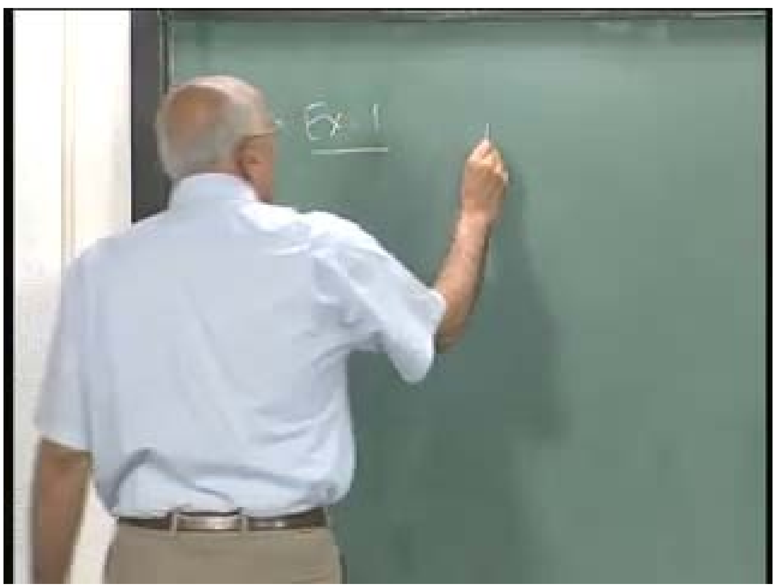
Classical physics
Prof. V. Balakrishnan
Department of Physics
Indian Institute of Technology, Madras

Lecture No. # 12

We ended last time by writing down the Liouville's Arnold criterion for integrability of a Hamiltonian system and I pointed out that, when a system is fully integrable you suppose to have for an n degree of freedom system, n constants of a motion in involution with each other, and then the statement was this is necessary and sufficient for you to find a canonical transformation, which would to action angle variables, after which the problem is in principle solved. Now, what I am going to do now is to give you number of examples and we apply this criterion and ask is this problem integrable or not integrable. And that will decide for us; what we should expect in the general case? So, let us start with the simplest of problems.

(Refer Slide Time: 01:54)



Example one, a Hamiltonian with one degree of freedom H of q p given, equal to p square over 2 m plus V of q say for example; so in this problem n equal to 1, 1 degree of freedom. Is this an integrable system or not?

Yes, it is. I know that H of q p is a constant of the motion and n is equal to 1 and I need just one of them, it is of course an involution with itself and that is it. So, every one degree of freedom problem is integrable, is solvable in principle. You can always write down the phase trajectories with simply our H of q p equal to constant; those are the constant energy curves. What about this example two, with n equal to 2, I have an H of q 1, q 2, p 1 and p 2, and this happens to be in the form H 1 of q 1 p 1 plus H 2 of q 2 p 2. So, it is a two degree of freedom system, which you could regard as two particles, for example, moving on an axis moving on the x axis for instance. But the Hamiltonian happens to be the sum of two Hamiltonians; one of which has nothing to do with the other pair of variables and vice-versa.

Is this an integrable system?

Yes, when are the two constants of the motion in involution with each other?

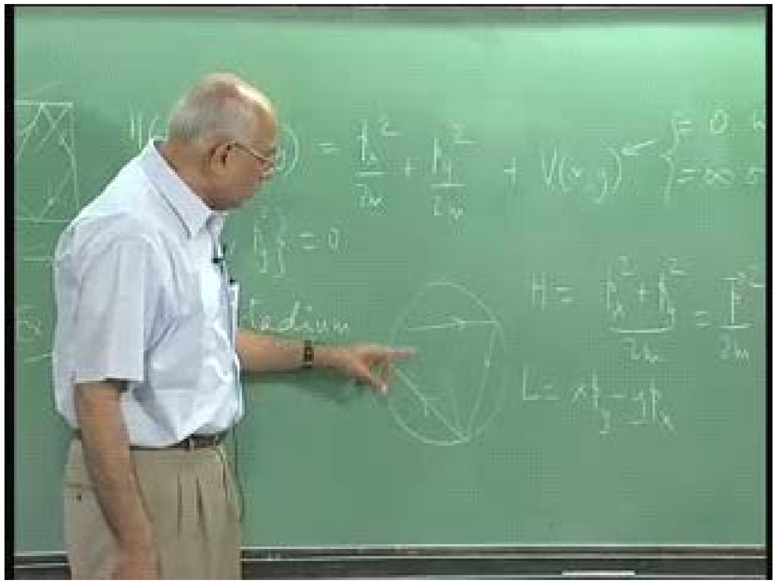
H one and H two, because it will turn out since q one pass on bracket q two is zero, q one with p two is zero, p one with q two is zero and p one with p two is zero. This set of this function here and that functions there are in involution, the pass on bracket it is a guaranteed to be zero. So, your guaranteed that H one, H two is equal to zero. So, you have two independent constants of the motion and this is sufficient, it is just like saying I have two separate particles, one of them here and one of them somewhere else and each of them is one degree of freedom system and its solved completely. What about a generalization to general n, the same thing H of q p equal to a summation i equal to one to n, H i of q i, p i.

Is this solvable? Is this integrable?

Yes, it is just n uncoupled one degree of freedom systems and this is immediately integrable. Where are the n constants of the motion in involution?

The different H i's are all in involution with each other and we are guaranteed, this is trivial and completely solvable. Of course, you see that there were gone to be a problems once you have interaction terms, where the different q’s interact to combine with each other and there are functions which involve q one and q two and so on. Then, it is different story. So, all such separable problems are integral, so it is no problem. Now, let us look at some other examples of interest.

(Refer Slide Time: 05:19)



Let us take a free particle, so this is an n equal to two cases. Let us take a free particle moving inside a square box in two dimensions. Now, you got constraints and so on. So, the particles inside the box and let us say this box is some kind of zero to L. For example, this is the x direction, that is the y direction and that is L and this particle is confined to remain inside this box and it is free, it moves on this plane and it is free no forces on it. What is the Hamiltonian of this problem? So, this is x y and let me just call it p x and p y this is q one, q two, p one and p two just the Cartesian components of the momento. And what is the Hamiltonian?

It is just kinetic energy, there is no potential energy. So, this is equal to p x square over two m plus p y square over two m, this is example three for instance. Is this the total Hamiltonian or is there some other term?

