Physics I: Oscillations and Waves
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Lecture - 07
Coupled Oscillations

In the past few lectures, we have been discussing a single oscillator. We started with a single simple harmonic oscillator. And we then discussed the effect of damping and its behaviour under the influence of an external force. Today, we shall extend our discussion to a situation where we have two oscillators.

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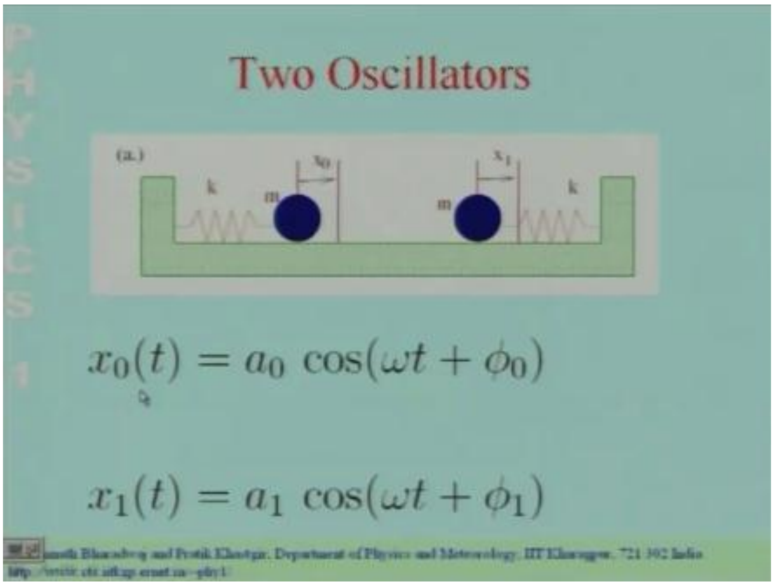

To set things in perspective, let us consider a situation where we have 2 oscillators of the same spring constant k and the masses have the same value m. So, we have 2 such oscillators, which are free to oscillate independently. So, as we have already studied each of them, will execute simple harmonic oscillations, at the angular frequency omega naught omega, which is square root of k by m and the amplitudes of the 2 oscillators. So, we are going to use x0 to denote the displacement of this oscillator over here, the 0th oscillator and we are going to use x1 to denote the displacement of the first oscillator over here. So, each of these oscillators will execute simple harmonic motion. The amplitude of these 2 oscillators' a0 and a1 are both independent they are in no way connected. And the phases a 5, 0 and 5,1 of these 2 oscillators are also independent, they are in no way connected.

So, if you disturb this oscillator and leave the second oscillator undisturbed, the first oscillator is going to continuant to oscillate and the second oscillator is going remain, where you had left it. Similarly, if you leave this at rest and disturb this one, it is not going exert any influence on this. And it is, this is going oscillate, this is going remain where you had left it. Now, this is nothing of great not of great interest. This is just an extension of what we have already studied.

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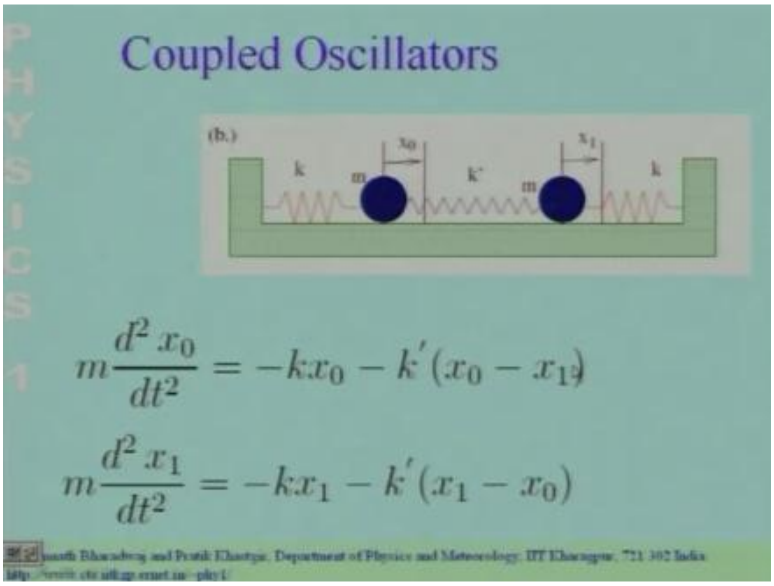

The situation which is of interest in today's class is where we introduce a third spring with spring constant k prime, the third spring attaches the 2 masses. So, it effectively couples these 2 oscillators which we have just discussed. So, if you introduce a third spring into the problem which couples 2 oscillators the behaviour gets completely modified. If I disturb the zeroth oscillator, these disturbances are also propagated to the first oscillated through the new spring which has been introduced here.

Similarly, if I disturb the first oscillator, the disturbance is propagated to the zeroth oscillator through the spring introduced here. So, the new spring couples the 2 oscillator and they no longer can oscillate independently. So, let us analyse this system. We proceed by writing down the equations of motion of these 2 masses. So, let us first write down the equation of motion, for the x0 for the displacement of the zeroth mass. So, the mass into the acceleration corresponding to this displacement is equal to the total forces acting on this, if I displace it.

So, if I displace m the zeroth mass the spring over here gets extended. And this gives rise to a force minus kx 0 a displacement of this mass and the zeroth mass also produces a deformation in the spring which couples the 2 oscillators. And if I hold the first mass fixed and displace this, there then will be a force which is proportional to the displacement of this and it is in to the left. So, it will have a plus sign so, I will have a force plus k prime, minus k prime. The force will be, will oppose the displacement. So, there will be a force minus k prime x0. Let me just go through this again, if I displace this mass slightly to the right. The spring over here is going exert a force opposing that displacement.

And that force is minus k prime x0; it is going to oppose the displacement. So, there is going to be a force if I move this over here. Now, there is also possibility that if I keep this mass fixed and if I were to displace this mass by an amount x1. So, this is fixed, this has move, this again causes this intermediate spring, the coupling spring to get extended and this is going to exert a force in the same direction as the displacement on this mass. So, I have a force minus k prime a plus k prime x1 which I can write here. So, I have combined the forces which arise due to the motion displacement of this and this, due to the spring and written them as 1 term and that is minus k prime x0 minus x1. So, this gives me the equation of motion for the zeroth particle x0 the displacement of the zeroth particle x0.

Let us also, consider the equation of motion of the first particle x1. So, going through a same analysis, the mass into the acceleration of x1. This is equal to the total forces that act on this particle. If I displace this particle, then there will be a force which opposes this displacement from this spring over here. And this force is minus kx 1, there will also be a force which will oppose this displacement form the spring over here. And this force is minus k prime x1. We have to also consider the situation where this particle is fixed and if I have moved this particle displace this particle by x0.

So, if I displace this particle by x0 is going to exert a force in the same direction. And that force is k prime x0. So, is going to exert a force k prime x0 on this particle, which I have to also take into account. So, the total force due to this spring k prime arises due to both the displacements of x1 and x0. And I combine these 2 terms and we can put it here. So, we now have the equations of motion of the 2 particles x0 and x1. Now, the point to note is that these are 2 second order differential equations. They are very much like the simple harmonic oscillator equation which we had studied earlier.

The only difference which arises here is that, these 2 equations are coupled. Since, the 2 oscillators are coupled; the 2 differential equations governing the oscillators are also coupled. By coupling what we mean is that, if you look at the equation for x0. The equation for x0 on the right hand side has a term which involves x1. And the differential equation for x1 has a term on the right hand side which has x 0. So, I cannot separately solve the equations for x0 and x1. The two variables, I have to somehow determine a method to solve them together.

So, the question is how do we solve these coupled differential equations? Now, the method you adopt to solve such coupled linear, ordinary linear differential equations is that you have to find linear combinations of the variables. In this case, you have to find linear combinations of x0 and x1. The 2 variables such that the differential equations that governed these linear combinations of x0 and x1 are not coupled or uncoupled. So, by finding suitable linear combinations of your variables you have to obtain differential equations which are no longer coupled. So in this case, the process is quite straight forward.

Let us look at the equation again, notice that if I add these two equations. The term, which arises due to the spring that couples the 2 oscillators, cancels out. This is essentially because of, the fact that the force which this spring exerts on this is equal an opposite to the force the same spring exerts on this. So, if I add the two equations for this mass and this mass these 2 terms cancel out. So, this gives me 1 of the linear combinations; the other linear combination in this case is, obtained by subtracting these 2 equations.

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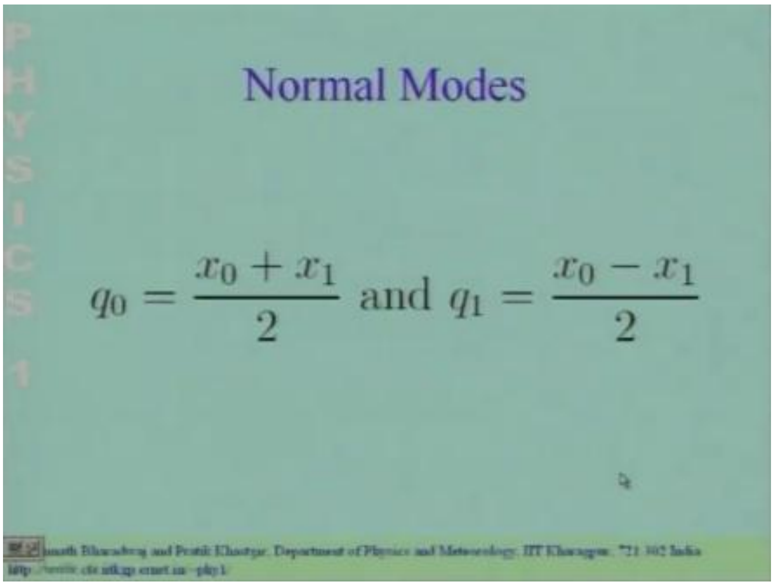

So, these are what are called the normal modes. So, in this particular case, we have 2 normal modes q naught and q1, q1 is x0 plus x1 divided by 2. The second normal mode q1 is x0 minus x1 divided by 2.You can obtain the differential equation, governing x0 by taking the difference of those 2 differential the sum of the those differential equations. So, the differential equation governing q0 is obtained by taking the sum of the 2 differential equations which we had earlier. And the differential equation governing q1 is obtained by taking the difference of the 2 differential equations which we had earlier.

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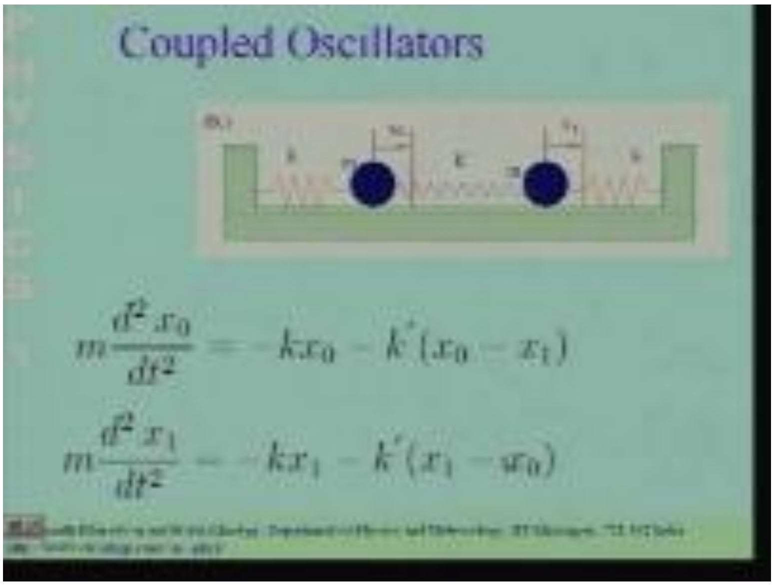

So, if you take the difference of these 2 differential equations. You will have the mass which is common throughout you will have x0 minus x1. And then I am going to subtract these 2 equations and divide by 2. So, I will have x0 minus x1 divided by 2. And here I am going to have x0 minus x1 divided by 2. Here, I am going to have 0 x minus 1 divided by 2 and here I am going to again have 0 minus x1 divided 2 with the minus sign.

