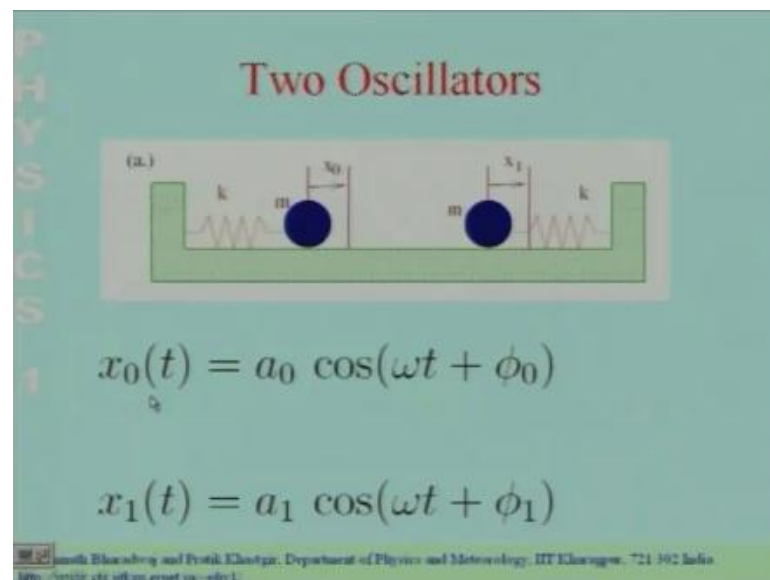


**Physics I: Oscillations and Waves**  
**Prof. S. Bharadwaj**  
**Department of Physics and Meteorology**  
**Indian Institute of Technology Kharagpur**

**Lecture - 07**  
**Coupled Oscillations**

In the past few lectures, we have been discussing a single oscillator. We started with a single simple harmonic oscillator. And we then discussed the effect of damping and its behaviour under the influence of an external force. Today, we shall extend our discussion to a situation where we have two oscillators.

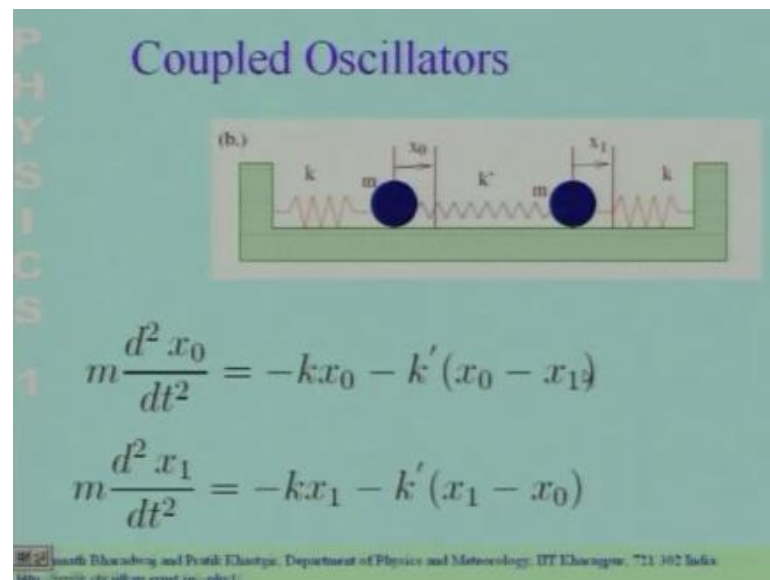
(Refer Slide Time: 01:16)



To set things in perspective, let us consider a situation where we have 2 oscillators of the same spring constant  $k$  and the masses have the same value  $m$ . So, we have 2 such oscillators, which are free to oscillate independently. So, as we have already studied each of them, will execute simple harmonic oscillations, at the angular frequency  $\omega$  naught  $\omega$ , which is square root of  $k$  by  $m$  and the amplitudes of the 2 oscillators. So, we are going to use  $x_0$  to denote the displacement of this oscillator over here, the 0th oscillator and we are going to use  $x_1$  to denote the displacement of the first oscillator over here. So, each of these oscillators will execute simple harmonic motion. The amplitude of these 2 oscillators'  $a_0$  and  $a_1$  are both independent they are in no way connected. And the phases  $\phi_0$  and  $\phi_1$  of these 2 oscillators are also independent, they are in no way connected.

So, if you disturb this oscillator and leave the second oscillator undisturbed, the first oscillator is going to continue to oscillate and the second oscillator is going to remain where you had left it. Similarly, if you leave this at rest and disturb this one, it is not going to exert any influence on this. And it is, this is going to oscillate, this is going to remain where you had left it. Now, this is nothing of great interest. This is just an extension of what we have already studied.

(Refer Slide Time: 03:06)



The situation which is of interest in today's class is where we introduce a third spring with spring constant  $k'$ , the third spring attaches the 2 masses. So, it effectively couples these 2 oscillators which we have just discussed. So, if you introduce a third spring into the problem which couples 2 oscillators the behaviour gets completely modified. If I disturb the zeroth oscillator, these disturbances are also propagated to the first oscillator through the new spring which has been introduced here.

Similarly, if I disturb the first oscillator, the disturbance is propagated to the zeroth oscillator through the spring introduced here. So, the new spring couples the 2 oscillators and they no longer can oscillate independently. So, let us analyse this system. We proceed by writing down the equations of motion of these 2 masses. So, let us first write down the equation of motion, for the  $x_0$  for the displacement of the zeroth mass. So, the mass into the acceleration corresponding to this displacement is equal to the total forces acting on this, if I displace it.

So, if I displace the zeroth mass the spring over here gets extended. And this gives rise to a force minus  $kx_0$  a displacement of this mass and the zeroth mass also produces a deformation in the spring which couples the 2 oscillators. And if I hold the first mass fixed and displace this, there then will be a force which is proportional to the displacement of this and it is in to the left. So, it will have a plus sign so, I will have a force plus  $k$  prime, minus  $k$  prime. The force will be, will oppose the displacement. So, there will be a force minus  $k$  prime  $x_0$ . Let me just go through this again, if I displace this mass slightly to the right. The spring over here is going to exert a force opposing that displacement.

And that force is minus  $k$  prime  $x_0$ ; it is going to oppose the displacement. So, there is going to be a force if I move this over here. Now, there is also possibility that if I keep this mass fixed and if I were to displace this mass by an amount  $x_1$ . So, this is fixed, this has moved, this again causes this intermediate spring, the coupling spring to get extended and this is going to exert a force in the same direction as the displacement on this mass. So, I have a force minus  $k$  prime  $x_0$  plus  $k$  prime  $x_1$  which I can write here. So, I have combined the forces which arise due to the motion displacement of this and this, due to the spring and written them as 1 term and that is minus  $k$  prime  $x_0$  minus  $x_1$ . So, this gives me the equation of motion for the zeroth particle  $x_0$  the displacement of the zeroth particle  $x_0$ .

Let us also, consider the equation of motion of the first particle  $x_1$ . So, going through a same analysis, the mass into the acceleration of  $x_1$ . This is equal to the total forces that act on this particle. If I displace this particle, then there will be a force which opposes this displacement from this spring over here. And this force is minus  $kx_1$ , there will also be a force which will oppose this displacement from the spring over here. And this force is minus  $k$  prime  $x_1$ . We have to also consider the situation where this particle is fixed and if I have moved this particle displace this particle by  $x_0$ .

So, if I displace this particle by  $x_0$  is going to exert a force in the same direction. And that force is  $k$  prime  $x_0$ . So, is going to exert a force  $k$  prime  $x_0$  on this particle, which I have to also take into account. So, the total force due to this spring  $k$  prime arises due to both the displacements of  $x_1$  and  $x_0$ . And I combine these 2 terms and we can put it here. So, we now have the equations of motion of the 2 particles  $x_0$  and  $x_1$ . Now, the point to note is that these are 2 second order differential equations. They are very much like the simple harmonic oscillator equation which we had studied earlier.

The only difference which arises here is that, these 2 equations are coupled. Since, the 2 oscillators are coupled; the 2 differential equations governing the oscillators are also coupled. By coupling what we mean is that, if you look at the equation for  $x_0$ . The equation for  $x_0$  on the right hand side has a term which involves  $x_1$ . And the differential equation for  $x_1$  has a term on the right hand side which has  $x_0$ . So, I cannot separately solve the equations for  $x_0$  and  $x_1$ . The two variables, I have to somehow determine a method to solve them together.

So, the question is how do we solve these coupled differential equations? Now, the method you adopt to solve such coupled linear, ordinary linear differential equations is that you have to find linear combinations of the variables. In this case, you have to find linear combinations of  $x_0$  and  $x_1$ . The 2 variables such that the differential equations that governed these linear combinations of  $x_0$  and  $x_1$  are not coupled or uncoupled. So, by finding suitable linear combinations of your variables you have to obtain differential equations which are no longer coupled. So in this case, the process is quite straight forward.

Let us look at the equation again, notice that if I add these two equations. The term, which arises due to the spring that couples the 2 oscillators, cancels out. This is essentially because of, the fact that the force which this spring exerts on this is equal and opposite to the force the same spring exerts on this. So, if I add the two equations for this mass and this mass these 2 terms cancel out. So, this gives me 1 of the linear combinations; the other linear combination in this case is, obtained by subtracting these 2 equations.

(Refer Slide Time: 11:24)

## Normal Modes

$$q_0 = \frac{x_0 + x_1}{2} \text{ and } q_1 = \frac{x_0 - x_1}{2}$$


$q_0$

Smith Bhattacharya and Pratik Chaudhary, Department of Physics and Meteorology, IIT Kharagpur, 721 302 India  
http://www.cse.iitkgp.ernet.in/~phy1

So, these are what are called the normal modes. So, in this particular case, we have 2 normal modes  $q_0$  and  $q_1$ ,  $q_0$  is  $x_0$  plus  $x_1$  divided by 2. The second normal mode  $q_1$  is  $x_0$  minus  $x_1$  divided by 2. You can obtain the differential equation, governing  $x_0$  by taking the difference of those 2 differential equations. So, the differential equation governing  $q_0$  is obtained by taking the sum of the 2 differential equations which we had earlier. And the differential equation governing  $q_1$  is obtained by taking the difference of the 2 differential equations which we had earlier.

(Refer Slide Time: 12:26)

## Coupled Oscillators



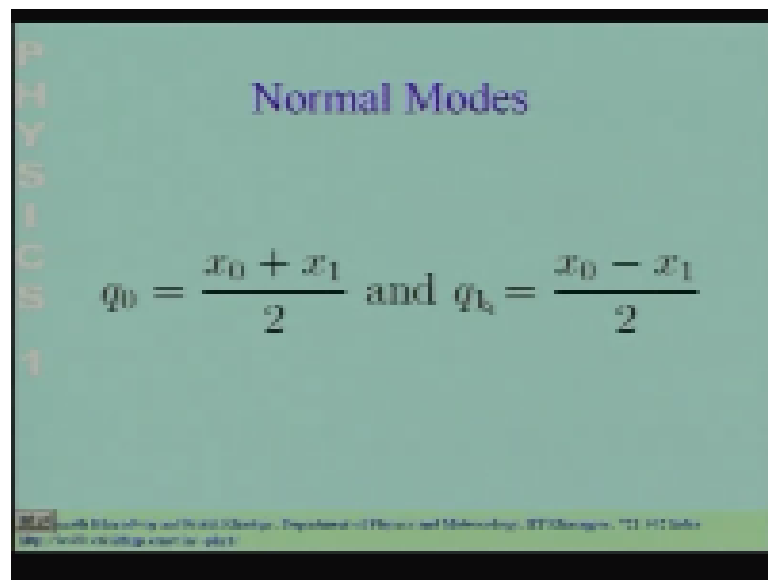
$$m \frac{d^2 x_0}{dt^2} = -kx_0 - k'(x_0 - x_1)$$

$$m \frac{d^2 x_1}{dt^2} = -kx_1 - k'(x_1 - x_0)$$

Smith Bhattacharya and Pratik Chaudhary, Department of Physics and Meteorology, IIT Kharagpur, 721 302 India  
http://www.cse.iitkgp.ernet.in/~phy1

So, if you take the difference of these 2 differential equations. You will have the mass which is common throughout you will have  $x_0$  minus  $x_1$ . And then I am going to subtract these 2 equations and divide by 2. So, I will have  $x_0$  minus  $x_1$  divided by 2. And here I am going to have  $x_0$  minus  $x_1$  divided by 2. Here, I am going to have  $0 \times$  minus 1 divided by 2 and here I am going to again have  $0$  minus  $x_1$  divided 2 with the minus sign.

