Classical Physics
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Lecture No. # 10

So, let us start today with slight digression, a mathematical digression, and then I get back to Lagrangian dynamics; I would like to move over to Hamiltonian dynamics; and this requires a mathematical transformation, with which you are already familiar in other context, but perhaps the name was not used and so let me do that and point out what the advantages are.

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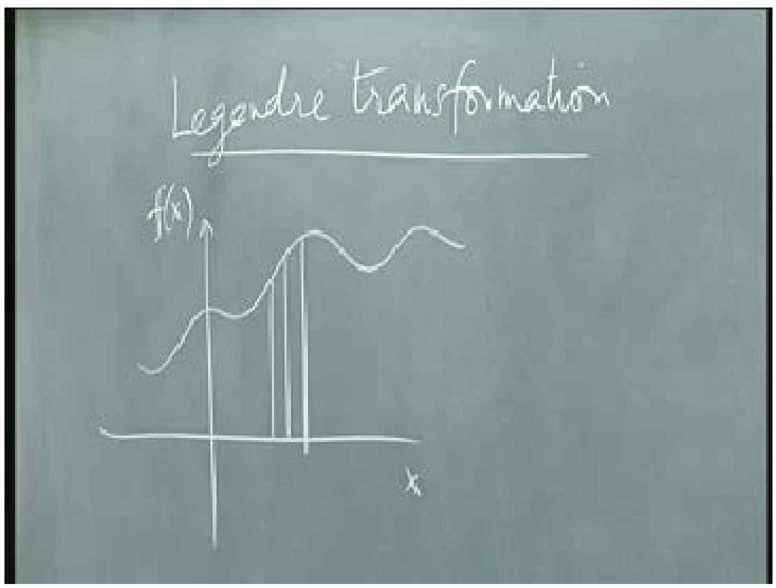

And this is the so called Legendre transformation; and the idea and essence is the following. If I give you a simple function of x, a single variable x, and there are two ways of looking at this function, one of them is to say that for every value of x, I assigned a value of f of x and therefore, if I assign this set of numbers corresponding to each value of x. Specify, numerical values of f of x for each x, I have specified the function.

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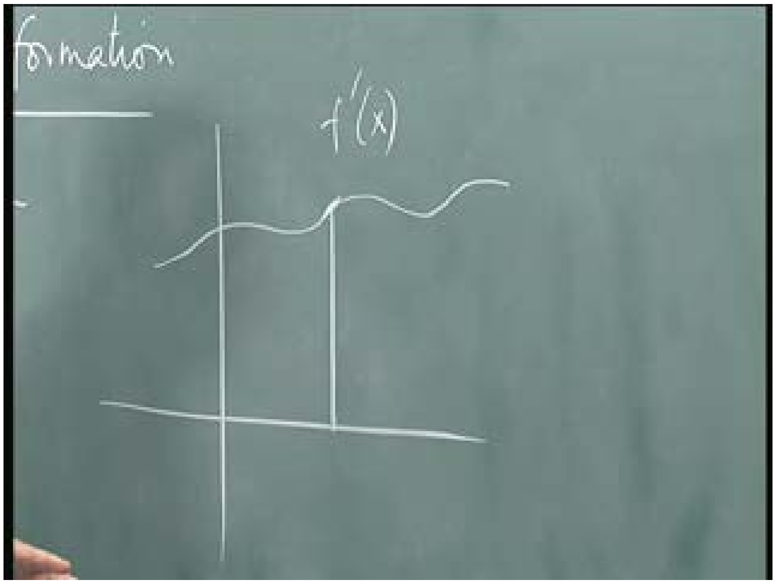

There is another way of doing it and that is to say, I assign at each point the value of the slope of the function. And at the next point, I assign the slope once again and the slope once again and so on and in this manner I can build up the function if I specify, at each point f prime of x at each point provided of course, I also specify f of x at some reference point otherwise of course you could shift this entire graph parallel to itself and it would be exactly the same as if I before just gave you the slope but, if I also tell you f of zero for instance then you know where the intercept is and after that you can reconstruct the function. This is the idea behind Legendre transformation. Except that is not very useful in 1 variable but when you have more than one variable it becomes more exceedingly useful.

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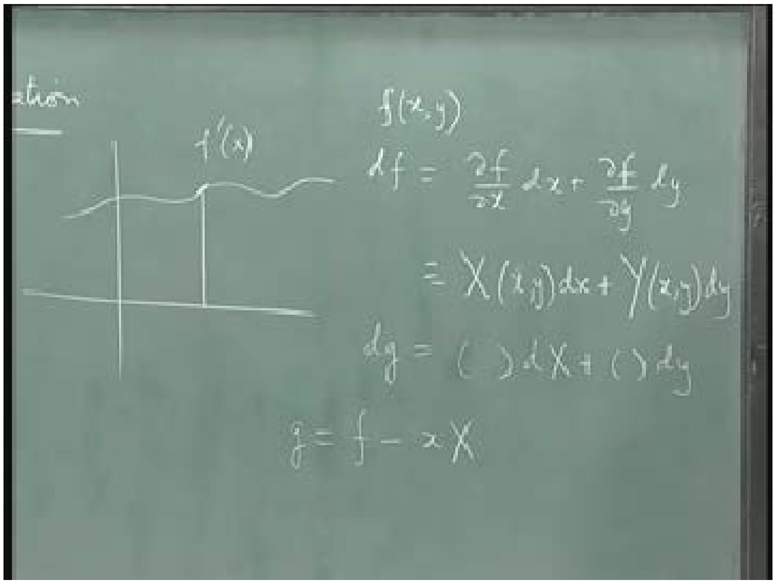

So, let me give you an example, suppose you have a function f of x, y then of course, you know that you can write its differential df as delta f over delta x dx plus delta f over delta y dy, and this quantity delta f over delta x is some function of x and y so, let me call this capital X of little x little x let me use curly x little x in this fashion. So, distinguish between these two quantities and capital Y of x y dy by definition capital X and capital Y are these partial derivatives. And now, I might want to say that instead of looking at little x and little y as the independent variables I look at capital X and little y as the independent variables, in other words, the slope with respect to x.

If I want to do that and I want to construct the function g such that dg is equal to something times d capital X plus something else times d little y in recognition of fact that this quantity this function g is a function of capital X and little y the trick is very simple all I have to do is to subtract from f minus little x times capital X. Then of course, if I take f lets define g as equal to this dg is df minus X capital dx minus capital X times little dx and of course, that term cancels out and you are left with dg to be a function of dx and dy combination of dx and dy. So, this is the idea behind Legendre transform.

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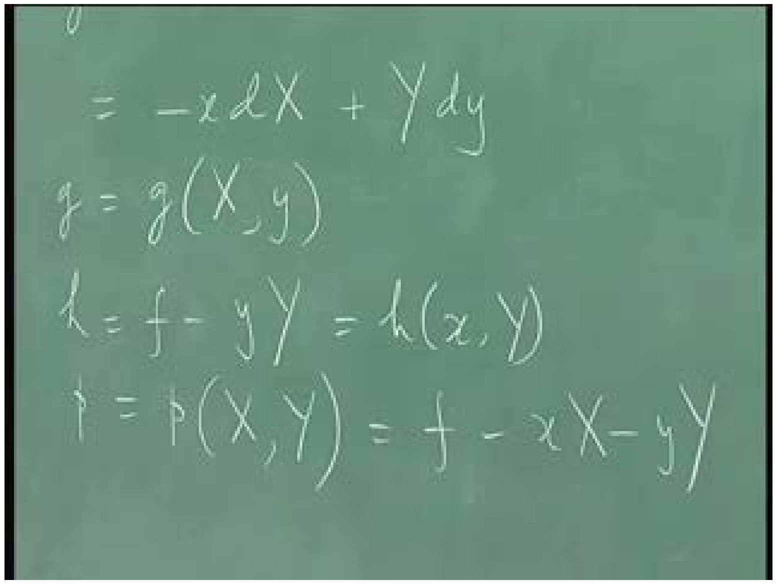

Invariably you have to subtract out so lets write down what dg is df minus x dX minus X dx and that is equal to minus x dX plus Y dy, because this term cancels out on both sides. But, you must remember that this little x must be now re-expressed in terms of capital X and little y.