Pardon me

There is infinite potential outside the box; it cannot get out of the box. So, we assuming it are in a box with perfectly reflecting walls coefficient of restitution is unity. There is elastic collusions with the walls of the box, but it is not allowed to go out of the box this is forbidden. So, those are constraints it says in this problem zero less then equal to x less then equal to L, zero less then equal to y less then equal to L. Those are constraints, but they are not holonomic constraints they do not decrease the number of degrees of freedom. So, you leave them as they are and then the problem really has a potential also but then you have to say, the potential is zero inside the box and infinite outside the box. So, there is a penitential V of x comma y and this guy is equal to zero inside box and infinite outside box. Now, where we interested in the motion inside the box?

Well, we simple saying it cannot go outside that is it. So, I do not want to have situation where I have infinite only on a line and may be it will tunnel through and so on we do not want to do anything, only this is just infinite outside. Now, is this an integrable system? We, interested in the motion inside the box nothing more than that; so, what would happen physically if I started with a particle here and I gave it an initial velocity in the y direction for instance, it start from here what would its subsequent path be?

It would just bounce off and come back, it would keep doing this if I start here, it would of course go down and bounce and then by the law of reflection it would do this and it would keep going. So, it can execute fairly complicated trajectories inside depending on what the initial conditions are; but the question asked is this integrable? By which I mean if I specify the initial positions and momenta at p equal to zero, can I predict analytically can I write down what the solution is at an arbitrary instant of time, no matter how long in the future. For this it is necessary and sufficient that you must have two constants of the motion that are in involution with each other. Are they two set constants?

p x square is not a constant of p x is not a constant of the motion, p x is not because as soon as you hit this wall, the vertical wall p x is reversed in sign, if you hit this wall p y is reversed in sign. So, p x and p y are not constants of the motion.

But, p x squared and p y squared are constants of the motion. So, certainly this is true and they are analytical constants of the motion. So, we know that p x square and this is sufficient, it is integrable this is of going to be of some interest. Because, you can know change the situation just a little bit and the system will become chaotic.

Suppose, the Hamiltonian is no longer differentiable then you are in trouble, I assume that all these are analytic constants of the motion. Yes, excellent. In this problem itself, the Hamiltonian is not differentiable it is got infinite discontinuities, that does not bother us so much. The real problem is what happens, if I shoot the particle directly into that corner. What happens now? How do I apply the law of reflection? So, this problem is set to be pseudo integrable, because there are sets of measure zero, initial conditions with a sets of measure zero for which you cannot write down what the solution is. You assume then for simplicity that anything that hits the corner is absorbed and that is the end of it; so, apart from that technicality this problem is solvable, it is integrable and so on.

Since, he does not like the idea of sharp corners; Let us look at a circle and put the particle inside a circular stadium this is like carom coin. So, let us look at example four, circular. These are called stadia, this is called circular stadium and you have a particle inside a circular box and it is confined to stay inside here. Now, the Hamiltonian is still p x squared over two m plus p y squared over two m, but what is there is there a problem integrable? Because, neither p x square nor p y square is going to be a constant of the motion; is this integrable? Because if you hit this is going to do that, and then it is going to do crazy things. So, at each stage you got to find the normal and then you have to find out what there angle of reflection is and so on. There may be some special trajectories some special initial conditions where this guy would just go through a diameter back and forth or it will go in an equilateral triangle and so on. But, in general of course for arbitrary initial conditions that is not guaranteed at all. Is this integrable?

Pardon me

We should go to polar coordinates, plain polar coordinates. Then, the problem appears to become integrable but what is the other constant of the motion? The Hamiltonian is a constant of the motion; the energy is conserved of course. Is the distance from the centre constant?

Not quite; not quite close but not quite. So, what is constant in this problem? Let us look at it from first principles; this problem has no potential it is a free particle. Is angular momentum conserved? Angular momentum about the centre is conserved, because this problem the boundary has circular symmetry. So, the boundary also matters if the potential is zero inside and infinite on the boundary; the boundary has circular symmetry then this problem has circular symmetry you can actually rotate the coordinate axis and nothing will change. So, what is the other constant of the motion? One of them is H, which is p x squared plus p y squared over two m that is equal to the total momentum squared over two m, that is a constant of the motion. What is the other constant of the motion? It is the angular momentum about the origin and what is that?

But, can you be write it in a Cartesian coordinates?

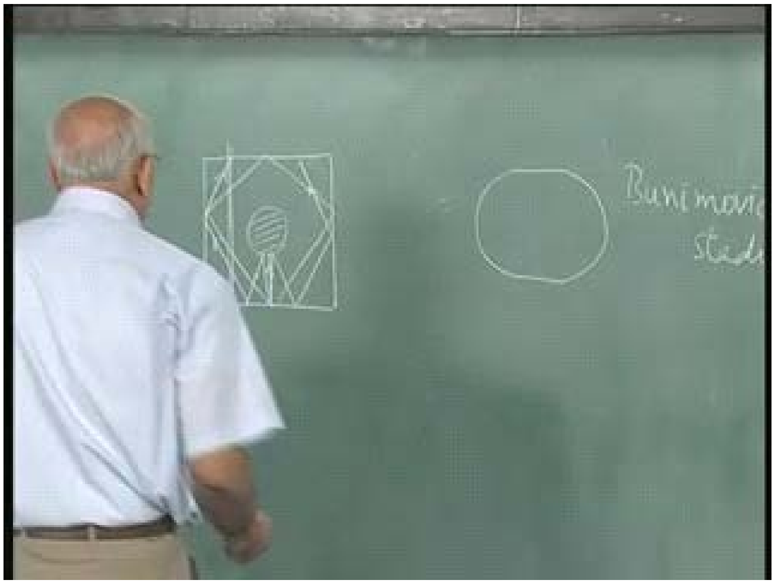
Since, it is a plainer problem angular momentum has only one component; in two dimensional angular movement is got only one component, it is not a vector there is no z direction, it is just a x p y minus y p x that is it. Now, I leave you to check that x p y minus y p x in this problem actually commutes with personal commutes with p x squared plus p y squared over two m. We have two constants of the motion and therefore this problem is solvable. Now, he mentioned something about the distance from the center; now the angular momentum as you can see the magnitude the speed of this particle is not going to change. So, if this is the distance of closest approach then you can actually find the magnitude of the momentum by multiplying this impact factor multiplied by the speed time's m that is going to be constant.

