitary real ma real matrix which is unitary is orthogonal. It says matrix transpose equal to inverse of the matrix that is an orthogonal matrix, this is not quite orthogonal.

Because, there is a J sitting there always, I call it pseudo orthogonal it is like an orthogonality condition. We should by now expect this J to appear everywhere to pop up everywhere in Hamiltonian mechanics it is acting like some kind of metric in the space of Q's and P's. So, this is the generalization of the condition of orthogonality you call it pseudo orthogonality, there is a name for all such matrices.

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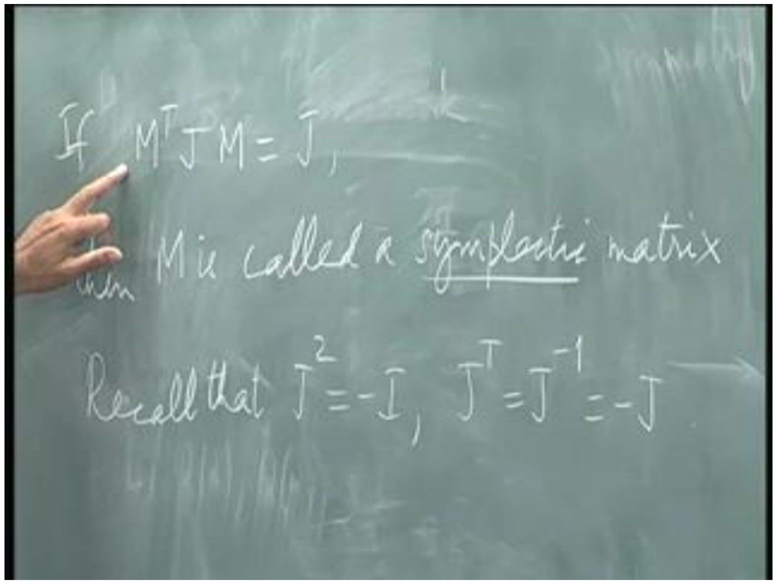

So, let us look at all matrices if a matrix M transpose J M equal to J, then M is called the symplectic matrix and of course, this J has got marvelous properties, because recall. So, any matrix that satisfies any 2 by 2 n by 2 n matrix face spaces always even dimensional in Hamiltonian mechanics at this level.

So, any even dimensional any 2 n by 2 n matrix, which satisfied this condition is called as symplectic matrix. And now it is a simple matter to prove that every simple symplectic matrix has an inwards that is obvious from here actually, what is the inverse of n well all you got to do is to use this once again and it is going to be related to be M transpose. Just as if it would have orthogonal M transpose would be M inverse. Now this is not quit true it is related to M M inverses related to M transpose with J's on either side and so on.

So, symplectic matrix has an inverse and we know canonical transformation must have an inverse transformation. So, it is better have an inverse, when the next thing you can show is that the set of symplectic matrix is from a group. The product of two symplectic matrices is also a symplectic matrix. Physically what does that imply for canonical transformation? I make one canonical transformation and then make a second canonical transformation it is equivalent to a single canonical transformation.

So, there is an inverse and there is a composition law and these transformations form a group. So, the set of canonical transformation for an n degree of freedom Hamiltonian system form a group, it is the group of 2 n by 2 n symplectic matrices.