So, you need to take this equation X of x, y equal to delta f over delta x and you have to solve this equation, for little x and express little x as a function of capital X and little y. Once, you do that then you guarantee that g equal to g of capital X and little y. You could go on and say can I define a function h which is a function of little x and capital Y, then of course you can do that by defining h is equal to f minus y times Y is equal to h of little x and capital Y like one more Legendre transformation. And finally, you could transform the capital X and capital Y all together so, you could define function say p equal to p capital X capital Y this would be f minus x X minus y Y. You can generalize this to any number of independent variables x y z etcetera.

So, the whole idea is simply to see what is the most convenient function for your purpose very often this fs and gs and so on. There will be some kind of potentials or some kind of functions like the laplacian, the lagrangian, which I would like to transform to a set of different set of independent variables for some reason or the other. You are already familiar with this in another context, what is that?

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Pardon me; the Gibb's potential in thermodynamics used to this thing because if you recall we are going to do thermodynamics but let me recall to you what happens.

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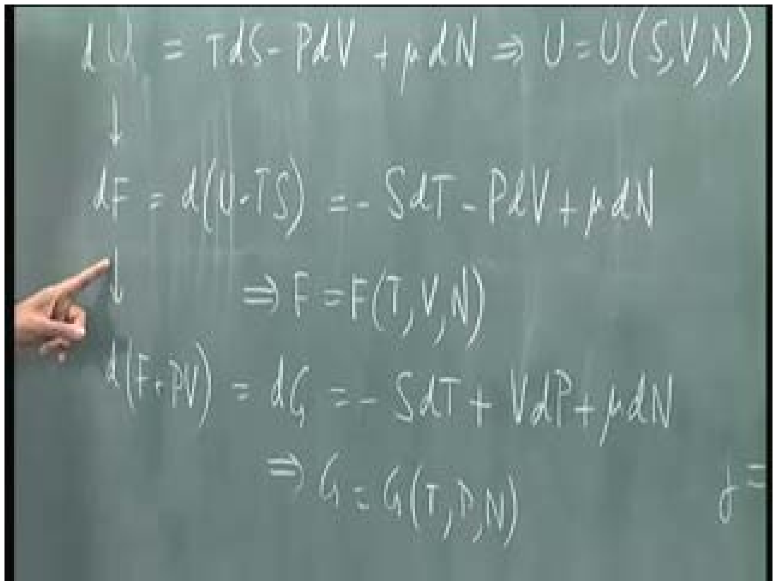

You start with the internal energy U and this internal energy you transform and what is this a function of what is dU a function of, well dU is T dS minus P dV, for fixed number of particles. So therefore, U is a function of S and V. So, this in general dU, the first law of thermodynamics can be written T dS minus P dV plus mu dN actually if you change the number of particles you also have a chemical potential times the differential in the number of particles. So, this of course implies that U is a function of S, V and N as the independent variable and it is obvious from this, that T is delta U over delta S keeping V and N constant, they are the partial derivatives.

Then you might want to consider a function dF, which is equal to d of U minus TS and that will now become a function of T V and N because dF the T dS part is going to cancel from here is going to be T dS and an S dT so, it becomes dF equal to minus S dT minus P dV plus mu dN. This implies, that of course F is a function of T, V, N. So, what you have done is Legendre transform going from the internal energy U to Helmoltz energy F because, you would like to have as controlled variables the temperature volume and number rather than the entropy volume and number.

Depending on your conditions, you might further want to change from P dV to V dP because, you might be able to control temperature, pressure and number. Then what you do? You go from this F by subtracting F minus PV plus PV because, there is already minus sign here and what would this give you dG here and that is equal to minus S dT plus V dP plus mu dN and that implies that G is a function of T, P and N. Because, dG is proportional to a combination of dT, dP and dN.

So, depending on what your external conditions are, what your experimental conditions are? If you can handle temperature, pressure and the number as the control variables, then you would like to have the Gibb's free energy as a thermodynamic potential. These are all thermodynamic potentials. And you go from one to other depending on your convenience, depending on the kind of processes you want to look at and then, of course you know in thermal equilibrium in a state of thermal equilibrium, if you are a constant entropy, volume and number then the internal energy is at a minimum.

If you are in a stage, where the temperature, volume and number are constant, then in thermal equilibrium the Helmoltz free energy is at the minimum and similarly, for the Gibb's free energy. So, this is the reason for going from 1 thermodynamic potential to another simply so, that you can handle first write the minimum principle and secondly you can, decide on what your experimental control variables are? Much the same thing is going to happen in the Hamiltonian framework, going from the lagrangian to the Hamiltonian, its going to be a Legendre transform so, how do I do that?

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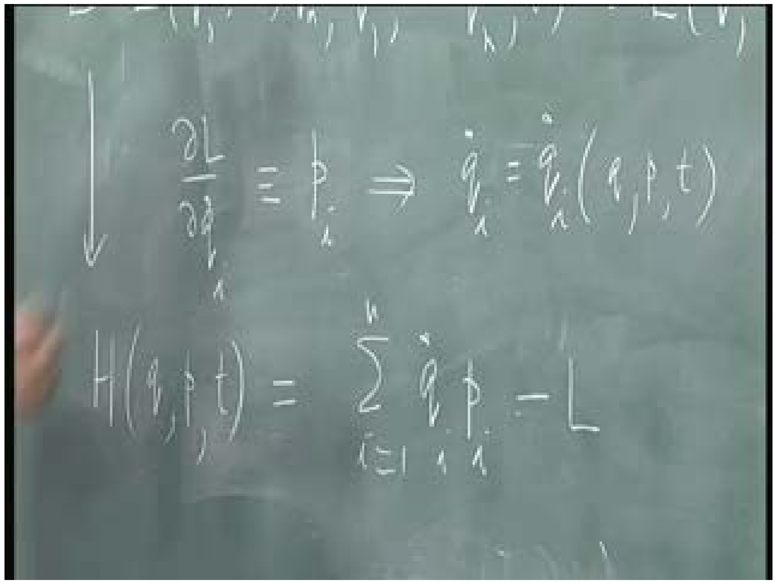

Again, let me simplify notational little bit, a Lagrangian L which is a function of q1 to q n q1 dot to q n dot and possibly t, I have written in abbreviated notation as L of q, q dot and t, its actually a function of 2n plus 1 variables in general, generalized coordinates, generalized momentum and possibly time if it is a non autonomous system. I would like to get rid of these velocities and use the partial derivatives of L with respect to these velocities as the independent variable.

So, I would like to use delta L over delta qi dot as the independent variable. And, let me call this and define this, this is equal to p sub i, and I call it a generalized momentum. You can see why I am doing this? Because we already saw from the Euler Lagrange equations that, if you have delta L or delta qi equal to d over dt of delta L over delta qi dot. Then we know that if qi is a cyclic coordinate or ignorable coordinate, this quantity here is the constant of the motion.

So, every time you have a cyclic coordinate, but corresponding momentum generalized momentum is a constant of the motion. And this is how invariance is start from again constants of the motion will be identifiable. So, let me define right now once and for all, delta L over delta qi dot as the momentum the generalized momentum conjugate to the corresponding generalized coordinate and I will explain the meaning of this word conjugate, but they go in pairs.

What happens if I do this? I would then like to go from L to a new function H which is a function of the qs and the ps collectively even if i write as p and possibly t and how should I define it? I should define this as L minus the product of the q dots with the corresponding ps but historically the reverse sign it doesn't matter. So, let me define this as summation over i equal to 1 to n qi dot pi minus L. I have done to change the sign and the reason is H is going to turn out to be the total energy of systems, mechanical systems in particular therefore, I would like to retain a plus sign, no other reason.

Could have done it L minus q dot qi dot pi dot let us leave it. However, H is a function of q, p and t velocities do not appear in the Hamiltonian, this quantity is called the Hamiltonian and velocities do not appear in it. So, how are you going to get rid of these velocities? They are sitting here right here so, what you are supposed to do is to solve the set of equations and write qi dot as a function of all the qs ts and possibly time.