So, it is evident that no matter what your initial condition is this particle either will have a close trajectory or will go on doing this. So, at some stage it will do this it will keep bouncing off and there would be an inner circle into which the particle can never come and outside like those thread work things you have seen the pins struck on the board and then you have an envelope curves. So, this circle inside will form like an envelope curve. But, the problem is integrable; it is solvable completely. So, the circular stadium is solvable.

Now, you can play this game and ask what happens, if you have an elliptic stadium. Next thing is to ask what happens, if you have an elliptic stadium. So, let us look at that, it is a non trivial problem really. So, I have a particle moving in an elliptic stadium with two foci here. Do you think this is an integrable problem? Angular momentum about the center of the ellipse is not conserved, definitely not because in this problem you definitely do not have circular symmetry.

The sum of the angular momenta with respect to each of the foci, this is conserved in this problem. So, this stadium is also solvable this thing is solvable. What happens if I do this, so another example I do not want to number it, because it is not something I am going to discuss now.

(Refer Slide Time: 16:18)



I have a square stadium and I put a circular obstacle inside at the center and I am not allowed to go through into that. So, whenever it hits the particle hits that obstacle it bounces off, by the laws of reflection. Do you think this is problem is solvable? It does not have Cartesian symmetry, because this obstacle does not have Cartesian symmetry it is got circular symmetry, but then the boundary does not have circular symmetry. So, there is a conflict here between circular symmetry and Cartesian symmetry. Do you think this problem is solvable?

You need further information, this is not an integrable system this simple looking; thing is not an integrable system and let me tell you the mechanism by which chaos appears here. We are not going to discuss it in great detail right now and that is the following. What happens if the system is integrable? Is that we saw you could go the action angle variables and once you go the action angle variables, we saw the action remains constant and the angle variable increase linearly in time; which means that if you start with two face trajectories one of them here and one of them here with an enabling initial condition, the distance between these two can only increase linearly in time in the angle variable. And when you go back to the original variables, it would increase in some prescribe fashion some known fashion, but the fact is errors do not amplify in this problem not exponentially any way, you just increase linearly.

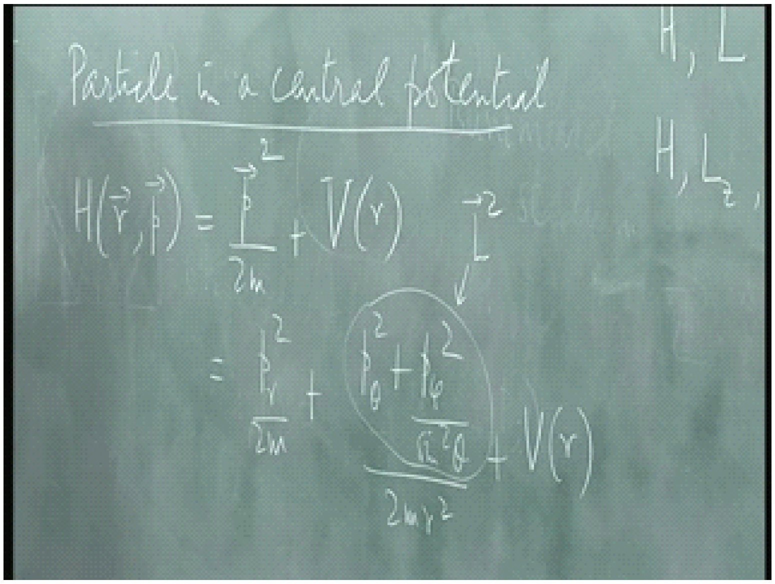
But, in this problem a very simple physical consideration shows you can be in deep trouble, because if I showed a particle at it like this it bounces back, but I shot it a little bit to the right, ever so little to the right. Then, the next time it does this and then it does this and then it does this etc, but ever so little to the left would cause it to take a totally different history and this spreads out. And, any error in that initial angle can actually become as big as a system size itself, any separation initially can become as big as a system itself. Due to the fact, that this Cartesian symmetry is not come insulate with this circulate symmetry here and it does deep focusing effect here. And because of this, the system becomes chaotic; it is not predictable. There are lots of trajectories in initial condition, which you can predict things. For example, if I shoot it like this it would do this or if I shoot it in this fashion, it would keep going in a trajectory of this kind no problem, but there are sets of non-zero measure initial conditions for which the trajectory in future cannot be predictable; it is not computable and this stadium has chaos.

Now, you might say this is very easy, because every time you have something like this a defocusing effect at a convex lens you are in good shape, you can immediately say this is going to defocus and produce chaos. So, to disabuse you of that notion let me point out that the following is going to happen. We saw that the square stadium was integrable and the circular stadium was integrable. So, suppose I take a circular stadium I cut it in two and separate the two pieces and then I have a semicircle and then a straight segment here and straight segment here. The slope at this point and slope at this point, there is no discontinuity. But the curvature has a discontinuity.

Because the straight line has zero curvature and this is finite curvature. So, this stadium very famous one; it is called a bunimovich stadium and it has no convex surface, it has two concave lenses. If you like concave mirrors here and then straight mirrors, plane mirrors, but this problem is also chaotic. As long as this ratio of this straight line segment to the radius of the circle is non zero, you have chaotic behavior. There is defocusing here in this problem and to give you an optics analogy, the defocusing occurs not because of the defocusing of a convex mirror, but it occurs due to other optical abrasions. So, you could have astigmatism for instances and you could have spherical abrasion.

So, those effects could because this is like ray optics as you can see and that produces chaos in this problem. And then, of course you can categorize various kinds of stereotracy which would produces chaos and which would not and most of the time systems are chaotic. So, even two degree of freedom systems because the face pace is four dimensional, there are four differential equations can produce chaos. But, these are sort of mathematical models let us go the physical models, let us go and ask next what happens and that is going to be our example five, I believe or four example five.

(Refer Slide Time: 21:32)



Example five; particle in a central potential in three dimensions; so now we trying to look for physical problems and so we look at the physical problems of a particle in three dimensions moving in a central potential. What is the Hamiltonian going to be? It is the function of r and p and this is equal to the kinetic energy plus a potential v, which is a function of little r alone no angular dependences. So, there are two cyclic coordinates here to start with and the potential there is no r depend, theta dependence, no pie dependence, but remember in p square there is theta dependence.

If I wrote this out in spherical polar coordinates, this is p r square over two m plus p theta square plus p pie square over sin square theta over two m r square plus V of r; this is what you call the square of the angular momentum. So, there is one constant of the motion, how many degrees of freedom does this problem have?