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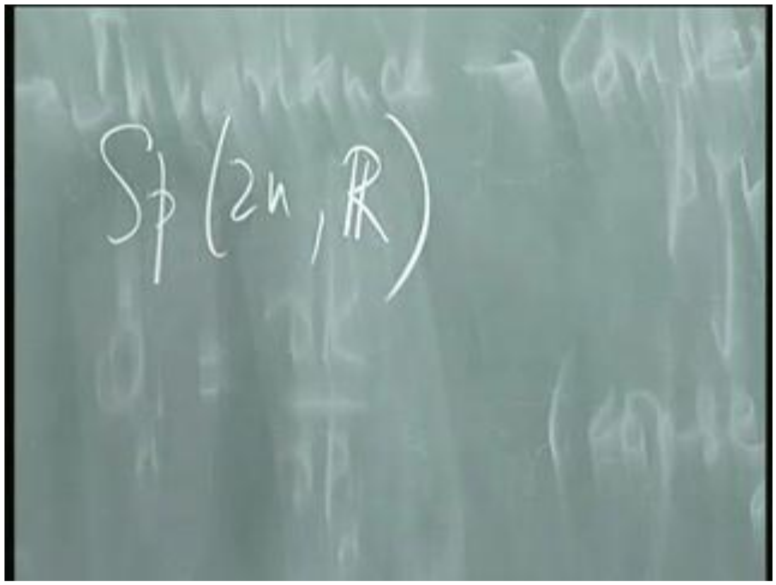

And this group has a name and this group is called Sp 2 n. And since we are working on the real because you cannot complex over the whole thing it is called as p 2 n comma R. That is the group of canonical transformation.

No a symplectic matrix cannot be singular it has an inverse it guaranty to have an inverse is J's symplectic matrix itself. J itself is one you can see that right away, because J transpose that is the same as minus J, but j square equal to minus identity.

So, this has J of course, it is true immediately what is the canonical transformation corresponding to choosing that Jacobean matrix to be J itself Q equal minus p, p equal to q the one we those with yesterday that is the transformation. So, that was a simplest of transformation the first canonical transformation you can think of, where the coordinates in momentum actually interchange minus sign.

And that corresponded to choosing M equal to J itself, this is a fairly large group every group of matrices has a certain number of parameters independent parameters the group of rotations in a plane has one parameters the rotation angle. The group of rotations in three dimensions has, in three dimensions three parameters.

The group of rotations in three dimension has three parameters there are three Euler angles which you need to go from any coordinate system to any other by rotation. So, it has three real parameters we say that the group of orthogonal matrices with determinant plus 1 S O 3 in three dimension, S O 3 has three parameter.

How many parameters does the group of rotation in n dimension have it is the group of orthogonal matrices in n dimensions with determinant plus 1. You would not would like look at proper rotations how many parameters does that have, how many angles are needed to specify a rotation in n dimensions. Any angles why do you say that, is a good guess why did you say n.

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Very good point with respect to each axis you rotate and each axis there are n independent axis and therefore, is argument is you have n angles. But, what happens in 2 dimensions there is only 1 angle, how do you say that, how do you define a rotation in 2 dimension. What is a rotation in two dimensions 2 euclidean dimensions rotation, where is the z, there is no z, about the origin, rotation about the origin. So, it is rotation about a point there is no axis it is rotation about a point, rotations are little trickier than you imagine, so let us do that.

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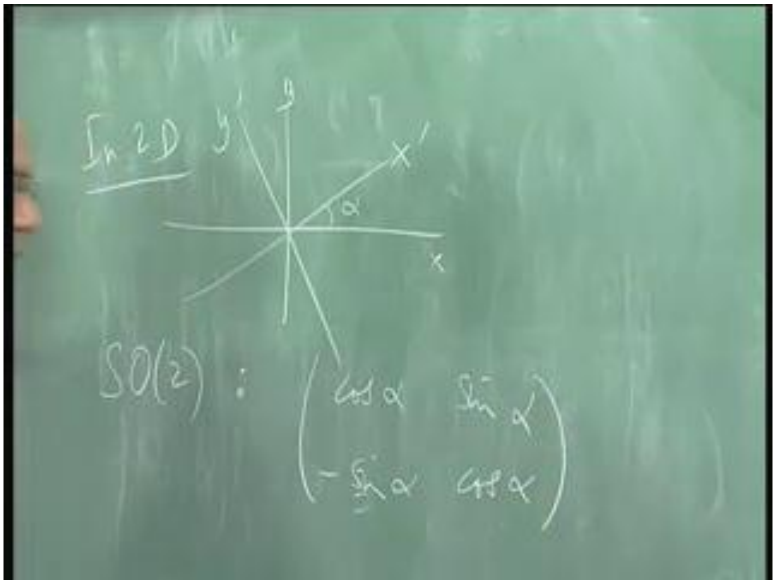

In 2 D I rotate from this set of axis to that set of axis, parameterize by single alpha here. So, here is x y here is x prime y prime. So, the group of rotations which comprises the group of matrices which are orthogonal, real entries and has determinant plus 1, can always be written in the form this contain.