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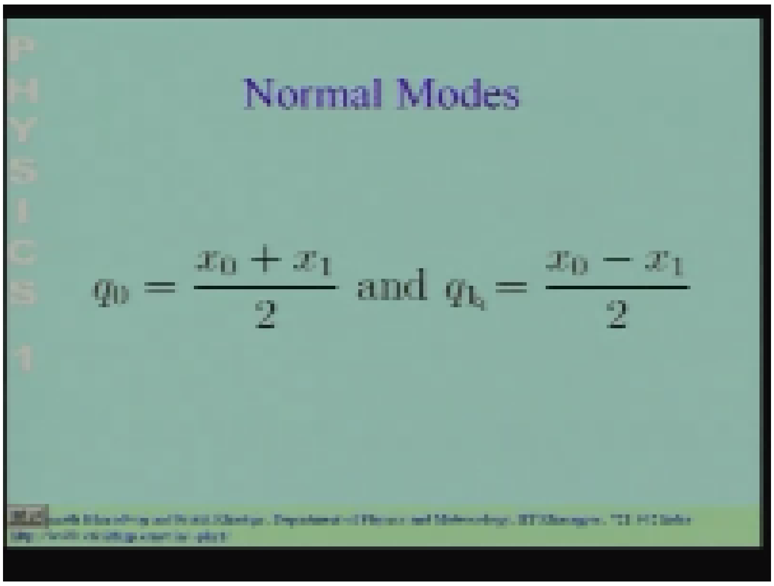

So, when I subtract these 0 equations I get an equation for the normal mode q1. So, the normal modes have a property that the differential equations governing the normal modes q0 q1 are not going to be coupled.

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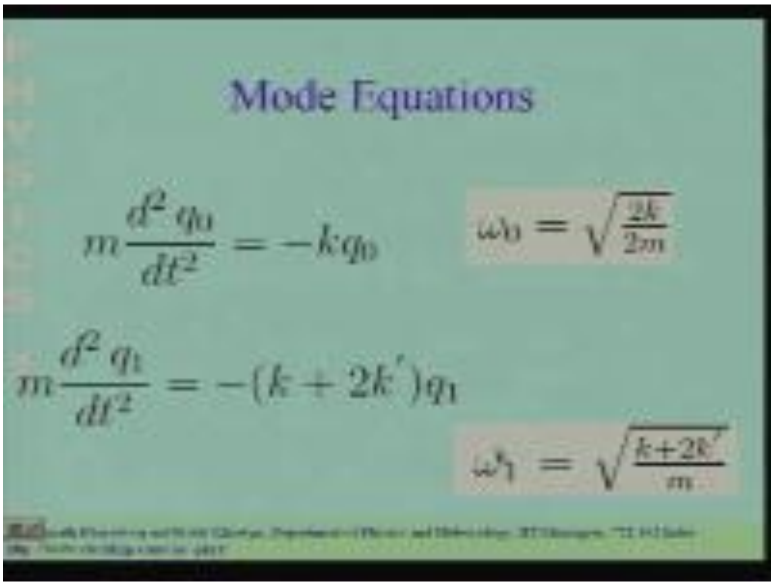

So, these are the differential equations, governing the normal modes q0 and q1. You can obtain this equation, by just adding the 2 equations which we had earlier. And you can obtain this equation by just subtracting the 2 equations which we had earlier. So, the normal modes, the equations governing the normal modes are not coupled. They are independent so, let me just summarize, what normal modes are, when I have a set of couple differential equations in this case we have 2, but if you had a set n couple differential equations with n variables. Then the method to solve these equations is, to find linear combinations of this variable. If you had n variables, you will have n linear combinations.

So, you have to find linear combinations of these variables, such that the differential equations governing the linear combination of the variables are not coupled, they are independent. Such linear combinations of the variables are called the normal modes of the vibration. So, the coupled vibration you can break up into normal modes; each normal mode executes a vibration which is not coupled to any of the other normal modes. So, you can decompose the whole problem into the vibration of normal modes. Notice that, each normal mode behaves like a simple harmonic oscillator which is not coupled.

So, the zeroth normal mode q0 is governed by the equation over here. The oscillations of this mode q0 have an angular frequency omega 0. Omega 0 is the square root of k by m which can also be written as 2 k by 2m. So, notice that omega 0 the angular frequency omega 0 is the angular frequency of the oscillators if they were not coupled. So, if these oscillators were not coupled, they would individually vibrate at the angular frequency omega naught. And 1 of the modes vibrates at exactly this frequency omega naught which is the angular frequency when they are not coupled.

The other mode q1 has an angular frequency omega 1which is k plus 2k prime by m the square root of that. So, omega one is the square of k plus 2k prime by m, you should note that omega 1 is always greater than omega 0. So, you have a new angular frequency which is introduce because of the coupling of the 2 oscillators. And the new angular frequency omega 1 is always larger than the angular frequency omega 0, which is the angular frequency that would be there if the 2 oscillators were not coupled.

So, we have from these coupled oscillators from the coupled differential equations, for these 2 oscillators we have gone over to a linear combination of the variables called the normal modes. We find that one normal mode the sum of the 2 variables; the normal mode corresponding to the sum of the 2 variables oscillates, at the frequency which the particles would oscillative if they were not coupled. And we have another normal mode, which oscillates at a higher frequency higher angular frequency. So, this is the faster mode and this is the slow mode.

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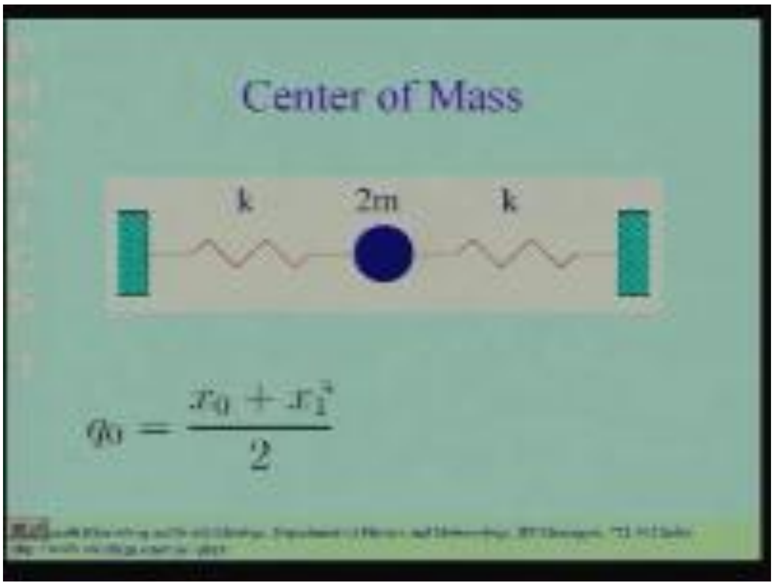

Now, let me give you an interpretation a physical interpretation of these 2 normal modes. The normal mode q0 is the sum of x1 and x2. So, it is the sum of x1 and x2 divided by 2. So, you can interpret q0 as representing the oscillations of the centre of mass of the system.

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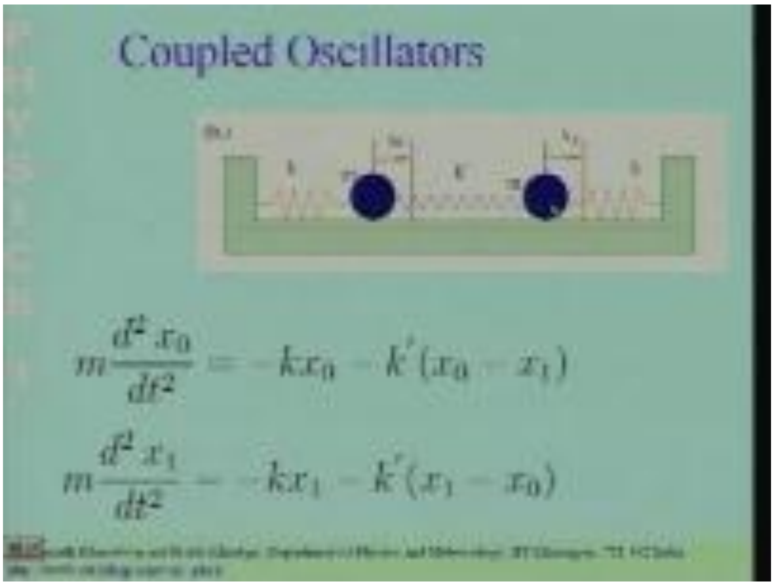

So, you can interpret q0 as representing the oscillations of the centre of mass of the system. So, my system has 2 particles which I can displace and if I were to represent these 2 particles by 1 particle. So, that is the role of the centre of mass, let me just digest a little and discuss the role of the centre of mass. If I were to ask you the question, where are you? You would then, give me some answer as to your position. I am located at such and such place now; if you look at your position carefully your nose is in the different position form the toes in your feet. And your fingers are in the different position form both of these. Now, when I ask you the question where are you, I am not interested in the individual positions of the different parts of your body.

I am interested in learning of 1coordinate, 1 set of coordinates, which will tell me where you are, which will give me an idea of where you are. If I were interested more details then I would ask you where nose where toe etcetera. But in many situations I am not interested in that information. I am interested in learning where in learning of I am interested in knowing a point. Which I can associate with your position. So then, the you would tell me that you are located at some point. And that point which you would tell me would be represent your average of the whole of your body where the average of whole of your body is and this is the centre of mass.

So, a centre of mass for a two mass system represents. Where the average on the average, if I look at this system form sufficiently far away. I would not be able to discern that there are 2 particles I would think of it is one big lump. And then you could tell me that, the big lump comprising of both these particles is located at a point, that point would be the centre of mass. It should be the average position of both of these particles. And if the masses were different then I would have to weigh the contribution from each of these with the masses, but here the masses are same. So, I could just average the displacements and replace these 2 masses with the single particles over there.

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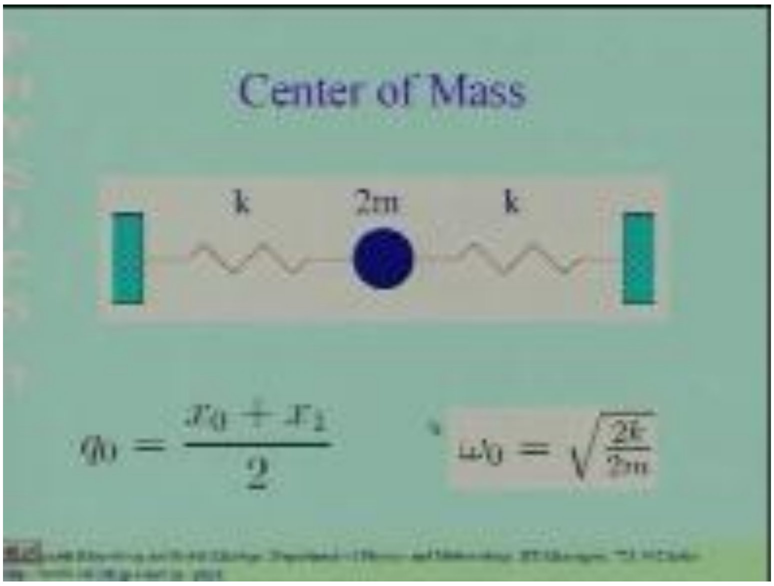

And that would have a position x0 plus x1 by 2. So, the normal mode, the 1 of the normal modes the normal q0 in the normal mode q0 we think of the 2 particles as 1 particle of mass 2m. So, if you think of the 2 particles as 1 single particle of mass 2m. So, you have collapse the 2 particles into a single particle and you forgotten, about the spring; which in this case, is something like an internal degree of freedom. So, you have forgotten about this spring that connects the 2 masses. So, you replace the 2 mass and the spring that connects them in between by a single mass 2m. Now, those 2 masses each had a spring attaching it to this rigid supports. So, there is 1 spring here of spring constant k, there is another spring here of spring constant k.