So, you have to take this quantity differentiate it and then of course you get a function of q q dot and t and you are supposed to eliminate if possible, the different qi dots and write them express them explicitly as functions of qs, ps and ts. And after that you pluck that in here and where ever q dots appear here, you again write them as functions of qs, ps and ts and if you can do this, then this is a function of q, p and t.

In principle, now when would you be able to do this? When can you actually do this? Its not obvious that you can always eliminate, you have 1 variable its not so easy its not so difficult you have single q, p and t, but if you have a multiple scheme multiple set of quantities, then its not so obvious, that you can do this and I put it to you that I will explain this, later on by example we will see this.

That if this, matrix of quantities qi dot delta qj dot, this is some function this is labeled by the indices i and j and its obviously written as 2 by 2 matrix, if that matrix is non singular then you can always make this change of variables. You can invert this set of quantities, this is called the Hessian matrix corresponding to this transformation we will come back we will look at examples, just want to point out here that you should be able to make this transformation otherwise there is no point in it.

Now, let us assume that we can do so and let us ask what happens next. All right I pretend I have done this, what happens next? What should I do? I should try to find out that H is indeed a function of qs, ps and ts, and what it implies? Let us do that, see what happens?

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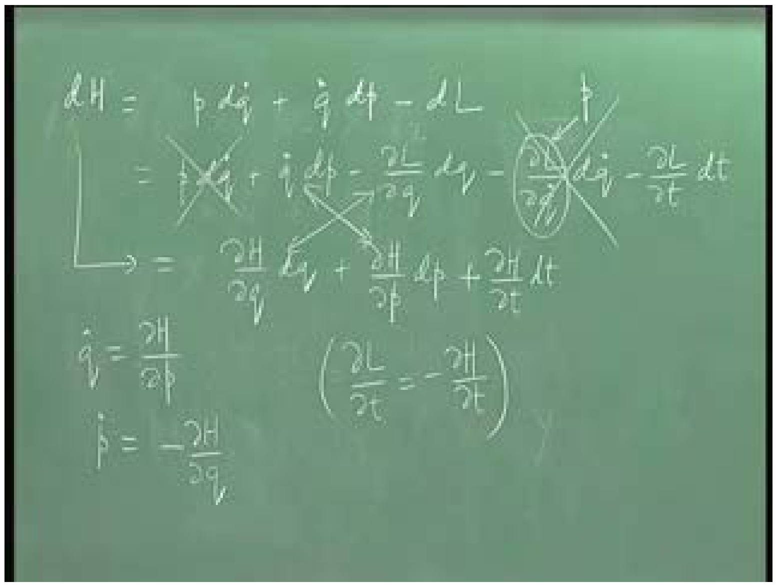

So, dH therefore becomes equal to again let me save myself a little bit of trouble by assuming that I write this as q dot p forget the i here you can put the summation and put the indices in specified and so on. So, dH becomes dq dot minus p plus q dot times dp minus dL and what is dL, well this is equal to p dq dot plus q dot dp minus delta L over delta q dq minus delta L over delta q dot dq dot minus delta L by delta t dt. I just take the differential in the right hand side L being a function of q q dot and p you can write dL in that form. On the other hand, dH itself should be writable as delta H over delta q dq plus delta H by delta p dp plus delta H by delta t dt.

Now lets compare both sides, we already know that delta L over delta q dot is in fact p. By definition, therefore, this term cancels out against this term and now we are in good shape because, p, q and t are independent variables and if this is equal to that the coefficients of dq, dp and dt must be equal to each other; since is the completely independent variables. What does that imply? This immediately gives us set of consequences if I compare for example, this with that it says q dot equal to delta H over delta p. So, that has matched this with that. Now, I match this with that it hears by the physical input is going to be, what do I get? I get delta L over delta q equal to delta H minus delta H over delta q. But remember, we are talking about solutions of the Euler Lagrange equations. We are going to use the equations of motion, on these solutions trajectories delta L over delta q by the Euler Lagrange equation is known to delta L over delta q equal to d over dt p, its delta L over delta q dot which is p.

And therefore, this equation here becomes t dot equal to minus delta H over delta q. By erase that and of course if I compare this with that for non autonomous systems it turns out that delta L over delta t is minus delta H over delta t. This therefore, is now the content of Euler Lagrange equations that is what we put in. And since, this is the set that we arrive at incidentally notice that, this definition p equal to delta L over delta q dot has a conjugate has a corresponding counter part which is this q dot is delta H over delta p.

Whereas, p is delta L over delta q dot, it is again the slope that is, just telling you that you can always go from the Hamiltonian to the Lagrangian back with the Legendre transform and this is precisely the Legendre transform. So, that is not very surprising, but this equation here contains dynamics because, the Euler Lagrange has the input in. This set of equations is now exactly what we want what is the advantage of this set of equations over the Euler Lagrange equations.

They are first order equations, they are explicitly first order equations in the 2n dynamical variables. So, the great advantage is that we reduce the second order set of equations to first order set of differential equations by going to the Hamiltonian framework. But you are going to do lot more you are going to get many more advantages what are they and lets start writing them down.

So far I have not defined the Hamiltonian anything more than p q dot minus L, I have not said its the total energy of the system, I have not identified it to the total energy, this depends on what L would be. But as far as I am concerned, you give me a system with Lagrangian and provided I can make the Legendre transform, I can always go to the Hamiltonian what ever the function that be. And then, once you have that function you end up with this set of first order differential equations and I now forget about the Euler Lagrange equations and focus on solving this set of equations.

But it has got magical properties, so again lets write this down q dot is delta H over delta p p dot is equal to minus delta H over delta q. The minus sign is crucial the most important minus sign in the universe is absolutely crucial. You will see what it does has to go along? The first point is that, if H is independent of time explicitly then H is the constant of the motion automatically, how do we find out?

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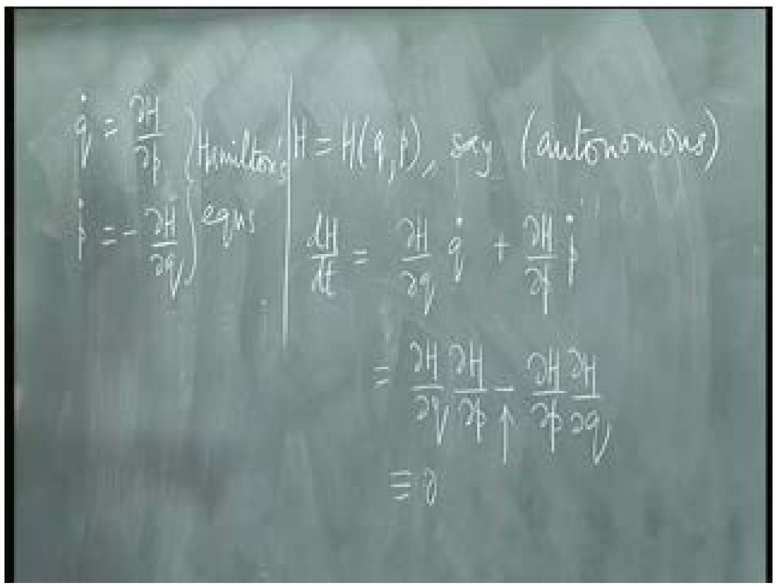

Well, if H is equal to H of q, p say this is autonomous system then what is dH over dt equal to delta H over delta q q dot plus delta H over delta p dot by definition. But on a solution trajectory, namely when the Hamiltons equations, when Hamiltons equations apply so you should call this Hamiltons eqations. This is called to be a starting point in some sense. When these equations apply, then I replace q dot by delta H by delta p q. So, this becomes delta H over delta q delta H by delta p minus delta H by delta p delta H by delta q and you see the role played by this minus sign so this becomes identicaliy 0.