(Refer Slide Time: 13:04)

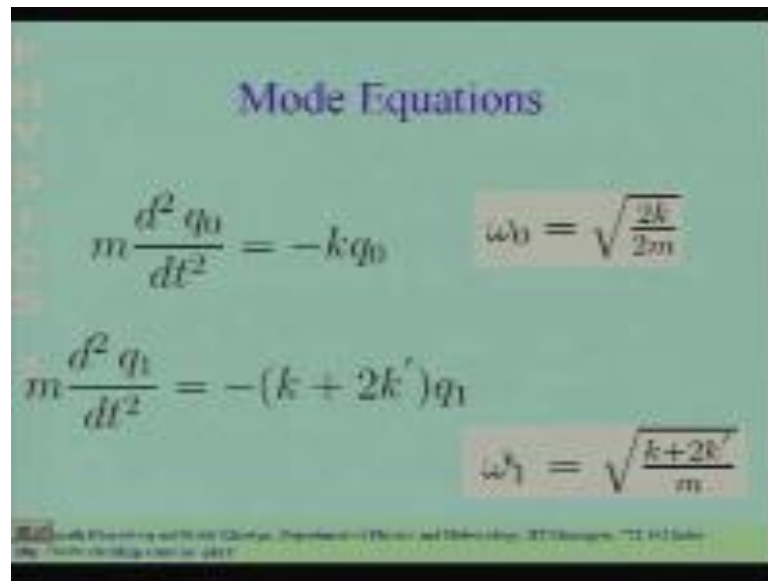


The slide is titled "Normal Modes" in purple text. On the left side, the word "PHYSICS" is written vertically in orange. The main content shows the equations for normal modes:  $q_0 = \frac{x_0 + x_1}{2}$  and  $q_1 = \frac{x_0 - x_1}{2}$ . At the bottom, there is a small logo and text: "© 2014 by the author and the publisher, Department of Physics and Meteorology, BITS Pilani, Hyderabad Campus. All rights reserved." followed by a URL.

$$q_0 = \frac{x_0 + x_1}{2} \text{ and } q_1 = \frac{x_0 - x_1}{2}$$

So, when I subtract these 0 equations I get an equation for the normal mode  $q_1$ . So, the normal modes have a property that the differential equations governing the normal modes  $q_0$   $q_1$  are not going to be coupled.

(Refer Slide Time: 13:19)



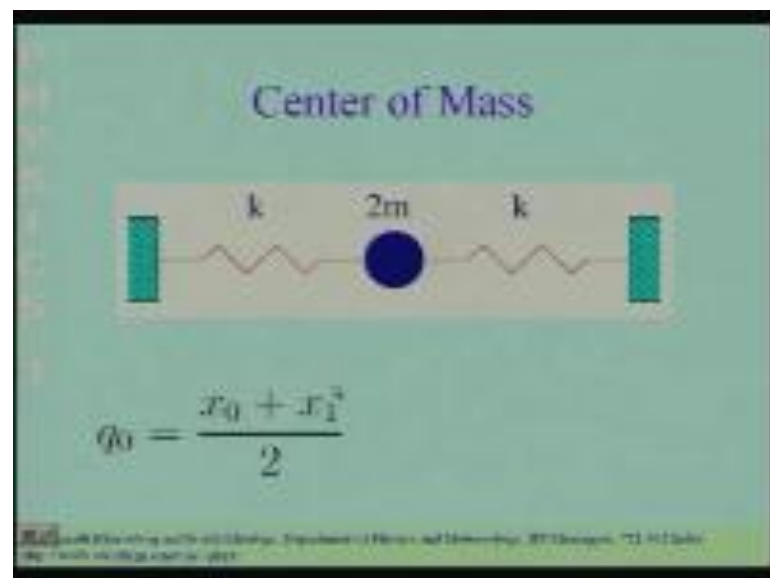
The slide is titled "Mode Equations" in a purple font. It contains two sets of equations. The first set shows the differential equation  $m \frac{d^2 q_0}{dt^2} = -k q_0$  and its angular frequency  $\omega_0 = \sqrt{\frac{2k}{2m}}$ . The second set shows the differential equation  $m \frac{d^2 q_1}{dt^2} = -(k + 2k') q_1$  and its angular frequency  $\omega_1 = \sqrt{\frac{k+2k'}{m}}$ . At the bottom, there is a small text line: "Lecture 13: Coupled Oscillations and Normal Modes, Department of Physics and Astronomy, UT Dallas, '12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459, 460, 461, 462, 463, 464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 496, 497, 498, 499, 500, 501, 502, 503, 504, 505, 506, 507, 508, 509, 510, 511, 512, 513, 514, 515, 516, 517, 518, 519, 520, 521, 522, 523, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 546, 547, 548, 549, 550, 551, 552, 553, 554, 555, 556, 557, 558, 559, 560, 561, 562, 563, 564, 565, 566, 567, 568, 569, 570, 571, 572, 573, 574, 575, 576, 577, 578, 579, 580, 581, 582, 583, 584, 585, 586, 587, 588, 589, 590, 591, 592, 593, 594, 595, 596, 597, 598, 599, 600, 601, 602, 603, 604, 605, 606, 607, 608, 609, 610, 611, 612, 613, 614, 615, 616, 617, 618, 619, 620, 621, 622, 623, 624, 625, 626, 627, 628, 629, 630, 631, 632, 633, 634, 635, 636, 637, 638, 639, 640, 641, 642, 643, 644, 645, 646, 647, 648, 649, 650, 651, 652, 653, 654, 655, 656, 657, 658, 659, 660, 661, 662, 663, 664, 665, 666, 667, 668, 669, 670, 671, 672, 673, 674, 675, 676, 677, 678, 679, 680, 681, 682, 683, 684, 685, 686, 687, 688, 689, 690, 691, 692, 693, 694, 695, 696, 697, 698, 699, 700, 701, 702, 703, 704, 705, 706, 707, 708, 709, 710, 711, 712, 713, 714, 715, 716, 717, 718, 719, 720, 721, 722, 723, 724, 725, 726, 727, 728, 729, 730, 731, 732, 733, 734, 735, 736, 737, 738, 739, 740, 741, 742, 743, 744, 745, 746, 747, 748, 749, 750, 751, 752, 753, 754, 755, 756, 757, 758, 759, 760, 761, 762, 763, 764, 765, 766, 767, 768, 769, 770, 771, 772, 773, 774, 775, 776, 777, 778, 779, 780, 781, 782, 783, 784, 785, 786, 787, 788, 789, 790, 791, 792, 793, 794, 795, 796, 797, 798, 799, 800, 801, 802, 803, 804, 805, 806, 807, 808, 809, 810, 811, 812, 813, 814, 815, 816, 817, 818, 819, 820, 821, 822, 823, 824, 825, 826, 827, 828, 829, 830, 831, 832, 833, 834, 835, 836, 837, 838, 839, 840, 841, 842, 843, 844, 845, 846, 847, 848, 849, 850, 851, 852, 853, 854, 855, 856, 857, 858, 859, 860, 861, 862, 863, 864, 865, 866, 867, 868, 869, 870, 871, 872, 873, 874, 875, 876, 877, 878, 879, 880, 881, 882, 883, 884, 885, 886, 887, 888, 889, 890, 891, 892, 893, 894, 895, 896, 897, 898, 899, 900, 901, 902, 903, 904, 905, 906, 907, 908, 909, 910, 911, 912, 913, 914, 915, 916, 917, 918, 919, 920, 921, 922, 923, 924, 925, 926, 927, 928, 929, 930, 931, 932, 933, 934, 935, 936, 937, 938, 939, 940, 941, 942, 943, 944, 945, 946, 947, 948, 949, 950, 951, 952, 953, 954, 955, 956, 957, 958, 959, 960, 961, 962, 963, 964, 965, 966, 967, 968, 969, 970, 971, 972, 973, 974, 975, 976, 977, 978, 979, 980, 981, 982, 983, 984, 985, 986, 987, 988, 989, 990, 991, 992, 993, 994, 995, 996, 997, 998, 999, 1000, 1001, 1002, 1003, 1004, 1005, 1006, 1007, 1008, 1009, 1010, 1011, 1012, 1013, 1014, 1015, 1016, 1017, 1018, 1019, 1020, 1021, 1022, 1023, 1024, 1025, 1026, 1027, 1028, 1029, 1030, 1031, 1032, 1033, 1034, 1035, 1036, 1037, 1038, 1039, 1040, 1041, 1042, 1043, 1044, 1045, 1046, 1047, 1048, 1049, 1050, 1051, 1052, 1053, 1054, 1055, 1056, 1057, 1058, 1059, 1060, 1061, 1062, 1063, 1064, 1065, 1066, 1067, 1068, 1069, 1070, 1071, 1072, 1073, 1074, 1075, 1076, 1077, 1078, 1079, 1080, 1081, 1082, 1083, 1084, 1085, 1086, 1087, 1088, 1089, 1090, 1091, 1092, 1093, 1094, 1095, 1096, 1097, 1098, 1099, 1100, 1101, 1102, 1103, 1104, 1105, 1106, 1107, 1108, 1109, 1110, 1111, 1112, 1113, 1114, 1115, 1116, 1117, 1118, 1119, 1120, 1121, 1122, 1123, 1124, 1125, 1126, 1127, 1128, 1129, 1130, 1131, 1132, 1133, 1134, 1135, 1136, 1137, 1138, 1139, 1140, 1141, 1142, 1143, 1144, 1145, 1146, 1147, 1148, 1149, 1150, 1151, 1152, 1153, 1154, 1155, 1156, 1157, 1158, 1159, 1160, 1161, 1162, 1163, 1164, 1165, 1166, 1167, 1168, 1169, 1170, 1171, 1172, 1173, 1174, 1175, 1176, 1177, 1178, 1179, 1180, 1181, 1182, 1183, 1184, 1185, 1186, 1187, 1188, 1189, 1190, 1191, 1192, 1193, 1194, 1195, 1196, 1197, 1198, 1199, 1200, 1201, 1202, 1203, 1204, 1205, 1206, 1207, 1208, 1209, 1210, 1211, 1212, 1213, 1214, 1215, 1216, 1217, 1218, 1219, 1220, 1221, 1222, 1223, 1224, 1225, 1226, 1227, 1228, 1229, 1230, 1231, 1232, 1233, 1234, 1235, 1236, 1237, 1238, 1239, 1240, 1241, 1242, 1243, 1244, 1245, 1246, 1247, 1248, 1249, 1250, 1251, 1252, 1253, 1254, 1255, 1256, 1257, 1258, 1259, 1260, 1261, 1262, 1263, 1264, 1265, 1266, 1267, 1268, 1269, 1270, 1271, 1272, 1273, 1274, 1275, 1276, 1277, 1278, 1279, 1280, 1281, 1282, 1283, 1284, 1285, 1286, 1287, 1288, 1289, 1290, 1291, 1292, 1293, 1294, 1295, 1296, 1297, 1298, 1299, 1300, 1301, 1302, 1303, 1304, 1305, 1306, 1307, 1308, 1309, 1310, 1311, 1312, 1313, 1314, 1315, 1316, 1317, 1318, 1319, 1320, 1321, 1322, 1323, 1324, 1325, 1326, 1327, 1328, 1329, 1330, 1331, 1332, 1333, 1334, 1335, 1336, 1337, 1338, 1339, 1340, 1341, 1342, 1343, 1344, 1345, 1346, 1347, 1348, 1349, 1350, 1351, 1352, 1353, 1354, 1355, 1356, 1357, 1358, 1359, 1360, 1361, 1362, 1363, 1364, 1365, 1366, 1367, 1368, 1369, 1370, 1371, 1372, 1373, 1374, 1375, 1376, 1377, 1378, 1379, 1380, 1381, 1382, 1383, 1384, 1385, 1386, 1387, 1388, 1389, 1390, 1391, 1392, 1393, 1394, 1395, 1396, 1397, 1398, 1399, 1400, 1401, 1402, 1403, 1404, 1405, 1406, 1407, 1408, 1409, 1410, 1411, 1412, 1413, 1414, 1415, 1416, 1417, 1418, 1419, 1420, 1421, 1422, 1423, 1424, 1425, 1426, 1427, 1428, 1429, 1430, 1431, 1432, 1433, 1434, 1435, 1436, 1437, 1438, 1439, 1440, 1441, 1442, 1443, 1444, 1445, 1446, 1447, 1448, 1449, 1450, 1451, 1452, 1453, 1454, 1455, 1456, 1457, 1458, 1459, 1460, 1461, 1462, 1463, 1464, 1465, 1466, 1467, 1468, 1469, 1470, 1471, 1472, 1473, 1474, 1475, 1476, 1477, 1478, 1479, 1480, 1481, 1482, 1483, 1484, 1485, 1486, 1487, 1488, 1489, 1490, 1491, 1492, 1493, 1494, 1495, 1496, 1497, 1498, 1499, 1500, 1501, 1502, 1503, 1504, 1505, 1506, 1507, 1508, 1509, 1510, 1511, 1512, 1513, 1514, 1515, 1516, 1517, 1518, 1519, 1520, 1521, 1522, 1523, 1524, 1525, 1526, 1527, 1528, 1529, 1530, 1531, 1532, 1533, 1534, 1535, 1536, 1537, 1538, 1539, 1540, 1541, 1542, 1543, 1544, 1545, 1546, 1547, 1548, 1549, 1550, 1551, 1552, 1553, 1554, 1555, 1556, 1557, 1558, 1559, 1560, 1561, 1562, 1563, 1564, 1565, 1566, 1567, 1568, 1569, 1570, 1571, 1572, 1573, 1574, 1575, 1576, 1577, 1578, 1579, 1580, 1581, 1582, 1583, 1584, 1585, 1586, 1587, 1588, 1589, 1590, 1591, 1592, 1593, 1594, 1595, 1596, 1597, 1598, 1599, 1600, 1601, 1602, 1603, 1604, 1605, 1606, 1607, 1608, 1609, 1610, 1611, 1612, 1613, 1614, 1615, 1616, 1617, 1618, 1619, 1620, 1621, 1622, 1623, 1624, 1625, 1626, 1627, 1628, 1629, 1630, 1631, 1632, 1633, 1634, 1635, 1636, 1637, 1638, 1639, 1640, 1641, 1642, 1643, 1644, 1645, 1646, 1647, 1648, 1649, 1650, 1651, 1652, 1653, 1654, 1655, 1656, 1657, 1658, 1659, 1660, 1661, 1662, 1663, 1664, 1665, 1666, 1667, 1668, 1669, 1670, 1671, 1672, 1673, 1674, 1675, 1676, 1677, 1678, 1679, 1680, 1681, 1682, 1683, 1684, 1685, 1686, 1687, 1688, 1689, 1690, 1691, 1692, 1693, 1694, 1695, 1696, 1697, 1698, 1699, 1700, 1701, 1702, 1703, 1704, 1705, 1706, 1707, 1708, 1709, 1710, 1711, 1712, 1713, 1714, 1715, 1716, 1717, 1718, 1719, 1720, 1721, 1722, 1723, 1724, 1725, 1726, 1727, 1728, 1729, 1730, 1731, 1732, 1733, 1734, 1735, 1736, 1737, 1738, 1739, 1740, 1741, 1742, 1743, 1744, 1745, 1746, 1747, 1748, 1749, 1750, 1751, 1752, 1753, 1754, 1755, 1756, 1757, 1758, 1759, 1760, 1761, 1762, 1763, 1764, 1765, 1766, 1767, 1768, 1769, 1770, 1771, 1772, 1773, 1774, 1775, 1776, 1777, 1778, 1779, 1780, 1781, 1782, 1783, 1784, 1785, 1786, 1787, 1788, 1789, 1790, 1791, 1792, 1793, 1794, 1795, 1796, 1797, 1798, 1799, 1800, 1801, 1802, 1803, 1804, 1805, 1806, 1807, 1808, 1809, 1810, 1811, 1812, 1813, 1814, 1815, 1816, 1817, 1818, 1819, 1820, 1821, 1822, 1823, 1824, 1825, 1826, 1827, 1828, 1829, 1830, 1831, 1832, 1833, 1834, 1835, 1836, 1837, 1838, 1839, 1840, 1841, 1842, 1843, 1844, 1845, 1846, 1847, 1848, 1849, 1850, 1851, 1852, 1853, 1854, 1855, 1856, 1857, 1858, 1859, 1860, 1861, 1862, 1863, 1864, 1865, 1866, 1867, 1868, 1869, 1870, 1871, 1872, 1873, 1874, 1875, 1876, 1877, 1878, 1879, 1880, 1881, 1882, 1883, 1884, 1885, 1886, 1887, 1888, 1889, 1890, 1891, 1892, 1893, 1894, 1895, 1896, 1897, 1898, 1899, 1900, 1901, 1902, 1903, 1904, 1905, 1906, 1907, 1908, 1909, 1910, 1911, 1912, 1913, 1914, 1915, 1916, 1917, 1918, 1919, 1920, 1921, 1922, 1923, 1924, 1925, 1926, 1927, 1928, 1929, 1930, 1931, 1932, 1933, 1934, 1935, 1936, 1937, 1938, 1939, 1940, 1941, 1942, 1943, 1944, 1945, 1946, 1947, 1948, 1949, 1950, 1951, 1952, 1953, 1954, 1955, 1956, 1957, 1958, 1959, 1960, 1961, 1962, 1963, 1964, 1965, 1966, 1967, 1968, 1969, 1970, 1971, 1972, 1973, 1974, 1975, 1976, 1977, 1978, 1979, 1980, 1981, 1982, 1983, 1984, 1985, 1986, 1987, 1988, 1989, 1990, 1991, 1992, 1993, 1994, 1995, 1996, 1997, 1998, 1999, 2000, 2001, 2002, 2003, 2004, 2005, 2006, 2007, 2008, 2009, 2010, 2011, 2012, 2013, 2014, 2015, 2016, 2017, 2018, 2019, 2020, 2021, 2022, 2023, 2024, 2025, 2026, 2027, 2028, 2029, 2030, 2031, 2032, 2033, 2034, 2035, 2036, 2037, 2038, 2039, 2040, 2041, 2042, 2043, 2044, 2045, 2046, 2047, 2048, 2049, 2050, 2051, 2052, 2053, 2054, 2055, 2056, 2057, 2058, 2059, 2060, 2061, 2062, 2063, 2064, 2065, 2066, 2067, 2068, 2069, 2070, 2071, 2072, 2073, 2074, 2075, 2076, 2077, 2078, 2079, 2080, 2081, 2082, 2083, 2084, 2085, 2086, 2087, 2088, 2089, 2090, 2091, 2092, 2093, 2094, 2095, 2096, 2097, 2098, 2099, 2100, 2101, 2102, 2103, 2104, 2105, 2106, 2107, 2108, 2109, 2110, 2111, 2112, 2113, 2114, 2115, 2116, 2117, 2118, 2119, 2120, 2121, 2122, 2123, 2124, 2125, 2126, 2127, 2128, 2129, 2130, 2131, 2132, 2133, 2134, 2135, 2136, 2137, 2138, 2139, 2140, 2141, 2142, 2143, 2144, 2145, 2146, 2147, 2148, 2149, 2150, 2151, 2152, 2153, 2154, 2155, 2156, 2157, 2158, 2159, 2160, 2161, 2162, 2163, 2164, 2165, 2166, 2167, 2168, 2169, 2170, 2171, 2172, 2173, 2174, 2175, 2176, 2177, 2178, 2179, 2180, 2181, 2182, 2183, 2184, 2185, 2186, 2187, 2188, 2189, 2190, 2191, 2192, 2193, 2194, 2195, 2196, 2197, 2198, 2199, 2200, 2201, 2202, 2203, 2204, 2205, 2206,