So, if you write down the equation of motion of this combined system. That is; if I replace the 2 masses by a single mass and then writes down the equation of motion for this. I will find that, the angular frequency of the system is the square root of 2k. I now, have 2 springs divided by the mass which is both the masses; which is the angular frequency omega naught which would be there if I had the 2 oscillators uncoupled. So, the normal mode q0 represents the motion of the centre of mass of the system. Now, I could have another kind of motion of these 2 particles. And the other the kind of motion would leave the centre of mass unchanged.

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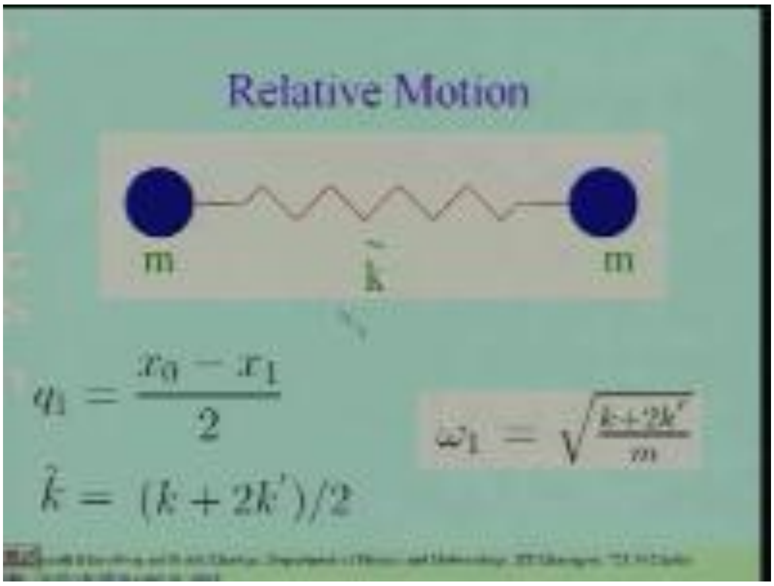

So, I have these 2 masses it could oscillate like this which would leave the centre of mass of the system unchanged, if it were to oscillate like this. So, this is the thing that I show over here. The other kind of motion possible is the relative motion of these 2 particles. The relative motion leaves the centre of mass unchanged. In the relative motion the centre of mass is fixed and this particle is displaced by exactly the same amount as this particle is displaced in this direction. So, they are if this particle is displaces this way, this particles will be displaced by exactly the same amount and when this moves away this also moves away by exactly the same amount. So, that the centre of mass remains fixed.

The relative motion is what q1 which is the difference x0 minus x1 tells you. So, the q1 refers to the relative motion of these 2 masses. The motion that leaves the centre of mass unchanged. And q1 is x0 minus x1 by 2 and if you write down the equation of motion for the relative motion. You will find that you can think of it, as this 2 masses being attached by a spring constant of effective by a spring of effective spring constant k tilde which has a value k plus 2k prime by 2. And the angular frequency of this system is the square root of k plus 2k prime by m, because the mass is also twice. So, the angular frequency corresponding to the system is, k plus 2k prime by m this is the angular frequency corresponding to this.

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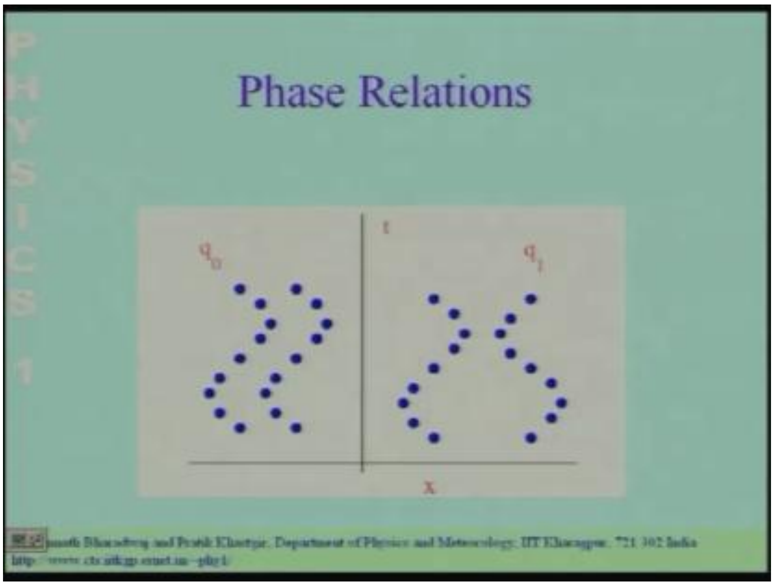

So, this shows you the same thing which I had been telling you. In terms of, the phase relation between the 2 different modes. So, in the mode x0 in the mode q0 the 2 particles x0 and x1 move with exactly the same phase that is what I show here. So, q0 is x0 plus x1. So, if x 0 moves in a certain way x1 also moves in exactly the same way. So, in the mode q0 the 2 particles x0 and x1 move exactly in the same phase, whereas in the mode q1 the 2 particles x0 and x1 move exactly out of phase. So you see that, if you couple 2 oscillators the phases of the motions are not, no longer independent, the phase of the 2 oscillators now get coupled.

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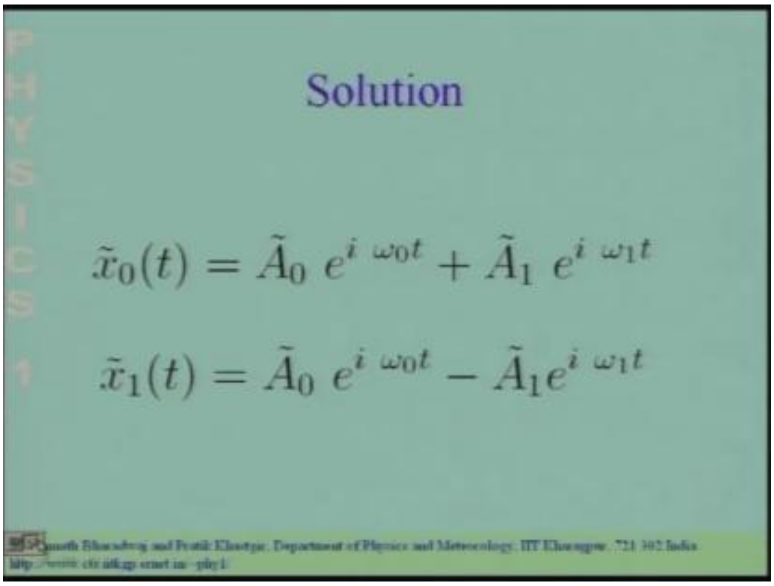

So, we can now, having given you a physical interpretation of these 2 different normal modes. We can now, write down the solutions to the differential equations governing the normal modes. The normal mode q0 has an angular frequency omega naught, the normal mode q1 has the angular frequency omega 1. So, we have the solutions, the solutions are q0 is equal to A0 e to the power I omega 0 t and q1 is A1 e to the power I omega 1t. Remember, that we are using the complex notation that is what the tilde tells us. So, this q0 and q1 are both complex variables A0 and A1 are the complex amplitudes.

The complex amplitudes have both the phase and the amplitude of this motion. So, these are the solutions to the 2 differential equations governing the 2 modes. We can now, combine these 2 solutions and obtain the solutions for x0 and x1. So, you have to invert the relation between x's and q's between the variables which we are the physical variables and the normal modes. If you invert these relations, you will get x0 is q0 plus q1 which gives us x0 to be A0 e to the power I omega 0t plus A 1 e to the power I omega one t. And x1 is the difference of the 2 variables q0 and q1 which is what we have here.

So, these are the expressions for x0 and x1 again in the complex notations. The question is what are the values of these different constants A0 and A1 that appear over here? How do we determine the values of these constants? The values of these constants are determined by the initial conditions. So, you will know the values for A0 and A1 only when you put in the initial conditions.

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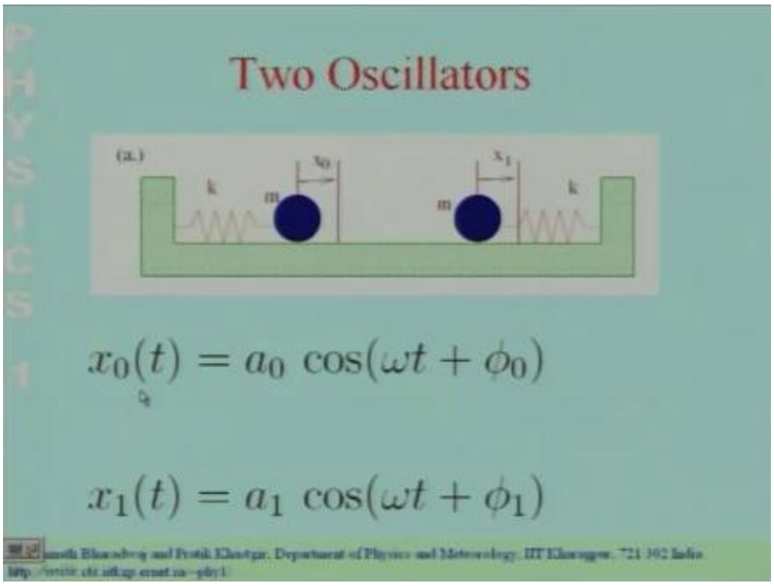

So, let us take up a particular situation here and the situation that we are going to take up is as follows;

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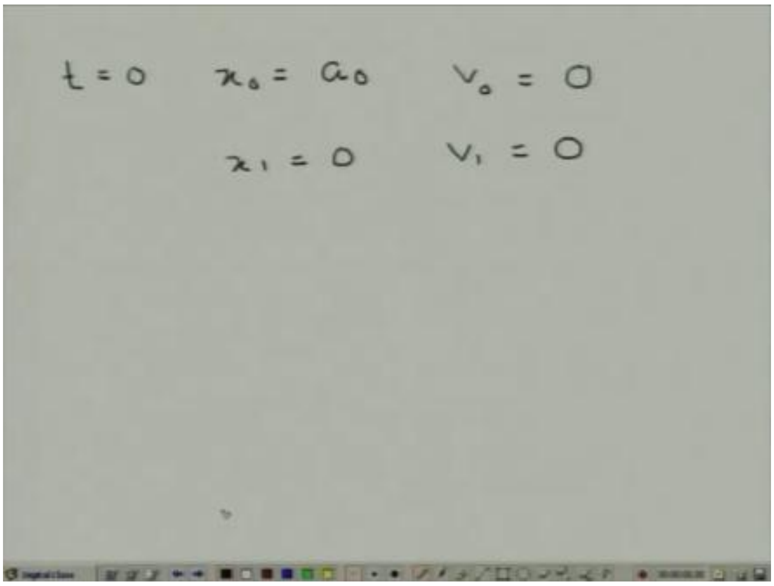

We have disturbed at t equal to 0, we have disturbed the zeroth particle. So, x0 is A0 at t equal to 0 and velocity of the zeroth particle v0 is equal to 0, x1 is equal to 0 v1 is equal to 0. So, what have we done we have these 2 masses which are coupled. So, these 2 oscillators which are coupled, the zeroth oscillator has displacement x0 the first oscillator has displacement x1. These oscillators are initially at rest and at equilibrium, what I have done is, I have displaced the zeroth oscillator from the equilibrium position by an amount

A0. And left it there and then I leave this whole system and watch how it oscillates. So, initially x1 is in the equilibrium position and both particles are at rest only the zeroth particle has been disturbed from the equilibrium position. You want to solve the motion for this.

