Physics I: Oscillations and Waves
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Lecture - 43
Quantum Tunneling

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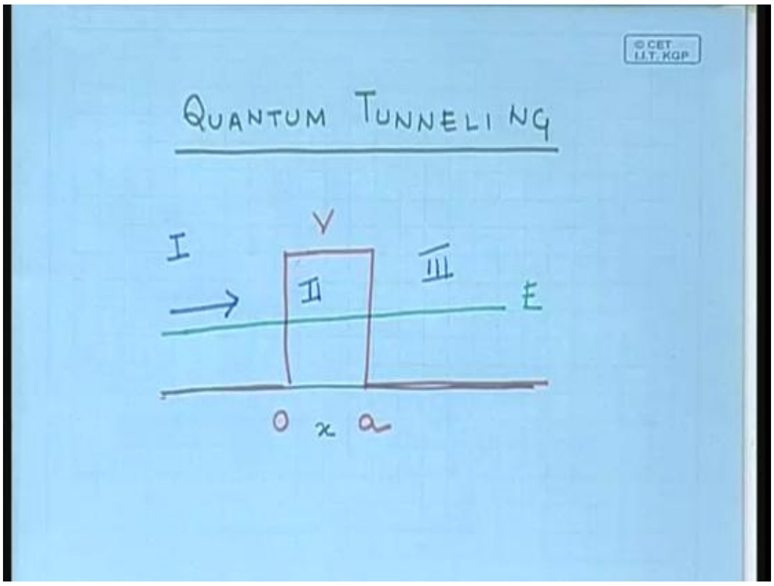

Good morning today we are going to discuss quantum tunneling. we have a particle which is free to move along the x axis, and there is a potential, the potential looks like this, it is zero in the region to the left, and then here it takes on a value which is constants for some range of x, then again it is zero on this side. And this constant value, let we denote by v. So, essentially there is a potential barrier v, which separates the region to left from the region to the right, and let us say that the potential barrier extends from x equal to 0 to x equal to a. Now, we will consider a particle of energy E, which is less than the height of this potential barrier. So, this is the energy of the particle E. And a particle of this energy E, is incident on this potential barrier from the left hand side. So, the particle is the incident like this. Now, if you were to analyze this situation, using classical mechanics, where we had a potential barrier v, and there was a particle incident from the left hand side, whose energy was less than the potential v.

In classical mechanics we would except that the particle would get reflected back. It would not be able to overcome this potential and reach the other side, the particle would get reflected back. So, if it were a macroscopic particle whose behavior is well described by classical mechanics, but we could say that for the definitely for sure the particle would come up to the potential barrier, and then get reflected back. It would not be able to overcome this potential barrier. But what happens in quantum mechanics; that is the question that we are interested in. So, in quantum mechanics, if you are dealing with microscopic particles, you have to apply quantum mechanics. And for microscopic particles, you have to use a wave to describe the particle. So, we have to write down a wave for the particle in this region.

We will refer to this region as region one. We have to write down a wave for the particle in this region, which we will refer to as region two. And we will write down a wave for the particle in this region, which we shall call region three. And we have already seen that if I have a particle of energy E, less than the potential, and if it is incident on a step potential like this. This is the step potential. We have already discussed this. We have seen that the wave corresponding to this particle, is going to penetrate inside this to some extent. So, in today's lecture, we are going to see what is the consequence of this. So, what happens in such a situation in quantum mechanics. The left hand side, the region one. In region one, the particle is essentially free, because there is no potential in region one. So, let me write down the wave function in region one, where the particle is essentially free, because the potential there is 0. Let me indicate that here, it is not require to indicated. So, in will region one, the particle is essentially free, because the potential is 0.

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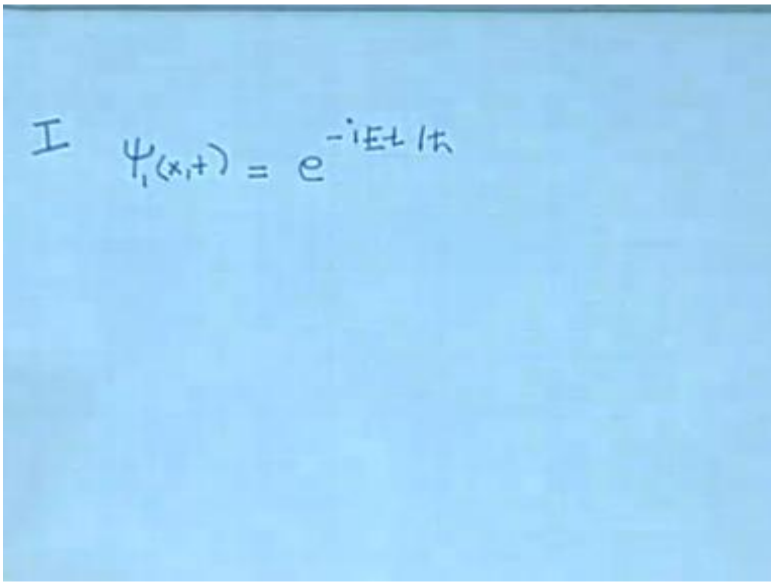

And I can write down the wave function in this region, the wave function in this region psi 1 x t is going to be e to the power minus i E t by h cross. The time part is going to be e to the power minus i E t by h cross.

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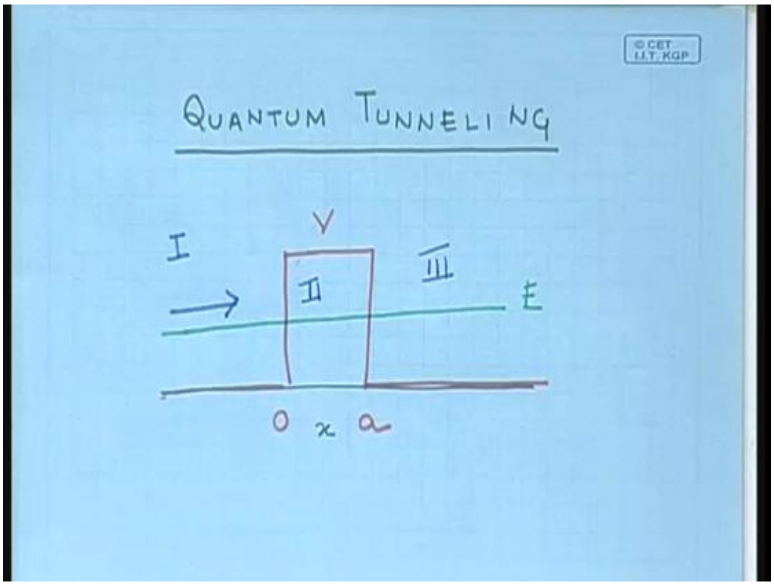

Remember that whenever I have a static potential, the time part of the wave function, is always going to be exponential minus i times the energy into t by h cross, where E is the energy of the particle. We have seen this in a lecture quite of few days ago, that the time part is always the same in a static potential. So, this is going to be the time part.

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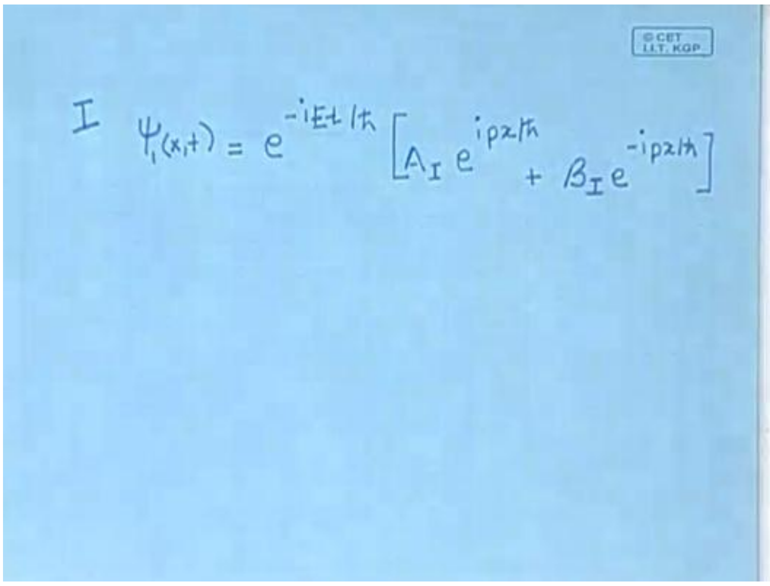

And the spatial part, is going to be A 1 e to the power i p x by h cross plus B 1 e to the power minus i p x by h cross. So, the spatial part of the free particle wave function has could have two parts; the first part when I combine with the time part, we can see that it is a right propagating wave. The second part, when I combine with this, we can see that it is a left propagating wave.

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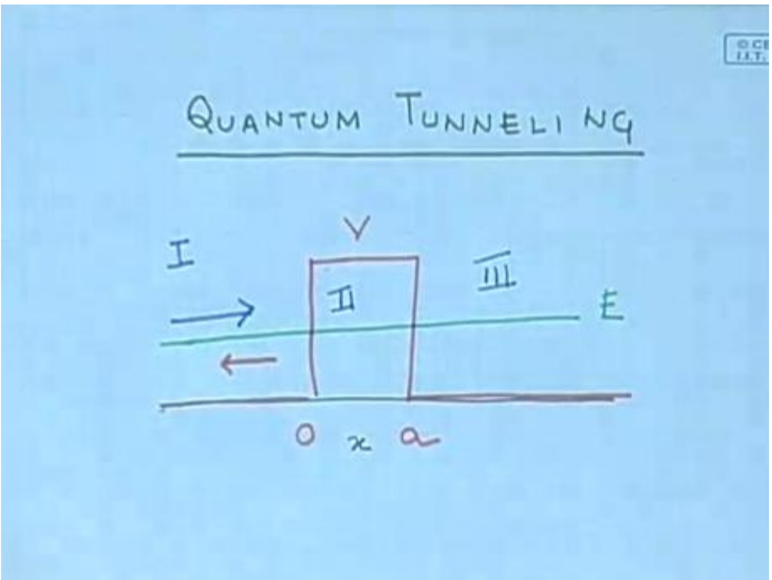

So, the term with A 1, is a right propagating wave, it represents a particle going to the right. The term with B 1 is a left propagating wave, it represents a particle going to left. So, the term B 1 represents a particle going to the left.

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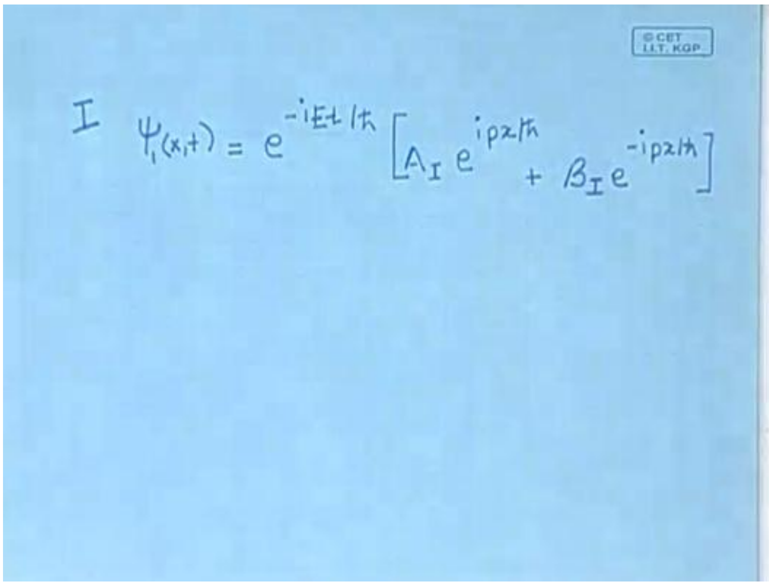

Both of these are Eigen functions of the momentum operator, we have seen that the momentum operator p is minus i h cross, the partial derivative with respect to x. So, it should be clear, that both this and this are Eigen functions of the momentum operator, when you act with the momentum operator on this, let me do it for you here.

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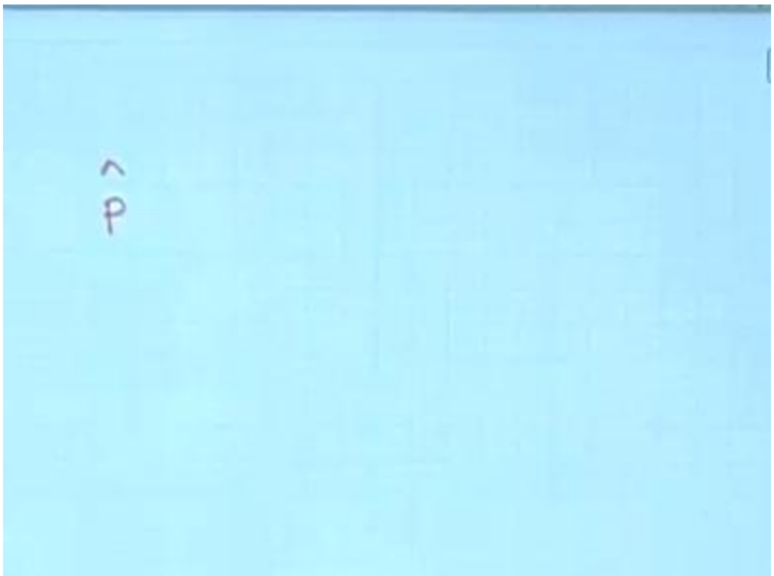

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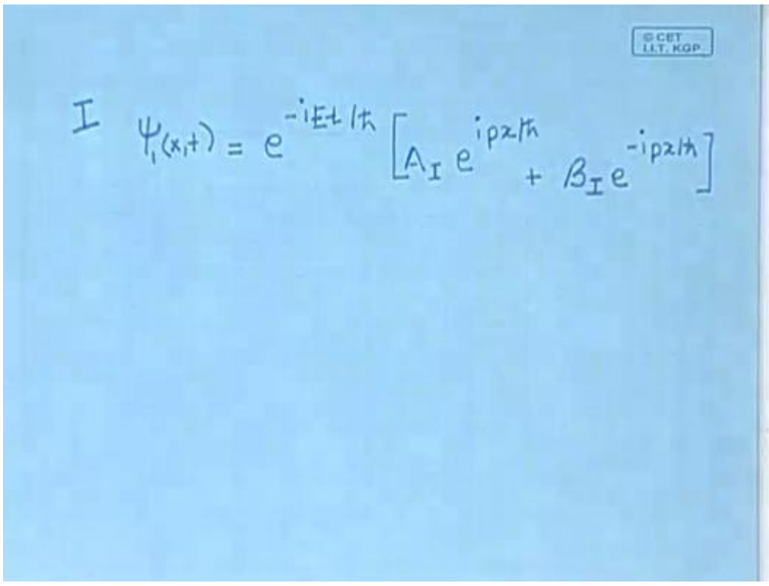

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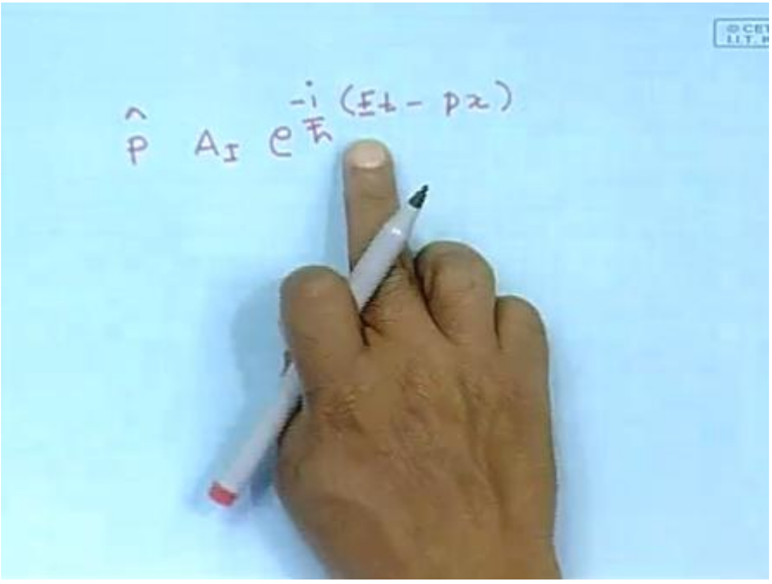

P, let me write out this thing, this part separately. So, the first part is A 1 e to the power minus i by h cross E t minus p x. This is the first part, this part.