$\omega_0$  is the angular frequency of the oscillators if they were not coupled. So, if these oscillators were not coupled, they would individually vibrate at the angular frequency  $\omega_0$ . And 1 of the modes vibrates at exactly this frequency  $\omega_0$  which is the angular frequency when they are not coupled.

The other mode  $q_1$  has an angular frequency  $\omega_1$  which is  $k + 2k$  prime by  $m$  the square root of that. So,  $\omega_1$  is the square of  $k + 2k$  prime by  $m$ , you should note that  $\omega_1$  is always greater than  $\omega_0$ . So, you have a new angular frequency which is introduced because of the coupling of the 2 oscillators. And the new angular frequency  $\omega_1$  is always larger than the angular frequency  $\omega_0$ , which is the angular frequency that would be there if the 2 oscillators were not coupled.

So, we have from these coupled oscillators from the coupled differential equations, for these 2 oscillators we have gone over to a linear combination of the variables called the normal modes. We find that one normal mode the sum of the 2 variables; the normal mode corresponding to the sum of the 2 variables oscillates, at the frequency which the particles would oscillate if they were not coupled. And we have another normal mode, which oscillates at a higher frequency higher angular frequency. So, this is the faster mode and this is the slow mode.

(Refer Slide Time: 17:17)

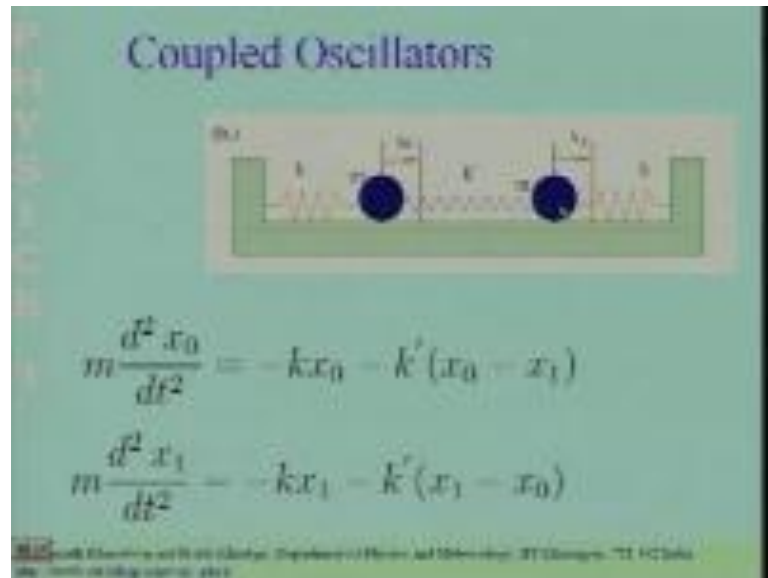


Now, let me give you an interpretation a physical interpretation of these 2 normal modes. The normal mode  $q_0$  is the sum of  $x_1$  and  $x_2$ . So, it is the sum of  $x_1$  and  $x_2$



divided by 2. So, you can interpret  $q_0$  as representing the oscillations of the centre of mass of the system.