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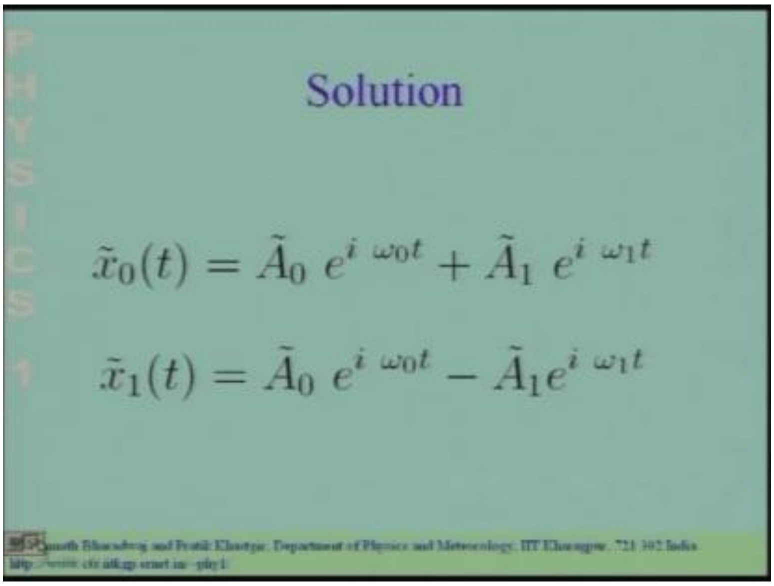

So, in to order to do this we have to determine we have to determine these coefficients A0 and A1 in order to fully get the solution. So, let us look at the equations for x0 first and at t equal to 0 let us put t equal to 0. So, at t equal to so, let me write down the equation for x0 over here.

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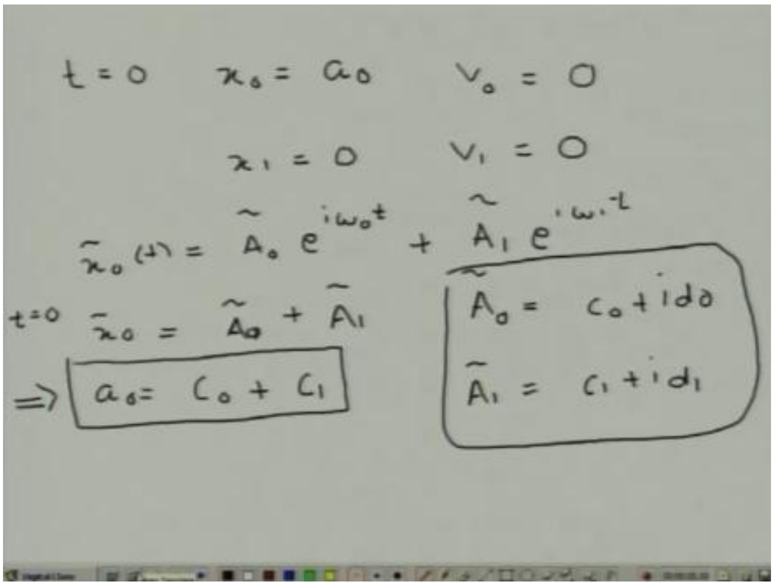

So, the equation for x0 is x0 t is equal to A0 tilde e to the power I omega 0 t plus A1 tilde e to the power I omega 1t. At t equal to 0 this is x tilde in the complex notation, at t equal to 0 we are given the fact. So, let us first calculate the position at t equal to 0 the complex position at t equal to 0, the complex variable corresponding to the position at t equal to 0. So, at t equal to 0 this is equal to A1 tilde A0 tilde at t equal to 0 the t to the power I omega t is1. So, I have A0 plus A 1 so, at t equal to 0 x0 is equal to A0 plus A1 both the left hand side and the right hand side are complex.

Now, when I want to equate this to the position of the particle is just the real part of this variable. So, we have to deal with only the real part of this and this. So, let me break up A0 into c0 plus I d 0 and A1 to c one plus I d1. So, I have done this decomposition; I have written these amplitudes A0 and A1 complex amplitude in terms of the real parts c0 and the imaginary part d0 A0 as c0 and d0 in terms of c0 and d0 A1 in terms of c1 and d1. Now, when I want to equate this to the position A0, this I should take only the real part of the right hand side.

So, the real part of the right hand side is c0 plus c1. So, this is A0 is equal to c0 plus c1. So, I have obtained 1 relation between c0 and c1 from the initial conditions. So, this is the first relation which arises from here, now, this arises from the initial position of the zeroth particle of the zeroth displacement. Let us now, apply the second information given which is initial velocity of the zeroth displacement. So, to put that condition we have to differentiate this expression, let me differentiate this expression in a new page. So, let me get a new page first.

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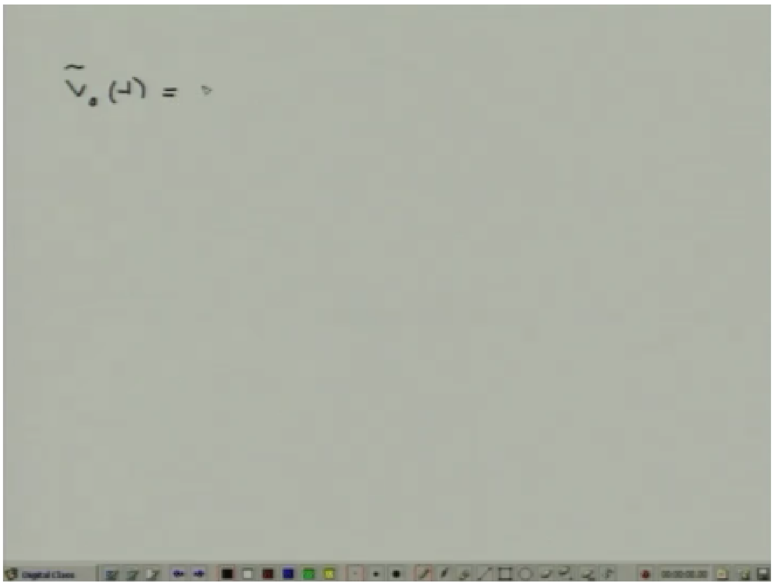

So, I am going to calculate V0 as a function of time and this you can,

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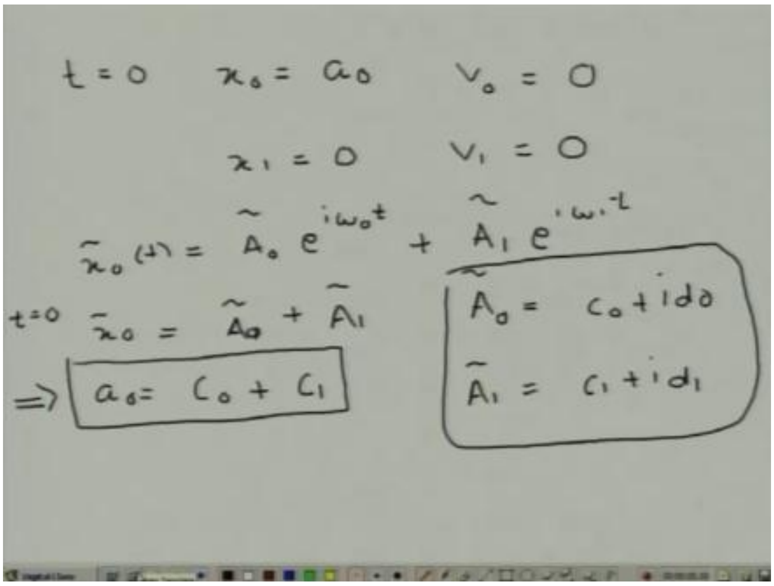

If I differentiate this, I will pick up a factor of I omega naught over here and I will pick up a factor of I omega 1 over here. So, let me write down the expression that I get when I differentiate it.

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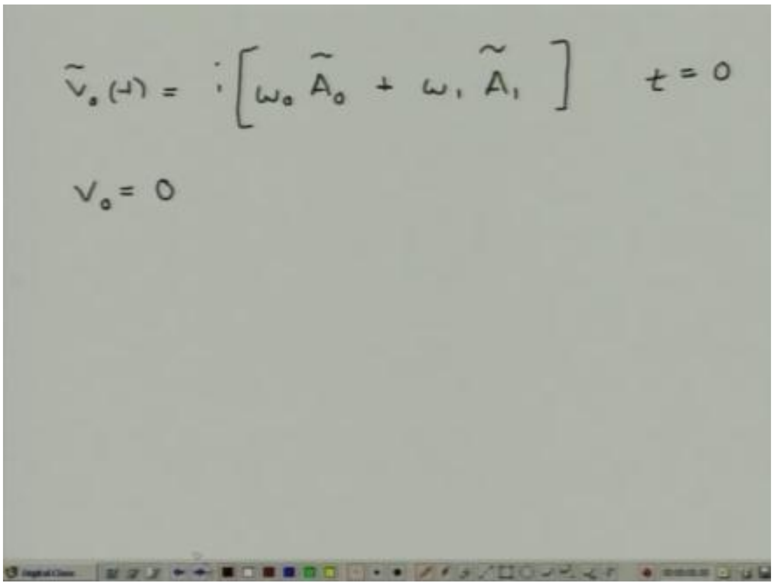

So, I will get I which I can take common outside, omega 0 A0 tilde plus omega 1 A1 tilde I have this is at t equal to 0. So, the e to the power I omega t which was there at t equal to 0, those become1. So, this is the complex velocity at t equal to 0 of the zeroth displacement at t equal to 0. Now, when I want to equate this to the real velocity of the part which I know is initially 0, I should take only the real part of this. So, the question is what is the real part of this?

So, there is an overall factor of I and so, I have to only look at the imaginary part of A0 and A1 because of this factor I over here. So, when this i multiplies the i which is there in the imaginary parts I get, let us look at, A0 and A1 A0 is c0 plus id 0 and A1 is c1 plus id 1. So, when I multiply these numbers with i, I will get a minus 1 and the imaginary parts of these now become the real part when I multiplied by i.

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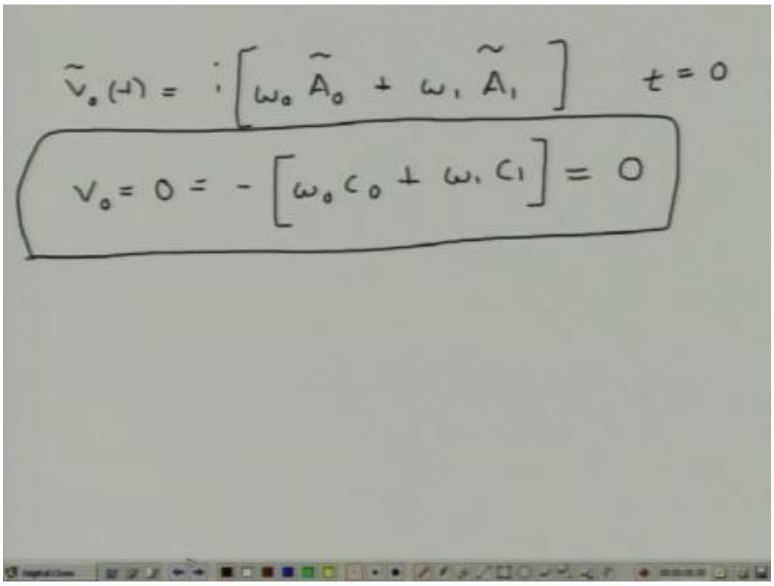

So, the expression that I get, when I take the real part of this; what I get is, this should be equal to minus i into the i which comes over here. And here gives a minus sign and then I have omega naught, c0 plus omega 1 c1 this should be equal to 0. So, the fact that the zeroth displacement is initially has no velocity the x0 has no velocity tells me this or it tells me that let us we could at we could leave at this. So, this is the information that we get from applying the velocity condition for the zeroth particle. Now, let us look at the first the particle the variable x1.