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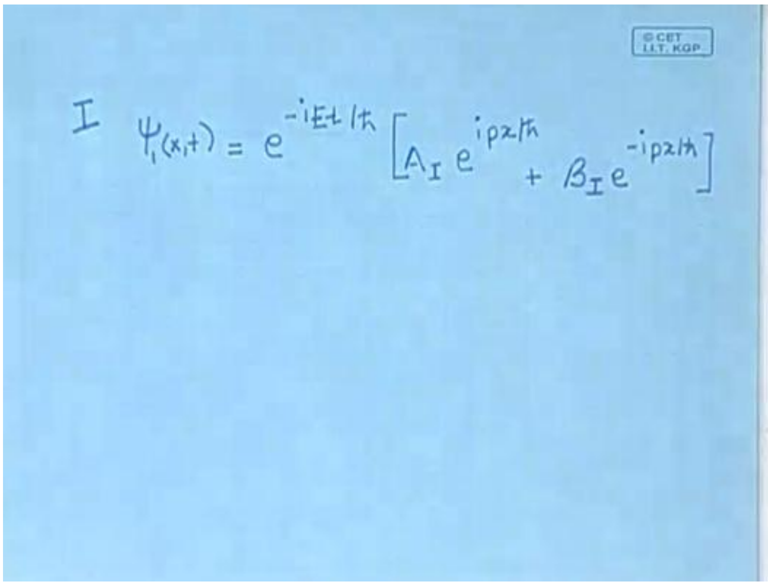

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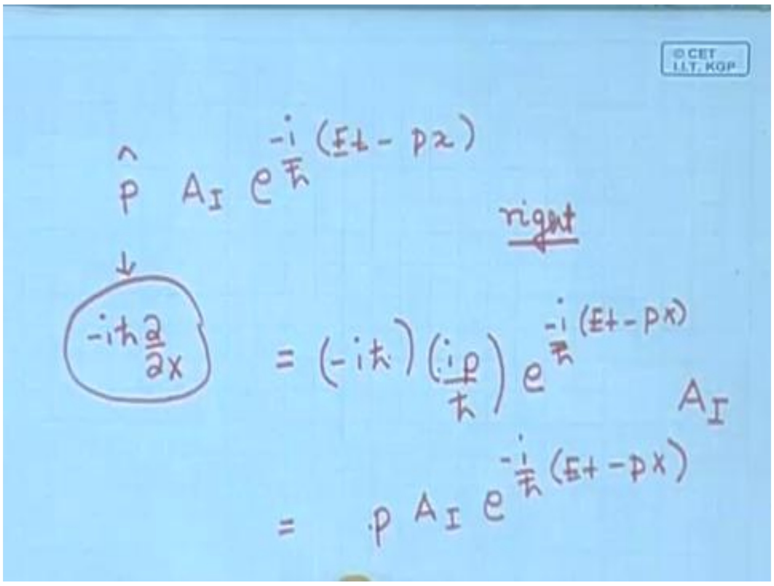

So, you can see that this is a right propagating wave, it is a right propagating wave, because as time evolves the point where the phase, has a constant same constant value, keeps on moving to the right. So, this is the right propagating wave. And when I act with the momentum operator on this, the momentum operator is minus i h cross del by del x. acts on this is what I get. See if I differentiate this with x I will pick up i p by h cross, and that is going to get multiplied. So, I already have minus i h cross over here, and if I differentiate this i will get i p by h cross, and then I will have this e to the power minus i E t minus p x by h cross into A 1.

And we see that this is going to be minus i into i is 1 h cross will cancel out. So, what we are going to get is p into A 1 e to the power minus i by h cross E t minus p x. So, what we see here, is that this function is an Eigen function of the momentum operator with momentum plus p.

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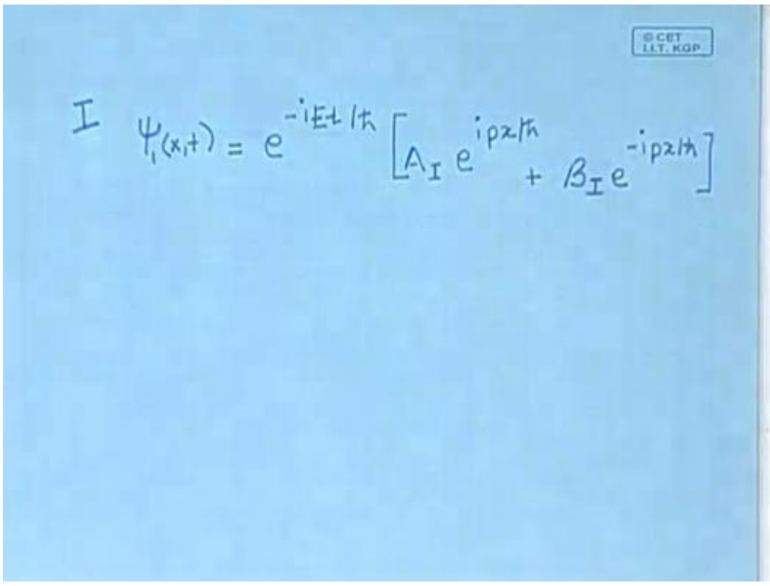

So, it should be cleared from this that the second part is a left travelling wave, and it is an Eigen function of the momentum operator with Eigen value minus p.

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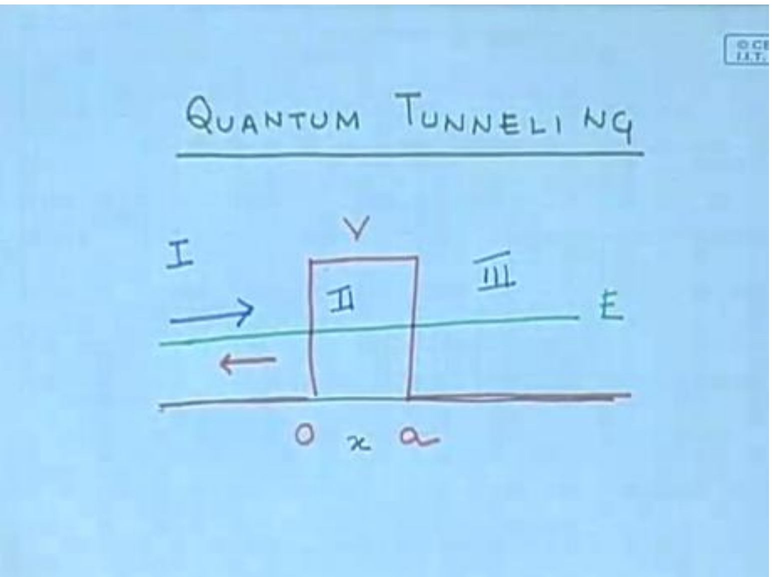

So, we can see that the first part represents a particle with momentum plus p. A positive momentum which is the incident particle, and the second part represents a particle going in the opposite direction, which is the reflected particle.

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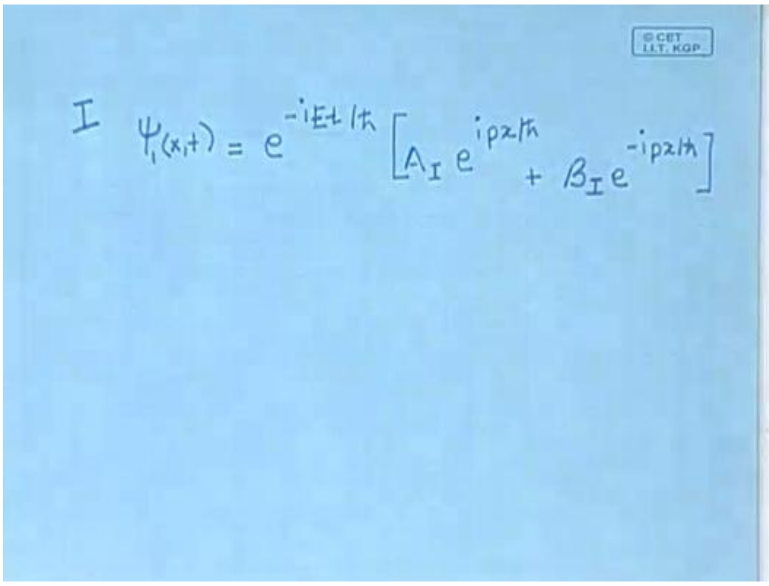

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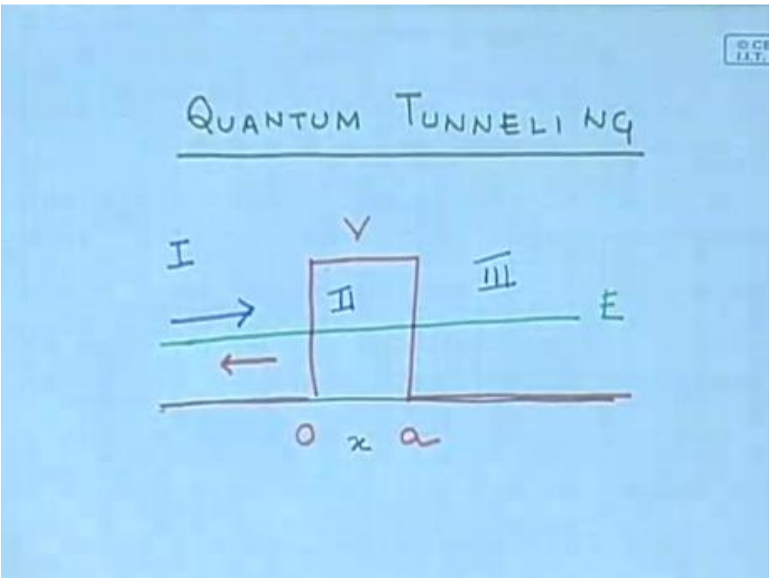

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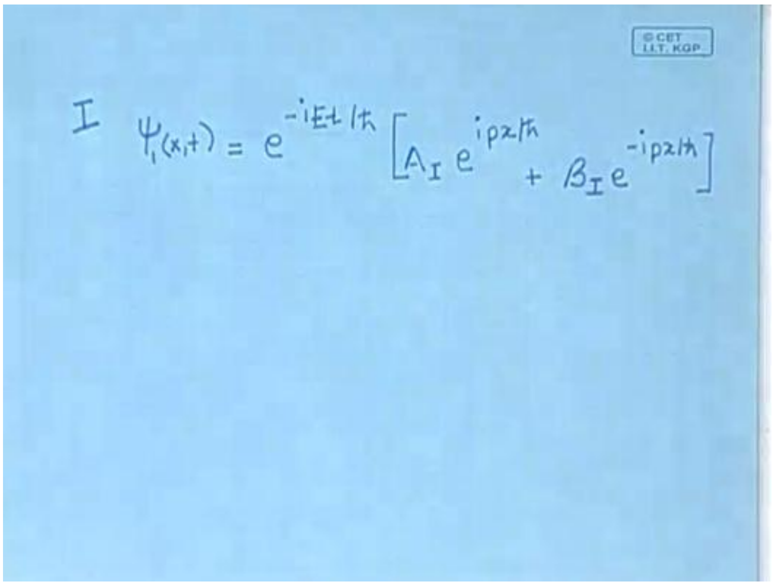

So, this is the wave function in region one, where the particle is free. And I have also told you the physical significance of these two parts of the wave function. This is the incident particle, this is the reflected particle. And remember that if we were doing classical mechanics, all particles would definitely be reflected back.

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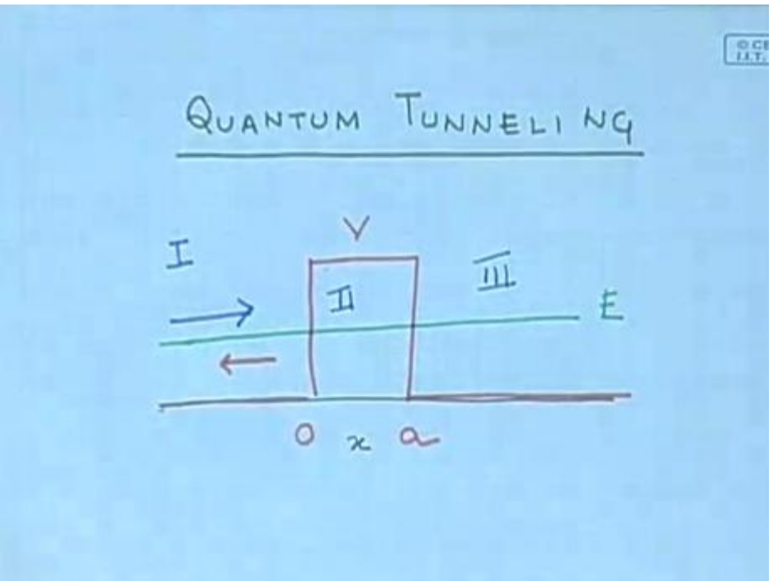

Now we have already discussed the behavior of the wave function in this region, where the potential is larger than the energy.

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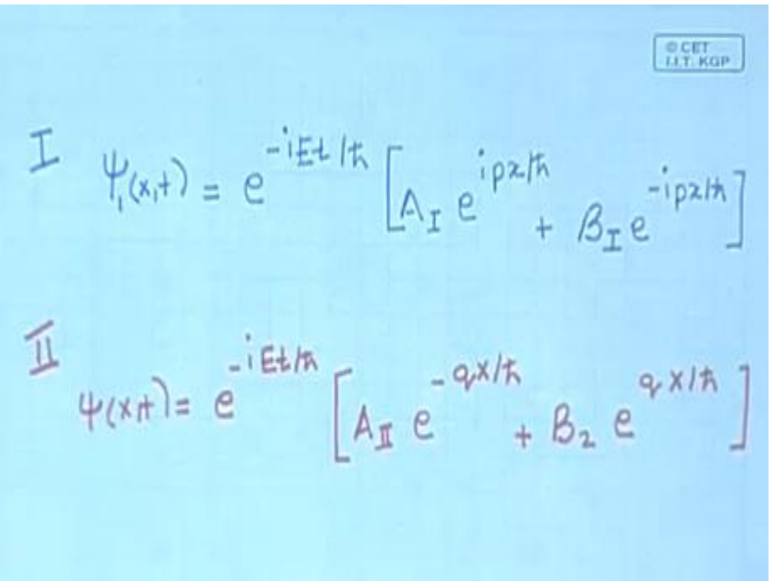

And in region two, the wave functions psi x t, is going to have the same time dependence. In all regions, it is going to have same time dependence. But in region two, instead of having an oscillatory behavior, the wave function is going to have exponential solutions of this form A 2 e to the power minus q x by h cross plus B 2 e to the power plus q x by h cross.

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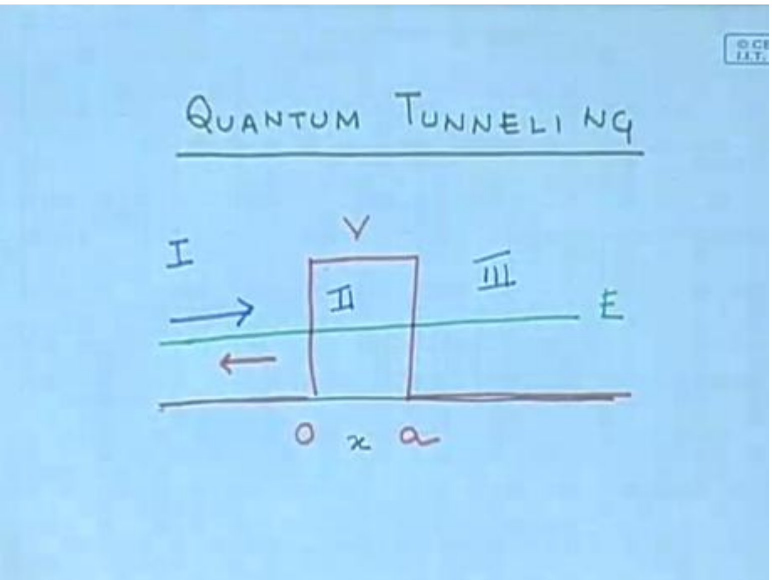

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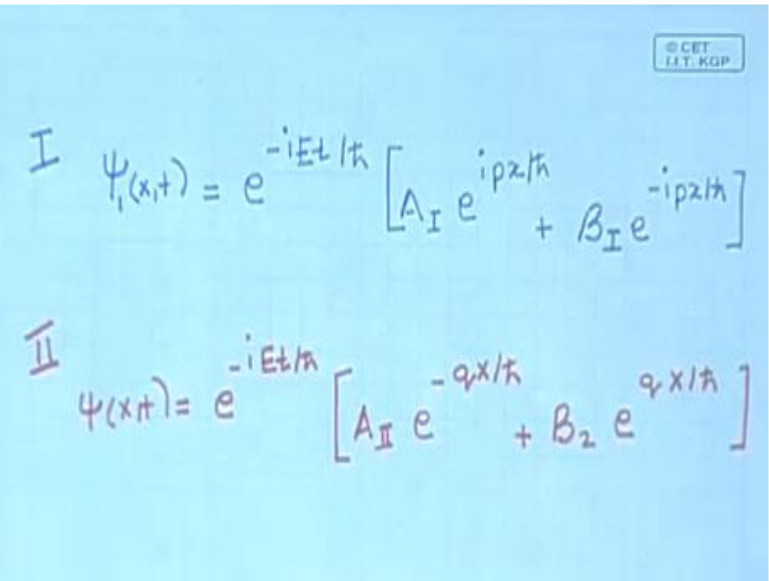

There are two possible exponential solutions in this region, and they are e to the power minus q x by h cross and e to power plus q x by h cross. Let me also tell you what this p and q are. We have discussed this in earlier lecture, let me remind you p and q are again.