(Refer Slide Time: 17:47)

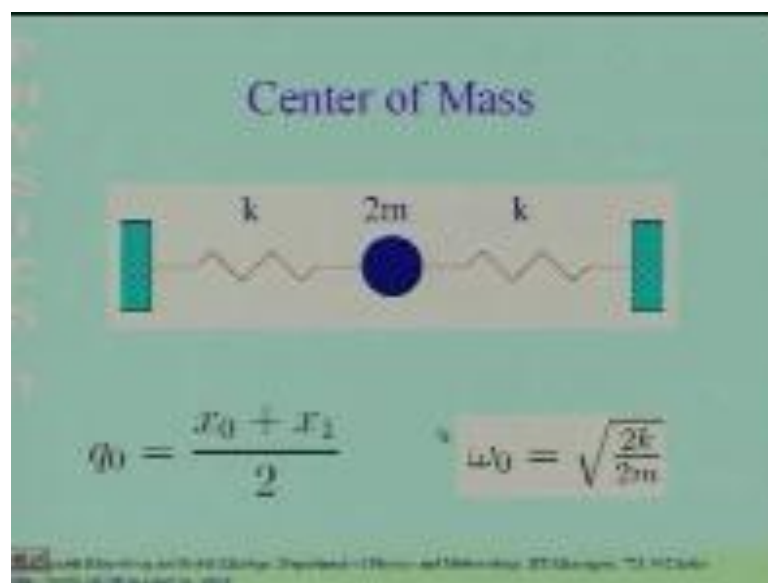


So, you can interpret  $q_0$  as representing the oscillations of the centre of mass of the system. So, my system has 2 particles which I can displace and if I were to represent these 2 particles by 1 particle. So, that is the role of the centre of mass, let me just digest a little and discuss the role of the centre of mass. If I were to ask you the question, where are you? You would then, give me some answer as to your position. I am located at such and such place now; if you look at your position carefully your nose is in the different position from the toes in your feet. And your fingers are in the different position from both of these. Now, when I ask you the question where are you, I am not interested in the individual positions of the different parts of your body.

I am interested in learning of 1 coordinate, 1 set of coordinates, which will tell me where you are, which will give me an idea of where you are. If I were interested more details then I would ask you where nose where toe etcetera. But in many situations I am not interested in that information. I am interested in learning where in learning of I am interested in knowing a point. Which I can associate with your position. So then, the you would tell me that you are located at some point. And that point which you would tell me would be represent your average of the whole of your body where the average of whole of your body is and this is the centre of mass.

So, a centre of mass for a two mass system represents. Where the average on the average, if I look at this system from sufficiently far away. I would not be able to discern that there are 2 particles I would think of it is one big lump. And then you could tell me that, the big lump comprising of both these particles is located at a point, that point would be the centre of mass. It should be the average position of both of these particles. And if the masses were different then I would have to weigh the contribution from each of these with the masses, but here the masses are same. So, I could just average the displacements and replace these 2 masses with the single particles over there.

(Refer Slide Time: 20:16)

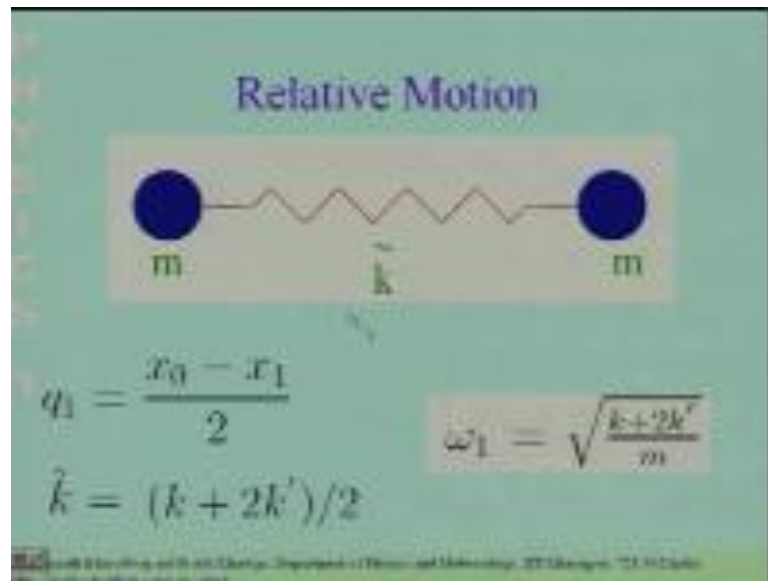


And that would have a position  $x_0$  plus  $x_1$  by 2. So, the normal mode, the 1 of the normal modes the normal  $q_0$  in the normal mode  $q_0$  we think of the 2 particles as 1 particle of mass  $2m$ . So, if you think of the 2 particles as 1 single particle of mass  $2m$ . So, you have collapse the 2 particles into a single particle and you forgotten, about the spring; which in this case, is something like an internal degree of freedom. So, you have forgotten about this spring that connects the 2 masses. So, you replace the 2 mass and the spring that connects them in between by a single mass  $2m$ . Now, those 2 masses each had a spring attaching it to this rigid supports. So, there is 1 spring here of spring constant  $k$ , there is another spring here of spring constant  $k$ .

So, if you write down the equation of motion of this combined system. That is; if I replace the 2 masses by a single mass and then writes down the equation of motion for this. I will find that, the angular frequency of the system is the square root of  $2k$ . I now,

have 2 springs divided by the mass which is both the masses; which is the angular frequency  $\omega_0$  which would be there if I had the 2 oscillators uncoupled. So, the normal mode  $q_0$  represents the motion of the centre of mass of the system. Now, I could have another kind of motion of these 2 particles. And the other the kind of motion would leave the centre of mass unchanged.

(Refer Slide Time: 22:12)

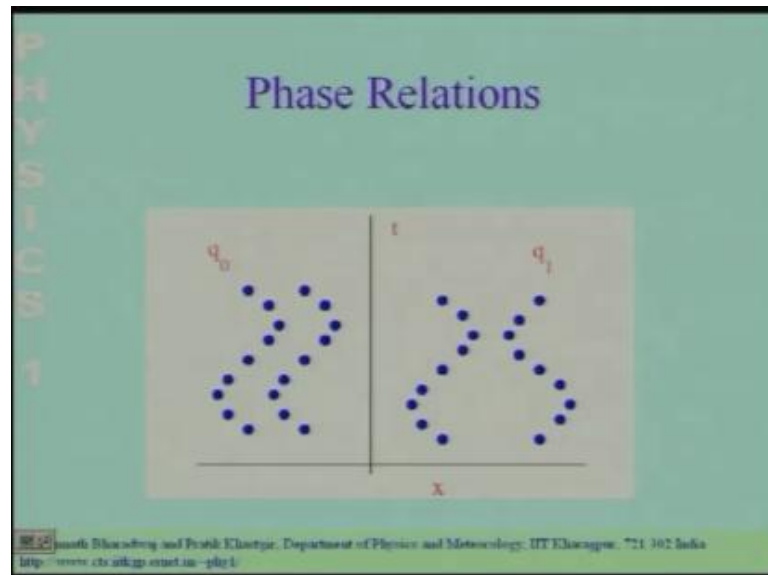


So, I have these 2 masses it could oscillate like this which would leave the centre of mass of the system unchanged, if it were to oscillate like this. So, this is the thing that I show over here. The other kind of motion possible is the relative motion of these 2 particles. The relative motion leaves the centre of mass unchanged. In the relative motion the centre of mass is fixed and this particle is displaced by exactly the same amount as this particle is displaced in this direction. So, they are if this particle is displaced this way, this particles will be displaced by exactly the same amount and when this moves away this also moves away by exactly the same amount. So, that the centre of mass remains fixed.

The relative motion is what  $q_1$  which is the difference  $x_0$  minus  $x_1$  tells you. So, the  $q_1$  refers to the relative motion of these 2 masses. The motion that leaves the centre of mass unchanged. And  $q_1$  is  $x_0$  minus  $x_1$  by 2 and if you write down the equation of motion for the relative motion. You will find that you can think of it, as this 2 masses being attached by a spring constant of effective by a spring of effective spring constant  $\tilde{k}$  which has

a value  $k$  plus  $2k$  prime by 2. And the angular frequency of this system is the square root of  $k$  plus  $2k$  prime by  $m$ , because the mass is also twice. So, the angular frequency corresponding to the system is,  $k$  plus  $2k$  prime by  $m$  this is the angular frequency corresponding to this.

(Refer Slide Time: 23:46)



So, this shows you the same thing which I had been telling you. In terms of, the phase relation between the 2 different modes. So, in the mode  $q_0$  the 2 particles  $x_0$  and  $x_1$  move with exactly the same phase that is what I show here. So,  $q_0$  is  $x_0$  plus  $x_1$ . So, if  $x_0$  moves in a certain way  $x_1$  also moves in exactly the same way. So, in the mode  $q_0$  the 2 particles  $x_0$  and  $x_1$  move exactly in the same phase, whereas in the mode  $q_1$  the 2 particles  $x_0$  and  $x_1$  move exactly out of phase. So you see that, if you couple 2 oscillators the phases of the motions are not, no longer independent, the phase of the 2 oscillators now get coupled.

(Refer Slide Time: 24:53)

**Solution**

$$\tilde{x}_0(t) = \tilde{A}_0 e^{i \omega_0 t} + \tilde{A}_1 e^{i \omega_1 t}$$
$$\tilde{x}_1(t) = \tilde{A}_0 e^{i \omega_0 t} - \tilde{A}_1 e^{i \omega_1 t}$$

Prof. Subashish and Prof. K. Srinivasan, Department of Physics and Meteorology, IIT Kharagpur, 721 102 India  
<http://www.cis.iitkgp.ac.in/~phy1/>

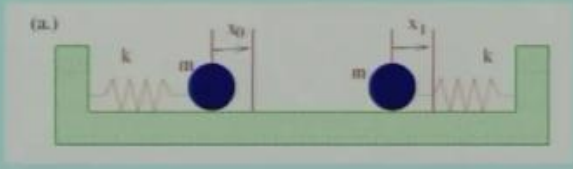
So, we can now, having given you a physical interpretation of these 2 different normal modes. We can now, write down the solutions to the differential equations governing the normal modes. The normal mode  $q_0$  has an angular frequency  $\omega_0$ , the normal mode  $q_1$  has the angular frequency  $\omega_1$ . So, we have the solutions, the solutions are  $q_0$  is equal to  $A_0 e^{i \omega_0 t}$  and  $q_1$  is  $A_1 e^{i \omega_1 t}$ . Remember, that we are using the complex notation that is what the tilde tells us. So, this  $q_0$  and  $q_1$  are both complex variables  $A_0$  and  $A_1$  are the complex amplitudes.

The complex amplitudes have both the phase and the amplitude of this motion. So, these are the solutions to the 2 differential equations governing the 2 modes. We can now, combine these 2 solutions and obtain the solutions for  $x_0$  and  $x_1$ . So, you have to invert the relation between  $x$ 's and  $q$ 's between the variables which we are the physical variables and the normal modes. If you invert these relations, you will get  $x_0$  is  $q_0$  plus  $q_1$  which gives us  $x_0$  to be  $A_0 e^{i \omega_0 t}$  plus  $A_1 e^{i \omega_1 t}$ . And  $x_1$  is the difference of the 2 variables  $q_0$  and  $q_1$  which is what we have here.

So, these are the expressions for  $x_0$  and  $x_1$  again in the complex notations. The question is what are the values of these different constants  $A_0$  and  $A_1$  that appear over here? How do we determine the values of these constants? The values of these constants are determined by the initial conditions. So, you will know the values for  $A_0$  and  $A_1$  only when you put in the initial conditions.