Three degrees of freedom; so you need three constants of the motion in involution with each other to solve this problem. What are they? The Hamiltonian is one. What are the others? Since, it has spherical symmetry there is no talk on this particle. Therefore, angular momentum is conserved. So, it is looks like you have an H and you have angular momentum itself L and that is got three constants of the motion in it L x, L y and L z. Are they in involution with each other? No, unfortunately no. So, that is not true. They are not in involution with each other. So, tell me do we have Constance of the motion in involution?

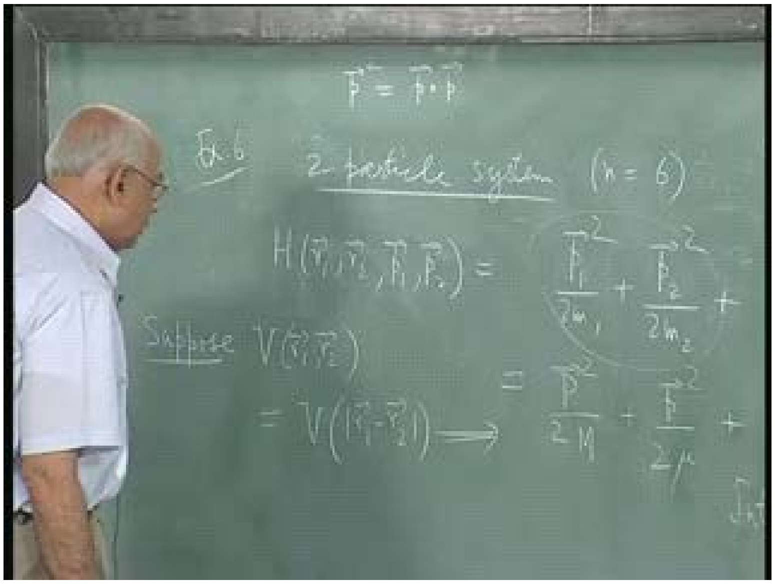
You can choose one component, what would you like to choose?

The z component, the e component does not have to be any component. So, let us choose z component so there is H there is L z, L x L y are out because as soon you choose this there are the two not in involution. Therefore, you cannot choose that. Anything else? Think of quantum mechanics in the hydrogen atom, you have this principle quantum number, you have the orbital angle momentum quantum number and you have this magnetic quantum number. Quantum numbers are like the analogs of constants of the motion here in classical physics. So, what is the other Constance of a motion?

Yes, L square. Absolutely, L square H L z L vector square this pass on commutes with every component you know that L x L y L z do not commute with each other, but each of them commutes with L square and you have to chose any one of them. Now, we have chosen an L z purely out of convenience, you could have chosen any other component could be having done this could we have chosen L dot some arbitrary unit vector n. Yes indeed. Any one direction in space you could have chosen that along with L square. So, these three form three independent Constance of the motion in involution with each other, they are functionally independent of each other; specifying two of them do not specify the third completely and therefore this problem is integrable. The central force problem is integrable in three dimensions.

Notice, I am not made use of the fact that this is of the form one over r it does not have to be every central force problem is integrable, the one over r and r square potential will be very special they have extra symmetries over and about this. There trajectories would have very very special properties, but in principle every central force problem in three dimensional is a single particle is integrable. The rest is beated writing down the solution, choosing the proper coordinates and generalizes momenta and so on. How about two particles interacting in the following way, so that is example six.

(Refer Slide Time: 26:17)



Two particle system, now I have H of r 1, r 2, p 1 and p 2 and let us suppose it is of the form p one square over two m one plus p two square over two m two plus a potential V which is a function of r one and r two in general arbitrary potential. Do you think this is integrable? Remember V is not a function of r one alone plus a function of r two alone then of course, it is separable. But, it is an arbitrary function of both these vectors do you think this is integrable first of all what is n in this problem, n is six; you need twelve dimensional face pace and you need six constants of the motion. We have one, the Hamiltonian is angular momentum concerned.

No, no about what the question you have asked is about what? What would you do to reduce this problem?

You go to the center of math's coordinates, you go to a system where you change variables from r one, r two to the center of math's and the relative coordinate. So, perhaps you do the following, you write R equal to m one r one plus m two r two divided by m one plus m two and little r equal to r one minus r two. Is this going to help? Is this going to help? It is going to help here, so what is happens to this thing? What happens to the kinetic energy, if I do that?

Well, conjugate to this R you would have total momentum P which is p one plus p two and conjugate to this you would have a relative momentum p which is equal to p one minus p two and then what happens to the kinetic energy? You change variables to these guys and then what happens, whether a certainly a contribution which is p square over twice total momentum squared no not reduce mass; this is the total system moving. Total mass M. We put M equal to m one plus m two, plus you see you cannot change the number of variables, does not r one and r two and p one and p two now you got a little r and big r little p and big p, that is it.

What happens to the remaining terms, this is p squared divided by twice capital m. This is the relative momentum squared, reduce mass here this is reduce the mass. So, it is simple exercises, you know how the reduced mass is defined?

m one m two divided by m one plus m two. So, that part is fine plus what happens to v you should do these exercises at some stage, you should take is not you know change these variables plug it in see, what happens; what happens to V? It some arbitrary functions something else some U of little r and capital R, there is nothing u can do about it; that is it. Is really nothing you can do about it in general? So, would you say this problem is integrable? You gone to the center of mass coordinate, but it has not produced any particular simplification. But, suppose V of r one comma r two equal to a function of the distance r one minus r two alone. Suppose, that is true then what happens? It instantly says this becomes V of r implies this. Is there cyclic coordinate? What is that? Capital R is the cyclic coordinate. So, capital R implies capital R is a cyclic coordinate.

What constancy of what constant of the motion does that yield at once? Capital P, that says the entire centre of mass moves either at constant speed forever or stays at rest depending on the initial condition; inside something is going on. So, this immediately implies P is a COM. Is this problem integrable? Would you say this problem is integrable? Where the Hamiltonian is the constant of the motion, p is the constant of the motion. So, is that integrable? Why not? How many constants of the motion you need now? So, you got six of see this problem with the capital P is completely solvable, it is totally solved then you have left with this because this is like saying I have two decoupled Hamiltonians; one which involves a capital variable, one which involves small variables, the relative variables. The capital variables is a free particles the center of max acts like free particles of course you can solve it. Hence, immediately solved it is just gone a move into a straight line at constant speed, whatever be the initial speed and that is the end of it.