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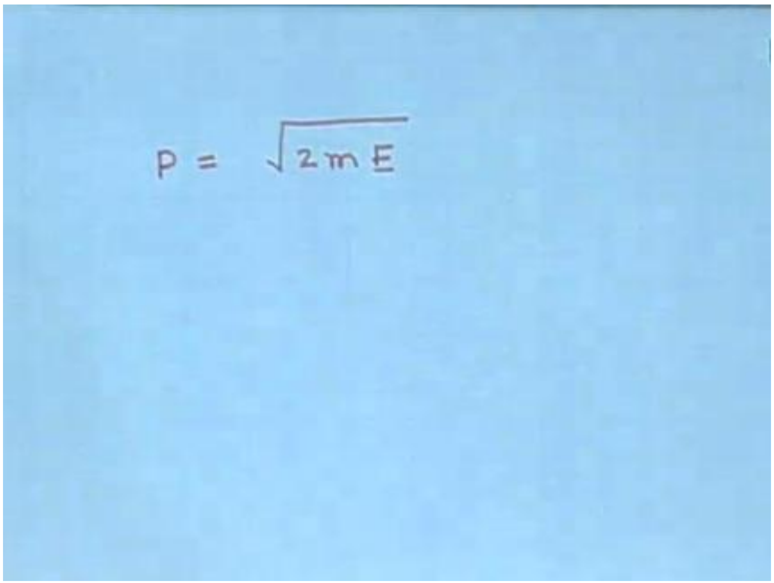

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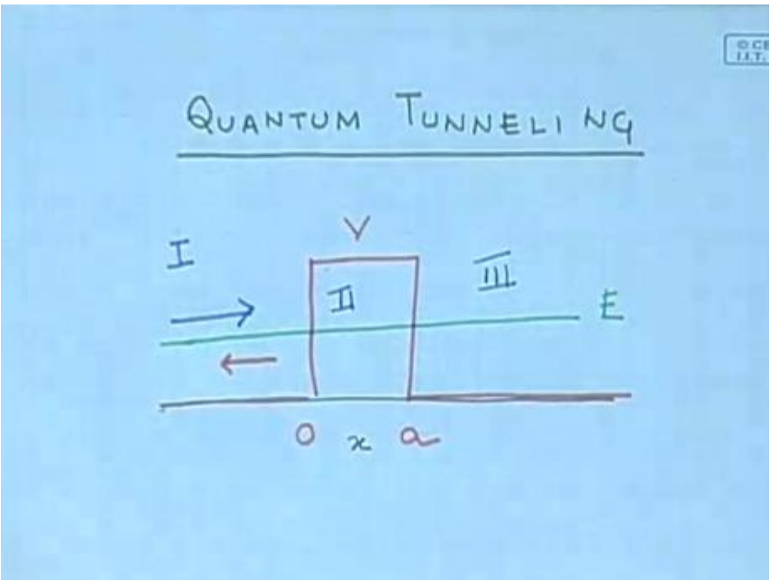

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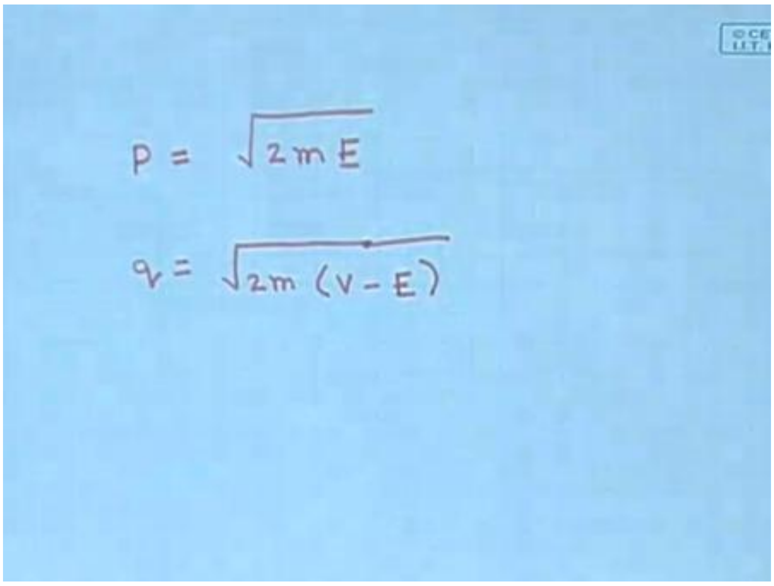

So, p is the momentum of the particle in the region where it is free, and this is 2 m E. So, the constant p which appears in the wave function in this region, can be interpreted as the wave function, as the momentum of the particle in the region where it is free. And in this region the momentum is 2 m E the square root of it, and q is square root of 2 m v minus E q is too used in the region where v is greater than E, and q is the square root of 2 m v minus. The difference in the potential, and the total energy of the particle, and it is this q which appears in the wave function in region two.

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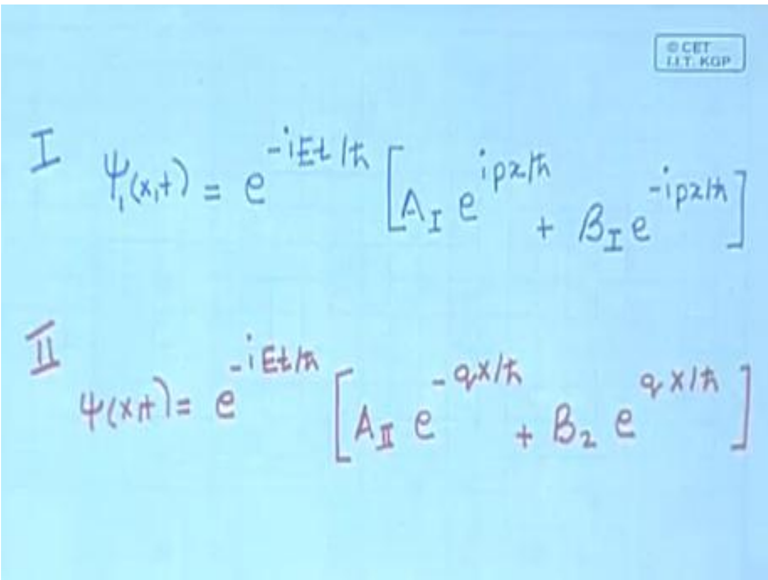

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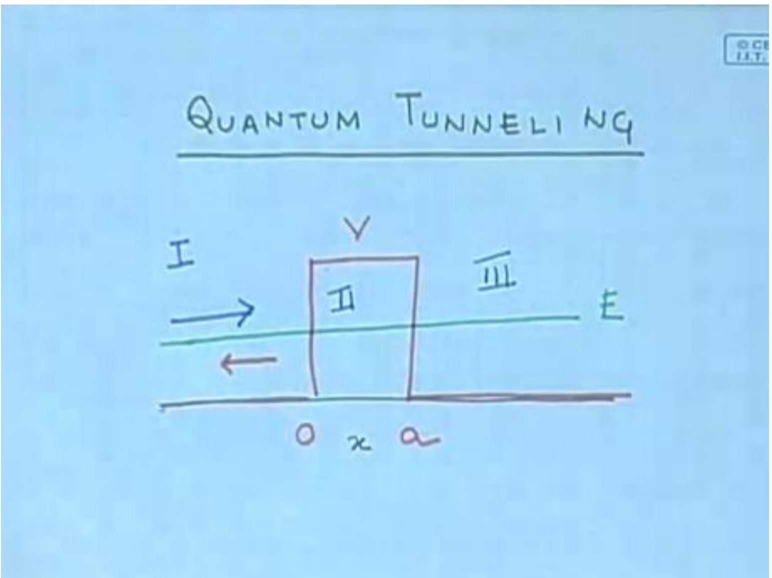

Now let us ask the question, what is the wave function of the particle in region three? There are three regions, the particle is free here, it is also free here, and it is under the influence of this high potential barrier in this region two. So, in region three also the particle is free, and we have already seen the free particle wave function. So, it is going to be the free particle wave function in region three also. So, I can write down, what the wave function is going to be in region three.

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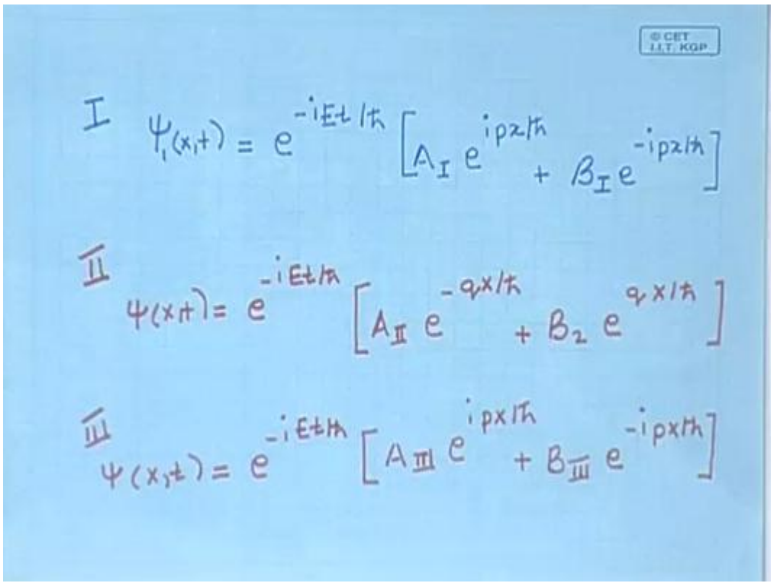

In region three, the wave function is going to be psi x t, is equal to the time part is going to exactly the same e to the power minus i E t by h cross. And I will have again two different amplitudes A 3 e to the power i p x by h cross plus B 3 e to the power minus i p x by h cross. So, in region three, the wave function is exactly the same as region one, except that these to coefficients of the right travelling wave and the left travelling wave are different. Now, let us again go back to the physical interpretation of these two parts. We have already discussed that this part, the first part represents a right travelling wave.

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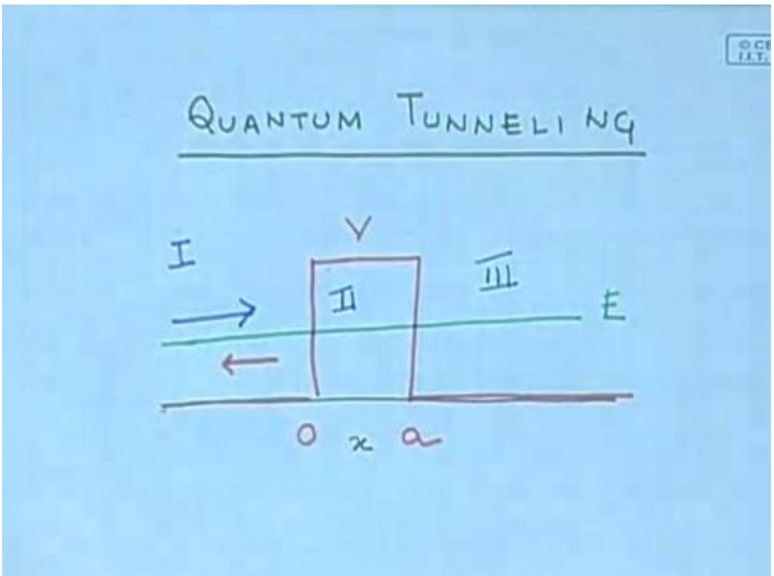

So the first part represents any part of the wave that may get through. So, the wave is incident, the particle is incident like this. So, we have a right travelling wave coming over here. And then the wave behaves exponentially, and it decays exponentially inside here. And the third part represents any part of wave, that may penetrate through this and come out in region through. So, the third part represents a right travelling wave in region three. And you can think of it as representing. This actually represents of a particle with momentum plus p in region three. So, it is any particle that can penetrate. So, that can penetrate through this barrier and get out in this region. So, it represents a particle going to the right in this region, and we can represent that as the particle penetrating through this and coming out here.

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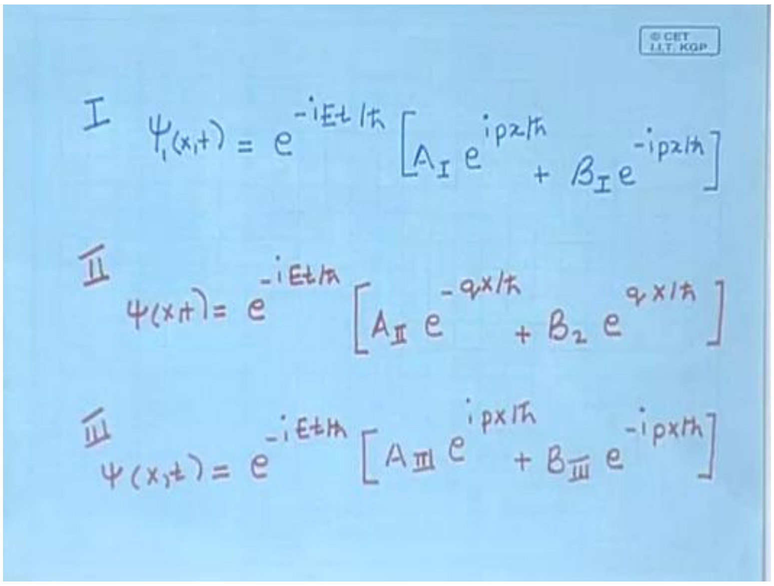

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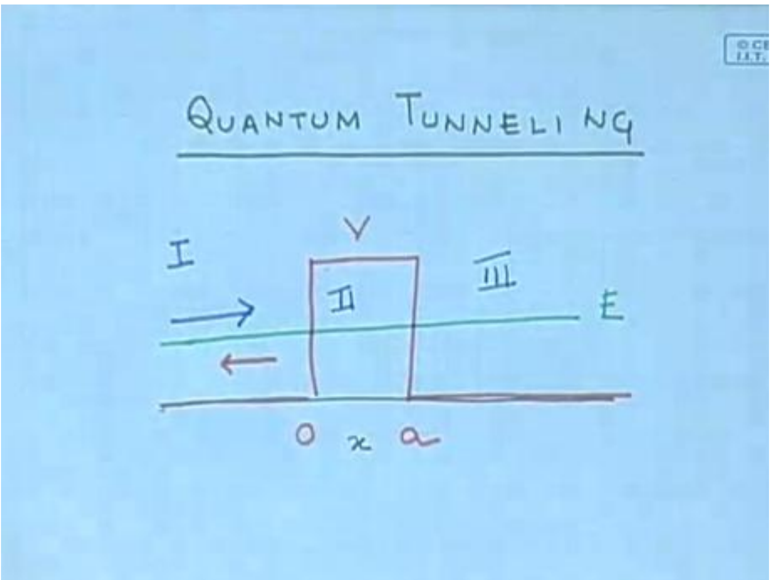

Whereas this term over here, the second term, term having B 3 represent is a left travelling wave in this region, wave travelling to the left like this.