(Refer Slide Time: 27:56)

**Two Oscillators**


$$x_0(t) = a_0 \cos(\omega t + \phi_0)$$
$$x_1(t) = a_1 \cos(\omega t + \phi_1)$$

Sanjiv Bhattacharya and Pratik Khosla, Department of Physics and Meteorology, IIT Kharagpur, 721 302 India  
http://www.its.iitkgp.ac.in/~phy1

So, let us take up a particular situation here and the situation that we are going to take up is as follows;

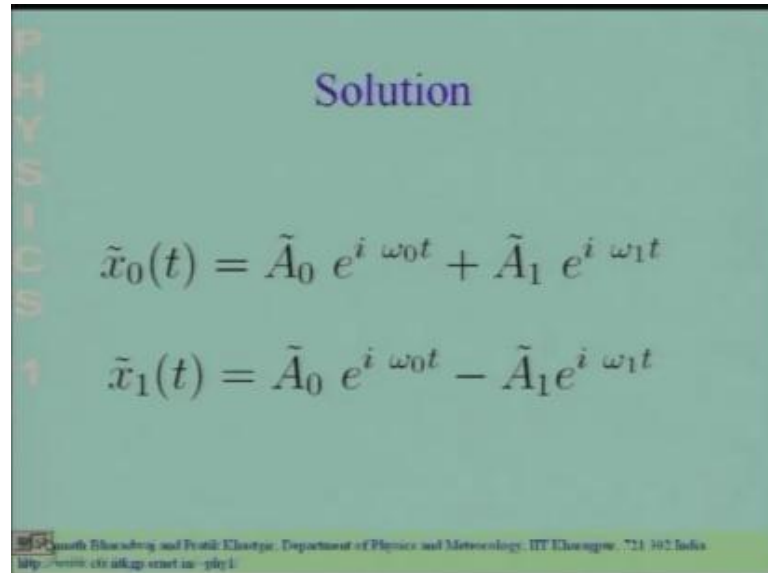
(Refer Slide Time: 28:18)

$$\begin{aligned} t = 0 \quad x_0 &= a_0 \quad v_0 = 0 \\ x_1 &= 0 \quad v_1 = 0 \end{aligned}$$

We have disturbed at  $t$  equal to 0, we have disturbed the zeroth particle. So,  $x_0$  is  $A_0$  at  $t$  equal to 0 and velocity of the zeroth particle  $v_0$  is equal to 0,  $x_1$  is equal to 0  $v_1$  is equal to 0. So, what have we done we have these 2 masses which are coupled. So, these 2 oscillators which are coupled, the zeroth oscillator has displacement  $x_0$  the first oscillator has displacement  $x_1$ . These oscillators are initially at rest and at equilibrium, what I have done is, I have displaced the zeroth oscillator from the equilibrium position by an amount

A0. And left it there and then I leave this whole system and watch how it oscillates. So, initially  $x_1$  is in the equilibrium position and both particles are at rest only the zeroth particle has been disturbed from the equilibrium position. You want to solve the motion for this.

(Refer Slide Time: 29:49)



**Solution**

$$\tilde{x}_0(t) = \tilde{A}_0 e^{i \omega_0 t} + \tilde{A}_1 e^{i \omega_1 t}$$

$$\tilde{x}_1(t) = \tilde{A}_0 e^{i \omega_0 t} - \tilde{A}_1 e^{i \omega_1 t}$$

© 2015 Anurag Bhattacharya and Pratik Khosla. Department of Physics and Meteorology, IIT Kharagpur, 721 302 India.  
http://www.cse.iitkgp.ac.in/~phy1

So, in to order to do this we have to determine we have to determine these coefficients  $A_0$  and  $A_1$  in order to fully get the solution. So, let us look at the equations for  $x_0$  first and at  $t$  equal to 0 let us put  $t$  equal to 0. So, at  $t$  equal to so, let me write down the equation for  $x_0$  over here.

(Refer Slide Time: 30:16)

Handwritten notes on a whiteboard:

$$t = 0 \quad x_0 = a_0 \quad v_0 = 0$$

$$x_1 = 0 \quad v_1 = 0$$

$$\tilde{x}_0(t) = \tilde{A}_0 e^{i\omega_0 t} + \tilde{A}_1 e^{i\omega_1 t}$$

$$t=0 \quad \tilde{x}_0 = \tilde{A}_0 + \tilde{A}_1$$

$$\Rightarrow a_0 = c_0 + c_1$$

$$\tilde{A}_0 = c_0 + i d_0$$

$$\tilde{A}_1 = c_1 + i d_1$$

So, the equation for  $x_0$  is  $x_0(t)$  is equal to  $\tilde{A}_0 e^{i\omega_0 t} + \tilde{A}_1 e^{i\omega_1 t}$ . At  $t$  equal to 0 this is  $\tilde{x}$  in the complex notation, at  $t$  equal to 0 we are given the fact. So, let us first calculate the position at  $t$  equal to 0 the complex position at  $t$  equal to 0, the complex variable corresponding to the position at  $t$  equal to 0. So, at  $t$  equal to 0 this is equal to  $\tilde{A}_0 + \tilde{A}_1$  at  $t$  equal to 0 the  $t$  to the power  $i\omega t$  is 1. So, I have  $\tilde{A}_0 + \tilde{A}_1$  so, at  $t$  equal to 0  $x_0$  is equal to  $\tilde{A}_0 + \tilde{A}_1$  both the left hand side and the right hand side are complex.

Now, when I want to equate this to the position of the particle is just the real part of this variable. So, we have to deal with only the real part of this and this. So, let me break up  $\tilde{A}_0$  into  $c_0 + i d_0$  and  $\tilde{A}_1$  to  $c_1 + i d_1$ . So, I have done this decomposition; I have written these amplitudes  $\tilde{A}_0$  and  $\tilde{A}_1$  complex amplitude in terms of the real parts  $c_0$  and the imaginary part  $d_0$   $\tilde{A}_0$  as  $c_0 + i d_0$  and  $\tilde{A}_1$  in terms of  $c_1$  and  $d_1$ . Now, when I want to equate this to the position  $a_0$ , this I should take only the real part of the right hand side.

So, the real part of the right hand side is  $c_0 + c_1$ . So, this is  $a_0$  is equal to  $c_0 + c_1$ . So, I have obtained 1 relation between  $c_0$  and  $c_1$  from the initial conditions. So, this is the first relation which arises from here, now, this arises from the initial position of the zeroth particle of the zeroth displacement. Let us now, apply the second information given which is initial velocity of the zeroth displacement. So, to put that condition we



have to differentiate this expression, let me differentiate this expression in a new page.  
So, let me get a new page first.

(Refer Slide Time: 33:36)

A whiteboard with the handwritten equation  $\tilde{v}_0(t) = 0$  in the top left corner. The rest of the board is blank.

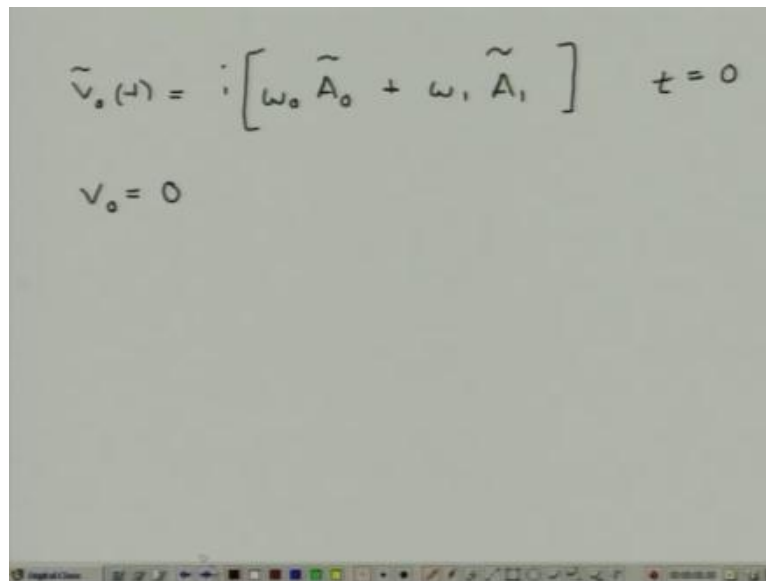
So, I am going to calculate  $V_0$  as a function of time and this you can,

(Refer Slide Time: 33:46)

A whiteboard with handwritten equations. At the top, it lists initial conditions:  $t=0$ ,  $x_0 = a_0$ ,  $v_0 = 0$ ,  $x_1 = 0$ , and  $v_1 = 0$ . Below these, the general solution for  $x_0(t)$  is given as  $\tilde{x}_0(t) = \tilde{A}_0 e^{i\omega_0 t} + \tilde{A}_1 e^{i\omega_1 t}$ . Then, at  $t=0$ , it shows  $\tilde{x}_0 = \tilde{A}_0 + \tilde{A}_1$ . This leads to two boxed equations:  $a_0 = c_0 + c_1$  and  $\tilde{A}_0 = c_0 + i d_0$ ,  $\tilde{A}_1 = c_1 + i d_1$ .

If I differentiate this, I will pick up a factor of  $i\omega_0$  over here and I will pick up a factor of  $i\omega_1$  over here. So, let me write down the expression that I get when I differentiate it.

(Refer Slide Time: 33:46)



The image shows a whiteboard with two handwritten equations. The first equation is  $\tilde{v}_0(t) = i [\omega_0 \tilde{A}_0 + \omega_1 \tilde{A}_1] \quad t=0$ . The second equation is  $v_0 = 0$ . The whiteboard has a standard toolbar at the bottom.

So, I will get  $i$  which I can take common outside,  $\omega_0 \tilde{A}_0 + \omega_1 \tilde{A}_1$ . I have this at  $t$  equal to 0. So, the  $e$  to the power  $i \omega t$  which was there at  $t$  equal to 0, those become 1. So, this is the complex velocity at  $t$  equal to 0 of the zeroth displacement at  $t$  equal to 0. Now, when I want to equate this to the real velocity of the part which I know is initially 0, I should take only the real part of this. So, the question is what is the real part of this?

So, there is an overall factor of  $i$  and so, I have to only look at the imaginary part of  $A_0$  and  $A_1$  because of this factor  $i$  over here. So, when this  $i$  multiplies the  $i$  which is there in the imaginary parts I get, let us look at,  $A_0$  and  $A_1$ .  $A_0$  is  $c_0 \cos \omega_0 t$  and  $A_1$  is  $c_1 \cos \omega_1 t$ . So, when I multiply these numbers with  $i$ , I will get a minus 1 and the imaginary parts of these now become the real part when I multiplied by  $i$ .

(Refer Slide Time: 35:46)

$$\tilde{v}_0(t) = i \left[ \omega_0 \tilde{A}_0 + \omega_1 \tilde{A}_1 \right] \quad t=0$$

$$v_0 = 0 = - \left[ \omega_0 c_0 + \omega_1 c_1 \right] = 0$$

So, the expression that I get, when I take the real part of this; what I get is, this should be equal to minus i into the i which comes over here. And here gives a minus sign and then I have omega naught, c0 plus omega 1 c1 this should be equal to 0. So, the fact that the zeroth displacement is initially has no velocity the x0 has no velocity tells me this or it tells me that let us we could at we could leave at this. So, this is the information that we get from applying the velocity condition for the zeroth particle. Now, let us look at the first the particle the variable x1.

(Refer Slide Time: 36:38)

$$t=0 \quad x_0 = a_0 \quad v_0 = 0$$

$$x_1 = 0 \quad v_1 = 0$$

$$\tilde{x}_0(t) = \tilde{A}_0 e^{i\omega_0 t} + \tilde{A}_1 e^{i\omega_1 t}$$

$$t=0 \quad \tilde{x}_0 = \tilde{A}_0 + \tilde{A}_1$$

$$\Rightarrow \boxed{a_0 = c_0 + c_1}$$

$$\boxed{\begin{aligned} \tilde{A}_0 &= c_0 + i d_0 \\ \tilde{A}_1 &= c_1 + i d_1 \end{aligned}}$$

(Refer Slide Time: 36:46)

**Solution**

$$\tilde{x}_0(t) = \tilde{A}_0 e^{i\omega_0 t} + \tilde{A}_1 e^{i\omega_1 t}$$

$$\tilde{x}_1(t) = \tilde{A}_0 e^{i\omega_0 t} - \tilde{A}_1 e^{i\omega_1 t}$$

So, the variable  $x_1$  is equal to  $a_0 e^{i\omega_0 t}$  minus  $A_1 e^{i\omega_1 t}$ .