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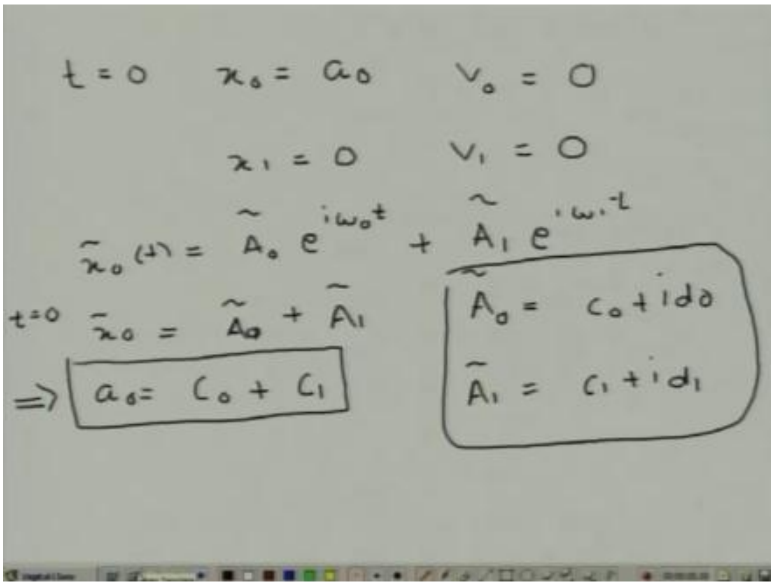

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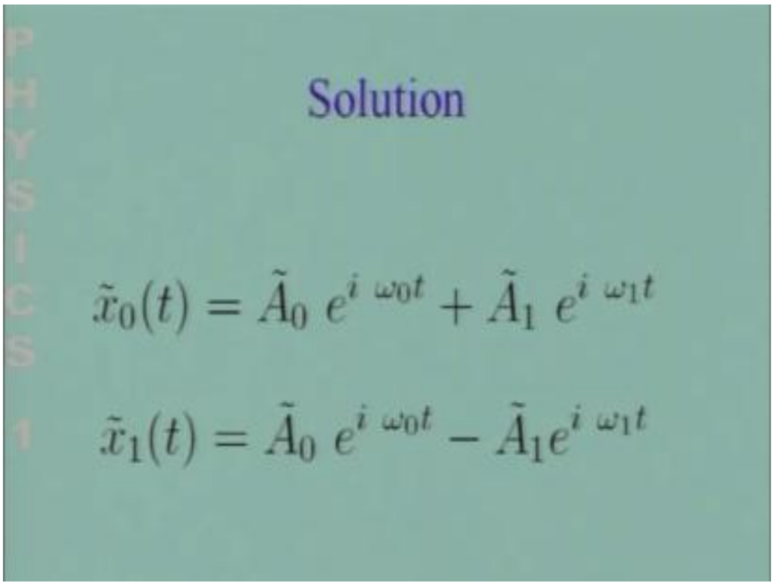

So, the variable x1 is equal to a0 e to the power I omega0 t minus A1 e to the power I omega 1t.

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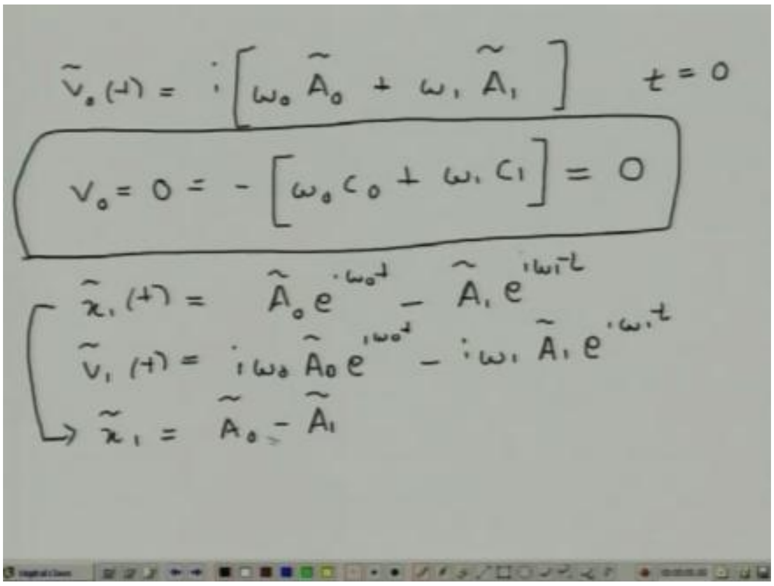

So, let us now look at this variable. So, let me write down the expression for x1 here, x1 tilde is equal to A0 tilde e to the power I omega 0t minus A1 tilde e to the power i omega 1t. At t equal to 0 let me also calculate the velocity first and then I can set everything to t equal to 0. So, the velocity as a function of t going to be i omega naught A tilde 0 i omega 0t minus i omega 1 A1 tilde e to the power i omega 1t. So, I have found the velocity; now, what we have to do is we have to set t equal to 0.

So, let us set t equal to 0 and then apply the initial conditions. So, at t equal to 0 the position variable gives me, x tilde 1 is equal to A0 minus A1. Now, when I apply the condition that the initial displacement is 0. I should take only the real part of this is the physical displacement. So, I should take the real part of this variable. So, if I take the real part of this, I should take the real part of the right hand side, let us just take a look again.

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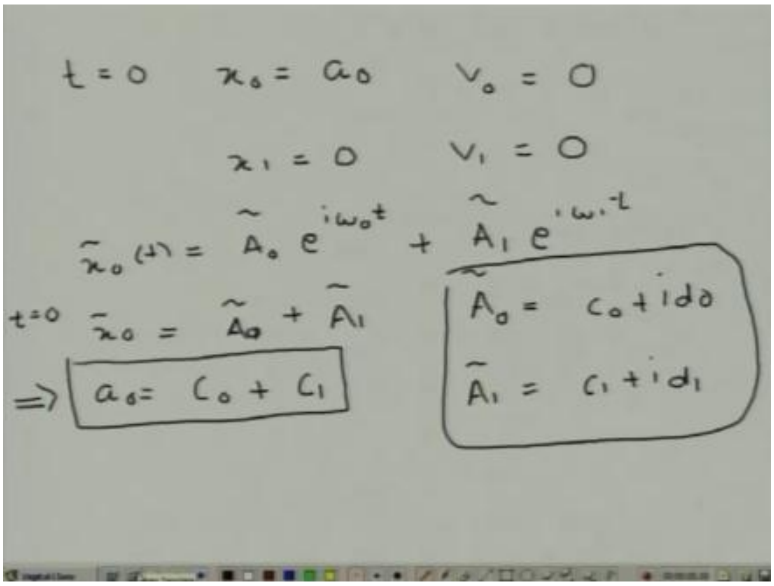

So, the real part of a 0 and A1 both are c0 and c1.

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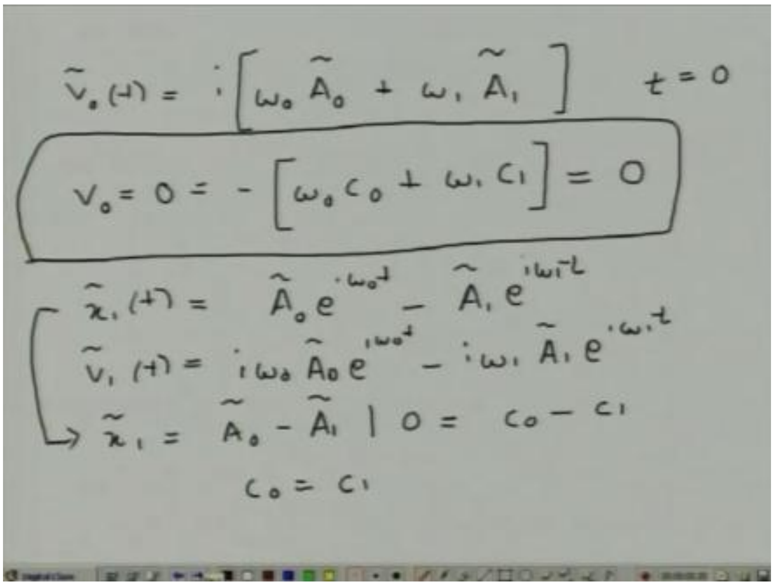

So, if I take the real part of the right hand side, I have so, the real part is 0 should be equal to c0 minus c1 the real part of this is c0 the real part of this is c1. So, the initial displacement 0 should be c0 minus c1 which straight away tells us that c0 should be equal to c1, if c0 has to be equal to c1.

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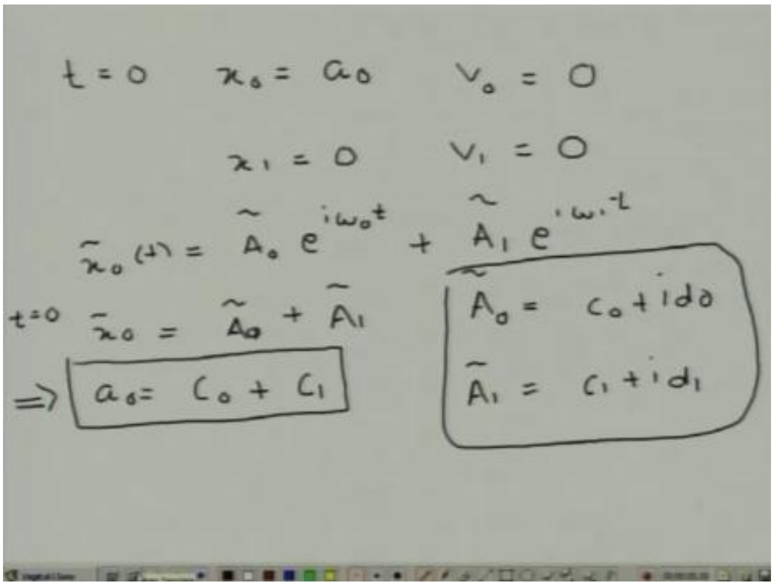

If c0 is equal c1 then you can see, that each of them must individually be A0 by 2.

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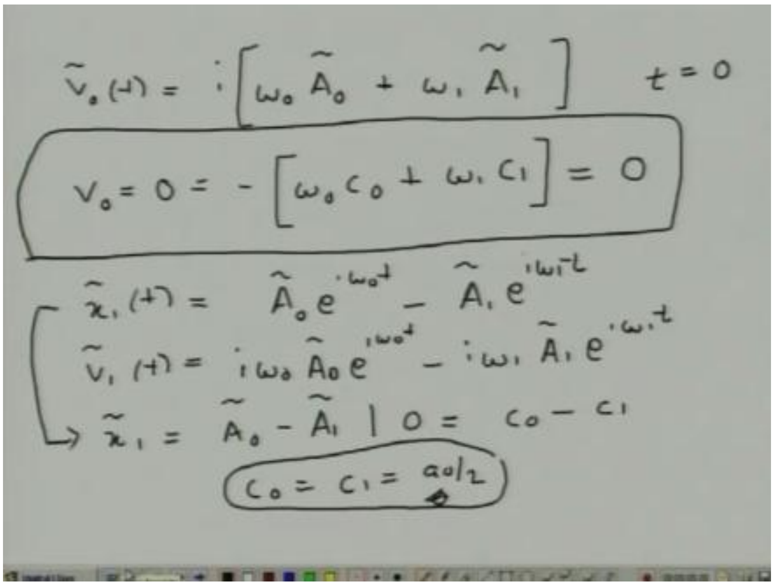

So, we find here that this should be equal to A0 by 2. So, we have obtained the real part of 2 unknown amplitudes and they are A0 by 2 they are both same and they are both A0 by 2. Now, let us look at the velocity information at t equal to 0, what does the velocity information tell us. So, at t equal 0 this tells us at we could just set t equal to 0 here and here and what we have is at t equal to 0.