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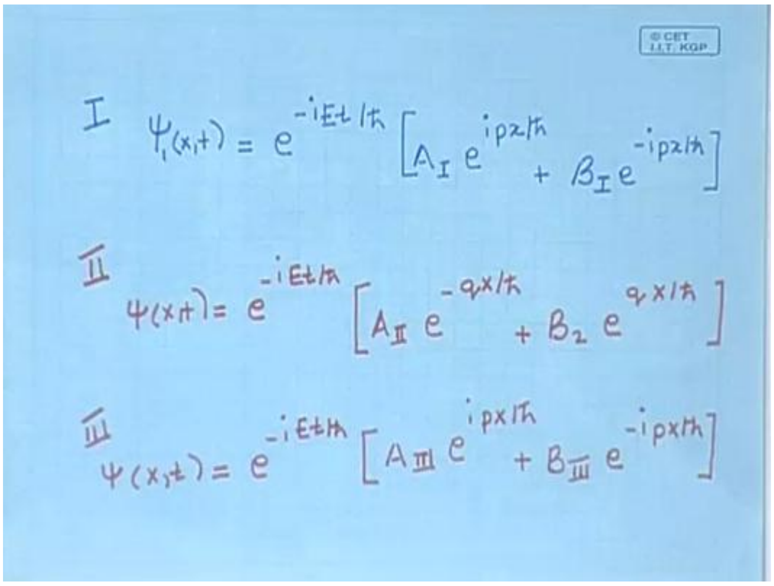

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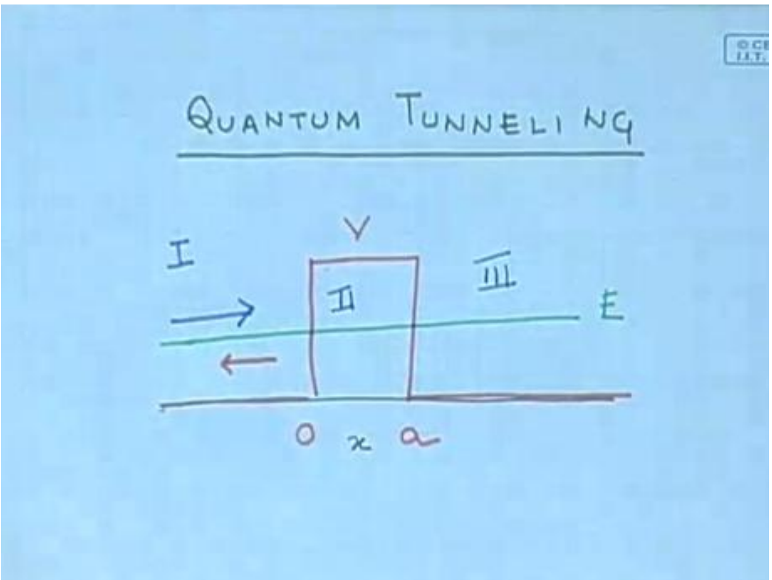

It is a particle which corresponds to a particle with momentum minus p. So, if I had a particle with momentum minus p over here, the particle would be travelling in this direction. So, it would represent particles which are incident on this potential barrier from the right. Now, I have already defined the physical situation that we are discussing. In this physical situation, we have a potential barrier, and there are particles incident on the potential barrier from only one direction, from the left. So, we have this potential barrier, and there are particles incident on it only from this side, and we would like to now study what happens to these particles. There are no particles incidents from this side. There are particles incident only from this side. Some of the particles will get reflected, some may get transmitted. In classical mechanics, all the particles would get reflected. In quantum mechanics, we see that the wave function corresponding to these incident particles can penetrate through the barrier, and some of it may come out, and then because of that we would have a finite probability of finding some particles here. So, these would be a particles that have managed to somehow get through this, and there would be particles going to the right. Particles going in this direction would. If I had of a part of the wave function going this way, they would correspond to particles coming like this. And there would be particles which are incident on this potential barrier from the right, but we are not sending in any particles on the potential barrier from the right. So, there are no particles coming in from the right, there are particles coming in only from the left.

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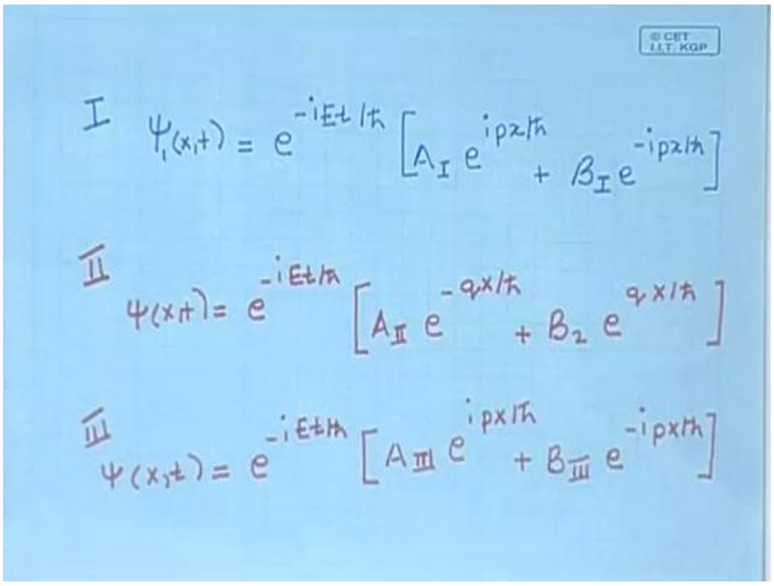

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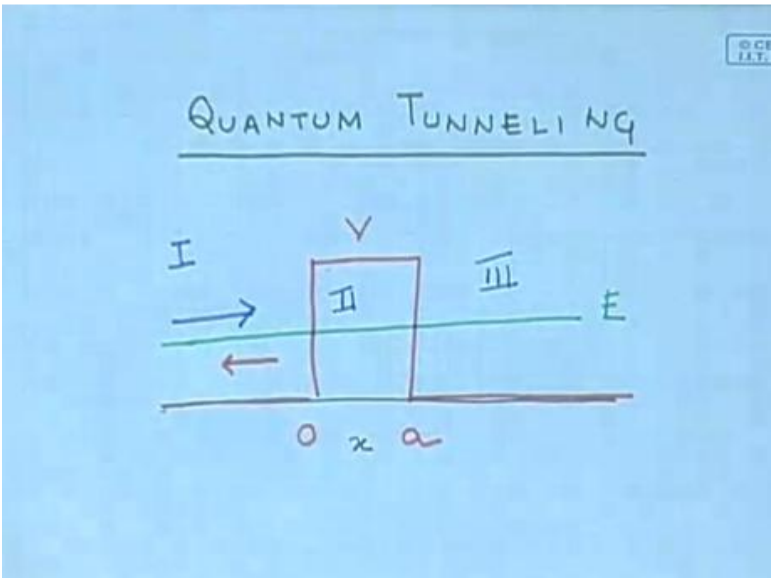

So, this essentially tells us that the amplitude of this part has to be 0, because this gives us the probability of finding some particles with momentum minus p over here. Momentum minus p means the particle going this way, but we are not sending in particles like this. We are only sending in particles in this direction, and some of the particles may get through and come out this way, which is what this would represent.

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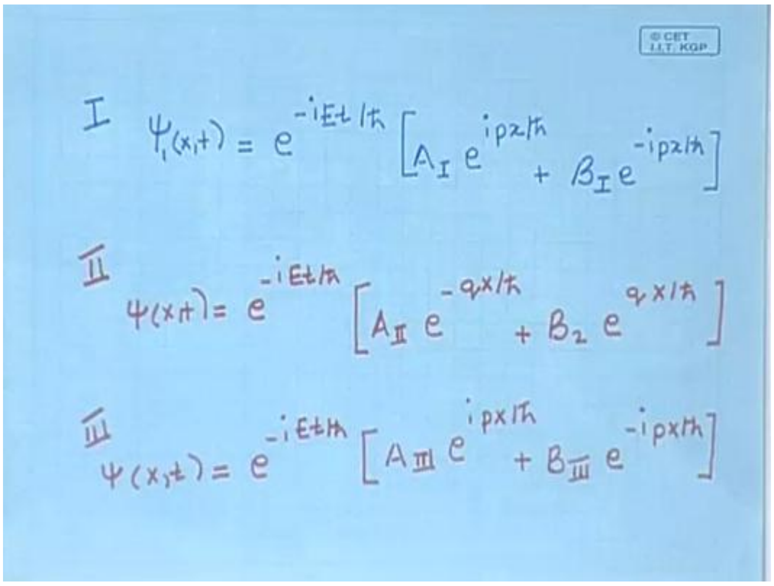

So, this tells us that this coefficient over here has to be 0. So, we have worked out the wave function in the three different regions. In our problem, we have worked out the wave function in the region one, in region two, and in region three.

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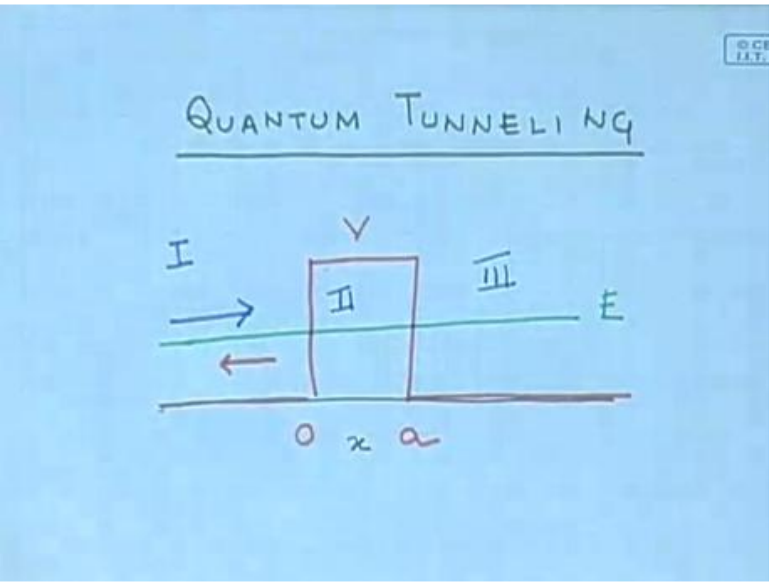

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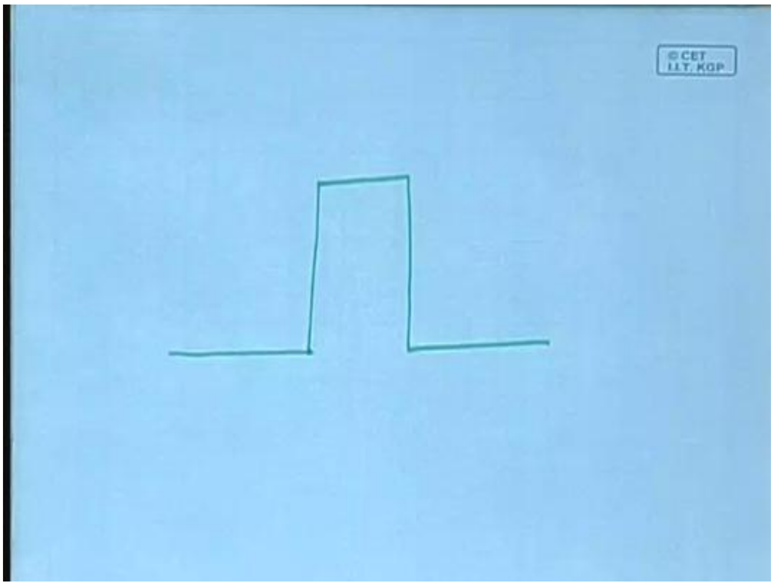

Let me graphically show you these wave functions in the three different regions. So, if I draw the potential here again, the potential looks like this. And what we see, is that on the left hand side we have.

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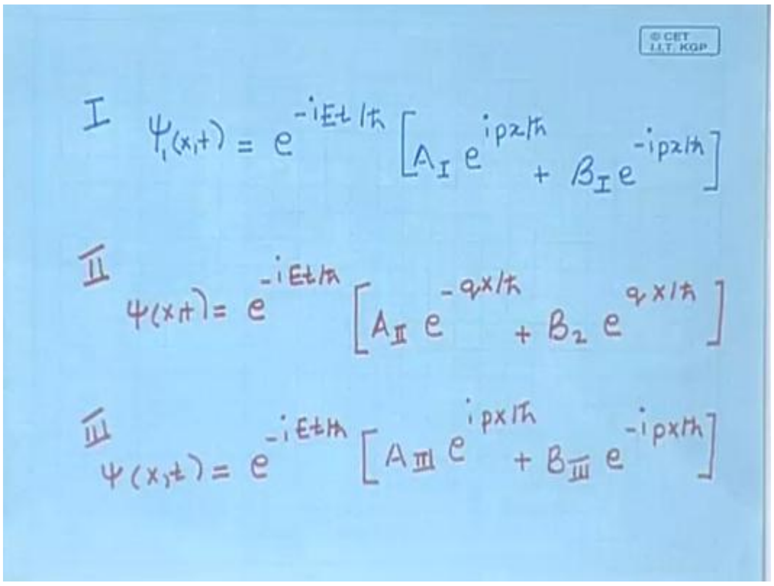

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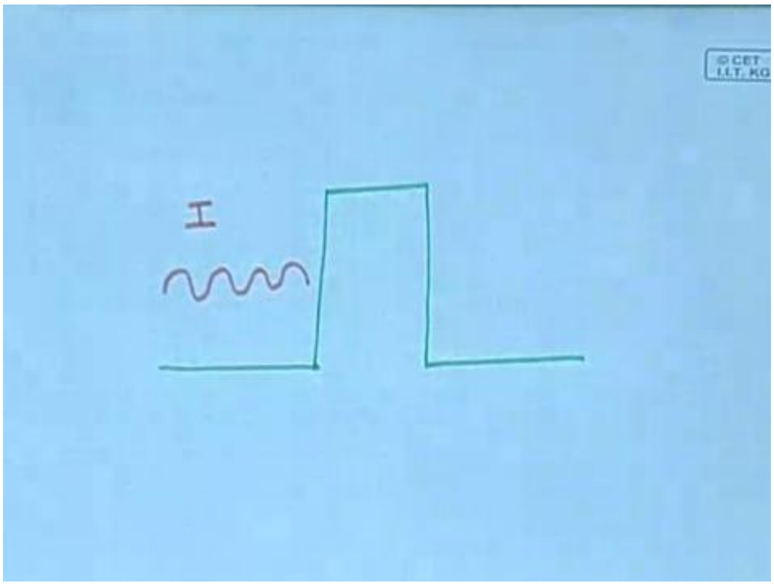

A plain wave solution with two possible parts; one representing a particle which is incident, this representing a particle which is reflected. The main point, is that we have plane wave solutions, whose wavelength is determined by p. So, we can schematically show that, like this that we have a plane wave solution in this region, region one.

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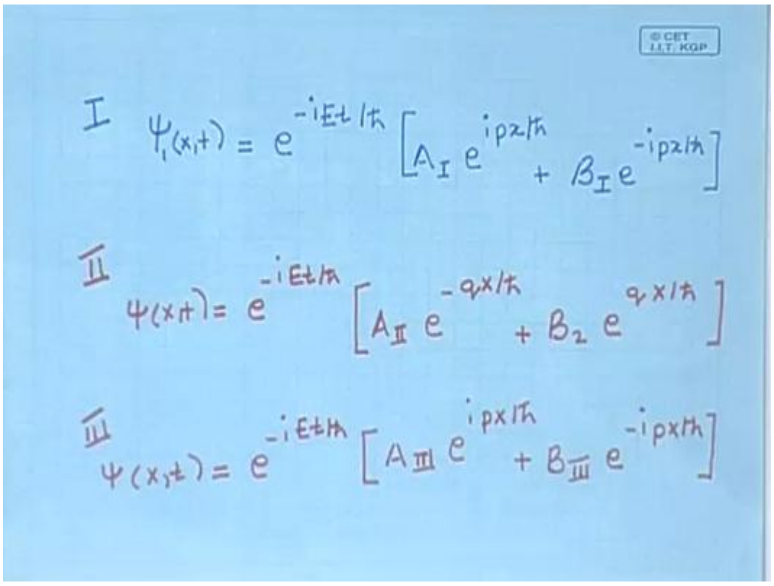

Now in region two, we see that the wave function behaves exponentially, and what we would expect, if is that the wave function actually decays exponentially.

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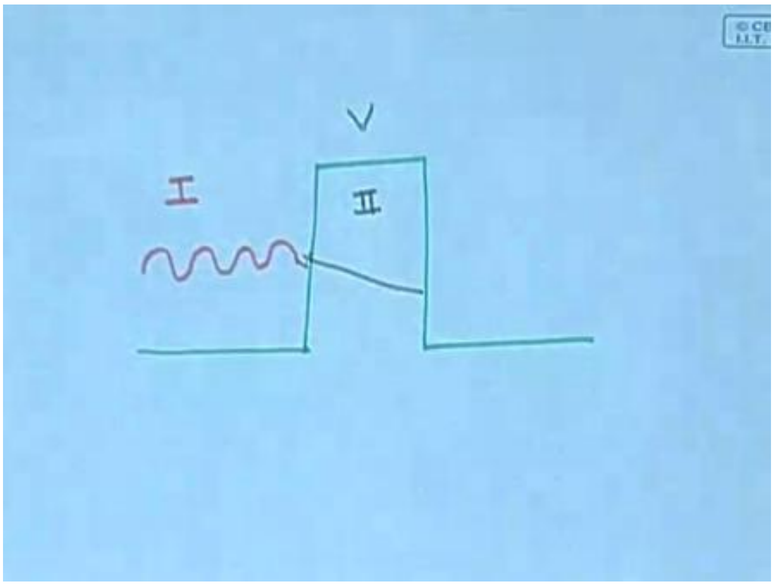

So, in region two the wave function goes. Some part of the wave function manages to penetrate into region two, where the potential is higher than the value of the energy, and the wave function in this region decays exponentially.