So, this is like reducing it in to a decoupled system. This is done it is finish, and then you left with a single degree of freedom in three dimension. So, three degrees of freedom here corresponding to little hours components and now this problem here is a central force problem, which we saw a solved. So, the orbital angle of the momentum is a constant.

So, that Hamiltonian together with the remaining two that L square the corresponding L square and the single component is like a central force problem and it solved. So, this is solvable, it is integrable. As long as the potential is a function of the difference between the distances between the two particles it is solved, it is integrable completely. Because, two body problems is reduced to two one body problems one of which is free motion and other is just simple central force. So, this is very much integrable.

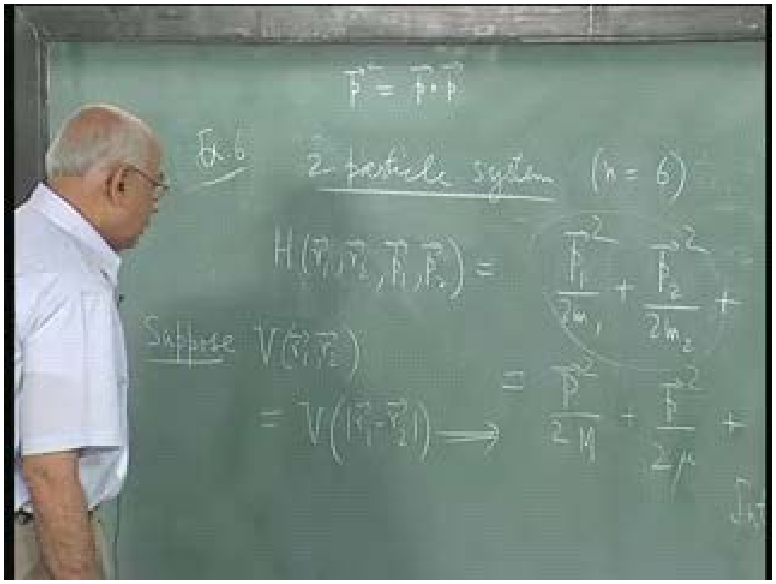
So, I am going to leave it to you an exercise to figure out, where are the constants of the motion you must checked there in involution with each other, six independent constants of the motion in involution. So, the hint I am giving you is that it is like two one particle problem and this set of three coordinates and that side of three coordinates are nothing do with each other anymore and therefore it solvable completely. So, this is very much integrable.

Yes

Yes, that is a vector. So, this stands for P x P one x P one y P one z etc it is moving in three-dimensional space.

I am not able to write bold face on the board so I put an arrow but then I put a square there I so stands for p dot p, which is a scale. So, it is a certainly a scale. So, this is solved. Now, suppose you have three particles this is the very important problem, because you have three particles and then let us assume that there interacting with each other by gravitational force for example, which depends on the distance between the two particles, so what would you say is happening, example seven.

(Refer Slide Time: 35:58)



Let us, in fact look at the n body problem. So, you look at the n body problem this Hamiltonian is a function of r one to r N, p one to p N and that is equal to a summation i equal to one to n p i square is put a bracket here over two mi that is the kinetic energy plus and now, yeah.

That is right pardon me, what is an potential yes No, the two particles experience potentials only due to each other, so there is no external potential added.

This is a function of r one and r two if you like you can add to this a constant potential, that is not gone to effect things. It might break the symmetry of course if I add a gravitational field might break the symmetry then you have a problem. So, you may not be able to right this anymore, but the assumption I made was that this is a set of particles interacting with each other by pair voice forces, in fact in this case just two of them and the force is entirely a function of the two coordinates.

Of course, I assume no velocity dependent forces then that was not good enough so, I said the force between these two particles is derived from a potential. So, it is a conservative force and that potential depends on the mutual distance between the particles. No non central force here and that immediately gave me a scalar it said this potential depends only on this distance and that got reduce to a central force form for the relative coordinates. And therefore, the problem was solvable. Now, with this understanding that we know this problem is reduce to in this fashion I try to do the n body problem.

So, now let us assume I know that the n body problem with the arbitrary forces is not integrable; even two body problem is not. So, let us assume that this potential here is V which is a function of modulus r i minus r j summed over i j equal to one to N and i not equal to j, simply to ensure that I am not adding unnecessary terms which corresponds to self interaction. Each point particles sees the force is due to all the other particles, but their pair voice interactions particle one interacts with two, three, four, etc always depend on the distance between the given particle and the other particle.

Excellent question, Are there two three body forces in, the answer is yes. This is a heart problem nuclear physics is happens all the time then of course, this whole things goes out of the window. There are three body true genuine three body effects to occur, we can give probabilistic arguments about how often they occur so on and but they do, they do happen.

Yes

Any number, yes this is possible in principle; yes it is possible an N body problem in quantum physics this is quite routine you really have N body potentials, but there is also another possibility the force between two particles need not depend on the just the distance between them. You could also depend on the vectors r one and r two, can you give me simple example of this? A simple example, with which you already familiar.

Yes, in electro magnetism where even an electro statics this such as a simple force, what happens if you got two points dipoles electric dipoles P one and P two? What happens to the potential energy between them so, you got a P one here to use the word the symbol P one, let us call the electrical dipole D one so, let D one here and little D two here and the distance between them is r. So, let us say this guy is at coordinate r one and this is at coordinate r two and what is the potential energy between these two dipoles; they are point dipoles. So, you assume that the length of the dipole goes to zero the charge goes to infinity. So, the product is one point is infinite, what is the potential energy?

What is a potential energy between two dipoles, I am studied electrostatic so there suddenly a term which is d one dot d two incidentally this must be a scalar, it must depend on the distance between then and so on. So, d one dot d two divided by r one two cube apart from one over four pi epsilon not which is matter of units. So, this is proportional to this term here it must linearly depend on d one, it must linearly depend on d two we got d one dot d two over r cube r one two cube, but that is not enough because I know that the force between two dipoles like this is different from the force between two dipoles like that.

So, it cannot just depend on the distance and certainly this force is different from that force. So, what is the next term? There is also a term which is three d one dot r d two dot r over let me just call r, let us call this vector r the difference between the two and what is the power here?

Five, because I put to factors is here so you see this depends on the relative orientation of the vector r, the distance between these loop the vector joining one to the other with d one and d two. It must be symmetric under the interchange of d one and d two, it must linear in d one, linear in d two and you got three vectors to play with d one d two and r, we cannot depend on the overall origin but only on r.