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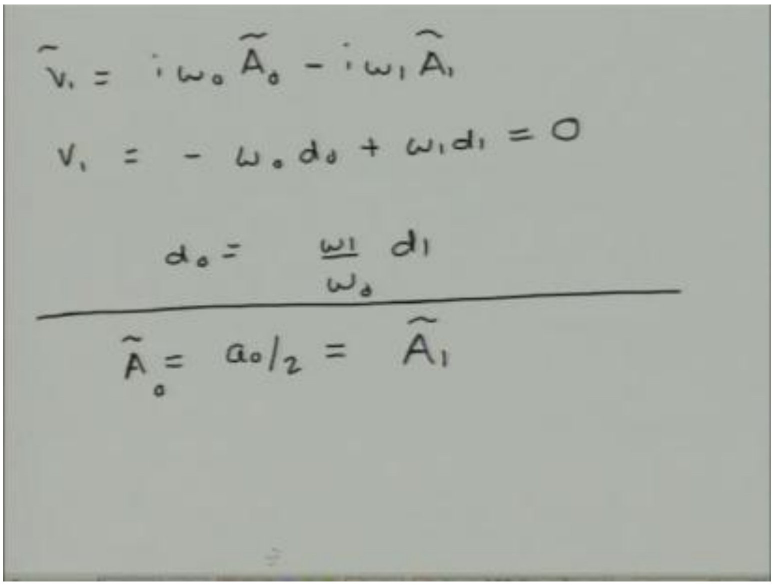

So, at t equal to 0 we have V1 is tilde is equal to i omega 0 A tilde minus i omega 1 A1. Now, again remember when a multiply by i, I only pick up the imaginary parts of A0 and A1. So, this is equal to minus omega 0 into d 0 plus this is a minus sign because of the i which comes, when I take the imaginary part over here. Plus omega 1 d1 this should be equal to, this is the real velocity and this should be of the first displace displacement 1 and this should be equal to 0. So, which tells us that d0 should be equal to omega 1 by omega 0 into d1. So, it gives us a relation between d0 and d1. And then they are just proportional to each other.

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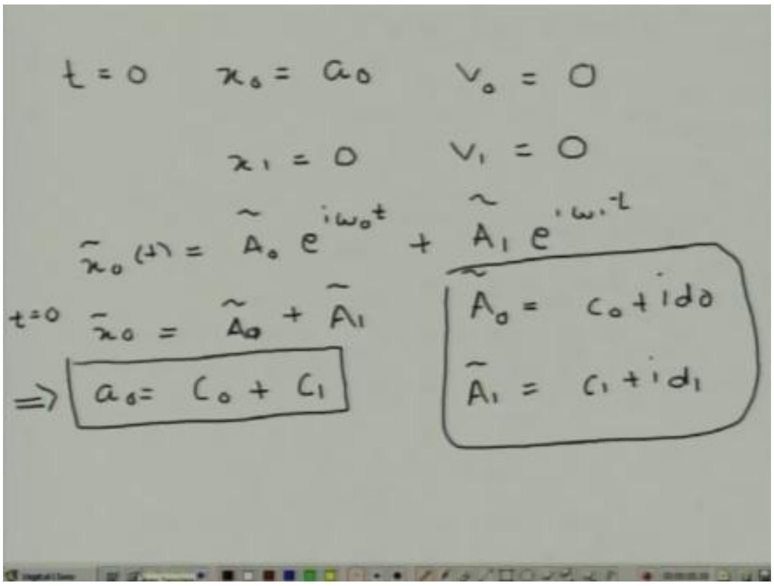

Now, if I plug this back in, into the condition which I get from the velocity of the first particle.

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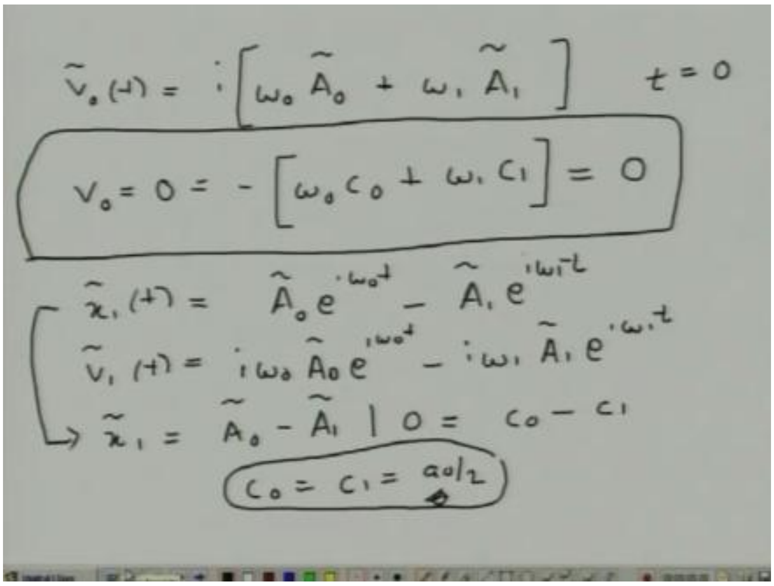

So, what is the condition that I get from the velocity of the first particle, from the velocity of the first particle, I get the condition. This should be d not c the imaginary part of A0 is d imaginary part of A1 is also d should have been d not c. So, the condition I have over here is that, d0 this d1can be written in terms of d0. So, it basically tells me that, d0 should be 0 and d1 should also be 0.

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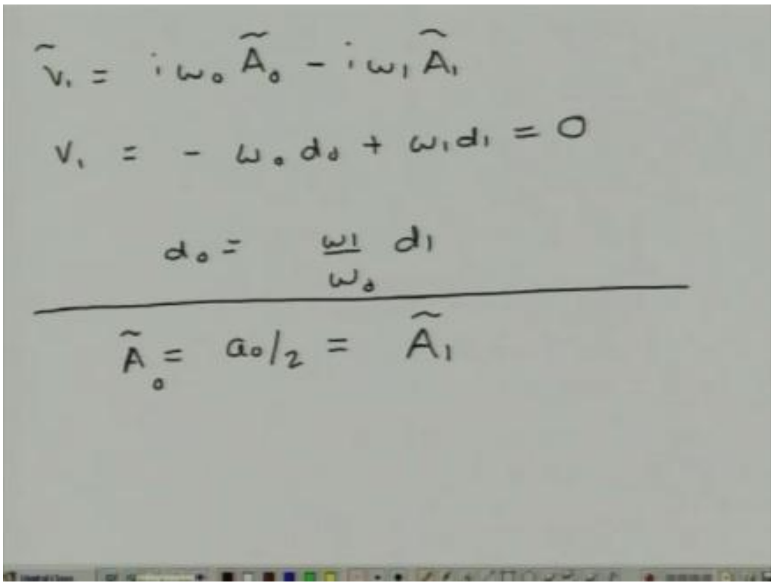

So, these together tell me that, A tilde is real it is a naught by 2 A tilde 0 and this is also equal to A1. So, both the amplitudes are real and they have a value A0 by 2.

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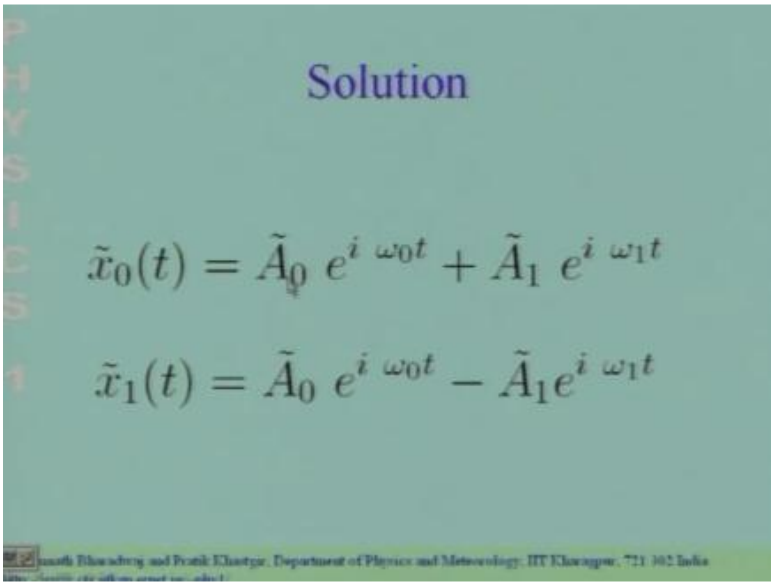

So, having worked out these coefficients for a particular case. Let us now, discuss the solution.

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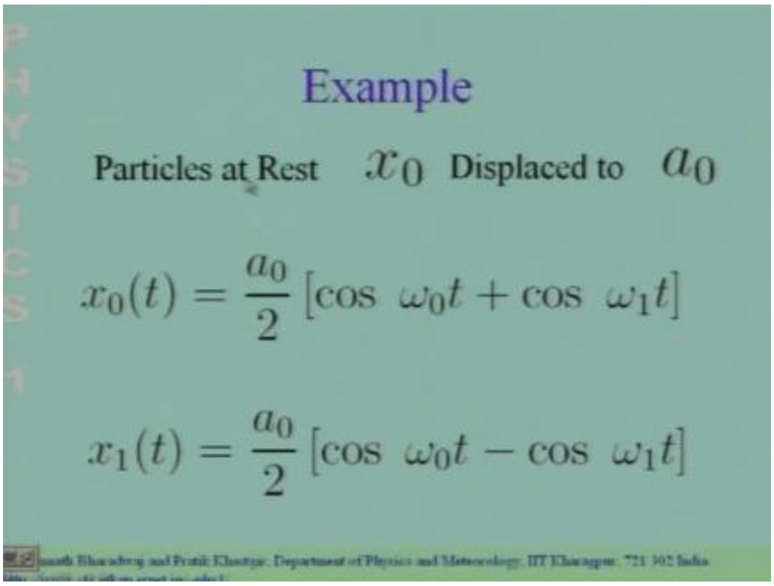

So, let me summarize what we have worked out, is we have considered the situation. Where x0 is initially displaced to A0 and the other particle x1 is initially not displaced and both the particles are at rest. So, I have displaced only 1 of the particles and given it no initial velocity other particle remains in equilibrium with no velocity and no displacement and I leave the system. Then the solution we worked out, the 2 unknown amplitudes that occur in the solution for this particular case and if you plug those unknown amplitudes over here.

Where this is real and this is also displacement by 2. You plug it in here and take only the real part, because that is the thing that gives the displacement the imaginary part has no information. So, take only the real parts, then you are led to this expression for the displacement x0 is A0 by 2 cos omega 0 by t plus cos omega 1t this is the displacement of the first particle which you gave an initial displacement. This is the displacement of the second particle, which you did not give an initial displacement. It moves because of the coupling with the this oscillators, which you had displaced.

So, this does not remain where it is, it gets disturb because of the motion of this through the coupling and this the x1 variable is governed by this particular expression. So, the point to note is that, both the motion, the motion of, both these particles have are simple harmonic oscillators are linear combination of simple harmonic oscillators of 2 different frequencies. These are the two different frequencies corresponding to the 2 different normal modes.

In general, there will be arbitrary coefficients and this particular case the coefficients are the same. So, both these particle motions are linear super positions of oscillations of two different frequencies. 1 is the angular frequency if there was no coupling between two oscillators, the other is the faster mode which has a higher angular frequency than omega naught. Let us now, investigate this particular solution in slightly more detail.