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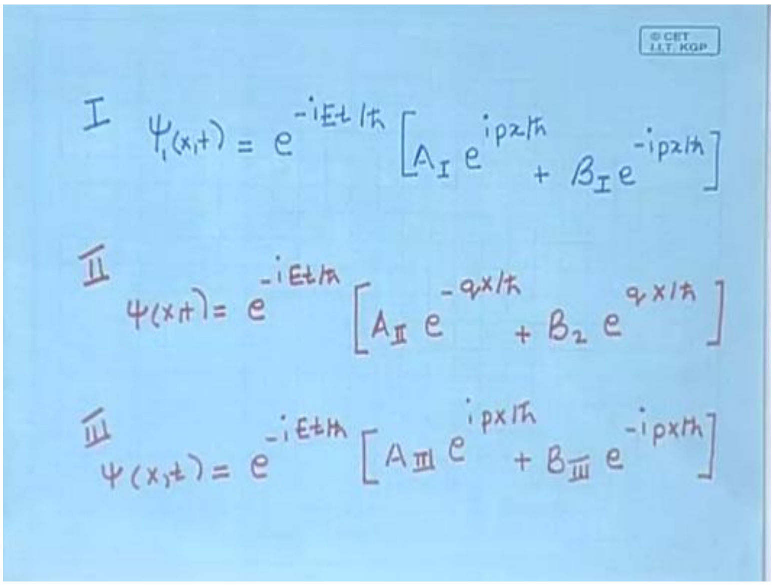

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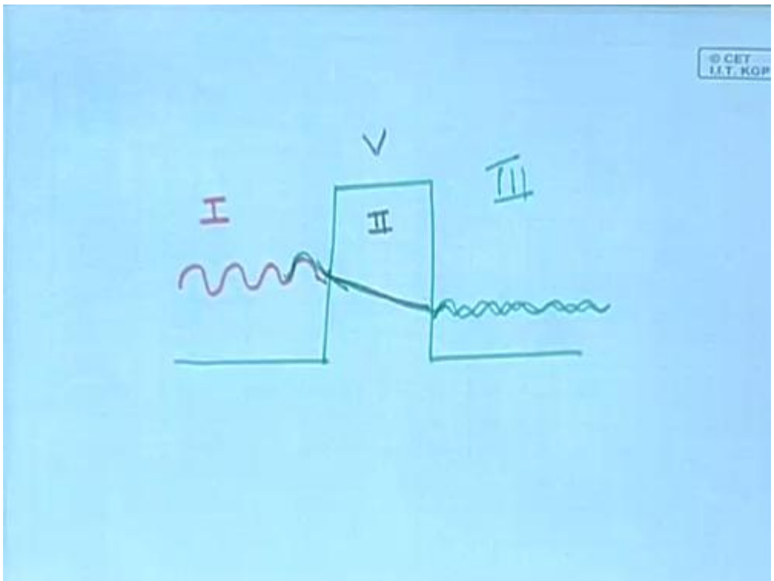

And then in region three, we again have oscillating solutions. So, we have oscillating solutions here again, but the amplitude of these, this of this. these oscillating wave functions here, is going to be much less than the amplitude of the wave function here, because the wave function decays exponentially inside this, and then finally, some of it does manage to come out on to this side. So, what we see, is that in quantum mechanics, there is a finite value for the wave function in region three. It is not going go to zero abruptly. And if you have a wave function finite wave function in this region, then there is a probability, the modulus square of the wave function in this region tells us the probability of finding a particle in region three. So, there is a finite probability of finding a particle in region three. So, in quantum mechanics when there are particles incident from this direction, there is a finite probability of finding some of the particles in region three. So, some of the particles can get through this potential barrier, and you may find them in region three. There is a probability, and this phenomena is the what is known as Quantum tunneling.

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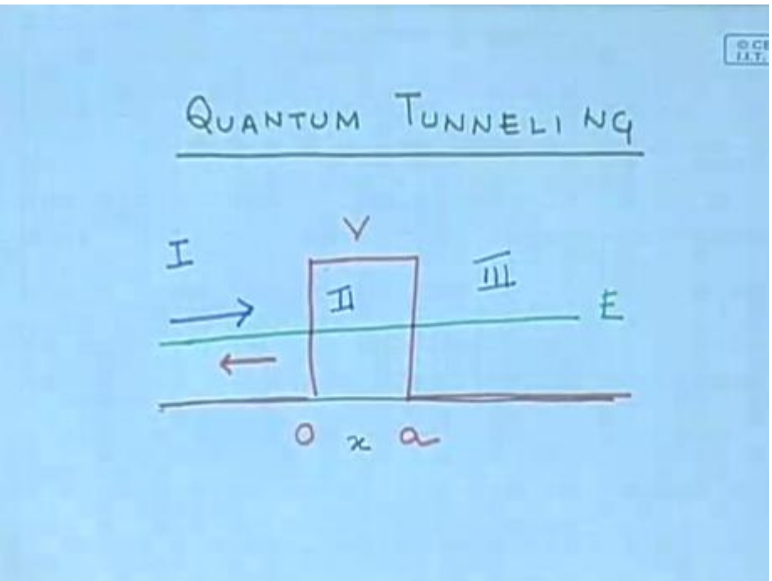

The analogy here is like a hill, if I have a hill on the surface of the earth. The hill represents a potential barrier; if a particle has to cross this hill, let us say that there is a car which would like to go over the hill. Then there are two possible ways which would like to get across on to the other side, there is a hill over here, the analogy is as follows.

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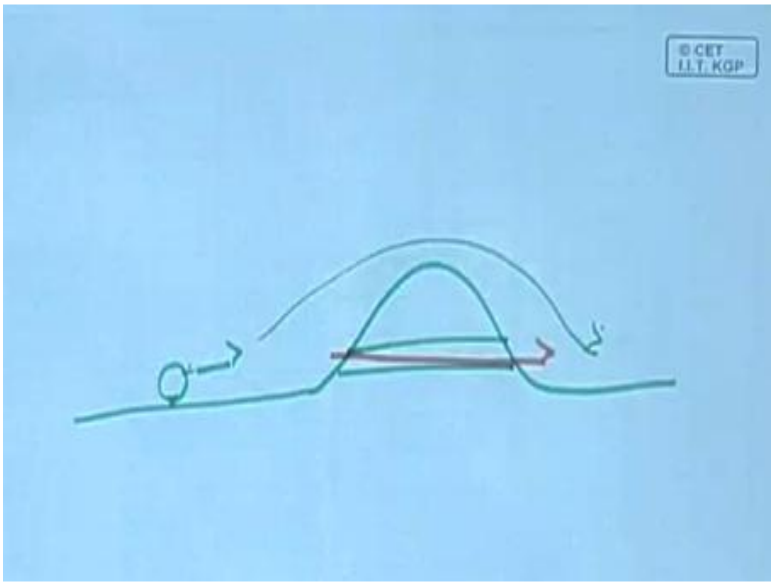

Say on the surface of the earth, there is a hill like this, and there is, let us say a particle over here. We would like the particle to get to the other side. There are two possibilities; one is give the particles sufficient kinetic energy so that it can roll over and get over there. The other possibility is if you have a tunnel which goes through this. So, then if the particle can go through this tunnel, it does not require to have the enough kinetic energy to get over, and the particle you can then go through this tunnel and reach the other side. If you are travelled by train to Mumbai for example, you will find that there are quite a few tunnels one the way, and these tunnels basically make it easier for the train to get to Mumbai, because there are quite a few hills in the western ghat, and it would require enormous amount of energy to overcome to get to the top of the hill and then come down, and this can be avoided if there is a tunnel through the hill.

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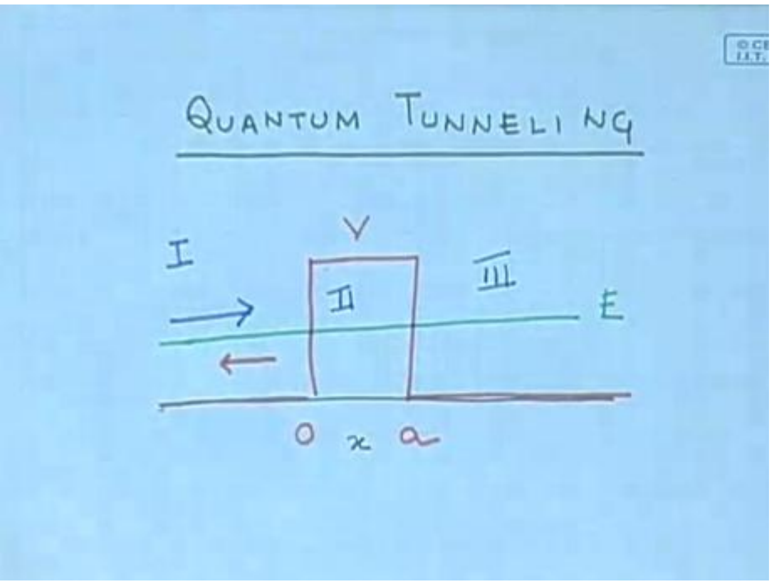

So, it is as if a particle which is being sent on this potential barrier. The particle which is being sent here, does not sufficient energy to go over this barrier, but when you use quantum mechanics to analyze such a situation, you find that there is a finite probability that the particle may get through to the other side.

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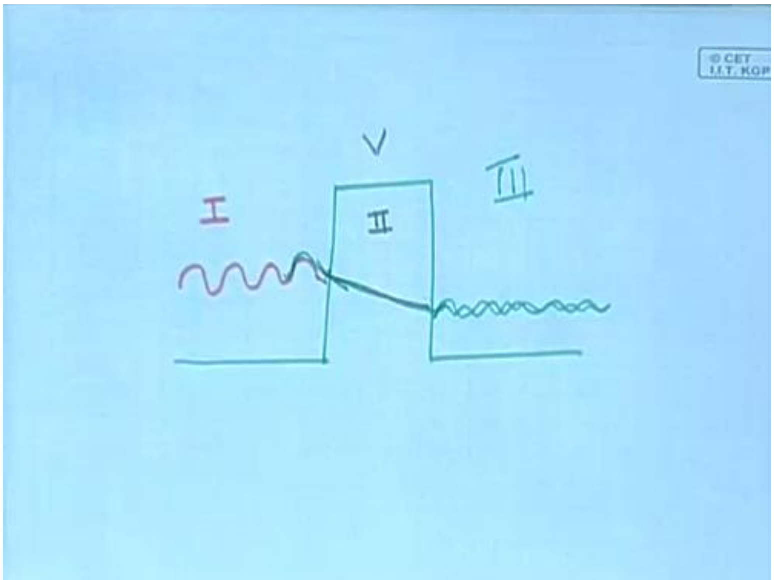

This arises, because you have now a wave description, and there is a possibility that a part of the wave actually penetrates through, because in this region the wave the function does not abruptly go to 0, it decays exponentially. And then once it comes here again you have this oscillating solution. So, there is wave function in this region, which tells you that there is a probability that even you if you have incident particles here, with energy less than the potential. There is a finite probability that you can find this particle on the other side.

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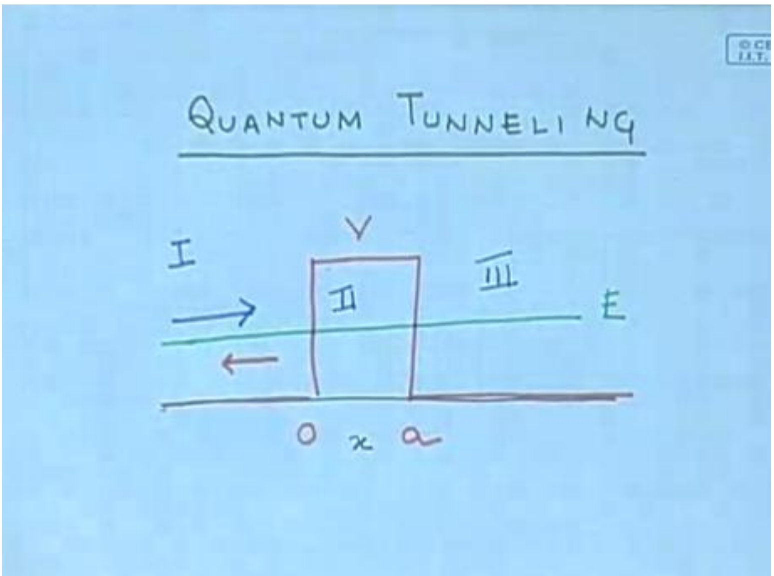

So, as if the particle has got through by making a tunnel, by somehow tunneling through this potential barrier, which is why this refers to as quantum tunneling. Now, the quantity of interest, which we would like to calculate, is the probability of finding the particle on the other side. So, we are sending in particles from the left, and the question which is of interest is, what is the probability that this of finding this particle which is incident from here, what is the probability of finding it in this region, what is the probability that it gets through the barrier, which we can referred to as the tunneling probability. So, we have incident particles coming in, what is the probability of finding the particle in region three. This is the question that we are in interested in, and how shall we address this question, let us go back to the wave function in the three different regions.

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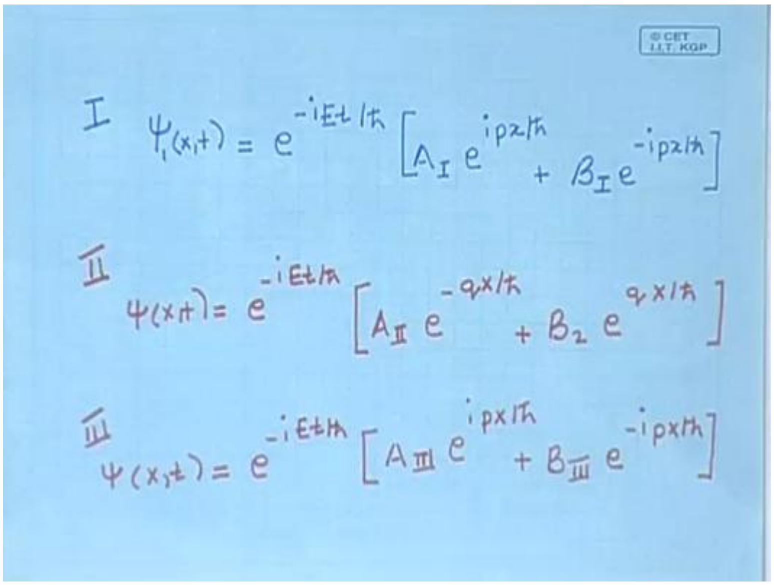

So, this is the wave function in the three different regions. This represents the incident particle. This represents the reflected particle. And this represents the particle that tunnels through and is found on the other side of the barrier. So, the probability of finding the particle on the other side, is there in the coefficient, is in the amplitude of this part of the wave function. The larger this amplitude, the more is the probability of finding the particle on the other side of the barrier. So, this is the amplitude of the incident wave, this is the amplitude of the transmitted wave. The ratio of this amplitude essentially tells us, we will get the probability amplitude from the ratio of these two amplitudes. The reflected wave amplitude is there in this. So, we will get the probability of the particle getting reflected from the ratio of these two amplitudes, how we will get it we shall discuss later. Now, the question is, how do we determine these coefficients which have been unknown till now. Remember earlier when I discussed potential step potentials, I did not go into the issue of how to relate the coefficients on the two sides. And I had told you that I shall be taking it up later. So, let us now discuss how to relate the coefficients, these amplitudes of the different parts of the wave on the two different sides of the potential, step potential.

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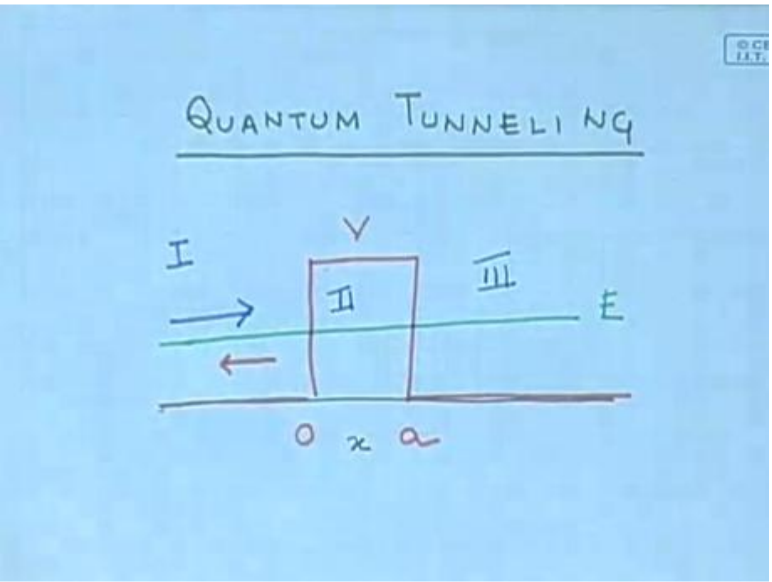

So, the governing principle is, that whenever we have a step potential like this. Let us whenever we have a step potential like this, or like this, the value of the wave function should on the left hand side, should match the value of the wave function on the right hand side. So, the wave function should be continuous, across this step in the potential. So, let me write down the boundary conditions.