(Refer Slide Time: 37:02)

$$\tilde{v}_0(t) = i \left[ \omega_0 \tilde{A}_0 + \omega_1 \tilde{A}_1 \right] \quad t=0$$

$$v_0 = 0 = - \left[ \omega_0 c_0 + \omega_1 c_1 \right] = 0$$

$$\begin{cases} \tilde{x}_1(t) = \tilde{A}_0 e^{i\omega_0 t} - \tilde{A}_1 e^{i\omega_1 t} \\ \tilde{v}_1(t) = i\omega_0 \tilde{A}_0 e^{i\omega_0 t} - i\omega_1 \tilde{A}_1 e^{i\omega_1 t} \end{cases}$$

$$\rightarrow \tilde{x}_1 = \tilde{A}_0 - \tilde{A}_1$$

So, let us now look at this variable. So, let me write down the expression for  $x_1$  here,  $x_1$  tilde is equal to  $A_0$  tilde  $e^{i\omega_0 t}$  minus  $A_1$  tilde  $e^{i\omega_1 t}$ . At  $t$  equal to 0 let me also calculate the velocity first and then I can set everything to  $t$  equal to 0. So, the velocity as a function of  $t$  going to be  $i\omega_0 A_0$  tilde minus  $i\omega_1 A_1$  tilde  $e^{i\omega_1 t}$ . So, I have found the velocity; now, what we have to do is we have to set  $t$  equal to 0.

So, let us set  $t$  equal to 0 and then apply the initial conditions. So, at  $t$  equal to 0 the position variable gives me,  $x_1$  is equal to  $A_0$  minus  $A_1$ . Now, when I apply the condition that the initial displacement is 0. I should take only the real part of this is the physical displacement. So, I should take the real part of this variable. So, if I take the real part of this, I should take the real part of the right hand side, let us just take a look again.

(Refer Slide Time: 38:53)

Handwritten notes showing initial conditions and complex amplitude equations:

$$t=0 \quad x_0 = a_0 \quad v_0 = 0$$

$$x_1 = 0 \quad v_1 = 0$$

$$\tilde{x}_0(t) = \tilde{A}_0 e^{i\omega_0 t} + \tilde{A}_1 e^{i\omega_1 t}$$

$$t=0 \quad \tilde{x}_0 = \tilde{A}_0 + \tilde{A}_1$$

$$\Rightarrow a_0 = c_0 + c_1$$

$$\tilde{A}_0 = c_0 + i d_0$$

$$\tilde{A}_1 = c_1 + i d_1$$

So, the real part of  $a_0$  and  $A_1$  both are  $c_0$  and  $c_1$ .

(Refer Slide Time: 39:03)

Handwritten notes showing velocity equations and solving for  $c_0$  and  $c_1$ :

$$\tilde{v}_0(t) = i [\omega_0 \tilde{A}_0 + \omega_1 \tilde{A}_1] \quad t=0$$

$$v_0 = 0 = - [\omega_0 c_0 + \omega_1 c_1] = 0$$

$$\tilde{x}_1(t) = \tilde{A}_0 e^{i\omega_0 t} - \tilde{A}_1 e^{i\omega_1 t}$$

$$\tilde{v}_1(t) = i\omega_0 \tilde{A}_0 e^{i\omega_0 t} - i\omega_1 \tilde{A}_1 e^{i\omega_1 t}$$

$$\rightarrow \tilde{x}_1 = \tilde{A}_0 - \tilde{A}_1 \mid 0 = c_0 - c_1$$

$$c_0 = c_1$$

So, if I take the real part of the right hand side, I have so, the real part is 0 should be equal to  $c_0$  minus  $c_1$  the real part of this is  $c_0$  the real part of this is  $c_1$ . So, the initial displacement 0 should be  $c_0$  minus  $c_1$  which straight away tells us that  $c_0$  should be equal to  $c_1$ , if  $c_0$  has to be equal to  $c_1$ .

(Refer Slide Time: 39:30)

Handwritten notes showing initial conditions and displacement equations for a two-degree-of-freedom system:

$$t=0 \quad x_0 = a_0 \quad v_0 = 0$$

$$x_1 = 0 \quad v_1 = 0$$

$$\tilde{x}_0(t) = \tilde{A}_0 e^{i\omega_0 t} + \tilde{A}_1 e^{i\omega_1 t}$$

$$t=0 \quad \tilde{x}_0 = \tilde{A}_0 + \tilde{A}_1$$

$$\Rightarrow a_0 = c_0 + c_1$$

$$\tilde{A}_0 = c_0 + i d_0$$

$$\tilde{A}_1 = c_1 + i d_1$$

If  $c_0$  is equal  $c_1$  then you can see, that each of them must individually be  $A_0$  by 2.

(Refer Slide Time: 39:38)

Handwritten notes showing velocity equations and solving for initial conditions:

$$\tilde{v}_0(t) = i \left[ \omega_0 \tilde{A}_0 + \omega_1 \tilde{A}_1 \right] \quad t=0$$

$$v_0 = 0 = - \left[ \omega_0 c_0 + \omega_1 c_1 \right] = 0$$

$$\tilde{x}_1(t) = \tilde{A}_0 e^{i\omega_0 t} - \tilde{A}_1 e^{i\omega_1 t}$$

$$\tilde{v}_1(t) = i\omega_0 \tilde{A}_0 e^{i\omega_0 t} - i\omega_1 \tilde{A}_1 e^{i\omega_1 t}$$

$$\rightarrow \tilde{x}_1 = \tilde{A}_0 - \tilde{A}_1 \mid 0 = c_0 - c_1$$

$$c_0 = c_1 = a_0/2$$

So, we find here that this should be equal to  $A_0$  by 2. So, we have obtained the real part of 2 unknown amplitudes and they are  $A_0$  by 2 they are both same and they are both  $A_0$  by 2. Now, let us look at the velocity information at  $t$  equal to 0, what does the velocity

information tell us. So, at  $t$  equal 0 this tells us at we could just set  $t$  equal to 0 here and here and what we have is at  $t$  equal to 0.

(Refer Slide Time: 40:23)

$$\begin{aligned}\tilde{v}_1 &= i\omega_0 \tilde{A}_0 - i\omega_1 \tilde{A}_1 \\ v_1 &= -\omega_0 d_0 + \omega_1 d_1 = 0 \\ d_0 &= \frac{\omega_1}{\omega_0} d_1 \\ \hline \tilde{A}_0 &= a_0/2 = \tilde{A}_1\end{aligned}$$

So, at  $t$  equal to 0 we have  $V_1$  is tilde is equal to  $i\omega_0 A_0$  tilde minus  $i\omega_1 A_1$ . Now, again remember when a multiply by  $i$ , I only pick up the imaginary parts of  $A_0$  and  $A_1$ . So, this is equal to minus  $\omega_0$  into  $d_0$  plus this is a minus sign because of the  $i$  which comes, when I take the imaginary part over here. Plus  $\omega_1 d_1$  this should be equal to, this is the real velocity and this should be of the first displace displacement 1 and this should be equal to 0. So, which tells us that  $d_0$  should be equal to  $\omega_1$  by  $\omega_0$  into  $d_1$ . So, it gives us a relation between  $d_0$  and  $d_1$ . And then they are just proportional to each other.

(Refer Slide Time: 41:24)

$$\begin{aligned}
 t=0 \quad x_0 &= a_0 \quad v_0 = 0 \\
 x_1 &= 0 \quad v_1 = 0 \\
 \tilde{x}_0(t) &= \tilde{A}_0 e^{i\omega_0 t} + \tilde{A}_1 e^{i\omega_1 t} \\
 t=0 \quad \tilde{x}_0 &= \tilde{A}_0 + \tilde{A}_1 \\
 \Rightarrow a_0 &= c_0 + c_1 \\
 \tilde{A}_0 &= c_0 + i d_0 \\
 \tilde{A}_1 &= c_1 + i d_1
 \end{aligned}$$

Now, if I plug this back in, into the condition which I get from the velocity of the first particle.

(Refer Slide Time: 41:38)

$$\begin{aligned}
 \tilde{v}_0(t) &= i \left[ \omega_0 \tilde{A}_0 + \omega_1 \tilde{A}_1 \right] \quad t=0 \\
 v_0 = 0 &= - \left[ \omega_0 c_0 + \omega_1 c_1 \right] = 0 \\
 \tilde{x}_1(t) &= \tilde{A}_0 e^{i\omega_0 t} - \tilde{A}_1 e^{i\omega_1 t} \\
 \tilde{v}_1(t) &= i\omega_0 \tilde{A}_0 e^{i\omega_0 t} - i\omega_1 \tilde{A}_1 e^{i\omega_1 t} \\
 \tilde{x}_1 &= \tilde{A}_0 - \tilde{A}_1 \mid 0 = c_0 - c_1 \\
 c_0 &= c_1 = a_0/2
 \end{aligned}$$

So, what is the condition that I get from the velocity of the first particle, from the velocity of the first particle, I get the condition. This should be d not c the imaginary part of A0 is d imaginary part of A1 is also d should have been d not c. So, the condition I have over here is that, d0 this d1 can be written in terms of d0. So, it basically tells me that, d0 should be 0 and d1 should also be 0.



(Refer Slide Time: 42:11)

$$\begin{aligned}\tilde{v}_1 &= i\omega_0 \tilde{A}_0 - i\omega_1 \tilde{A}_1 \\ v_1 &= -\omega_0 d_0 + \omega_1 d_1 = 0 \\ d_0 &= \frac{\omega_1}{\omega_0} d_1 \\ \hline \tilde{A}_0 &= a_0/2 = \tilde{A}_1\end{aligned}$$

So, these together tell me that,  $\tilde{A}$  is real it is  $a_0/2$  and this is also equal to  $\tilde{A}_1$ . So, both the amplitudes are real and they have a value  $a_0/2$ .

(Refer Slide Time: 42:37)

**Solution**

$$\begin{aligned}\tilde{x}_0(t) &= \tilde{A}_0 e^{i\omega_0 t} + \tilde{A}_1 e^{i\omega_1 t} \\ \tilde{x}_1(t) &= \tilde{A}_0 e^{i\omega_0 t} - \tilde{A}_1 e^{i\omega_1 t}\end{aligned}$$

So, having worked out these coefficients for a particular case. Let us now, discuss the solution.

(Refer Slide Time: 42:44)

Example

Particles at Rest  $x_0$  Displaced to  $a_0$

$$x_0(t) = \frac{a_0}{2} [\cos \omega_0 t + \cos \omega_1 t]$$
$$x_1(t) = \frac{a_0}{2} [\cos \omega_0 t - \cos \omega_1 t]$$

Ananth Bhattacharya and Pratik Khastgir, Department of Physics and Meteorology, IIT Kharagpur, 721 302 India

So, let me summarize what we have worked out, is we have considered the situation. Where  $x_0$  is initially displaced to  $A_0$  and the other particle  $x_1$  is initially not displaced and both the particles are at rest. So, I have displaced only 1 of the particles and given it no initial velocity other particle remains in equilibrium with no velocity and no displacement and I leave the system. Then the solution we worked out, the 2 unknown amplitudes that occur in the solution for this particular case and if you plug those unknown amplitudes over here.

Where this is real and this is also displacement by 2. You plug it in here and take only the real part, because that is the thing that gives the displacement the imaginary part has no information. So, take only the real parts, then you are led to this expression for the displacement  $x_0$  is  $A_0$  by 2  $\cos \omega_0 t$  plus  $\cos \omega_1 t$  this is the displacement of the first particle which you gave an initial displacement. This is the displacement of the second particle, which you did not give an initial displacement. It moves because of the coupling with the this oscillators, which you had displaced.

So, this does not remain where it is, it gets disturb because of the motion of this through the coupling and this the  $x_1$  variable is governed by this particular expression. So, the point to note is that, both the motion, the motion of, both these particles have are simple harmonic oscillators are linear combination of simple harmonic oscillators of 2 different frequencies. These are the two different frequencies corresponding to the 2 different normal modes.

In general, there will be arbitrary coefficients and this particular case the coefficients are the same. So, both these particle motions are linear super positions of oscillations of two different frequencies.  $\omega_0$  is the angular frequency if there was no coupling between two oscillators, the other is the faster mode which has a higher angular frequency than  $\omega_0$ . Let us now, investigate this particular solution in slightly more detail.