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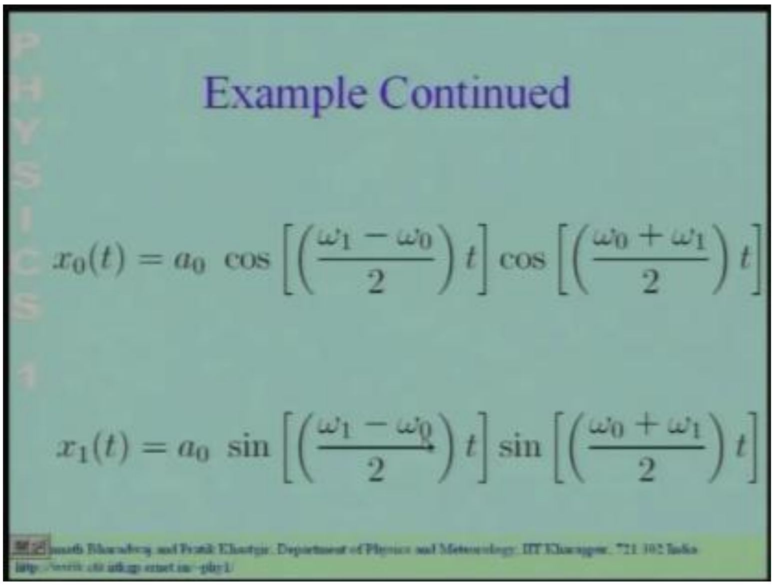

So, we could take this solution and write it in this way also, this is nothing just nothing, but an trigonometric transformation. So, cos a and plus cos be can be written as cos a minus b by 2 into cos a plus b by 2. Similarly, cos a minus cos b which is what we have here.

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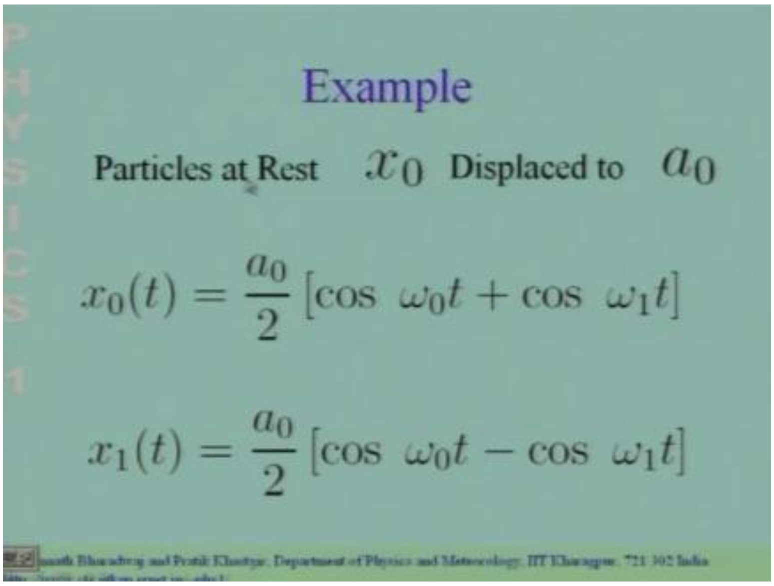

Cos a minus cos b can be written in as sin b minus a by 2 into b plus a by 2. So, we have used these trigonometric identities to write the sum of 2 cosines and the difference of 2 cosines as product of 2 cosines and product of sins. Now, you may be wondering why we have done this. Well this tells us something quite interesting it, the let me go little further. The interesting thing is that the sum of 2 cosines these are 2 oscillations. So, I have 2 oscillations of different frequencies. If I superpose 2 oscillation of different frequencies I can think of it as.

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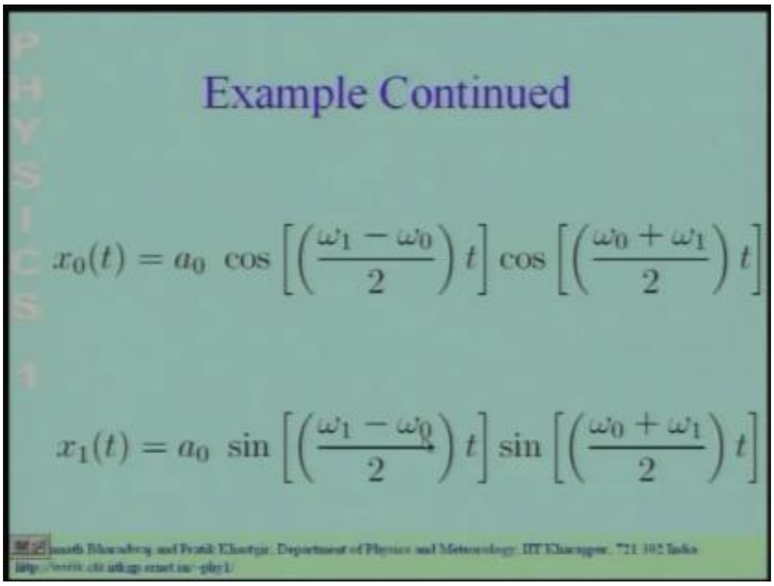

The product of 2 oscillations of different frequencies, the sum of 2 oscillations of different frequencies the superposition can also be thought of as the product of 2 oscillations of different frequencies. 1 oscillation is, at the average frequency of these at the average of these 2 frequencies. So, 1 oscillation is at the angular frequency omega 0 plus is omega 1 by 2 and the other is at the difference. Now, if omega 0 and omega 1 are very close then this term over here, becomes extremely small and we can think of the motion x0 and z1, as being a fast component. So, these are the fast components who's amplitude is under goes a slow modulation.

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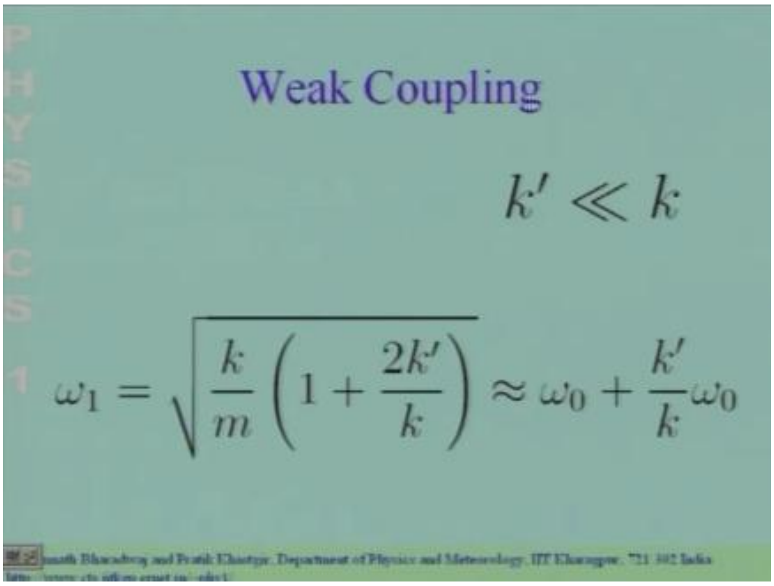

So, let me go into this in a little more detail, let us look at the situation for the spring mass coupled spring mass system when the coupling is weak. What we mean by, weak coupling is if the intermediate spring that we have introduced to couple that 2 oscillators. If the spring constant of that spring is much smaller than the spring constants of the 2 oscillators that I had to start with so, let me go back to the picture which we had to start with.

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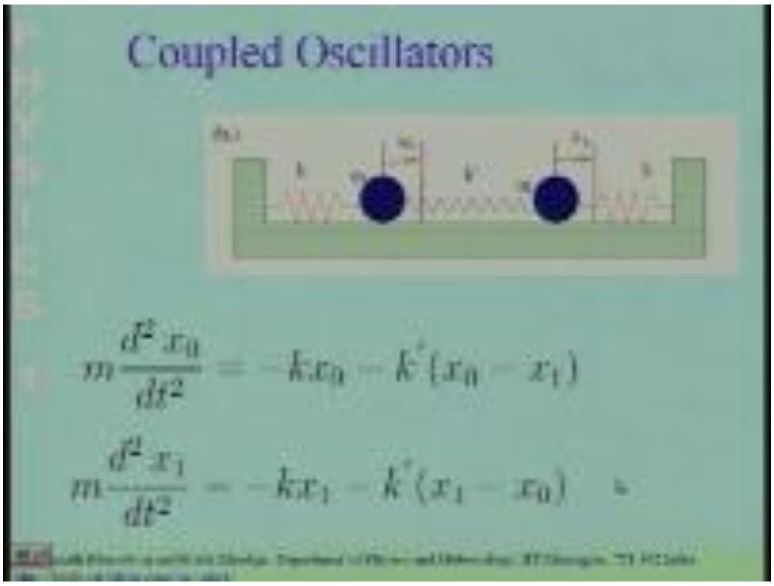

So, this is the situation which we are discussing and to start with

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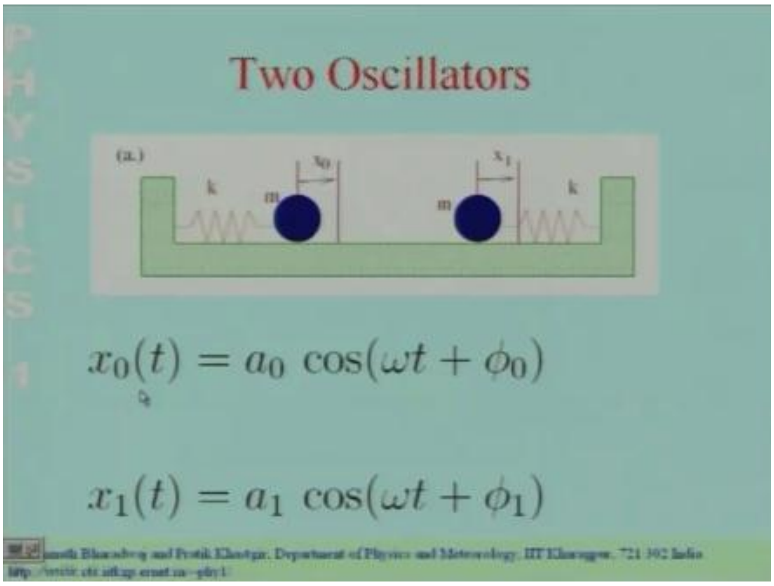

We had two oscillators which were free.

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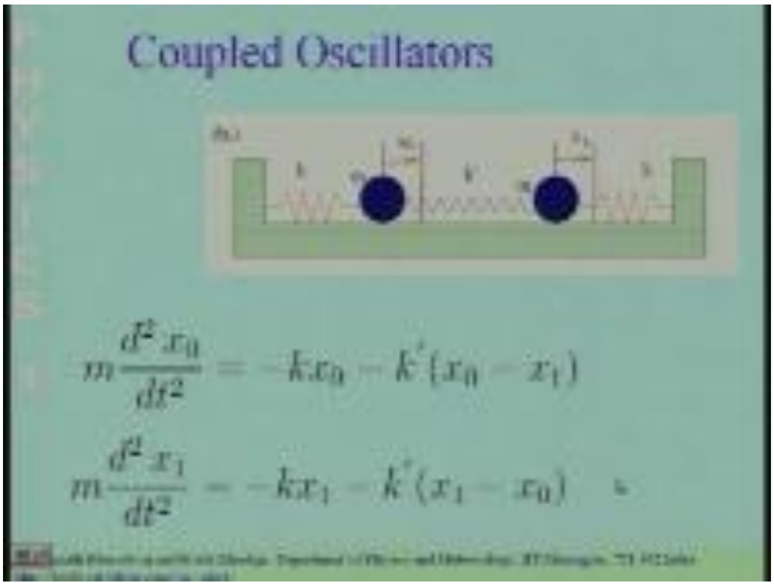

Now, we have introduced a spring of spring of a different spring constant k prime, each of these springs have a spring constant k. And we have introduced the third spring of a different spring constant k prime to couple this. Now, we will assume that this coupling is much weaker. So, this spring constant is much weaker, than the spring constants here and here. So, it is a very weak coupling is what we are going to assume in this assumption.