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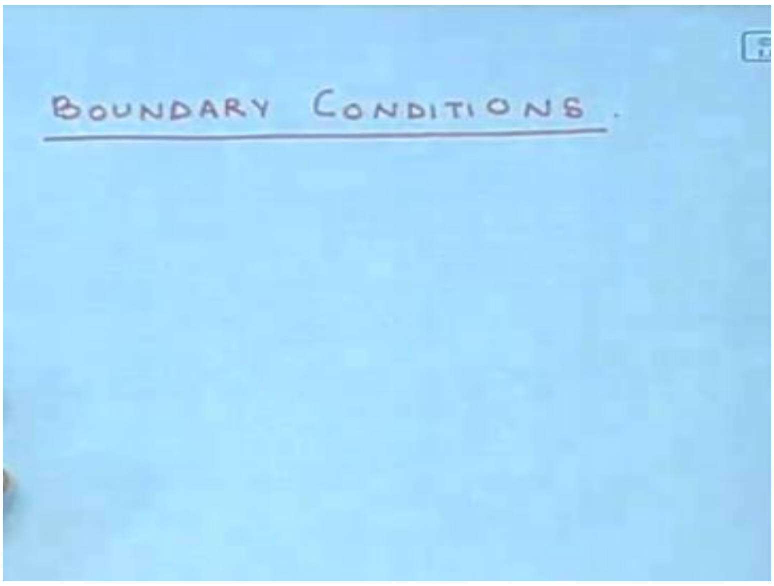

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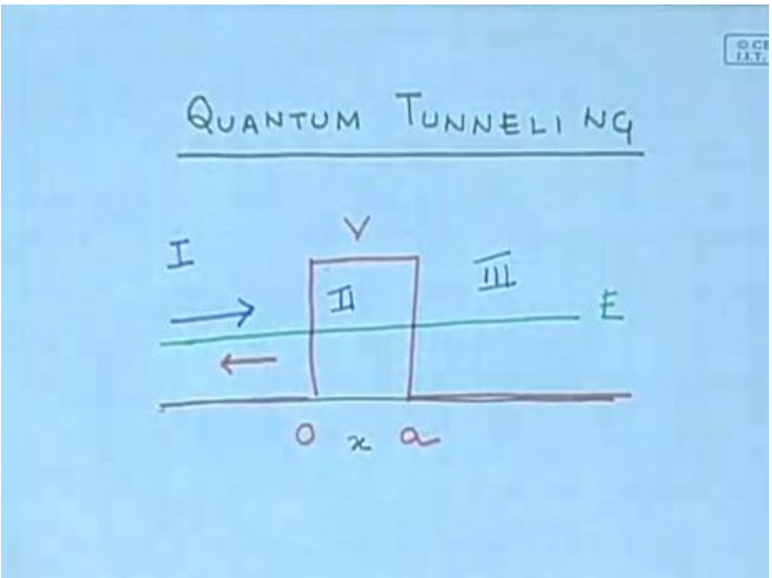

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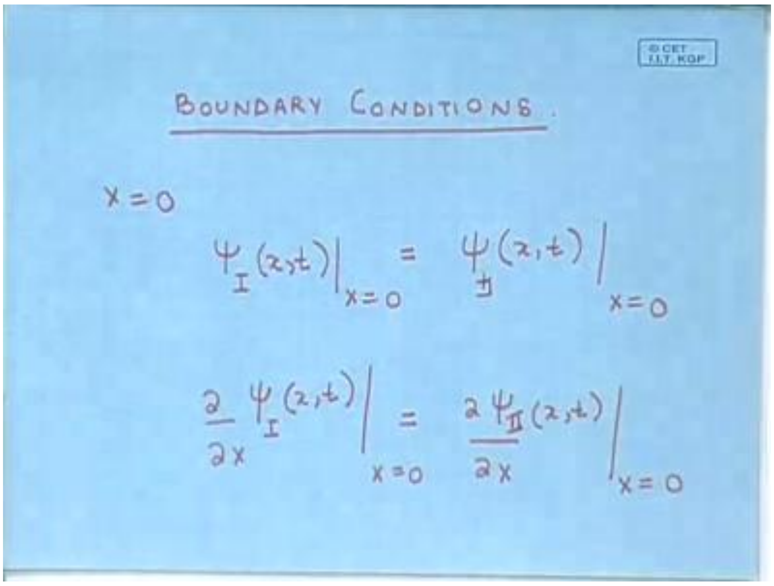

So, the first boundary condition as I mentioned, is that the wave function should be continuous across this boundary. So, let us consider the boundary at x equal to 0; the step at x equal to 0. And the value of the wave function on the left hand side psi 1 x t at x equal to 0 should be equal to psi in region two at x equal to 0. So, at the boundary the wave function should be continuous. Also the first derivative of the wave function, first spatial derivative of the wave function, at the boundary which is x equal to 0 should be continuous.

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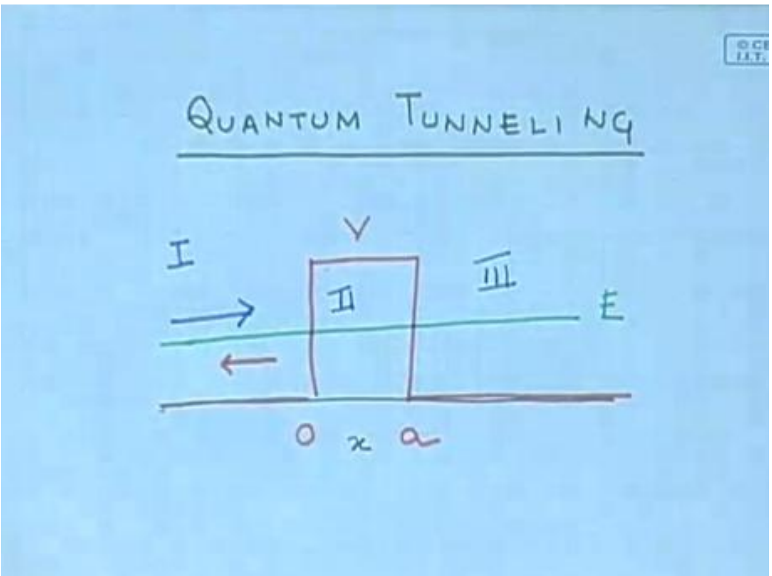

So, whenever we have a boundary, whenever we have a step in the potential at any boundary, any such boundary the wave function, and it is first derivative should be continuous. So, this is one region, this is another region. We have two different wave forms of the wave function here and here. At this interface at this boundary, the wave function and its first derivative should be continuous. Now you may ask the question why should the wave function, and it is first derivative be continuous, why not the second derivative, why not required that the second derivative also be continuous, the third derivative also be continuous. The fact why, the reason why we require only the wave function and it is first derivatives to be continuous, and the reason why we require this is, because the wave function is governed by the Schrodinger differential equation. The Schrodinger differential equation, remember.

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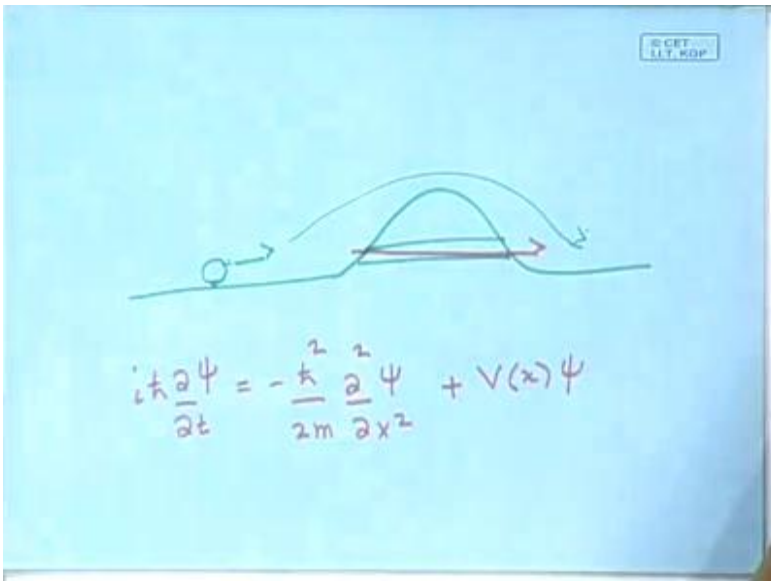

So, let we write down the Schrodinger differential equation here. the Schrodinger differential equation is i h cross del by del t of psi is equal to minus h cross square by 2 m del square del x square psi plus V x psi. This is the Schrodinger wave equation, which governs this wave function, and this equation has to be satisfied at every point. Now, the potential changes abruptly at the boundary, but this equation has to be satisfied everywhere.

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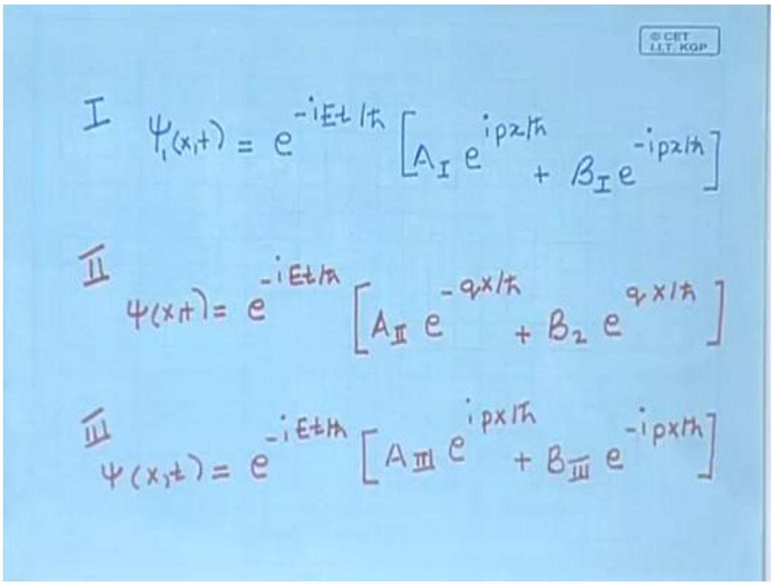

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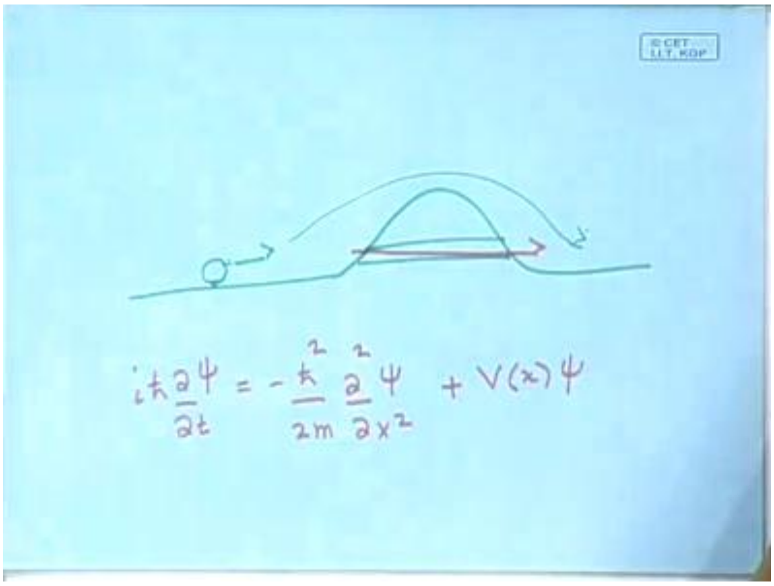

Now, let us first take the time part. The time part is the same throughout, the time part of the wave function is the same throughout. So, we really do not have to bother about it. We have to be concerned about the spatial part.

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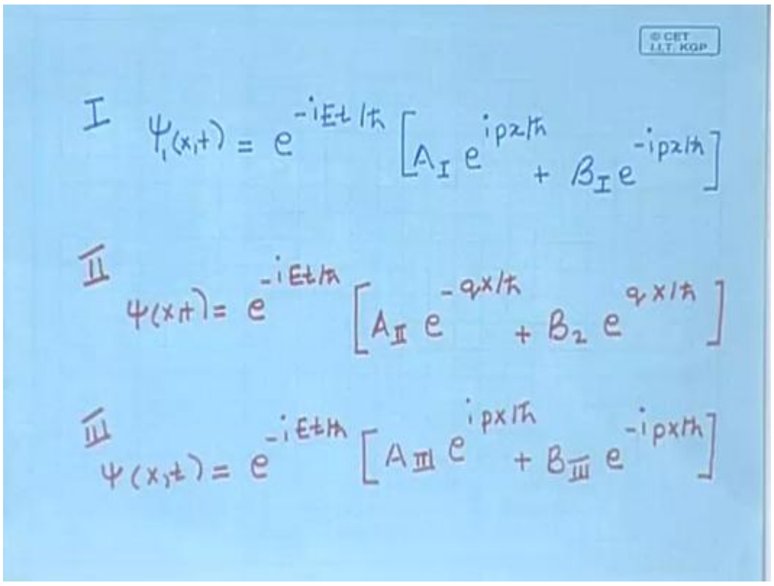

The spatial part is different in region one and region two. We are considering the boundary between these two. The spatial part is different in these two regions, and here we have the second spatial derivative

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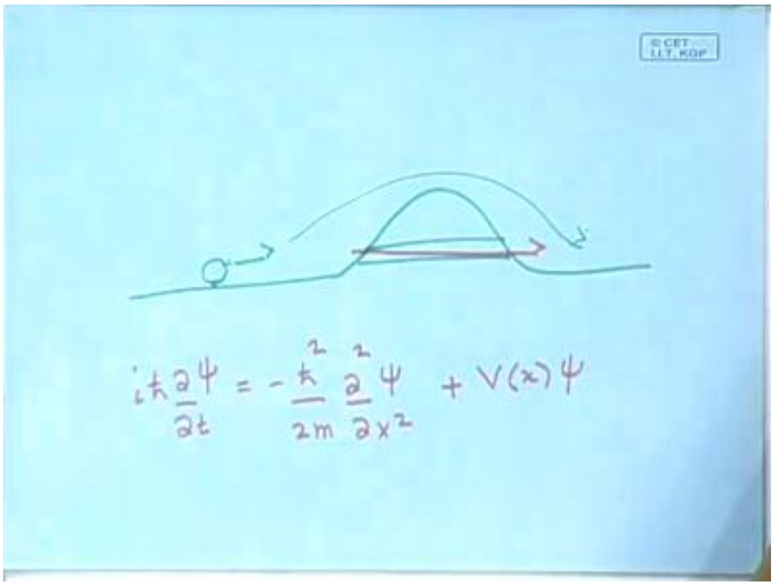

Now to evaluate this second spatial derivative at the boundary, we need to first evaluate the first spatial derivative at the boundary; the first derivative with respect to x. Now, if the function psi has different values on the left hand side and the right hand side, at the boundary if it has a different value, at the left and the right hand side. If is discontinuous, then you cannot evaluate the second the first derivative. The function has to be continuous for the first derivative to be defined. So, the fact that you have to evaluate the first derivative at the boundary tells us that this boundary condition has to be imposed this has to be satisfied.

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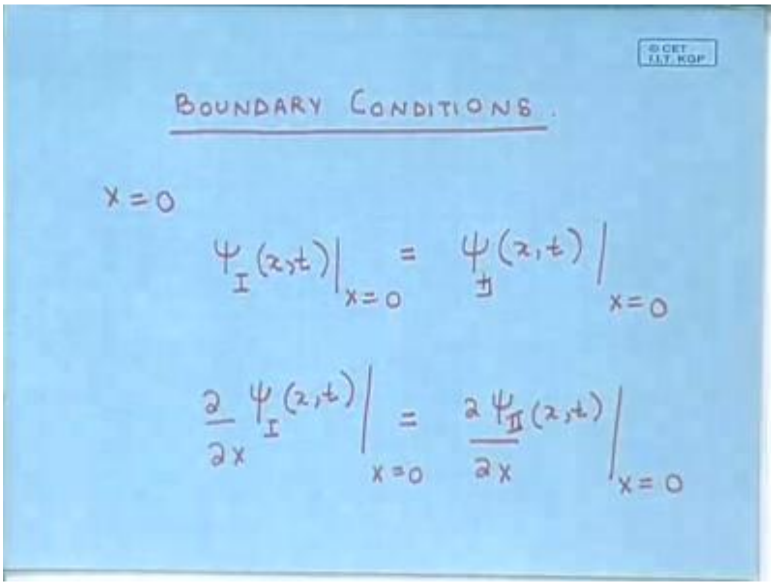

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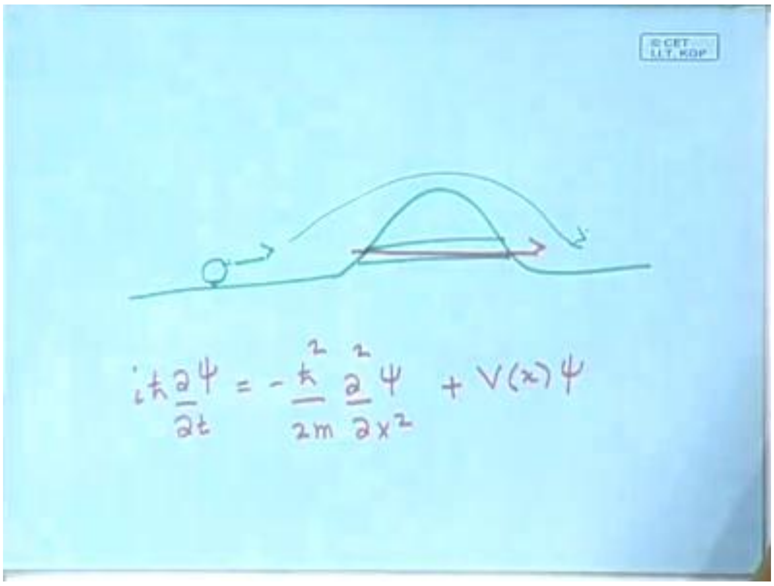

Now not only do you evaluate the first derivative, we also have to evaluate the second derivative at every point, so also at the boundary. To evaluate the second derivative, the first derivative, the first spatial derivative itself should be continuous. You cannot differentiate a function that is not continuous.