(Refer Slide Time: 45:13)

**Example Continued**

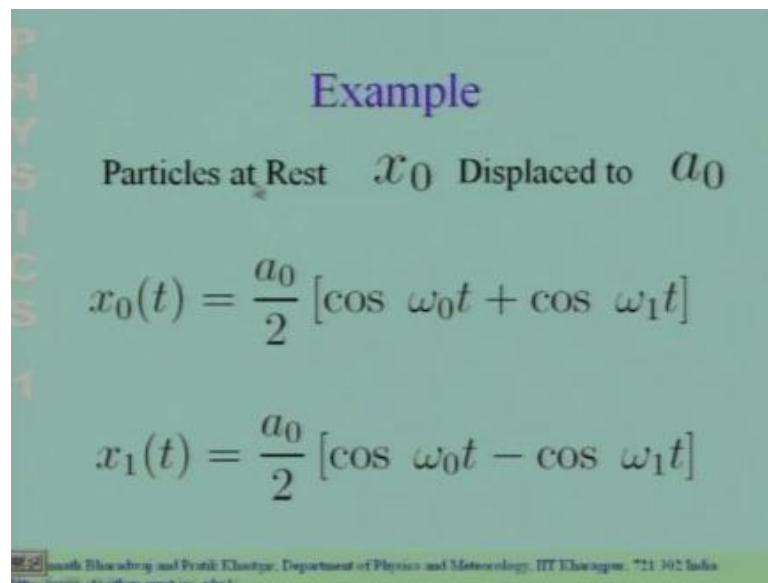
$$x_0(t) = a_0 \cos \left[ \left( \frac{\omega_1 - \omega_0}{2} \right) t \right] \cos \left[ \left( \frac{\omega_0 + \omega_1}{2} \right) t \right]$$

$$x_1(t) = a_0 \sin \left[ \left( \frac{\omega_1 - \omega_0}{2} \right) t \right] \sin \left[ \left( \frac{\omega_0 + \omega_1}{2} \right) t \right]$$

Dr. Ananth Kumar and Dr. Pratik Khosla, Department of Physics and Meteorology, IIT Kharagpur, 721 302 India  
<http://www.cmis.iitkgp.ac.in/~phd/>

So, we could take this solution and write it in this way also, this is nothing just nothing, but an trigonometric transformation. So,  $\cos a \pm \cos b$  can be written as  $2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$  or  $2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$ . Similarly,  $\cos a - \cos b$  which is what we have here.

(Refer Slide Time: 45:48)



**Example**

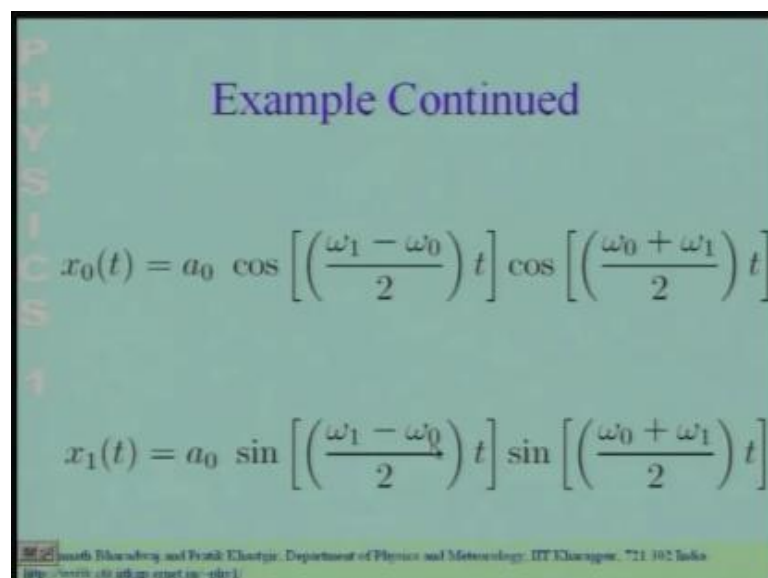
Particles at Rest  $x_0$  Displaced to  $a_0$

$$x_0(t) = \frac{a_0}{2} [\cos \omega_0 t + \cos \omega_1 t]$$
$$x_1(t) = \frac{a_0}{2} [\cos \omega_0 t - \cos \omega_1 t]$$

© 2011 Ananth Bhaskar and Pratik Khosla, Department of Physics and Meteorology, IIT Kharagpur, 721 302 India  
<http://www.cse.iitkgp.ac.in/~phys/>

$\cos a$  minus  $\cos b$  can be written in as  $\sin b$  minus  $a$  by  $2$  into  $b$  plus  $a$  by  $2$ . So, we have used these trigonometric identities to write the sum of  $2$  cosines and the difference of  $2$  cosines as product of  $2$  cosines and product of sines. Now, you may be wondering why we have done this. Well this tells us something quite interesting it, the let me go little further. The interesting thing is that the sum of  $2$  cosines these are  $2$  oscillations. So, I have  $2$  oscillations of different frequencies. If I superpose  $2$  oscillation of different frequencies I can think of it as.

(Refer Slide Time: 46:43)



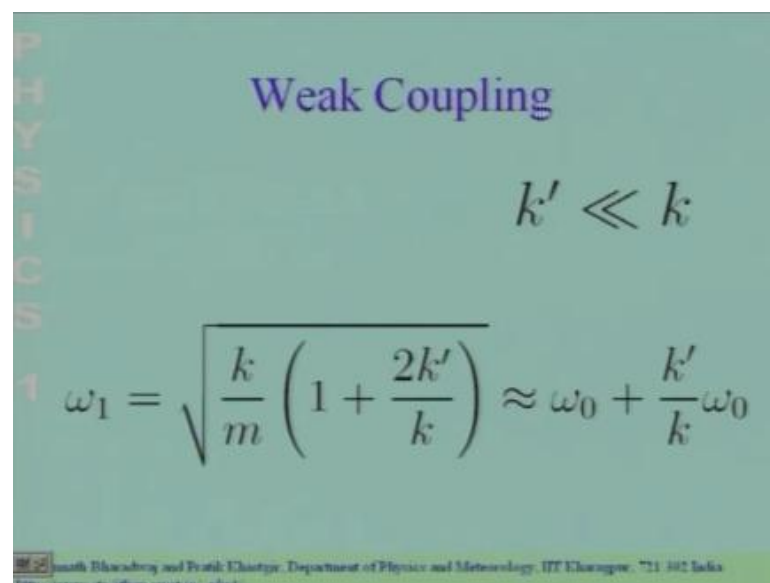
**Example Continued**

$$x_0(t) = a_0 \cos \left[ \left( \frac{\omega_1 - \omega_0}{2} \right) t \right] \cos \left[ \left( \frac{\omega_0 + \omega_1}{2} \right) t \right]$$
$$x_1(t) = a_0 \sin \left[ \left( \frac{\omega_1 - \omega_0}{2} \right) t \right] \sin \left[ \left( \frac{\omega_0 + \omega_1}{2} \right) t \right]$$

© 2011 Ananth Bhaskar and Pratik Khosla, Department of Physics and Meteorology, IIT Kharagpur, 721 302 India  
<http://www.cse.iitkgp.ac.in/~phys/>

The product of 2 oscillations of different frequencies, the sum of 2 oscillations of different frequencies the superposition can also be thought of as the product of 2 oscillations of different frequencies. 1 oscillation is, at the average frequency of these at the average of these 2 frequencies. So, 1 oscillation is at the angular frequency  $\omega_0$  plus is  $\omega_1$  by 2 and the other is at the difference. Now, if  $\omega_0$  and  $\omega_1$  are very close then this term over here, becomes extremely small and we can think of the motion  $x_0$  and  $z_1$ , as being a fast component. So, these are the fast components who's amplitude is under goes a slow modulation.

(Refer Slide Time: 47:35)



**Weak Coupling**

$$k' \ll k$$

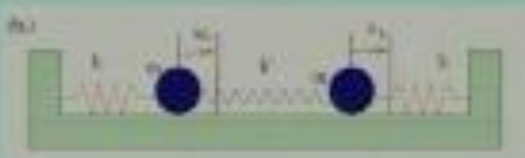
$$\omega_1 = \sqrt{\frac{k}{m} \left( 1 + \frac{2k'}{k} \right)} \approx \omega_0 + \frac{k'}{k} \omega_0$$

© Ananth Bhaskar and Pratik Chatterjee, Department of Physics and Meteorology, IIT Kharagpur, 721 302 India  
http://www.cds.ernet.in/~cds/

So, let me go into this in a little more detail, let us look at the situation for the spring mass coupled spring mass system when the coupling is weak. What we mean by, weak coupling is if the intermediate spring that we have introduced to couple that 2 oscillators. If the spring constant of that spring is much smaller than the spring constants of the 2 oscillators that I had to start with so, let me go back to the picture which we had to start with.

(Refer Slide Time: 48:11)

### Coupled Oscillators



$$m \frac{d^2 x_0}{dt^2} = -k x_0 - k' (x_0 - x_1)$$

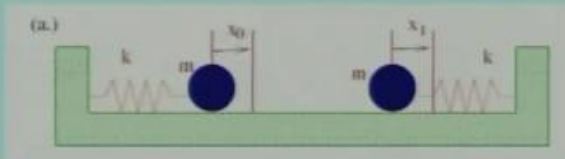
$$m \frac{d^2 x_1}{dt^2} = -k x_1 - k' (x_1 - x_0)$$

Dr. Anshu Bhattacharya and Probal Khanday, Department of Physics and Meteorology, IIT Kharagpur, 721 302 India  
http://www.cds.iitkgp.ernet.in/~phy1

So, this is the situation which we are discussing and to start with

(Refer Slide Time: 48:16)

### Two Oscillators



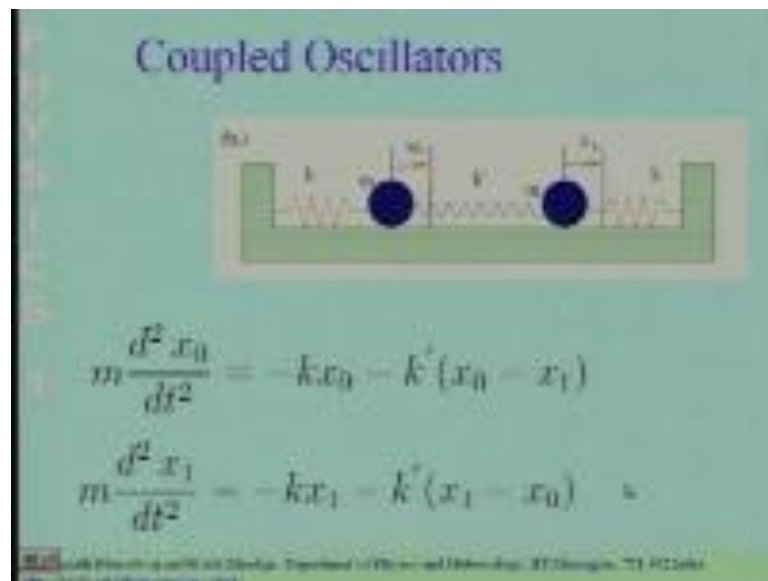
$$x_0(t) = a_0 \cos(\omega t + \phi_0)$$

$$x_1(t) = a_1 \cos(\omega t + \phi_1)$$

Dr. Anshu Bhattacharya and Probal Khanday, Department of Physics and Meteorology, IIT Kharagpur, 721 302 India  
http://www.cds.iitkgp.ernet.in/~phy1

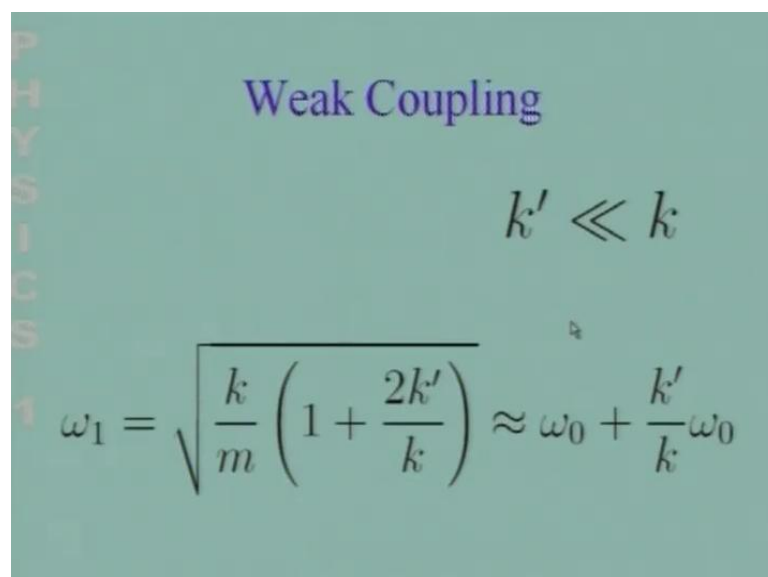
We had two oscillators which were free.