So, these are the only possibilities. So, it is linear dependents and incidentally it should be the same whether you check r to minus r or not. Because, whether you call r the vector going from dipole one to two or two to one does not matter and that too is preserved here; this is not a central force.

Well, lots a question of you know it is a question of resolution, if I have an atom then on distance is much larger then the atom it would certainly have a dipole could have a dipole moment or magnetic dipoled moment. So, it is not a question of whether something exists physically or not. I know for instance if I have an arbitrary charge distribution I could resolve it the potential due to this from, as if you had an effective monopole and then a dipole and a Quadra pole and so on and so forth.

Since, the resolution into spherically into components whose transformation properties under rotations are known to be a question of approximation; I mean a ceiling fan is a dipole in some approximation, because what was the force line due to the fan due to a dipole look like; they come out like this and they go back in this fashion is not it. This is what dipole force is going to look like. The field lines are going to look like this they go out this way and they come in this way. So, you could imagine the center of the ceiling fan it is sucking in air from an above and pushing it through and then going back.

So, and some approximation it is really like a dipole source of a velocity field. So, this is a non central, this is an example of a non central. But, our interest now is in trying to find out whether this problem is solvable or not. Because now we made all assumptions we need you said you got n particles and a c forces is only due to each and the force between any pair of particles is dependent only on the distance between them. It is directed along the line joining these two particles, so that is a scalar v of mode r i minus r is scaled. And these whole thing is spherically symmetrical in the sense that if I a rotate the coordinate system mode r i minus r j does not change. Therefore, the potential does not change, so it has spherical symmetry. Would you say this is integrable?

In the two body case, it was but in the three body case let us right down all the constants of the motion that we can right down. So, COM's the first one is a Hamiltonian itself and the second one would be the total angular momentum of this system, that is certainly a constant of the motion. What is the total angular momentum? Summation over i equal to one to n r i cross p i r cross p is the angular momentum for each particle, the orbital angular momentum about some origin incidentally; this Hamiltonian is invariant under shift of the origin.

Because, it only depends on the distances so, you can choose the overall origin wherever you like and certainly L equal to r cross p is a constant of the motion. There are three Constance of the motion there; three components. Anything else is a constant of the motion? How about the total momentum of the system? Is that? That is a constant that is external force so that certainly a constant generalization of Newton's third law totals momentum. Because, there is no external force on the system, so three capital P equal to summation i equal to one to N P i that is a constant of the motion. How many do we have now? We got one plus three four plus three seven. What is the little N equal to?

No, then I am only labeling now right now, I am only listing the constants of the motion after that we got examine how many of them are going to be in involution with each other. So, that is for integrability you need that but what is n equal to three N. Keep that as the back of our minds we found grand total of seven so far. What about you see I know this guy, I know this is the constant of the motion and therefore I know that R of t equal to R of zero plus P t over M. I certainly know that I know the center of mass of this entire system is going to move at uniform speed a constant velocity simply, because there is no external force on it and that is depends on the initial condition. So, other constants of the motion there?

There are. So, we have four R minus P t over M the time dependent, but then we yesterday we saw that we could have time dependent constants of the motion. How many are there now? Ten. They call the ten Galilean constants of the motion; the Galilean invariance. In the absence of further information, that is it is really nothing else you can do. Now, we still have to worry about the her problem which is how many of them are going to be involution with each other, it is clear the three components of L are not in involution with each other. They have to be involution with each and with the Hamiltonian of course, each of them is involution with Hamiltonian as a constants of the motion.

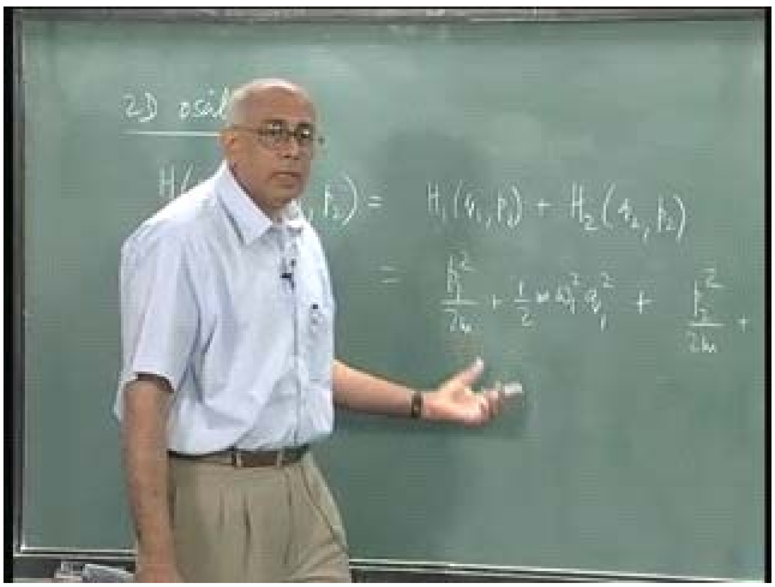
There is the three components of P they would certainly be in involution with everything. p with L is not true immediately it is called in any case you need 3 N constants of the motion, because we talk of integrability and you have a grand total of ten an even they are not involution with each other. For n equal to three, the three body problem only nine constants of the motion and you are far below that, even for three particles. Let us forget about ten to the twenty-three particles. So, the system is badly chaotic even the three body problem is not integrable except in very special cases, then of course you have ten to the twenty-three particles is gone. We saw infact that if you have a single particle inside a stadium with a fix scatterer who is got a different symmetry in two dimensions even, that is chaotic.

So, certainly what is happening to the gas in this room is highly chaotic there is no possibility of solving it. So, the reason why you need statistical mechanics partly is not just that you have ten to the twenty three particles and even god cannot write all those equation down. But even if you were three particles the situation is bad enough it is gone; it is cannot be integrated in general.

It is a very good question, how do I know I found out everything. It is possible to find out we can make a for the statement, the point is it is a very good question because we have to ask where are these constants of the motion coming from? What is the represent and the answer lies in theorem which I am going to do tomorrow, mention this. This constant of the motion associated with a symmetry of the system always symmetry. We saw if got spherical symmetry or circular symmetry, it had angular momentum as a constant of the motion. So, symmetry is going to imply invariance which is going to imply a conservation principle or conservation law and that is going to give you conserved quantities. So, this is where the constants of the motion come from and then you have to ask for what is the symmetry of this Hamiltonian and the most general Hamiltonian does not have any much more symmetry than spherical symmetry over all that is it.