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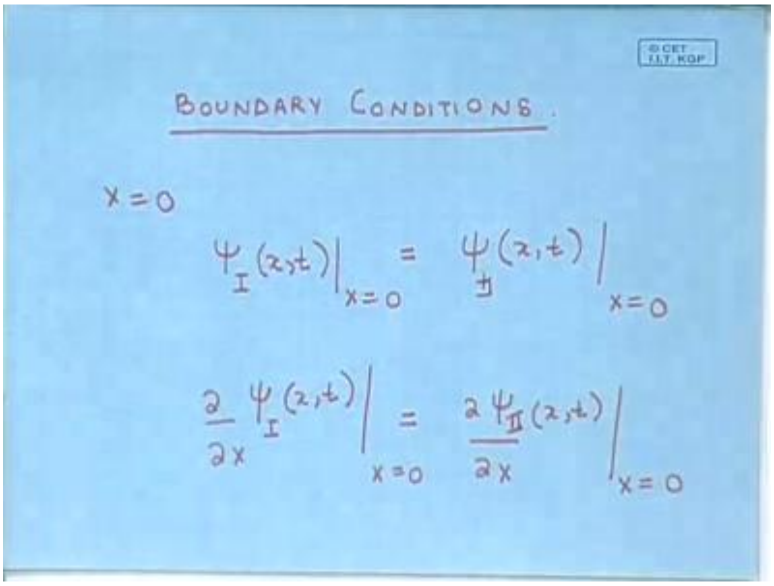

So in order that the second derivative is well defined, you require the first derivative to be continuous at the boundary. It is continuous elsewhere; you also require it to be continuous at the boundary which is why you have to also impose these boundary conditions. So, we now have two boundary conditions which have to impose at all boundaries. The time part is guaranteed to satisfy this. it is the spatial part the spatial part that we have to be concerned about.

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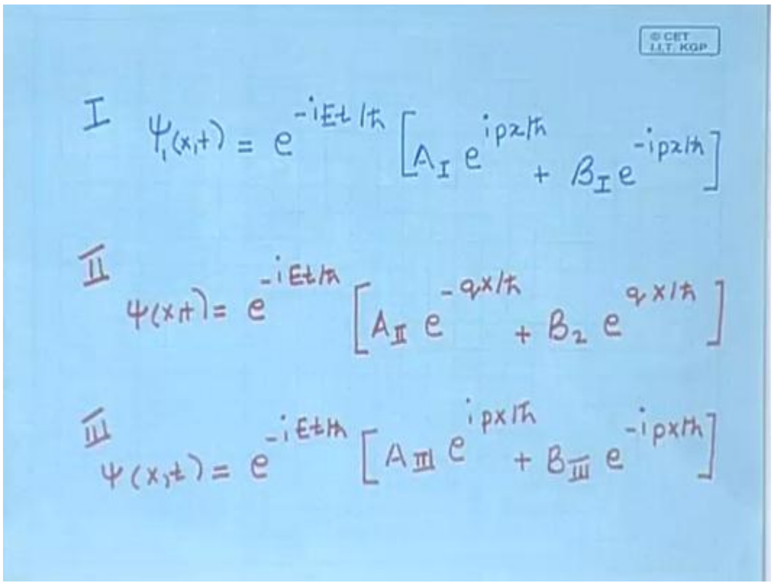

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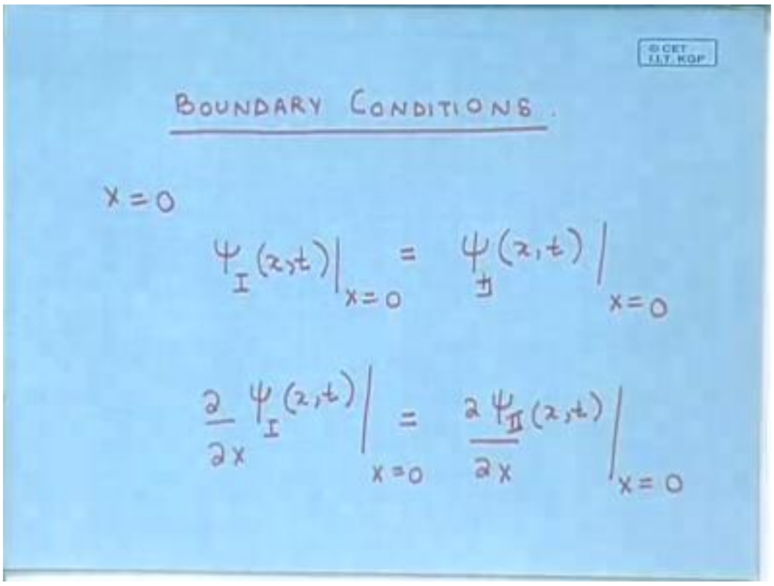

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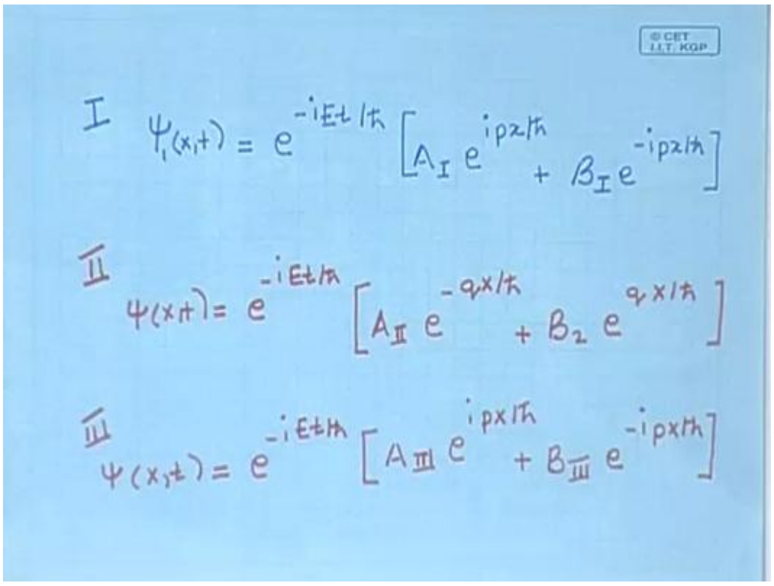

So, we have to impose these boundary conditions, and it is these boundary conditions that are going to tell us, how the coefficients the different on the two different sides are related. You have two different boundary conditions, and these two different boundary conditions are going to allow us. In principle, if I look at the interface between this region and this region. These two different boundary conditions will allow me in principle to eliminate A 2 and write in terms A 1 B 1. It will also allow me to eliminate B 2, because there are two boundary conditions. So, I can eliminate A 2 and B 2, and write them in terms of A 1 and B 1. Similarly, if I look at the boundary between regions two and three, I can eliminate B 3 and A 3 and write them in terms of this. So, finally, I can express the coefficients in all three regions, solely in terms of the coefficients in region one. So, there are two unknown coefficients which are going be there; only two unknown coefficients. All the other coefficients can be expressed in terms of these.

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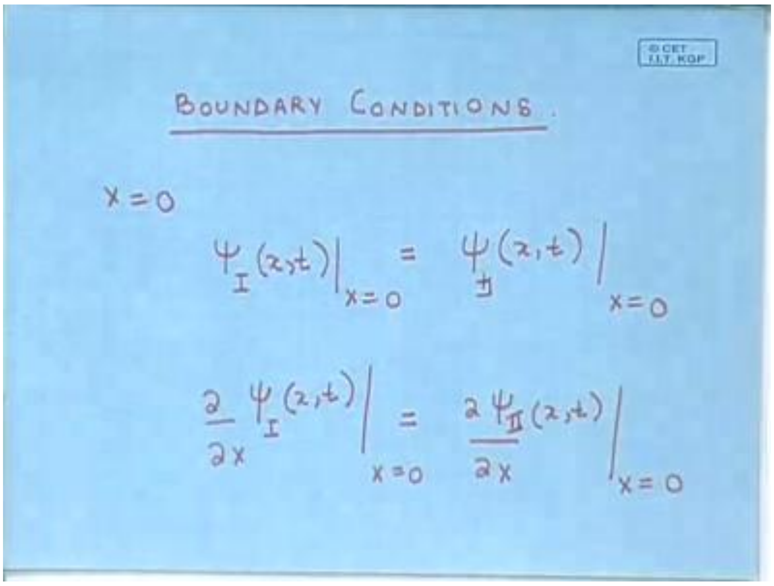

So, this is what we are going to do in the rest of the exercise. We are going to find relations between these coefficients, and then finally, we are going to use this to calculate the probability, that the particle can get through this barrier.

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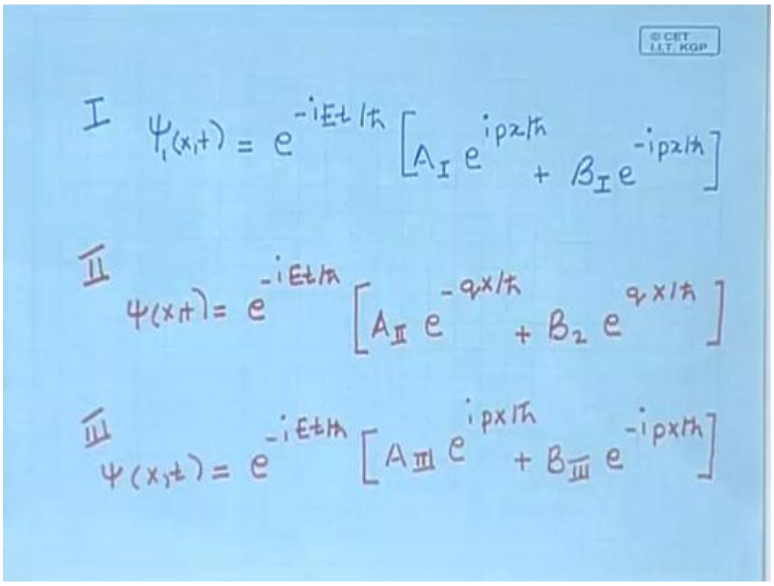

The probability that the particle can get through this barrier is going to be decided by the ratio of this coefficient to this coefficient. In doing this calculation, we are going to make a simplification, and the simplification that we are going to make is as follows.

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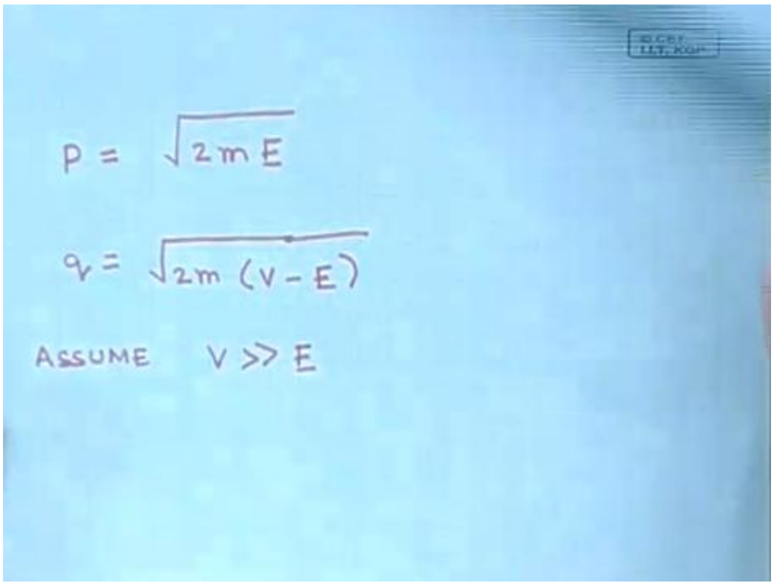

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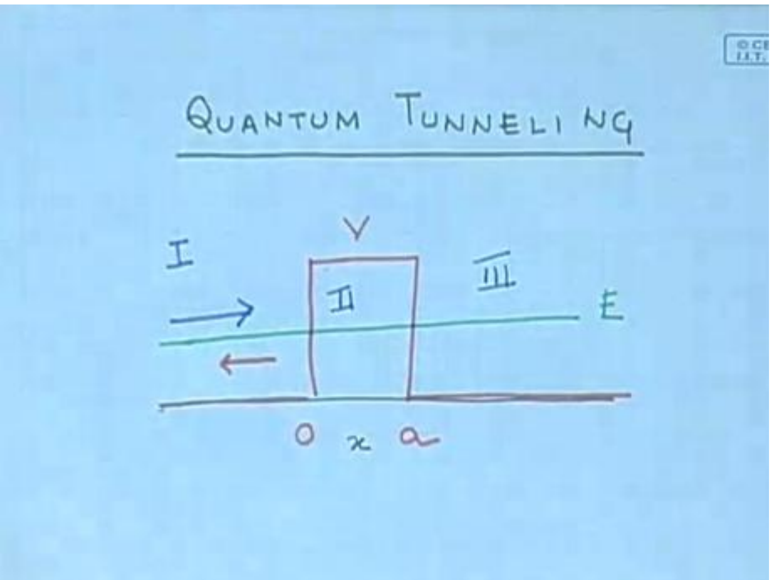

We will assume that the potential v is much larger than the energy. So, we are going to assume the height of the barrier is much larger than the energy of the particle. So, in this picture, we will assume that this v this height of this potential, is much larger as compared to this energy of the particle. We will make this assumption.

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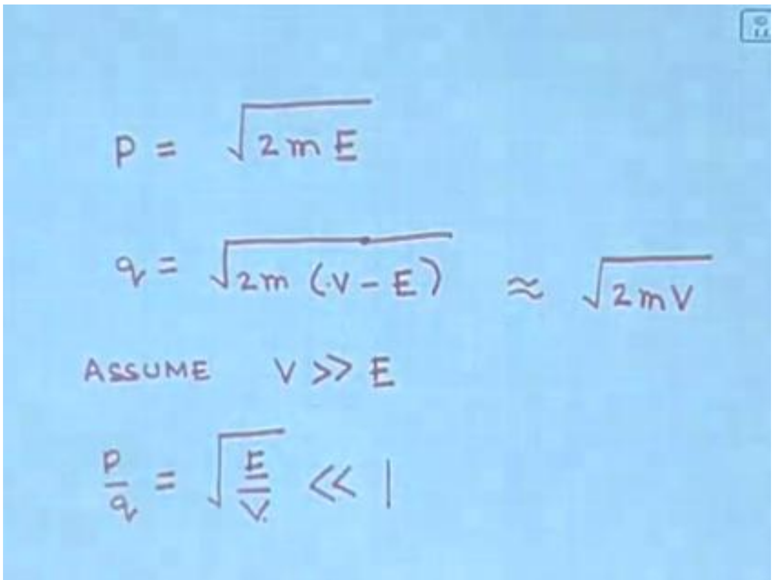

Under his assumption, if you assume this then q becomes approximately equal to v is much larger than E, I can ignore this E, so q will be approximately equal to 2 m v. And the ratio p by q, is now equal to the square root of E by v. And since we have assumed that v is much larger than E this ratio is much smaller than 1. So, in our calculation, we are going to make this assumption throughout. Not that this assumption is essential, to calculate tunneling probabilities etcetera. Tunneling, such tunneling calculations can be done in a very general situation, where the potential energy is comparable to the energy of the particle.