The Hamiltonian is therefore constant of a motion as long as it is not explicitly dependent on time. So, this is the way you discover a constant of the motion this is always true for every Hamiltonian system, which is autonomous system, explicitly time independent Hamiltonian you guaranteed that the Hamiltonian is a constant of the motion. It is going to play a very very fundamental role and what is that you already begin to see that it has a favored role over all other constants of the motion, which we are going to now try to deduce? The reason being that the way in which the free space point changes its velocity is governed by the Hamiltonian.

So, in that sense the Hamiltonian governs the time evolution of a system. Therefore, we call it the infinite decimal generation of time translations, because its telling if you give me qs and ps at any instant of time p zero a t zero plus delta t the Hamiltonian tells you where the system goes, where the point go and so on. So, this is going to have a very fundamental role to play in all that they want to do. We will write down the Hamiltonian for simple systems, but in general please notice that the Hamiltonian as written here this function very, very special role to play. You could ask what about other constants of motion? What does a general constant of motion look like? And let us do that it can be related to the Hamiltonian.

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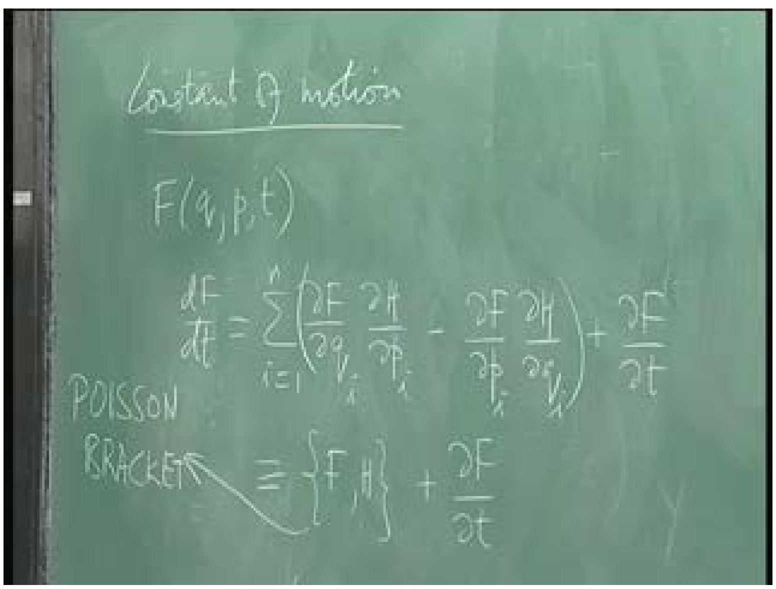

So, constant of the motion, suppose F is a constant of the motion in general, of course it is a function of all the qs ps and it could be explicitly time dependent. Please notice that the constant of the motion could be explicitly time dependent, because the time dependence which comes from the explicit time dependence could be cancelled by the time dependence of ps and qs.

So, in general of course the given system dynamical system will have constants of the motion which are time dependent some of them would be time dependent. I will give you a simple example in a minute, but what does this imply says dF over dt equal to delta F over delta q q dot, but q dot should be replaced by delta H over delta p because, you are on a solution trajectory plus delta F over delta p times p dot but p dot is minus delta H over delta q so, there is a minus here plus of course delta F over delta t.

Now, this little structure here has very profound implications have very deep meaning which I will come to very shortly, this set of partial derivatives please notice I abbreviated my notation, if you have n degrees of freedom it really stands for i equal to 1 to n return i here. Really it stands for that and this object which is constructed out of 2 functions F and H is called the Poisson of F with H.

By definition, it is denoted as F curly bracket H plus delta F by delta t. Its the Poisson bracket of the F with H, it plays the same role in classical mechanics, as the commutative of matrices or operators would in quantum mechanics and in fact the translation from classical to quantum mechanics through the Poisson bracket is very immediate we will see that. So, you agree this is some function of qs ps and ts after you finish this set of complicated set of partial derivatives, with some function of qs ps and ts and when is F a constant of the motion if dF over dt vanishes identically.

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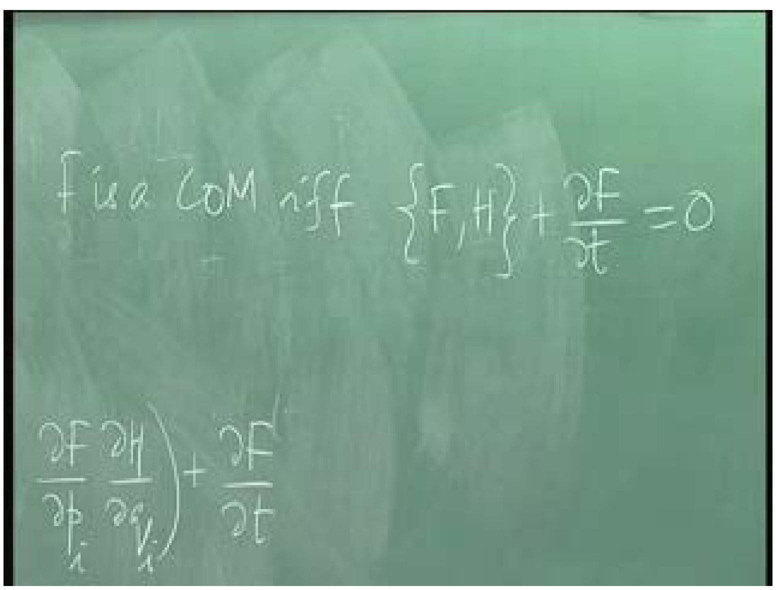

This means F is the constant of motion if and only if the Poisson bracket of F, H plus delta F by delta t is equal to zero. If F should turn out to be explicitly time independent then the last term goes away and then F is the constant of the motion if and only if its Poisson bracket with the Hamiltonian vanishes identically. And I would then say Poisson commutes with H, in analogy with what happens in quantum mechanics whereas 2 operators commute with each other in that sense I would say this Poisson bracket is zero I would say Poisson commute.

The other word that often use is F and H are said to be in involution with each other, I will write that down a little later when to write down the criterion for integrability, but at the moment I want you to focus on the point that the way you test if some quantity is a constant of motion now, is simply to calculate the set of partial derivatives and check if it is zero or not. If it is zero identically, it is the constant of the motion, if it is not then it is not a constant of motion. (Refer Slide Time: 30:43)

Now, of course this structure here has some deep implications you could define the Poisson bracket of any two functions of free space variables. So, you could have some A which is a function of all the qs and ps and ts perhaps and some B and this Poisson bracket here is defined as delta A over delta q dot delta B over p dot minus the reversed partial derivatives. It has some properties and the first property is that this is equal to minus B, A. That is immediately obvious you reverse A into change A and B and you get a minus sign (( )).

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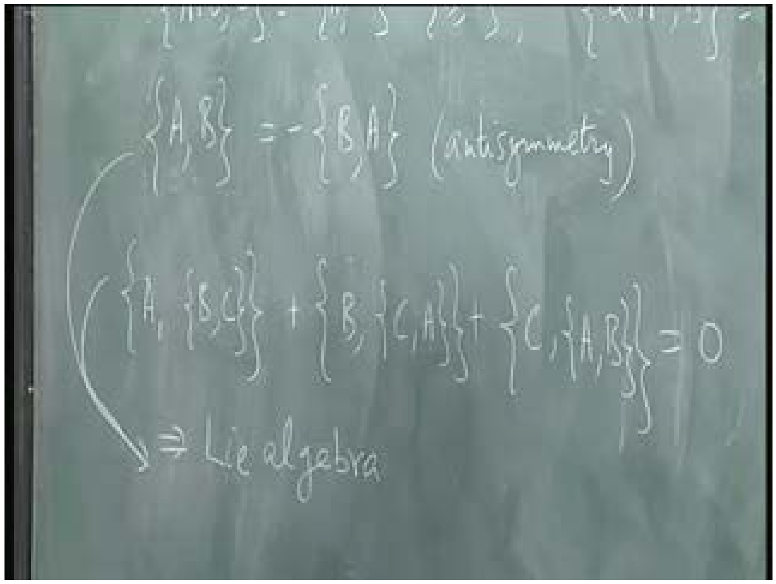

By the way, there is an even more trivial property which essentially says that A plus B with C is A with C plus B with C that is of course true, immediately all you have to do is to plug it in there and this is at once true. And similarly, if you took A and multiplied it by some constant B this becomes alpha times A, B. So, multiplication by a constant does not do anything it just comes out of the partial derivatives. So, this property here is anti symmetry is crucial property of the Poisson bracket.