So, all matrices of the form S O 2 contain all matrices of the form cos alpha, sin alpha minus sin alpha cos alpha. And what does a rotations do, it takes the x y takes the original coordinates makes each goes to new set of coordinates each of the new components is a linear combination of the old variables. And it is homogeneous the set of transformations is homogenous, because the origin remains unchanged, no constant is added not a translation rotation.

So, linear homogeneous transformation, which leave a point unchanged the origin unchanged. Therefore, it is homogeneous is a rotation it must have determinant plus 1 otherwise the axis get flipped the handedness gets flipped. So, linear homogeneous transformation, which it has determinant plus 1 it is called unimodular is what a rotation is; now how do you define a rotation in three dimension, what is the basic property of a rotation.

It will a give lengths unchanged distance between any 2 points is unchanged, that is the basic property. So, all you need to define a rotation is it should be a linear transformation, it should be homogeneous without loss of generalities we say let us the origin be unchanged. It should leave distances unchanged therefore, it must be orthogonal the transformation must be orthogonal that is the necessary and sufficient condition. It must be specified by an orthogonal matrix that is necessary and sufficient to ensure that distances are unchanged.

And it must have determinant plus 1 that is it that is all you need for defining a rotation. Now, suppose I am in four dimension and the coordinates are x 1 x 2 x 3 and x 4 that is, it I rotate in the x 1 x 2 plain. And I leave x 3 and x 4 unchanged is this a rotation or not yes it is, but which axis have I rotated by, about which axis have I rotated, I have left the whole of the x 3 x 4 space unchanged. So, rotation is not defined by saying you rotate about an axis.

Rotation is a linear transformation, which is homogeneous, orthogonal and unimodular that is it, that is the general definition of a rotation. It applies to all dimensionality; that is why we define a rotation.

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That is not part of a rotation that is a separate transformation all together right. So, when 2 translations group get enlarge to what is call the Euclidean group. Right now I am talking only about rotation about homogenous transformation. And I am trying to define a rotation appropriately in a rigorous way.

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yes yes.

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No no no no no that is a good question I am not saying that at all I am not saying a trans translation would leave change lengths and so, on. I am not saying that, I am saying what is the set of homogeneous transformations that leaves lengths unchanged you have to have orthogonality. And if I want to live orientations unchanged, you must have handedness unchanged you must also have unimodularity. Every orthogonal matrix has determinant either plus 1 or minus 1, you choose the plus 1 for the proper transformations, if you include reflections then it is minus 1 alright.

So, that is my definition of a rotation you cannot always associate an axis with a rotation, in three dimensions, what you can do is associate a plane you can always say every rotation can be regarded as rotation. In some plane does not have to be 1 of the Cartesian planes could be something tilted an angle, but there is always a plane. So, 2 variables are affected at a time; now, therefore, tell me since 2 variables are affected at a time how many rotations are there NC 2.

So, it is n times n minus 1 over 2, that is the number of angles you need to specify a rotation in n dimensions and what is it for n equal to 2, 1 and then equal to 3, 3.