So, really you cannot do much more so we have to now go back and ask what simple Hamiltonian looks would like. What kind of symmetry they do have and then we take lesson from that and see what happens in the general case. So, we now back track and let us go back to a problem which we can slow completely which is integrable and look at what is symmetry is and how that symmetry can be broken and the simplest of these examples is to go back and look at two simple harmonic oscillators, because one simple harmonic oscillator is doable completely.

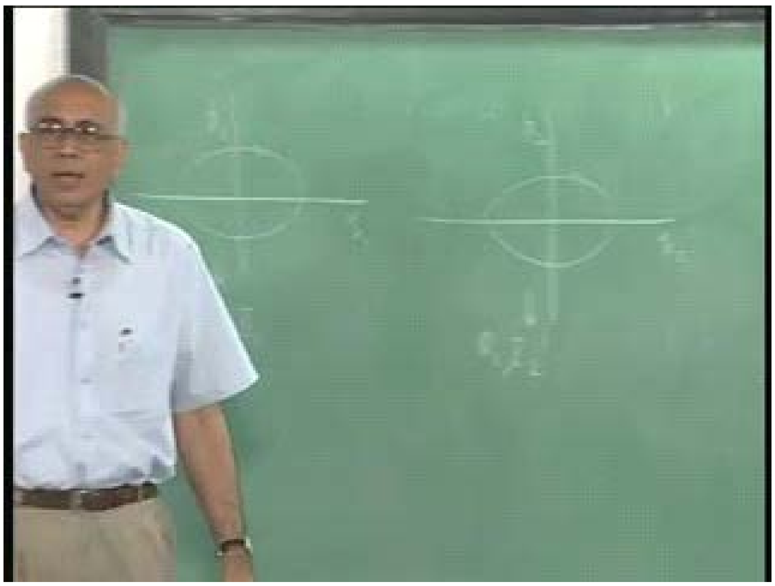
(Refer Slide Time: 52:41)



So, now let us go back and look at 2 D oscillator and by that I mean the situation in which H of q one q two p one p two is just two linear harmonic oscillators uncouple to each other. So, you could regard this as a single oscillator which is got a force in both x and y directions or you could regard it as two simple harmonic oscillators uncouple from each other does not matter. So, let me right as H one of q one p one plus H two of q two p two and let simplify things a little bit by saying this is equal to one over two m p one square; let us right it out p one square over two m plus one half m omega l square q one square plus I know that the actual property of oscillators basically the ratio of spring constant to the mass that is the relevant parameter in the frequency.

So, let me call the frequency omega one and let us take the second oscillator to have the same mass, but perhaps a different frequency q two square this fashion. I can go to action angle variables this problem is totally solvable, we know this is an independent oscillator that is an independent oscillator and each of them has a phase trajectory in it is q p plain which is an ellipse of some kind. But the real system has four a four dimensional phase space and all we can do we cannot draw the phase trajectories, because I cannot draw four dimensions. But, I can draw projections of the phase trajectory on to various plains so I could do the following and start by saying asking what happens in the q one q p one plane.

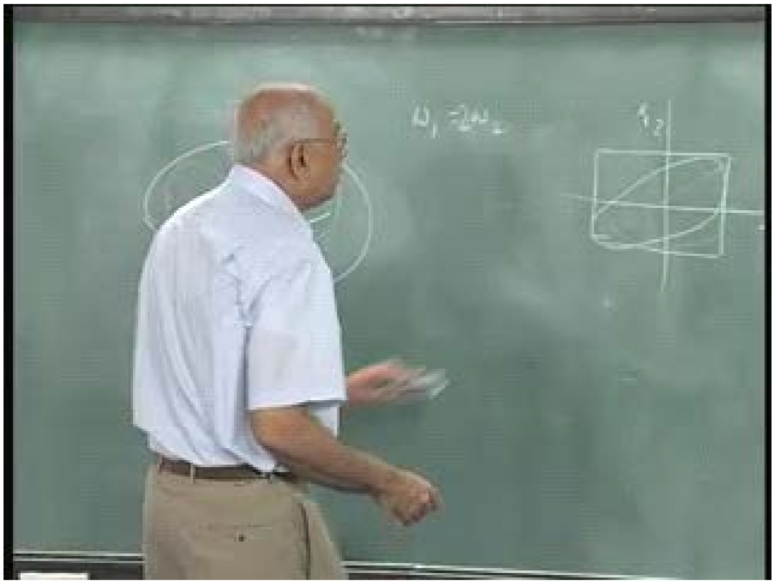
(Refer Slide Time: 54:32)



So, here is q one p one and what happens in the q 2 p 2 plain well, this guy would go round in this fashion and this fellow would also go round in this fashion. And, any given instant of time you are on some point on this ellipse and some point on that ellipse and they need not be in phase. These two oscillators need not been phase at all; this problem is solvable. I could if with a little bit effort draw the trajectory in the p one q two plain q two p one plain and so on q one q two plain and so on and so forth. What you call the trajectory drawn on the q one q two plain?

The two oscillators they could regard them as right angles they like figures. Now, we have got sophisticated we are not gone look at it in these coordinates, we know I can we can go to action angel variables and this action angle variables I define, so from here I go to an i one the theta one i one and form here I go theta two i two and I know that i one and i two are constants. Therefore, let us just look at it as a function of theta one theta two. What would they look like? What would the space look like?

(Refer Slide Time: 56:06)



Well, to describe theta one I would like to take an angle and to describe theta two I take another angel and the space is the space which is the direct product of two circles if you like and that is a two dimensional torus. So, let us draw this torus in this fashion and as I move along this tube in this direction, I thought say it is theta one and as move in the transverse direction, I call it theta two. And the point the phase space point you could go a little further you could say let us take the diameter the radius of this tube to be proportional to i two and the radius of this guy to be proportional to i one. So, you can see this entire four dimensional space is laminated by these torie and each torus specifies what i one and i two are and the motion on the i one direction has time period two pi over omega one and the other direction has 2 pi over omega 2.

Now, tell me what would the trajectory look like in general? Suppose, omega one equal to omega two a simplest case then it means that as the representative point moves once around in this direction, it moves once in this direction also. So, it is sort of winds round and comes back and what is the figure looks like. If this is the amplitude in the two direction and that is the amplitude in the one direction what would the Lisa to figure look like.

In general, this would be a periodic curve it closes so what would it look like well, if these two fellows are in phase then of course, it will just do this. But if they are not in phase, then in generally it will be an ellipse of some kind, if there exactly 90 degree out of phase it would be an ellipse of this kind. If there is some arbitrary angel out of phase it would be an ellipse of this kind, this is when the two frequencies are equal. Is the motion periodic?