There is another property, which is also a crucial property and I will explain why that so but for that we will need a preliminary, you should ask what is A with BC, where A B C are functions of the free space variables qs ps and possibly t, what is this equal to? Well put that in here you have a A here and a BC here and then you expand this out. If you do that you discover this is equal to B times A with C plus A with B times C.

So, it has this chain rule kind of property and of course, if these are functions this is a function and that is a function of free space variables everything commutes so, I could have written this B on the right hand side that C on the left hand side it will make a difference. But I have in mind always going over to quantum mechanics where these would be operator matrices or operators then of course A B is not equal to B A and this is the correct order. Since, B appears on the left here and C on the right that is, exactly how it appears everywhere B always appears on the left C on the right and that is why I put this B on the left here and C on the right here.

Now, I leave you to figure out what A B with C D is you have to apply this formula repeatedly so, this is the way you would find Poisson brackets of various complicated functions of the free space variables given a few elementary Poisson brackets and the first of these, before I do that let me, mention the property that I was interested here this property is easy to verify, first take the Poisson bracket of B with C with some function take the Poisson bracket with another function A and do this in cyclic order B with C A plus C with A B and answer turns out to be zero identically.

To do this, you have to verify this property using that I expression, using this relationship, using this formula you can verify that this is actually branded and this is called Jacobi identity. We will see what it uses in half a second, but you see this tells you that if you take 3 of these functions and you do this in cyclic order repeatedly you add it all up you get zero. Now, this property together with this property for a set of elements, any abstract set of elements A B C D etcetera, elements of a class if it should so, turn out that you can define the bilinear operation between them.

In other words, something which involves two of them at a time and produces another member of this class and here there is free space functions so, you take 2 free space functions find the Poisson bracket you produce another free space function, function of the free space variables. If you can define a bilinear operation among them which is anti symmetric which has a minus sign when you interchange 2 numbers and for which the appropriate identity is valid such a set of elements is said to form a Lie algebra, this plus this together imply that the set of elements is Lie algebra.

You are familiar with other examples, 1one of them is immediately matrices, if you took n by n matrices and define the bilinear operation to be the commutator A B minus B A then of course that is equal to minus of B A minus A B and the triple commutator is zero identically. So, matrices n by n matrices under the operation of commutation form a Lie algebra. Functions of free space variables for a system form a Lie algebra under the Poisson bracket operation.

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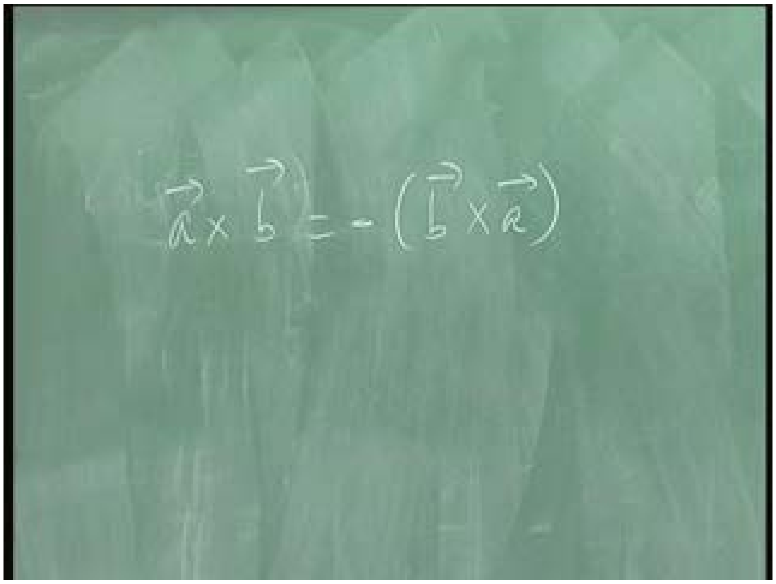

Can you give me another example, 1 which you are very familiar with in three dimensions very very familiar with A cross B ordinary vectors would do exactly this, because you know that if you took 2 vectors, when you take the cross product you find another vector this is certainly true. a cross b is equal to minus b cross a, and you know that triple product A cross B cross C plus B cross C cross A plus C cross A cross B is also zero. So, ordinary 3 dimensional vectors under the cross product operation form a Lie algebra

There are numerous other examples but, these are the simplest 1s and they are going to have this is going to have very profound properties we will see very, very shortly; so any questions (( )). So, you have seen what the constant of motion is, something which in the autonomous case is independent Poisson commutes with the Hamiltonian and we have also seen that the Hamiltonian itself is the constant of motion. Now, the next question one should ask is, given these properties can we write this down in more compact form what is the real meaning of the Poisson bracket? What is the geometrical meaning of the Poisson bracket? This is worth asking.

Why does this particular combination of things really turn out to be so fundamental, that is the first question. The other question is, is there something I can do to put this whole business in the language of general dynamical systems, what makes Hamiltonian systems so special so very special? The answer is the following.

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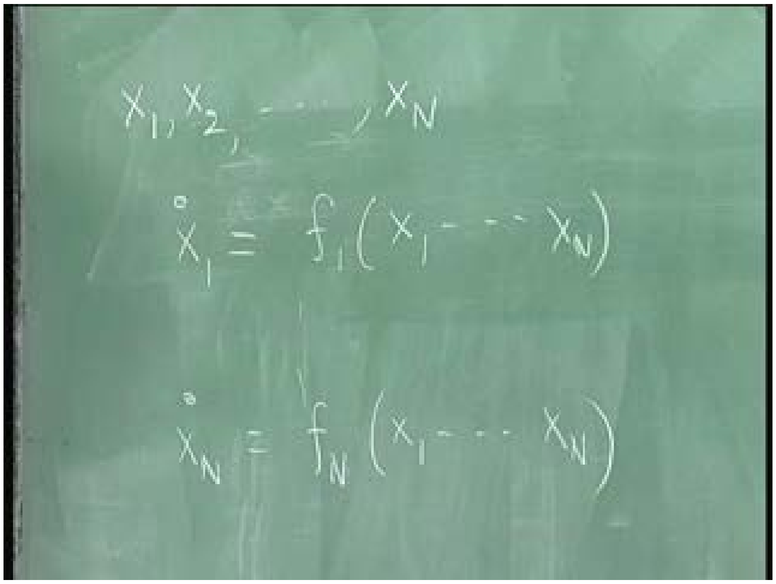

Recall that our general dynamical system involve variables x1 x2 up to x N and we had an equation x1 dot is f1 of all the variables x N and so on up to x N dot is f N of all the variable x N. In our case, instead of a vector field f1 to f N here a single scalar function namely the Hamiltonian is determining the right hand side.

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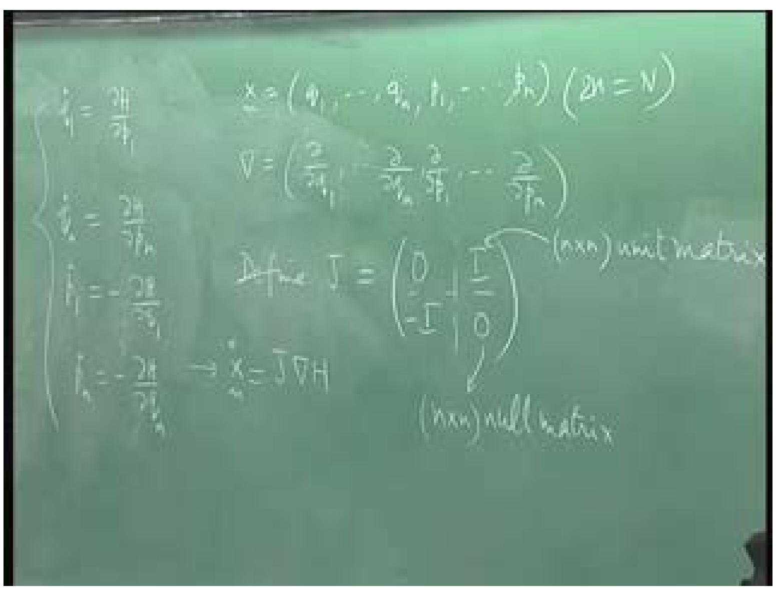

So, that is a great simplification because instead of having a vector field on the right hand side you have q1 dot is delta H over delta p1 right up to qn dot delta H by delta pn and then you have p1 dot as minus delta H over delta q1 up to pn dot with minus delta H over delta qn. Of free space variables x in this case are really q1 to qn p1 to pn so, we have a situation where 2n equal to N the free space is even dimensional for Hamiltonian system.