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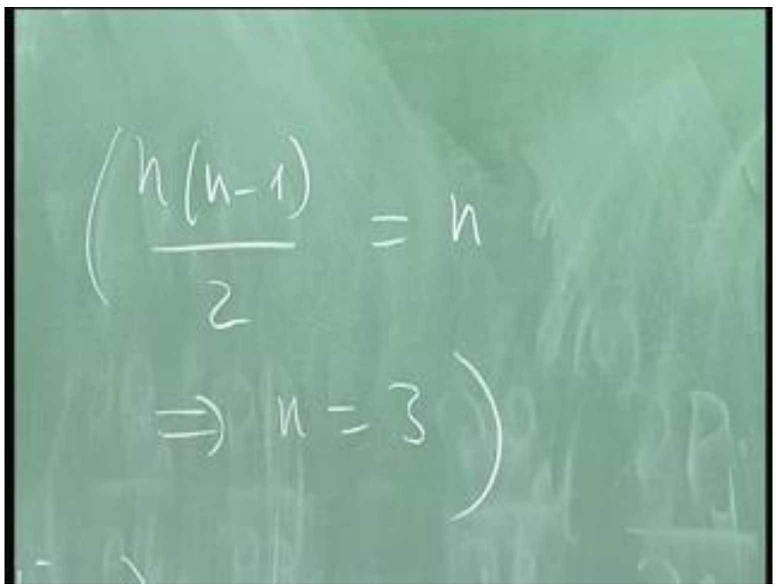

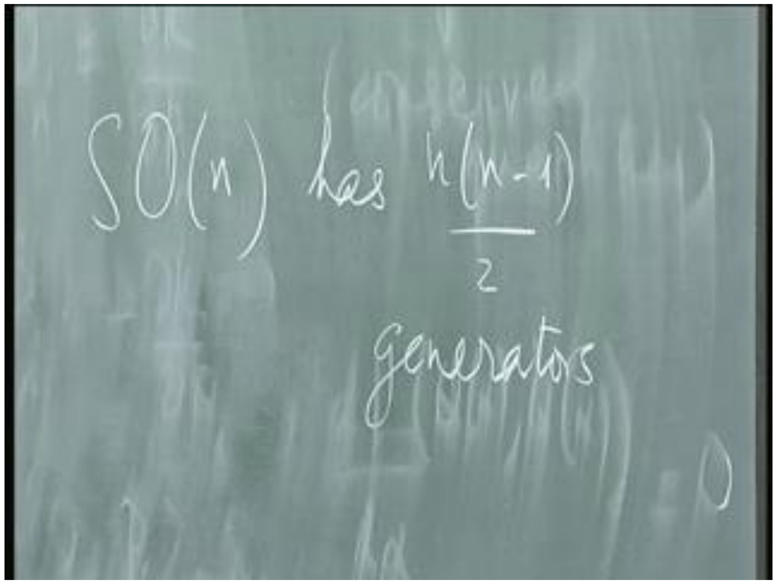

It turns out that this quantity n times n minus 1 over 2, which is a number of independent planes that you can have orthogonal planes, happens to be equal to n when n is equal to 3.

This is a great accident well may be not such a great accident after all, that is the reason you say's rotation about the z axis, the about the x axis about the y axis. Because, you really saying, rotation about the y z in the y z plane z x plane and x y plane, you cannot do this any other dimensionality including two. This has a solution in integers only for n e q implies and equal to 3, so one of the reasons why three dimensions is very special.

So, how many generators thus S O n have, in exactly the same way you could ask how many generators does the symplective group have; how many possible parameters do you need to specify or possible canonical transformations. And the way to do this is exactly what you did for rotations. Then we ask if I make an infinite decimal transformation how many parameters are there how many independent ways can I make infinite decimal transformations. So, let us make an infinite decimal transformations of out here.

So, let us say M is equal to the identity transformation plus some epsilon some small parameter like a delta alpha times G, G some other 2 n by 2 n matrix called the generator of a transformation.