(Refer Slide Time: 48:20)



Now, we have introduced a spring of a different spring constant  $k'$ , each of these springs have a spring constant  $k$ . And we have introduced the third spring of a different spring constant  $k'$  to couple this. Now, we will assume that this coupling is much weaker. So, this spring constant is much weaker, than the spring constants here and here. So, it is a very weak coupling is what we are going to assume in this assumption.

(Refer Slide Time: 48:51)



So, this assumption is  $k'$  is much smaller than  $k$  the spring, the coupling spring is much weaker than other 2 springs. In this assumption you can simplify the expression for  $\omega_1$ . So, recollect that  $\omega_1$  let me do it for you here.

(Refer Slide Time: 49:23)

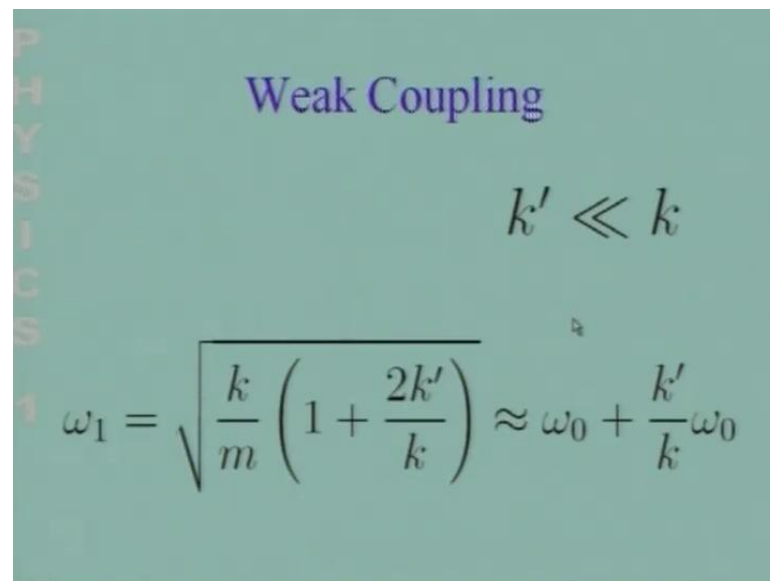
$$\begin{aligned}\omega_1 &= \sqrt{\frac{k + 2k'}{m}} \quad \omega_0 = \sqrt{\frac{k}{m}} \\ &= \sqrt{\frac{k}{m}} \left[ 1 + \frac{2k'}{k} \right]^{1/2} \\ &\approx \omega_0 \left[ 1 + \frac{k'}{k} \right] = \omega_0 + \frac{k'}{k} \omega_0\end{aligned}$$

So, recollect that  $\omega_1$  is the square root of  $k$  plus  $2k'$  by  $m$ . And this is the fast mode, the slow mode is  $\omega_0$  is equal to root  $k$  by  $m$ .  $k$  is the spring constant. If there was no coupling each oscillator would at oscillate  $\omega_0$  angular frequency  $\omega_0$  not because of the coupling there is a new mode which is a little faster, its  $k$  plus  $2k'$  by  $m$   $\omega_1$  is a square root of  $k$  plus  $2k'$  by  $m$ . Now, if  $k'$  is much smaller than  $k$  then, I could write  $\omega_1$  as square root of  $k$  by  $m$ .

So, I have taken this common so, what I have is  $1 + 2k'$  by  $k$  to the power half. If  $k'$  is much smaller than  $k$ , then I can do a Taylor expansion of  $1 + 2k'$  by  $k$  to the power half. And this Taylor expansion if I could retain only the first order term in  $k'$  by  $k$  because we are assuming that  $k'$  by  $k$  is a small number. We then have this is approximately equal to  $\omega_0$  into  $1 + \frac{1}{2}$  into  $2k'$  by  $k$ . This is the Taylor's first term, in the Taylor series and this gives us one plus  $k'$  by  $k$ . Which is equal to  $\omega_0$ , that is the original frequency of the oscillator plus a small change which is  $k'$  by  $k$  into  $\omega_0$ .



(Refer Slide Time: 51:27)



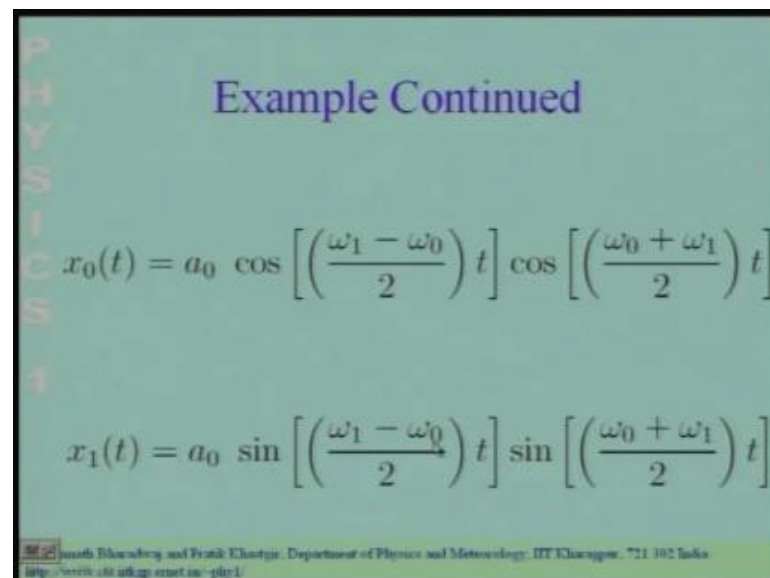
PHYSICS 1

### Weak Coupling

$$k' \ll k$$
$$\omega_1 = \sqrt{\frac{k}{m} \left(1 + \frac{2k'}{k}\right)} \approx \omega_0 + \frac{k'}{k} \omega_0$$

So, omega 1 and omega naught differ only by a small amount which is k prime by k into omega naught which is what we have over here. So, omega one is approximately equal to in the weak coupling limited is approximately equal to the frequency of the uncoupled oscillator omega naught plus a small correction which is k prime by k into omega naught.

(Refer Slide Time: 51:55)



PHYSICS 1

### Example Continued

$$x_0(t) = a_0 \cos \left[ \left( \frac{\omega_1 - \omega_0}{2} \right) t \right] \cos \left[ \left( \frac{\omega_0 + \omega_1}{2} \right) t \right]$$
$$x_1(t) = a_0 \sin \left[ \left( \frac{\omega_1 - \omega_0}{2} \right) t \right] \sin \left[ \left( \frac{\omega_0 + \omega_1}{2} \right) t \right]$$

© 2014 Ananth Bharadwaj and Frank Hees, Department of Physics and Meteorology, IIT Kharagpur, 721 302 India  
<http://www.cis.iitkgp.ac.in/~phys1/>

So, if you make this assumption and use it over here. So, then omega 1 minus omega naught, now, becomes k prime by k into omega naught by 2.

(Refer Slide Time: 52:06)

Continued

$$x_0(t) = \left[ a \cos \left( \frac{k'}{2k} \omega_0 t \right) \right] \cos \omega_0 t$$
$$x_1(t) = \left[ a \sin \left( \frac{k'}{2k} \omega_0 t \right) \right] \sin \omega_0 t$$

Dr. Anurag Bhattacharya and Pratik Chatterjee, Department of Physics and Meteorology, IIT Kharagpur, 721 302 India  
Email: anurag@iitkgp.ac.in, pratik@iitkgp.ac.in

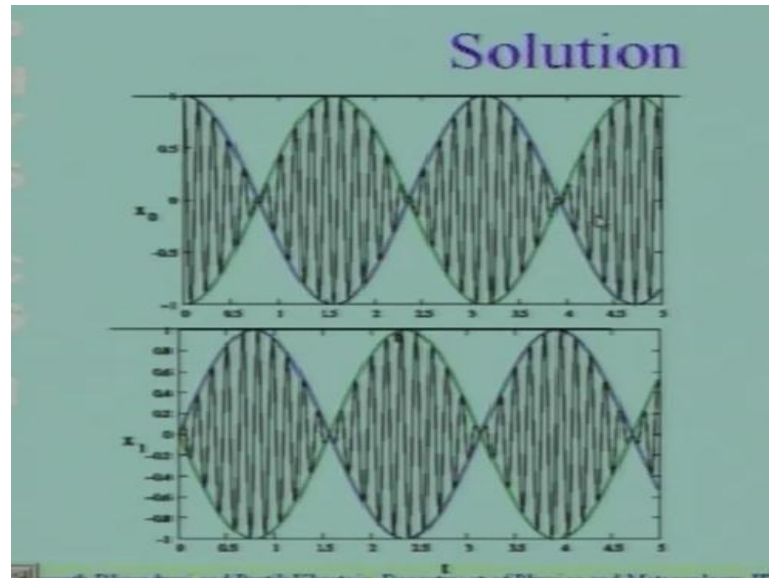
So, we will put in that and what we have is this. So, we have these 2 expressions for  $x_0$  and  $x_1$ . Let us look carefully at these 2 expressions so,  $\omega_0$  is much larger than  $k'$  by  $k$  into  $\omega_0$ . So, the oscillation of both  $x_0$  and  $x_1$  has 2 parts, there is the fast oscillation. So, I can think of it as a fast oscillation at an angular frequency  $\omega_0$ . So, both  $x_0$  and  $x_1$  execute fast oscillations at the angular frequency  $\omega_0$  which would have been there had there be no coupling at all.

So, the 2 masses in the weak coupling limit, the 2 masses continue you; can think of the 2 masses continuing to oscillate. As if there was no coupling so, the each of them oscillates with an angular frequency  $\omega_0$ . The effect of the coupling is through these 2 terms over here. So, the effect of the coupling is that it modulates the amplitude of the oscillations, the amplitude of the oscillations themselves do oscillations. So, the amplitude of the oscillation, so, this the oscillation and this is the amplitude.

The amplitude here, itself does oscillations at a much slower frequency compare to  $\omega_0$ . The slow angular frequency is  $k'$  by  $2k$  into  $\omega_0$ . So, its factor  $k'$  by  $2k$  slower. And both of these particles, the motion of both of these particles show a similar behaviour. In this particular case, we have started the whole system with just displacing  $x_0$   $x_1$  is at rest. So, if I start the whole system by just displacing  $x_0$ . Initially  $x_0$  is going to oscillate  $x_1$  is going to be at rest, but slowly the oscillations in  $x_1$  are going to pick up and the oscillations in  $x_0$  are going to fall. Because as cosine the amplitude here is, going to fall and the amplitude here is going to pick up.

When a the face approaches  $\pi$  by 2 and then again when the face approaches  $\pi$ , this is going to fall and this is going pick up and the amplitude of this is going to go back and forth.

(Refer Slide Time: 54:22)



So, the motion the motion of  $x_0$  and  $x_1$  is shown in the curves plotted over here. So, there are these fast oscillation that the frequency  $\omega_0$  and the amplitude of the fast oscillation gets modulated at the frequency  $k$  prime by  $2k$  into  $\omega_0$ .