And I write all the qs one after the other and then write ps 1 after the other and I call that a vector in free space a position vector in free space. Then this quantity here, tells you that the velocity of the free space point is determined by certain partial derivative with respect to a scalar function H. It almost looks like a gradient it almost looks like a gradient because, what would be the gradient operator be in free space? This would be delta over delta q1 delta over delta qn delta over delta p1 delta over delta pn, this is my gradient operator.

And it would be very nice, if I could write x dot equal to the gradient of H. Is this true? It is not true, because this is not q1 but p1 unfortunately and this here is not p1 but q1 and I have minus sign to both so, it is not quite the gradient. However, we can make it look like the gradient as follows, lets define a matrix J to be equal to 0, the unit matrix minus the unit matrix and 0, where this thing is a n by n unit matrix. And this here stands for the n by n null matrix.

Let me define matrix big matrix capital J, we have 2n by 2n matrix in 4 blocks the top block the top left hand corner block is just a null matrix so, as the bottom right hand corner block and then the identity matrix in the off diagonal positions and minus the identity the lower right hand side. Once I do this, then it is a trivial matter to check that this set of equations this entire set of equations could be written as x dot equal to J times the gradient of H.

So, this says q1 dot on top would be the first component delta H over delta q1 right up to delta H over delta pn, but then you apply the J matrix on the left hand side and you would pick out just the right quantity. Because out here would be a 1 top can be here that 1 would pick out for q1 dot precisely this quantity here delta H over delta p1. So, this makes it look almost almost not quite, but almost like a gradient system. But the fact is all the simplicity that you gain by saying that your vector field on the right is derived from a scalar field are going to be present (( )) except it is a sort of twist at gradient, it is not quite the gradient but it twist the gradient.

Incidentally, this minus sign came from the Euler Lagrange equations so, we cannot wish it away, sitting there and it is a good thing it was there because, it gave us the information that the Hamiltonian is the constant of the motion itself. As long as we have autonomous systems so, this gradient here is called a symplectic gradient. I will explain why this word symplectic is used right here, but right now it is terminology. It is called the symplectic gradient of H.

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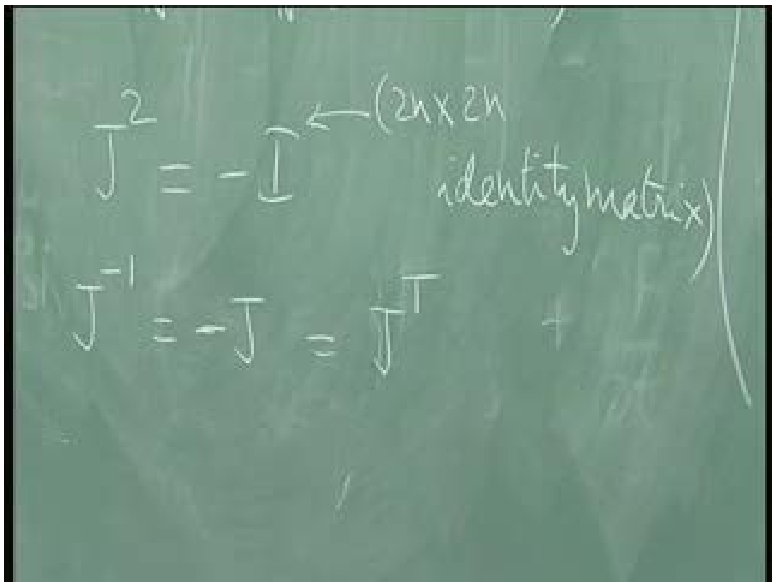

This matrix J has interesting properties, which will need J square is equal to minus the identity matrix, that is the easy to verify and this by the way stands for 2n by 2n identity matrix. And that immediately tells us, that J inverse equal to minus J. Because, J J inverse is minus sign so, J inverse is minus J. So, J has a inverse and its equal to minus itself and incidentally, it is a trivial matter to see, that this is also equal to J transpose if I transpose, if interchange the rows and columns this matrix. If I write the I J element as the J I element then, that is just going to be minus J very useful property. Can we say something about the Poisson bracket now?

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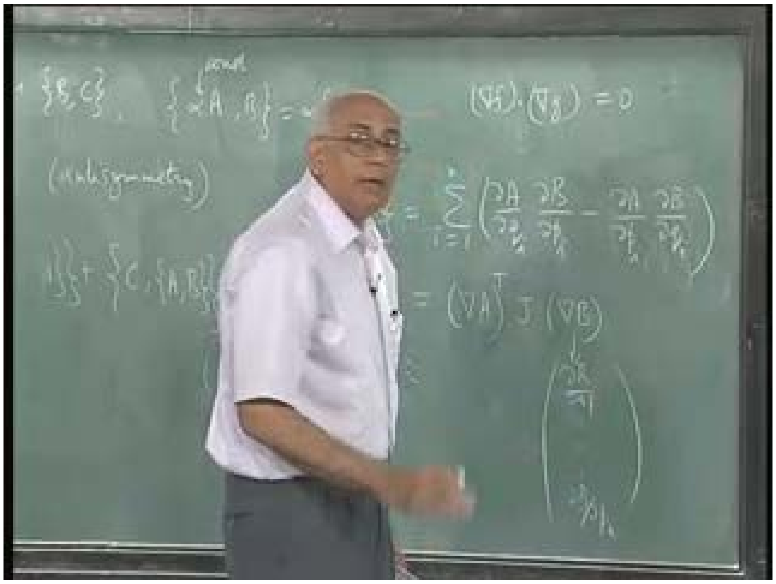

The answer is yes, because as you can now see that what we wrote as A, B as summation i equal to 1 to n delta A by delta qi delta B over delta pi minus delta A over delta pi delta B over delta qi this is what Poisson bracket was by definition, can also be written as well delta A over delta qi and then delta A over delta pi suggests that you are taking the gradient in free space of this quantity A the scalar function A.

So, what is happening is that this quantity is equal to gradient A transpose by gradient A I mean a vector with elements delta A over delta q1 delta A over delta q2 etcetera written as a column vector, by transpose I mean the corresponding row vector and that is getting multiplied by gradient of B, which is a column vector. But because you have this minus sign not surprisingly you have a J in between. Now what would you say if I wrote a column vector and wrote a row vector on the left hand side.

So, in ordinary space if I have a column vector a1 a2 up to an and then I have p1 p2 pn and this stands for the vector a and this stands for the vector b, then of course once I write it in this notation all I have done is to take b dot a the dot product, because its b1 a1 plus b2 a2 etcetera that is just the dot product the ordinary dot product. That has been replaced by a symplectic dot product, if this is a symplectic scalar product and its telling you that a and b these 2 functions in free space, they are symplectic dot product of the gradients is the Poisson bracket.

Now, go back to ordinary Euclidian space, what happens if I tell you that you have 2 functions f of x f and g functions of the space coordinate. Such that the gradient of f dot product with the gradient of g is 0. What would you say about the functions f and g? So, gradient of f dotted with gradient of g is equal to 0 or in matrix notation grad f transpose grad g is equal to 0. What geometrical information do you get by saying the gradient of this function and the gradient of that function dot product is zero, yeah their level surfaces are perpendicular to each other.