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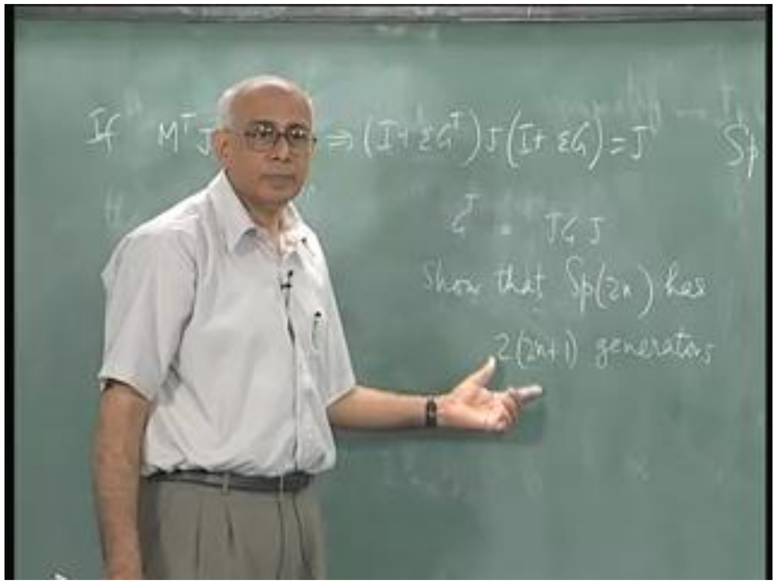

Plug that in here and what you get this implies I plus epsilon G transpose J I plus Epsilon G equal to J.

And let us open it out and you have to retain to first order in epsilon and what does it tell you it say's. So, if you multiply this through its J plus epsilon G transpose J time's I plus J epsilon G equal to J; dropping the Epsilon square term and the J cancels out and the epsilon trans. So, symplective transformations are generated by matrices which satisfy this property G transpose is J minus G transpose J is minus J G. And let us take the J to the right hand side, so this becomes J inverse but, J inverse is minus J.

So, the generator of a simplected transformation, the infinite decimal generator must satisfy this condition the transpose of this generator must be equal to J G J. And now you can count how many parameters it must have.

So, keep this as an exercise to you to show that S p 2 n twice 2 n plus 1 generators it is much larger than the .

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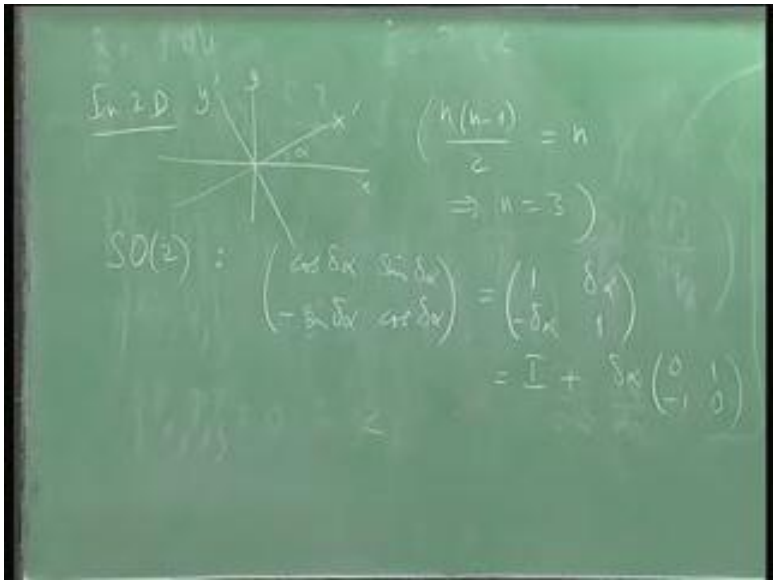

Well, I have not quite defined what a generator is but, where like to look at it is if I look at what happens here for in infinite decimal rotation. When let us replace this cos alpha by delta alpha sin alpha by a delta alpha delta alpha and what to first order in delta alpha and what is cos delta alpha to first order in delta alpha.

1 minus delta alpha whole square over 2 factorial, but then I am working to first order. So, cos alpha is just to 1 right, and this is equal to 1 delta alpha minus delta alpha 1 to first order in delta alpha, which is equal to the identity matrix plus delta alpha times 0 1 minus 1 0.