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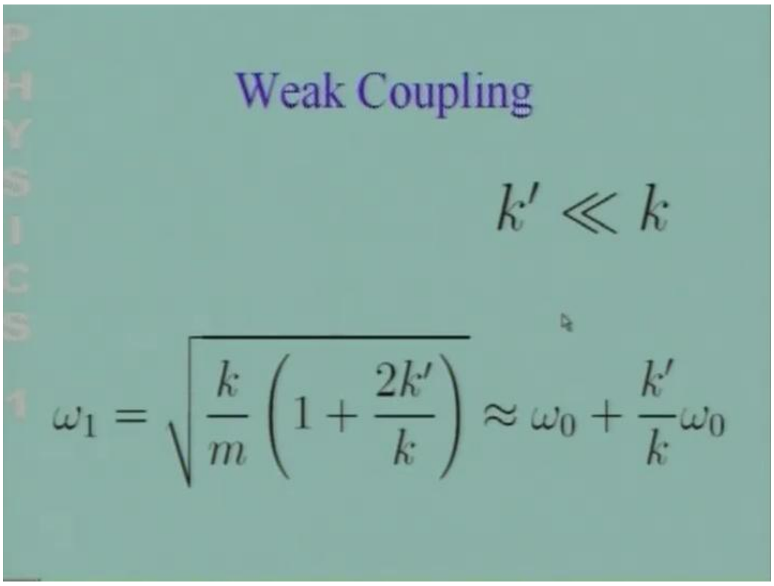

So, this assumption is k prime is much smaller than k the spring, the coupling spring is much weaker than other 2 springs. In this assumption you can simplify the expression for omega1. So, recollect that omega 1 let me do it for you here.

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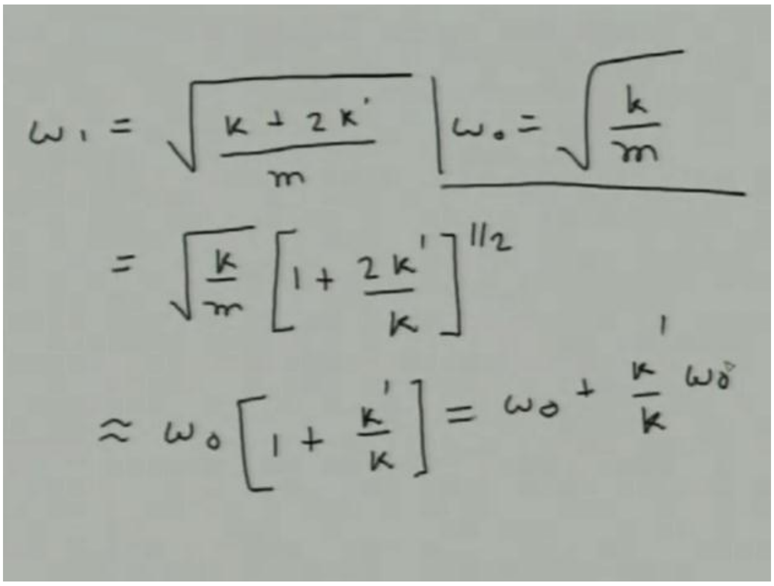

So, recollect that omega 1 is the square root of k plus 2 k prime by m. And this is the fast mode, the slow mode is omega naught is equal to root k by m. k is the spring constant. If there was no coupling each oscillator would at oscillate omega naught angular frequency omega not because of the coupling there is a new mode which is a little faster, its k plus 2k prime by m omega 1 is a square root of k plus 2k prime by m. Now, if k prime is much smaller than k then, I could write omega 1 as square root of k by m.

So, I have taken this common so, what I have is 1 plus 2k prime by k to the power half. If k prime is much smaller than k, then I can do a tailor expansion of 1 plus 2k prime by k to the power half. And this tailor expansion if I could retain only the first order term in k prime by k because we are assuming that k prime k is a small number. We then have this is approximately equal to omega naught into 1 plus half into 2k prime by k. This is the tailors first term, in the tailor series and this gives us one plus k prime by k. Which is equal to omega naught, that is the original frequency of the oscillator plus a small change which is k prime by k into omega naught.

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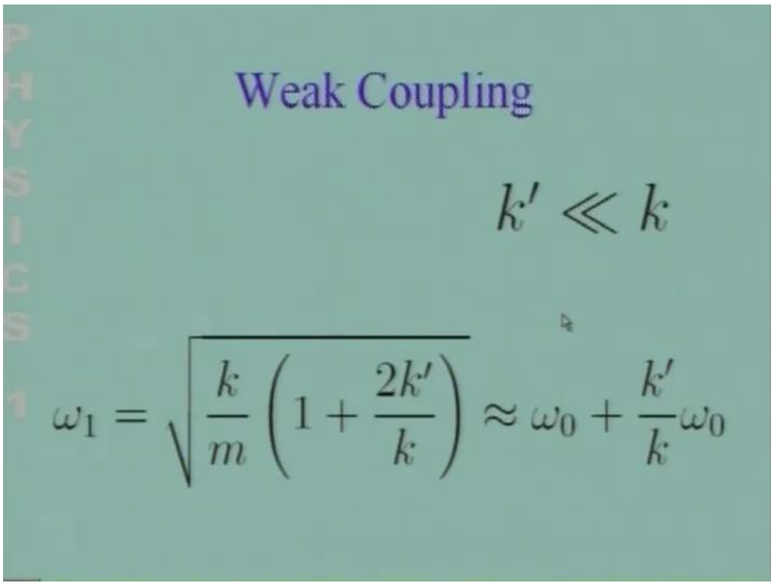

So, omega 1 and omega naught differ only by a small amount which is k prime by k into omega naught which is what we have over here. So, omega one is approximately equal to in the weak coupling limited is approximately equal to the frequency of the uncoupled oscillator omega naught plus a small correction which is k prime by k into omega naught.

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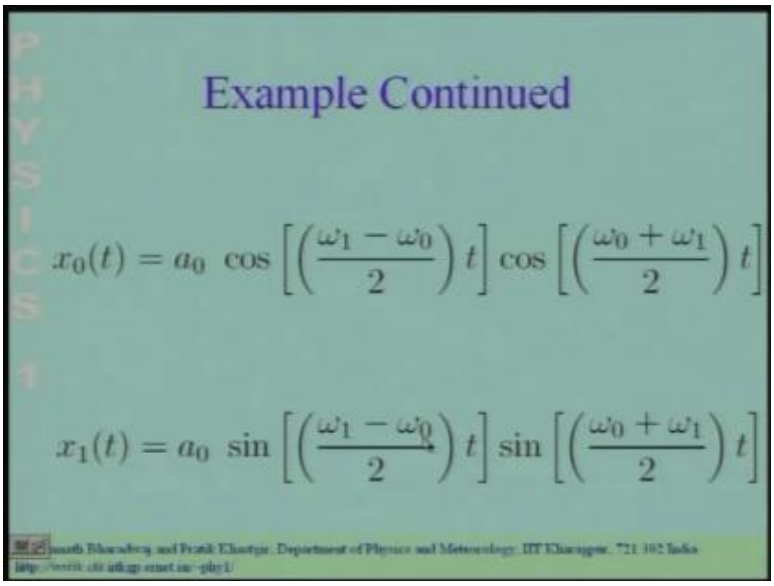

So, if you make this assumption and use it over here. So, then omega 1 minus omega naught, now, becomes k prime by k into omega naught by2.

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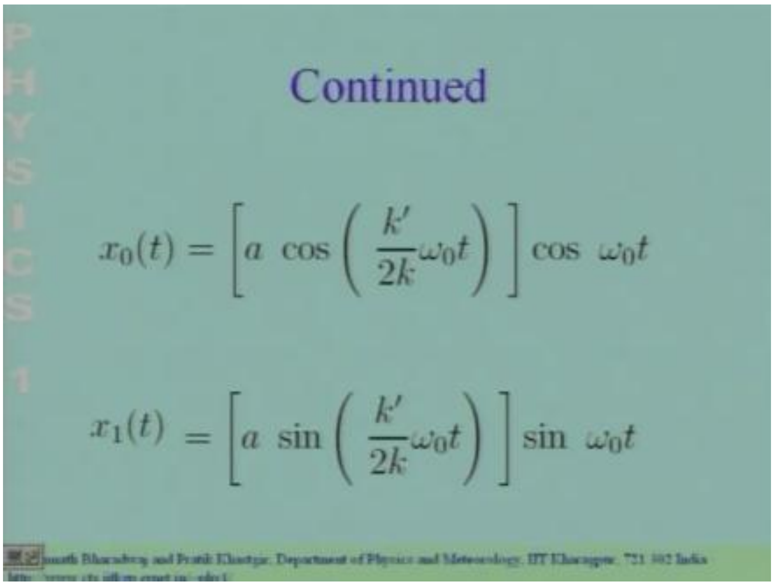

So, we will put in that and what we have is this. So, we have these 2 expressions for x1 x0 and x1. Let us look carefully at these 2 expressions so, omega naught is much larger than k prime by k into omega naught. So, the oscillation of both x0 and x1 has 2 parts, there is the fast oscillation. So, I can think of it as a fast oscillation at an angular frequency omega naught. So, both x0 and x1 execute fast oscillations at the angular frequency omega naught which would have been there had there be no coupling at all.

So, the 2 masses in the weak coupling limit, the 2 masses continue you; can think of the 2 masses continuing to oscillate. As if there was no coupling so, the each of them oscillates with an angular frequency omega naught. The effect of the coupling is through these 2 terms over here. So, the effect of the coupling is that it modulates the amplitude of the oscillations, the amplitude of the oscillations themselves do oscillations. So, the amplitude of the oscillation, so, this the oscillation and this is the amplitude.

The amplitude here, itself does oscillations at a much slower frequency compare to omega naught. The slow angular frequency is k prime by 2k into omega naught. So, its factor k prime by 2k slower. And both of these particles, the motion of both of these particles show a similar behaviour. In this particular case, we have started the whole system with just displacing x0 x1 is at rest. So, if I start the whole system by just displacing x0. Initially x0 is going oscillate x1 is going to be at rest, but slowly the oscillations in x1 are going to pick up and the oscillations in x0 are going fall. Because as cosine the amplitude here is, going to fall and the amplitude here is going to pick up.

When a the face approaches pi by 2 and then again when the face approaches pi, this is going to fall and this is going pick up and the amplitude of this is going to go back and forth.

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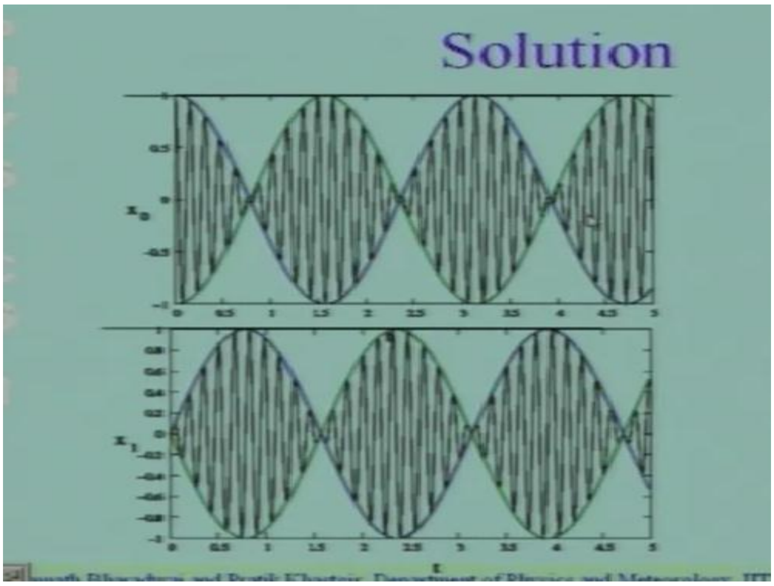

So, the motion the motion of x0 and x1 is shown in the curves plotted over here. So, there are these fast oscillation that the frequency omega naught and the amplitude of the fast oscillation gets modulated at the frequency k prime by 2k into omega naught.