So, this is telling you that if the Poisson bracket of a with b is 0, then it says the level surface of a and level surface of b are orthogonal to each other in the symplectic sense, with this metric, with this metric put in (( )). So, this is why I said that the people often say that Hamiltonian dynamics is a study of symplectic geometry. So, you have to pretend that your free space the qs and ps have this extra J sitting in their, when you want to define their dot products, when u want to define gradient and so on.

Once you do that, then you have what is called symplectic geometry or a symplectic metric and the whole of Hamiltonian dynamics has very elegant geometrical interpretation. With this metric, now I am going to come back to this and explain when we do relativity what I mean by orthogonality? When you have non Euclidian metrics? So, this is something which we will explain a little later. But right now, you can see that once I put in this J appropriately at various parts these things start looking like Euclidian geometry, except that I have to put in this J once in a while you take care of all these minus signs.

So, this is in fact the significance of the Poisson bracket, this is the way to look at it. This is by the way an algorithm to calculate the Poisson bracket but its real meaning is here in this place. Now, let us come back step back a little bit and do one more thing which is absolutely crucial for a normal Hamiltonian system and that is to define, what I meant to this word conjugate momentum it did not define that yet, so let us do that. Pardon me what is meant by the order means?

This guy here by this, I mean a column vector which is delta B over delta q1 up to delta B over delta pn. Everything here includes p you cannot talk about the qs and ps separately. The whole free space is 2n dimensional always, but by convention set I am going to write the first n components as the qs and the remaining n components as ps I could have inverted it see later on but it does not matter about I am going to stick to this convention.

This is my definition of free space point; this is my definition of gradient the del operator in free space obviously. Very shortly, we are going to see that it does not matter which variables you call the qs and which variable you call the ps you can actually exchange. These are generalized coordinates in generalized momentum and there is a deep symmetry between them. I am not saying the physical dimensions of a length and the linear momentum are the same, but apart from physical dimensionality we can always interchange variables.

So, the beauty of it is although we started out from very very humble beginnings this is telling us something extremely deep going on in major, the fact that dynamics is happening in free space is not a accident; it is got a very deep geometric structure.

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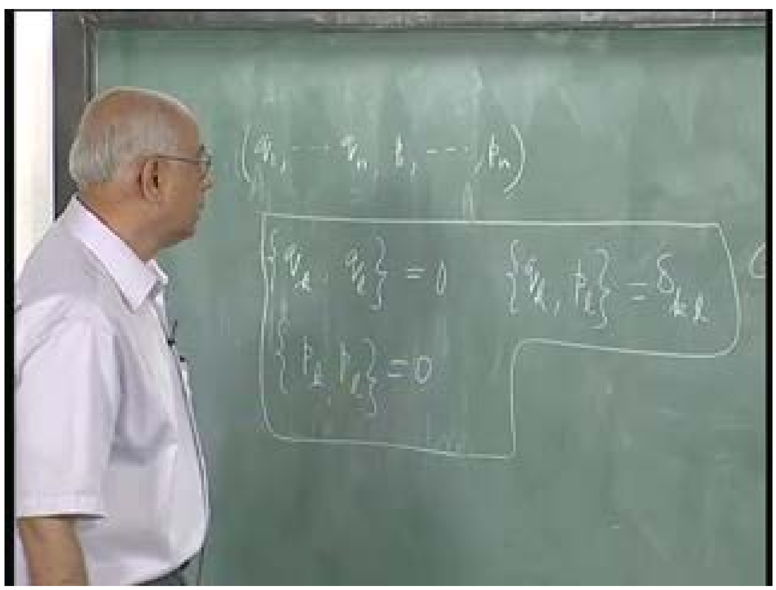

Now, let me define what is meant by conjugate we call that our independent variables q1 to qn p1 up to pn. And I could ask, what is the Poisson bracket of say qk with ql? What is this equal to? Certainly, qk and ql are dynamical variables, free space variables and they could ask, what is the Poisson bracket of this with that?

So, this is i equal to 1 to n delta qk over delta qi delta ql over delta qi minus in the reverse sorry, pi minus the reverse term. But you see that, qs and ps are independent variables completely independent variables so this is zero therefore zero this is completely zero so this is equal to 0, identically 0. In exactly the same way, pk with pl is identically zero and the reason is at some stage when you do this partial derivative you are going to take this function and differentiate with respect to q and that is identically zero.

But it is another matter, if you ask what is qk with pl? This is not going to be zero, what is this going to be? Summation i equal to 1 to n delta qk over delta qi delta pk over delta pi pl minus delta qk over delta pi delta pl over delta qi. And this term is 0, because the ps do not get differentiated with qs nor do the qs with respect to the ps so, that is gone. When is this term equal to non-zero quantity, when k equal to i otherwise its 0, because these are independent variables.

Therefore, this is equal to summation over i delta ik the chronicle delta ik and this of course is chronicle delta il and you have to sum over all possible i s and this is going to be 0 unless k is equal to i, this is going to be zero unless l equal to i, the free indices here are k and l therefore, this is equal to delta kl, just the chronicle delta kl. So, that defines for me, the canonical Poisson bracket relations this set of relations are called the standard or canonical Poisson bracket relations.

Now, once these relations are satisfied, then I say that the momentum pk is conjugate to the momentum the generalized coordinate qk or qk and pk form the conjugate pair. So, all the free space variables get broken up into pairs and notice here that each qk Poisson commutes with all the other ps except its conjugate, when the Poisson bracket is equal to 1 on the right hand side.

This is very familiar in some sense, because if you look at positions and momentum in quantum mechanics, the commutator of position coordinates 1 component with another is zero, momentums similarly is zero, but position with momentum the right hand side would give you an ih cross and the units operator if these are conjugate variables. This is the germ of that, its starting with that. So, that relation in quantum mechanics has actually arrived at from here at some sense follows from in this place.

So, there is in fact a correspondence which will tell you how to go from Poisson brackets to commutator does in quantum mechanics? And the Hamiltonian framework lays the groundwork for everything. Now of course, once you are given these relations and the Poisson bracket formulas you can find the Poisson brackets of any functions what so ever. And lets write down, what a Hamiltonian is for a simple system like the simple harmonic oscillator and see how this works?

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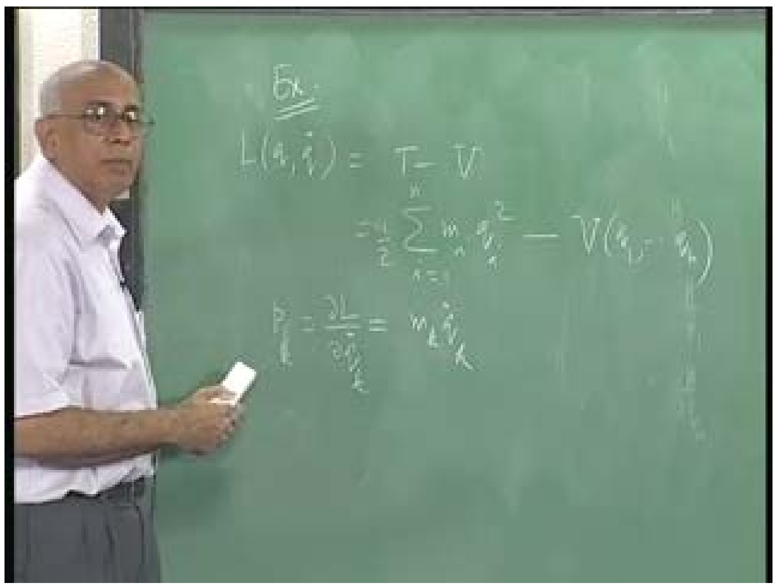

So, I have for instance you could do this in general, you could do this for a set of particles so, if I have a system for which the lagrangian as a function of q q dot is equal to I will look at a autonomous system now, is equal to 1 half the kinetic energy minus the potential energy and the kinetic energy is equal to it will take like a general function, but let me take 1 half mi qi square summed over i over all the particles I to n minus V of q1 to qn and this is an example.