This is the parameter, you can change it but, that is a given matrix and that matrix is what I call the generator. Question is find out how many such independent matrices are there in general, in three dimension it is not hard see that there be three of them for rotations for the symplectic group, that be twiced 2 n plus 1 this is a large group; so if at all kinds of transformations possible as infinite decimal as canonical transformations.

So, fairly large group we will, we will say a little more about it here I will come back and spend some time on rot the rotation group and it is generator and so, on. Because that is not quite the way, I would like to write the generator, I did like write the generator as a Hermitian matrix. And the reason is I would like to exponentiate this generator, when you exponentiate the Hermitian matrix, what do you get a to the power I times the Hermitian matrix is a unitary matrix.

And a unitary matrix has interesting properties the analog of canonical transformations in classical physics. So, we will do a little bit of this subsequently but, right now the point I want to make is that the group of canonical transformations can be related to a group of matrices, you can see the generated by this symplective group.

But now I pointed out coming back to our original problem, the symmetry group of a dynamical system is that set of transformations, which leaves the set of equations of motions unchanged, therefore, the set of solutions unchanged. Now, in the Hamiltonian framework you first needed to ensure that the structure of Hamilton's equations did not change. So, you needed to ensure that they were canonical transformations but, you also need to ensure that these transformations do not change the Hamiltonian itself, that the k was equal to the h itself.

And what would you expect that to be I had expect that this dynamical group the symmetry group, would be a certain sub group of a symplectic group. Some canonical transformations would not change the Hamiltonian and that would be me symmetry group. So, this is the expectation and the best way to do, this is to do this in terms of an example, so let me do that.

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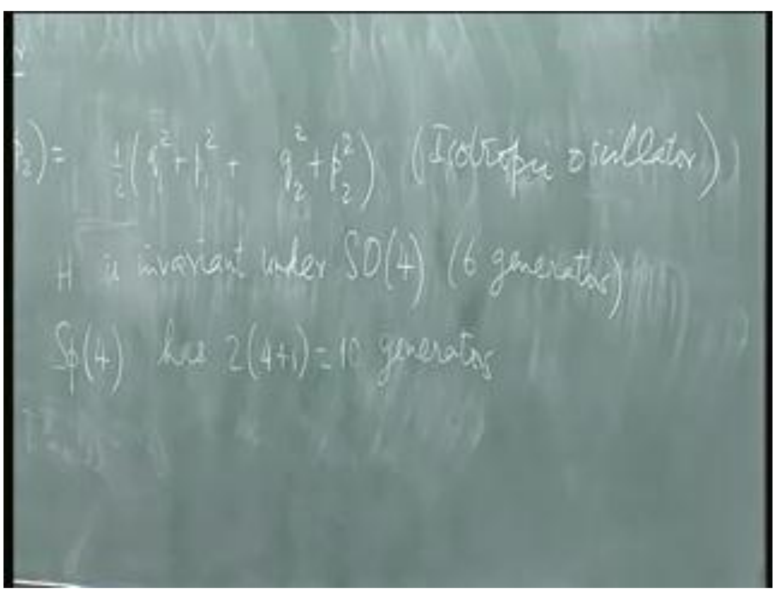

And let us look at the simple example, example 2 D oscillator, let me choose again two dimension simple harmonic oscillator for which we understood the motion. It is going to be some torus and it could be periodic and it could be cos periodic but, look at what happens let me choose the mash to be equal to 1. And let me choose the frequency of 1 of the oscillator to be 1.

So, I write the Hamiltonian q 1, q 2, p 1, p 2 to be equal to 1 half q 1 square plus p 1 square plus omega square q 2 square plus p 2 square. So, I choose the frequency have to be 1 and the other one to be omega and this in general let to immediately let to cos i periodic motion. In fact, it was not periodic even if omega was irrational then omega became rational it is periodic motion but, still fairly complicated is it a central force is it a central force.