Yes, it is periodic. What if one frequency is twice another frequency? What if omega one equal to twice omega two? Then, it says as the system goes around here once this it makes two curves on this side, but it would do I cannot draw this two well but do some crazy thing. But it would come back to the same point and the Lisa to figure in this case again very imperfectly, drawing it very imperfectly could perhaps do something like this like a figure of eight but it would close on itself.

What if omega one over omega two is a rational number P times omega one is q times omega two r and s for example in r times omega one is s times omega two, where r and s are integers. Then again, the Lisa to figure gets more complicated but it closes on itself. What if the frequency ration is irrational? What happens then when you see you might have seen these Lisa to figures they densely fill up this rectangle and the system never returns to it is original point, no matter where you start. So, it is periodic in q one in q one p one periodic in q two p two, but the two periods are not commenced with each other. So, therefore the face trajectory on this torus will never close, it will never close it will be like a bowl of thread going like round and round on top of this but never coming back to it is original point.

And, we can simplify matters little bit like by saying well let us write this Hamiltonian as p one square plus over two m plus half m q so, let us put p one square over two m plus half m, I put omega one equal to one so q one square plus p two square over two m plus half m omega square q two square. And, this is an irrational number the ratio of frequencies is a rational omega is irrational, then this Hamiltonian is integrable it is completely solvable just the some two oscillators; but on this torus the trajectory does not close. Would you say this is periodic motion? Because for me, periodic motion is when all the phase space variables, come back to their original values after finite amount of time.

So, the motion is said to be quasi periodic it is not periodic in general. So, omega irrational omega one omega two irrational implies quasi periodic motion. And, it has strange properties and you could do the following you could say let me set this frequency equal to one and look at it as a function of this second angle. So, I put a cross section here and this cross section is called a Poincare section, is an example of Poincare section. Now, I ask when does the trajectory hit this circle so what is happing is that is going round and round an every time it hits the circle I note it down.

Therefore, that circle if I look it at separately it started here then it went round in the other direction it came back here and then it went round in other direction, it came back some where here and so on. This is like saying each time you add an irrational number if this as unit circumference your adding an irrational number to the circumference. Of course, if you add a rational number then every point will come back to itself after certain amount of time. But, if you add an irrational number then there is a theorem due to wire stars with says if you take unit circle, circle of unit circumference add an irrational number.

So, what are you doing you saying theta n plus one equal to theta n plus an irrational number omega modulo one modulo two pie or one it does not matter this modulo one and this is irrational. Modulo one means you remove the integer and you come back. So, you start here and then you go here, you go here and you keep going this you never come back to this point and then you never over shoot or under shoot it and then it does this and so on.

And what happens is that this point of intersection given enough time will fill up the entire circle densely and uniformly. No particular point on it is preferred over anything else and this trajectory on the two dimensional torus will fill up eventually any initial condition will eventually fill up this entire torus. And the system is now set to be quasi periodic and since any initial point visits the neighborhood of the entire torus the system is said to be ergotic on the torus.

You are adding on irrational number.

I do not care what theta is it does not matter, any theta. So, do this experiment on the pocket calculator on a simple computer. You cannot add an irrational number because you will always end with a finite precision on your computer. So, you really cannot add an irrational number so what would you do, this is simple problem start with number theta not between zero and one it does not matter any number, add two it an irrational number throw away the integer and keep doing this and keep track of all the iterates and you will discover that they form histogram on the unit interval which is uniform, completely uniform never comes back.

But now I leave it to as a problem as to how you would add an irrational number on a computer, because any number you specify on the computer would be a rational number it terminates after certain stage, the decimal point terminates after certain stage. So, what would you do?

You have finite precision and this now leads us into very certain questions you should try and add as irrational number as possible, you should add a number which would be rational eventually, but as irrational as possible by that I mean the periodicity will be very large. In other words, if you express this number as a fraction you express it completely as a fraction exactly you are gone but the best approximation to this would be you should be fraction with very large denominator. Then, of course you know the period is very long, if the period is twenty-five thousand and six hundred and forty-two then you do not really care on the finite mode of time.

So, you should try to add a number which is very got a very large it is a very irrational square root of two minus one, good number for a reason I have explain later on. A square root of five minus one divided by two this is a very good number to add.

So, I wanted to say that it becomes ergodic where that I meant that any initial condition is going to visit arbitrarily close to every other point on the torus infinitely often as you keep going and with equal frequency.

No, it is unit circle so otherwise you write modulo two pi add an irrational number. So, that is why I said one because otherwise it is two pi irrational modulo two pi it becomes so I take a unit circumference. And, this is a very remarkable theorem were it has it is magic property and you see how number theory is getting into this whole game, you see how the property of irrational numbers, rational numbers are getting in to this game. So, you should remember what irrational numbers are they numbers at cannot have a decimal expansion, which terminates or recurs or if you like they are numbers which cannot be expresses as a ratio of two integers p over q that p in q is not zero.

What was that? I like choosing one twice, I am saying add an irrational number I said I choose the circumference to be equal to one rather than two pie. So, in his question would be equal to I am saying what happens if I add one, then of course you back but one is not irrational, you got it choose an irrational number between zero and one; number which is an element this range between 0 and 1. So, do not choose 0 or 0 1 of course, is trivial. Zero is the identity map, it does not do anything and one is also the identity map its say every point goes back to itself.

But, any number in between will see there are no periodic orbits. There are no exceptions, there are no numbers which you come back to themselves no initial conditions will come back to itself. Now, in general therefore if I have an n dimensional system, n oscillator the same thing would happen and the big lesson is that if I take a general Hamiltonian system and the system is integrable, the motion in general is quasi periodic. Bounded motion is in general quasi periodic, but the big difference between the harmonic oscillator and the general system is that the time period of oscillation would depend on the energy or the amplitude or the action variables, where as in this problem the frequency of going around that does not depend on the size of this torus, does not depend on the action variable.

But, in the general case of course that been non linearity then it will depend on the size. And then now you have to start imagining I have the n dimensional phase space, two n dimensional phase space in which I have n dimensional tori. So, the question is what happens in between the tori, if the system is full integrable then every point is on a torus. But, if it is not there are regions of chaotic behavior in between and then the next question is can it is region escape and this will again require a little bit of high dimensional of imagination, but it is not so difficult to come to that we need this we will get to it. So, let we stop here.