So, I have in mind the Lagrangian which is just, whose independent generalized coordinates are just the Cartesian coordinates of the set of particles, call them q1 up to q n and the potential which is function only of the coordinates, no explicit time dependence here. What is the Lagrangian in this case? First of all, what is pi? This is delta L over delta pk delta qk dot and this is equal to mk qk dot, because only that qk is going to get differentiated and it gives this mk qk dot.

And indeed in this simple example, it is clear that the momentum is mass times the velocity and vice versa, the velocity is 1 over the mass times the momentum does it always have to be true, the real definition is sitting here in this place.

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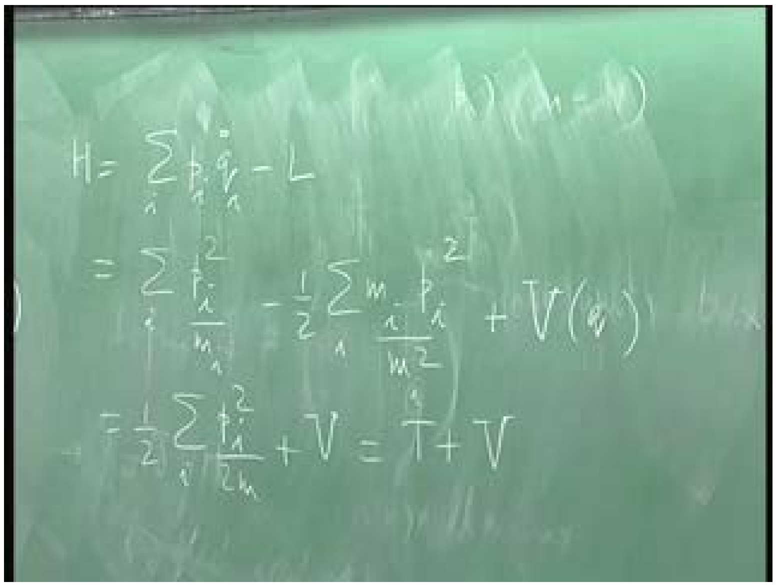

And therefore, what is the Hamiltonian? The Hamiltonian is equal to summation over i pi qi dot minus L, but that is equal to summation over i pi but for qi dot I have to put in this here and invert this expression here write the q dot in terms of the p, because remember the Hamiltonian is the function of qs ts and time possibly.

So, this becomes pi dot and another pi dot, so divide it by mi minus that takes care of this part minus 1 half summation over i mi, this is q dot square qi dot square but I have to write qi dot as pi dot divided by mi and then I prove that pi square over mi square plus the potential that gives you the familiar expression 1 half pi square over 2m plus V equal to T plus V. This is how you identify for such systems the Hamiltonian and the total energy of the system. So, the minus sign which we used in defining L has disappeared and it is become a plus sign.

And this is the reason I chose to define Hamiltonian as pq dot minus L rather than L minus pq dot so, that I would get the total energy. Now, since we know the Hamiltonian as the constant of the motion for autonomous systems, we know immediately that this combination here provides you with the first constant of motion. And in many cases, we simply write down the Hamiltonian as T plus V, but you must express in terms of moment.

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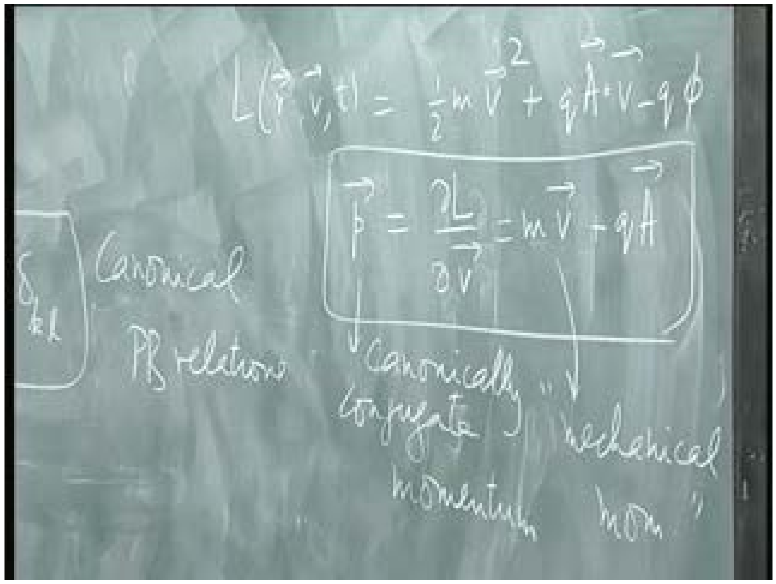

Now, let us look at the other example very quickly of the particle in a charged particle in an electromagnetic field, what happened there? What happened to the momentum? Remember the Lagrangian as a function of r v and possibly t was equal to 1 half mv square plus qA dot v minus q phi. So, what is the momentum p conjugate to the position r? This is delta L over delta v, I am sorry to abuse notation, by this I mean I do not mean to differentiate with respect to vector, by this I mean a vector formed by differentiating L with respect to each component separately.

So, short hand for it slight the gradient operator and the velocities you will write it in casually in this form and this is equal to mv, but here you have to differentiate with respect to v and it turns out to be qA. So, that gives us a very, very important relationship, which says that the momentum conjugate to the position for a charged particle in an electromagnetic field is not the mechanical momentum, which would be mass times the velocity. But it also involves the field explicitly, there are many many names given for this but I would like to simply call it the conjugate momentum or canonically conjugate momentum and this is sometimes called the mechanical momentum also sometimes called the kinematic momentum.

Which is what it is? For normal motion without presence of field. But it is important to remember, that p is not equal to mv, that it is not in most cases it is not if particles moving sufficiently fast certainly not equal to mv itself, but here you also have the field appearing potential appearing. Now, you should ask a very deep question immediately, once I tell you p equal to mv plus qA what would that be? Pardon me say this slowly A is not unique, A is not unique it is not gauge invariant, it looks like we ended up with trouble. Because, you have a momentum which I said is a free space variable in physical, and now I am telling you it involves A, its seems to be gauge dependent this is going to lead us to some trouble, but let see what happens? Go ahead with this.

What is the corresponding Hamiltonian and then I stop here I take it up from that point, H of r and p and may be t in this problem is equal to, you plug this back in here you can solve for this of course put it back in here and then it will turn out that this becomes p minus qA whole square over 2m plus q phi. That is the Hamiltonian of a charged particle in a electromagnetic field. p is just replaced by p minus q, the p of the free particle is replaced by p minus q.

And you guarantee absolutely that absolutely guaranteed that the Poisson bracket of px with x or x with px is 1, x with y with py is 1 and so on. Why did we use this symbol p for this combination in this case? I leave it to you as an exercise now to write down the Hamiltonian equations of motion and verify that its exactly the same as Euler Lagrange equation the Lawrence force equation that you got, but notice the plus sign appeared here.

And A is appeared here in a strange way, although it appeared linearly there, there is a quadratic term here. The term which depends on A square so, it is almost as if A is inducing itself. This is related to the phenomenon of diamagnetism, the fact that you have his A square term it is going to be non trivial term, there are many other non trivial terms going to happen. If i square this as a p dot A term, there is a A dot p term that does not matter in classical mechanics.

But in quantum mechanics, A is going to depend on r therefore, p dot A is different from A dot p, you have to keep track of them. But this is going to give us the right answers, this is the correct Hamiltonian for the charged particle in the electromagnetic field and we will see, what happens if I make a gauge transformation? What is going to happen this stage? And it is going to be related back to the fact, that you can add a total time derivative to the Lagrangian nothing happens. But we have to ask, what is the meaning of that statement in the Hamiltonian framework? What is the corresponding thing? And it will turn out it corresponds to something called a canonical transformation which does not change the physics.

So, again all these things are closely linked to each other and this is a very intricate machinery, but I will start tomorrow next time, I will start with this point and we will look at a few problems in Hamiltonian mechanics especially, if we have stability and so on. Good question let me give the detailed answer to that next time, says if the Hamiltonian more physical than the Lagrangian, no I do not say that at all. I will list the advantages and the disadvantages of each of the two formalisms then we will see just as physical.