If I imagine q 1 and q 2 to be the x and y coordinates of a particle and it is attach by springs, two springs here at right angles then when is it a central force. When both are exactly the same spring constant, then the force is always directed towards the origin right and what would that imply here what should. Omega equal to 1 have to put omega equal to 1. So, I do this and this is called the isotropic oscillators, why do we call it isotropic.

The same force in all directions independent of the direction it is always got the same spring constant, so same in all directions on the plane. So, this isotropic oscillator has an extra symmetry and what is a symmetry it is obvious here, look at it, it has circular symmetry, rotational symmetry not true for the non isotropic and isotropic oscillator. So, it is got an extra degree of symmetry this problem, what set of transformations leaves the Hamiltonian unchanged.

It is obvious from this form q 1, q 2, p 1, q p 2 are four real numbers running from minus infinity to infinity what transformations would leave this unchanged. Rotations not just in physical space, rotations in face space, the four dimensional face space. So, the q's and p's could get mixed up does not matter but, the Hamiltonian would remain unchanged. What group of rotations is that, I mean what what what group is that set of rotations it is S O 4. So, H has is is invariant under S O 4, this is not a group of transformations in physical space.

It is a set of rotations in the four-dimensional face space. So, the first lesson that comes out is that you could have dynamical symmetry arising as a consequence of transformations not just of the potential not just in physical coordinates but, it could also involve the momenta. So, that is the real set of transformations.

So, how many generator does SO 4 have 6 generators right lot's of transformations; what is the set of canonical transformations of this system it is Sp what? Well n equal to 2 in this problem. So, this is Sp 4 4 by 4 symplectic matrices for the face spaces four dimension how many generators does that have, twice 4 plus 1 equal to 10 generators.

So, it is clear these are different groups. So, it is clear immediately from this that all transformations which leave the Hamiltonian unchanged need not be canonical transformations. And similarly all canonical transformations on the system need not leave the Hamiltonian unchanged for a symmetry, you need both of the both these properties.

So, you need some sub groups of this symplectic group which contains the other group but, all members of the other group need not to be sitting here at all.

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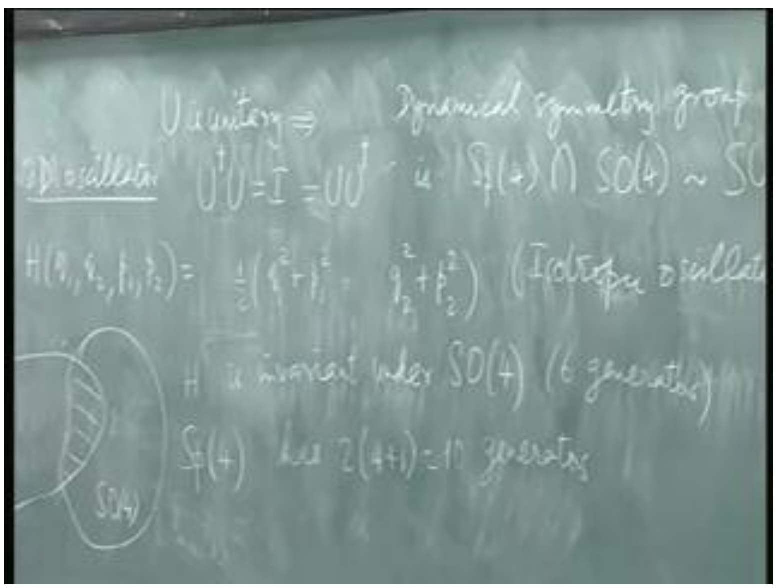

So, it is like this symbolically this is Sp 4 and this is SO 4. So, what should I choose to be the symmetry group, the intersection of these groups. Let us guaranty to the sub group of the symplectic group, I am not going to prove this but, it turns out the symmetry group of this system here the dynamical symmetry group, is the intersection of Sp 4 intersection with SO 4.

And that is isomorphic to or at least 1 to 1 correspondence with the following set of transformations. This is isomorphic to a group called SU 2 it is a set of 2 by 2 matrices which are unitary and have determinant plus 1. The matrices could have complex entries does not matter but, it is set of unitary matrices, what is the definition of unitary matrix?

Implies complex conjugate transports call the hermitian conjugate U on U must be equal to the identity matrix. In other words the inverse is equal to the Hermitian conjugate not the transpose alone, but complex conjugate transposes that is the unitary matrix. Is this also true is this also true for finite dimensional matrices this is true, because the left inverse is equal to the right inverse for finite dimension matrices, but if these are infinite dimension matrices. Then there is no guaranty that U, U dagger equal to I implies you dagger U equal to I does not imply that at all but, we are concerned with finite matrices.

So, it does not matter finite dimension matrices incidentally just for information U dagger U equal to I simply say's that you have property call partial isometric. And similarly U U dagger equal I says you have partial isometry one of them says you have a left inverse but, not a right inverse and vice versa on the other case.

But, if both are true then you have complete isometry which is also called unitarity, then you have a unitary matrix but, otherwise it is partial in our finite case it does not matter this implies that. Now, we are saying that the dynamical symmetric group of this two dimensional harmonic oscillator is that set of matrices which is is that set of transformations which is an 1 to 1 correspondence with the group of 2 by 2 unitary matrices with unique determinant. And then the next question to ask this how many generator does that have and what are these generators. And next time since I have gone out of time we will write down what the generators are.

Mean while I will leave it to as an exercise to verify that the following is true. Verify that in this problem the following are constants of the motions.

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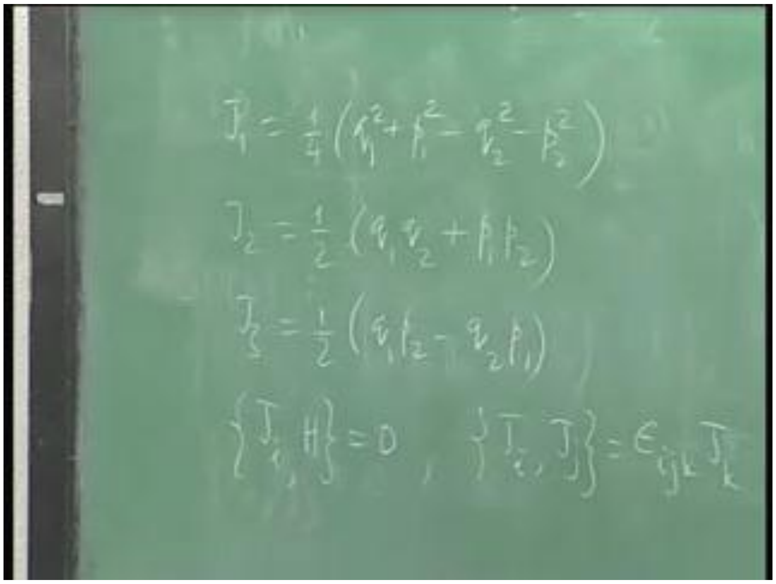

J 1 is 1 quarter q 1 square plus p 1 square minus q 2 square minus p 2 square, J 2 is 1 half q 1 q 2 plus p 1 p 2 and J 3 equal to 1 half q 1 p 2 minus q 2 p 1. Verify that J i H equal to 0, these are constants of the motion, but they not involution with each other. So, verify that J i J j equal to epsilon i j k J k. So, cos of J 1 with J 2 J 3 and so on; in cyclic presentation, they are going to turn out to be the generators of our dynamical symmetric group. They are not an involution with each other we will see where that fits in here. So, even the humble two dimensional oscillator has a great deal of deep mathematics unit at some level and our aim actually is to use this as an example. And then go over to the Keplar problem the 1 over our potential which has an even better symmetry than this, so that is my next target. But this is a preliminary required for that, so let me stop here.