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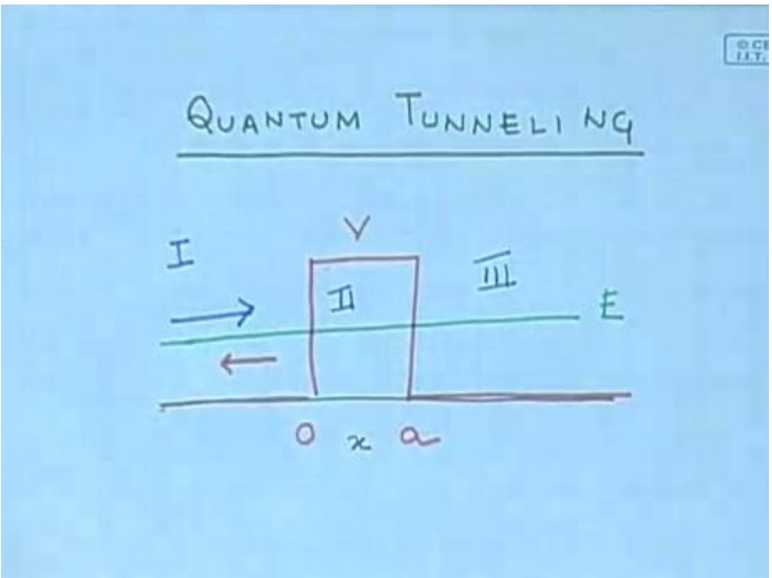

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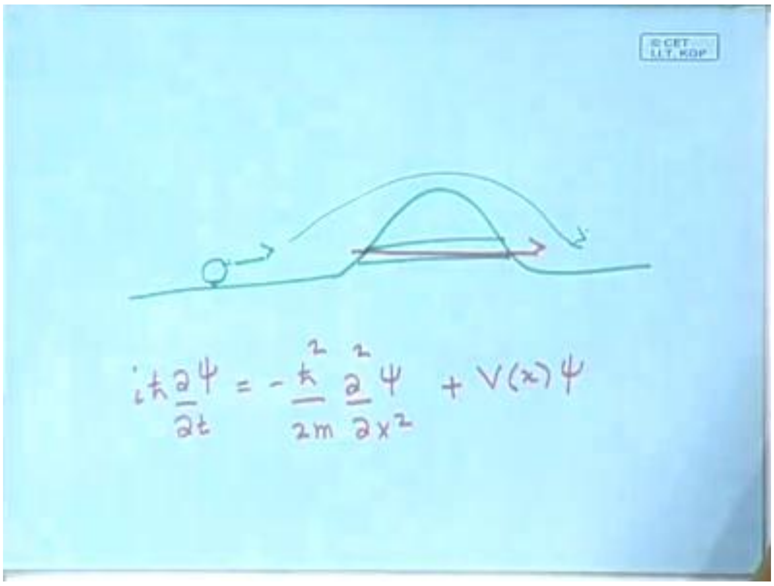

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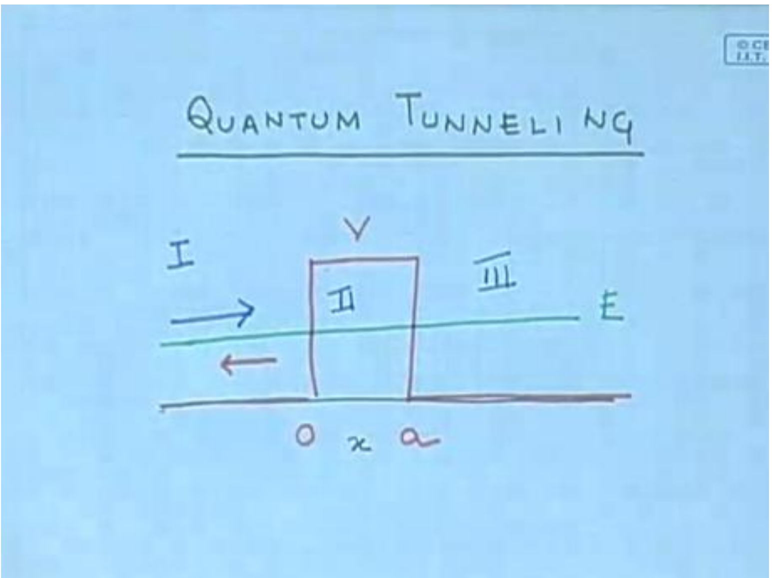

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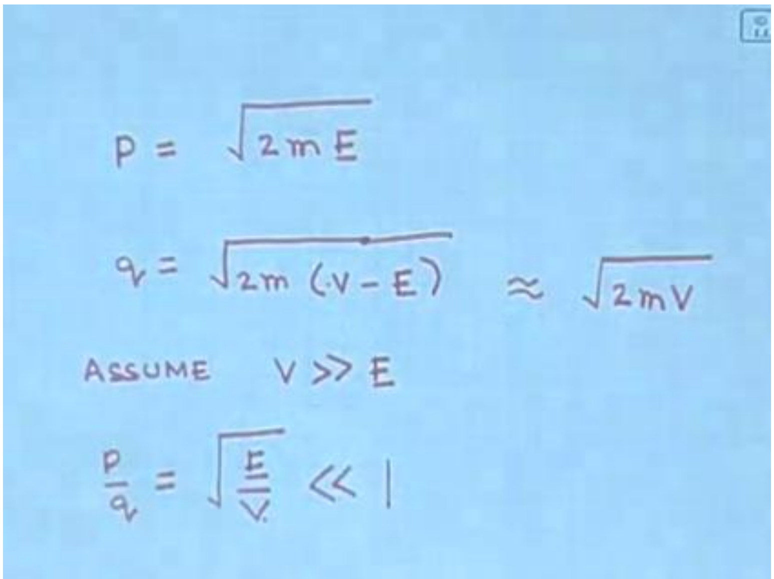

It can also be done in a more general situation where we do not have a step barrier, but we have something more general, which looks like this, and the particle has a smaller energy. But for mathematically simplicity we shall make these assumptions. We shall assume a step barrier, and we shall assume that the energy, is much smaller than the height of the potential. The potential is much larger than the energy of the particles. So, it is a very high barrier, we are going to assume, make these two assumptions. These two assumptions make the calculation a little simpler.

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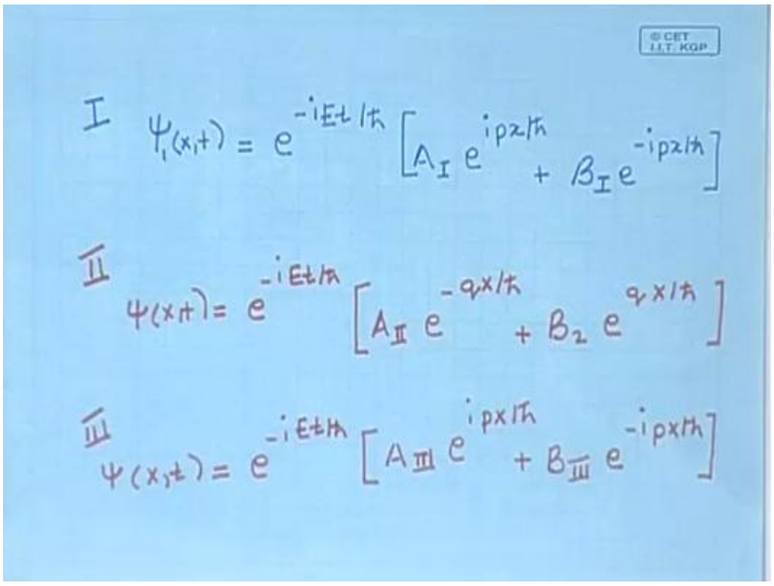

So, with these assumptions we now have to apply the boundary conditions, and that will allow us to relate, these give us relations between these coefficients. So, let us write down the boundary conditions; one for the boundary between region one and two, and another for the boundary between region two and region three. So, let me write down these boundary conditions.

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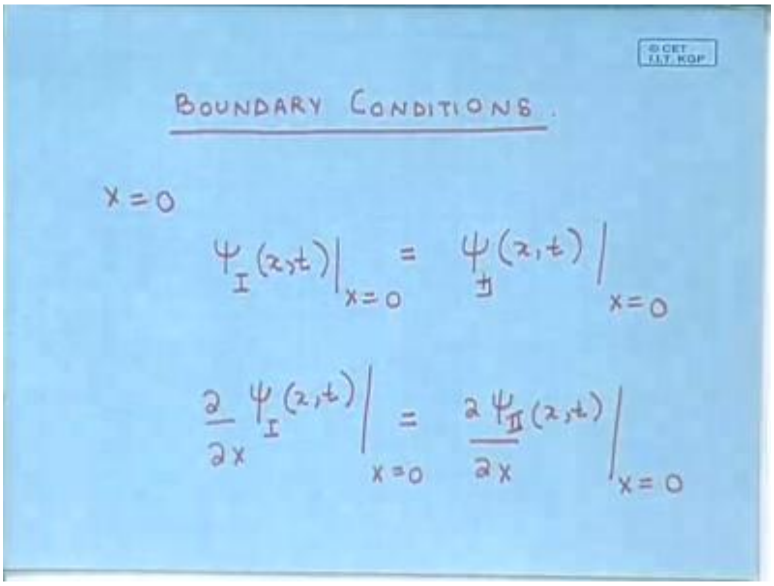

So, we have written the boundary conditions, I shall referred to this an as boundary condition one. I shall refer to this as boundary condition two. So, let us apply these boundary conditions; first to the boundary at x equal to 0. The boundary between region one and two.

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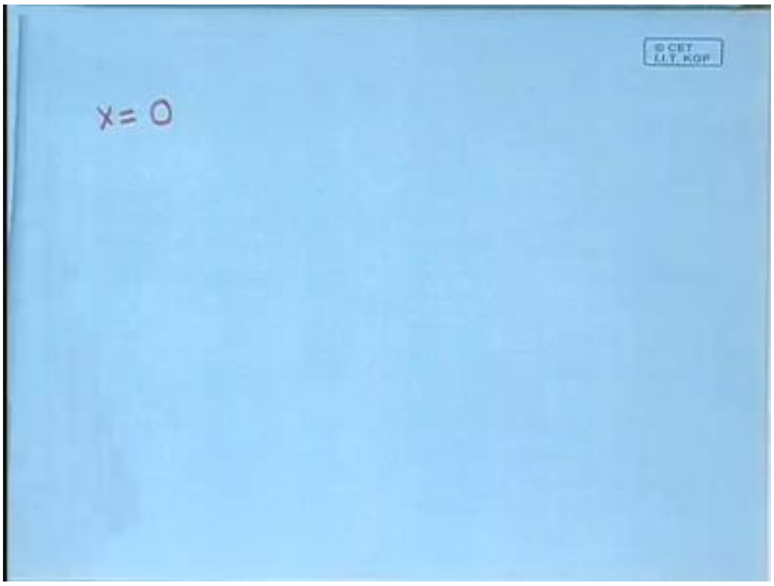

So, we will consider the interface between region one and region two, the boundary at x equal to 0. So, the first boundary condition is, that the wave function should be continuous across the boundary.

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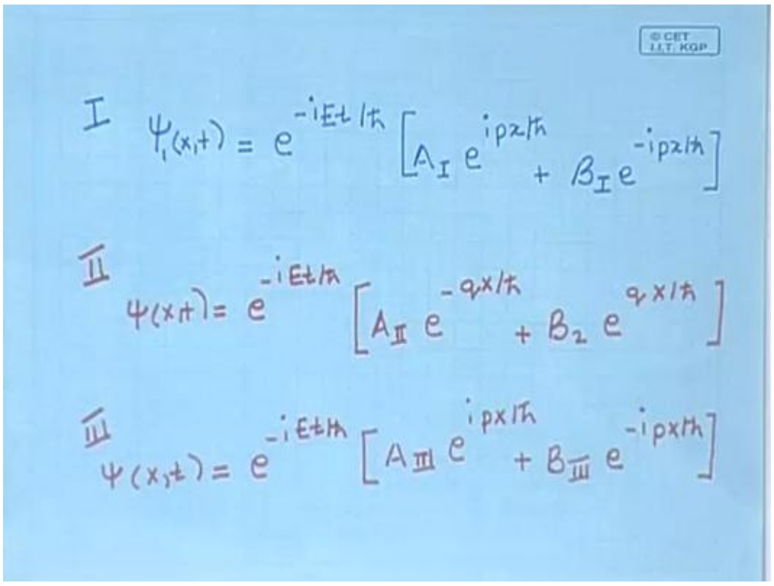

So, the first boundary condition, is that the wave function should be continuous. So, at x equal to 0 psi 1 should be equal to psi 2. We really do not have to bother about the time part, because they are already the same. So, we have to set x equal to 0 here and here. Time part is a same, so it cancels out. So, let us set x equal to 0 in psi 1. So, what it tells us is that A 1. If i set x equal to 0, this becomes A 1 this becomes B 1. So, at x equal to 0 psi 1 is A 1 plus B 1. I am not bothering about the time part, because it is exactly the same for all 1 2 and 3.

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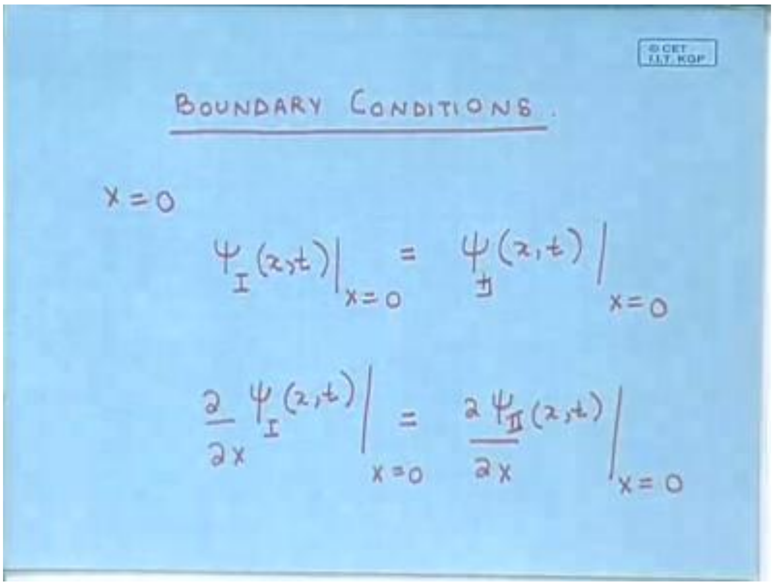

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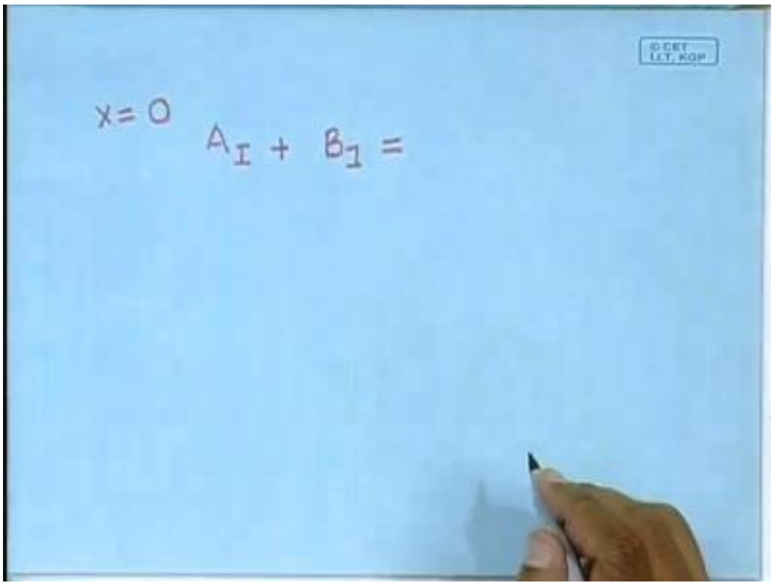

So, on the left hand side, the first boundary condition matching the wave function, at the boundary x equal to 0. On the left hand side, the wave function has a value A 1 plus B 1, and this should be equal to.

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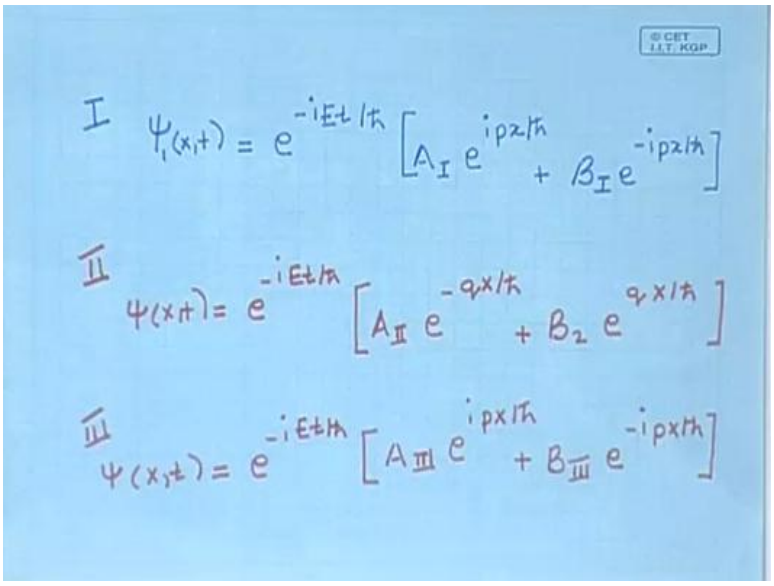

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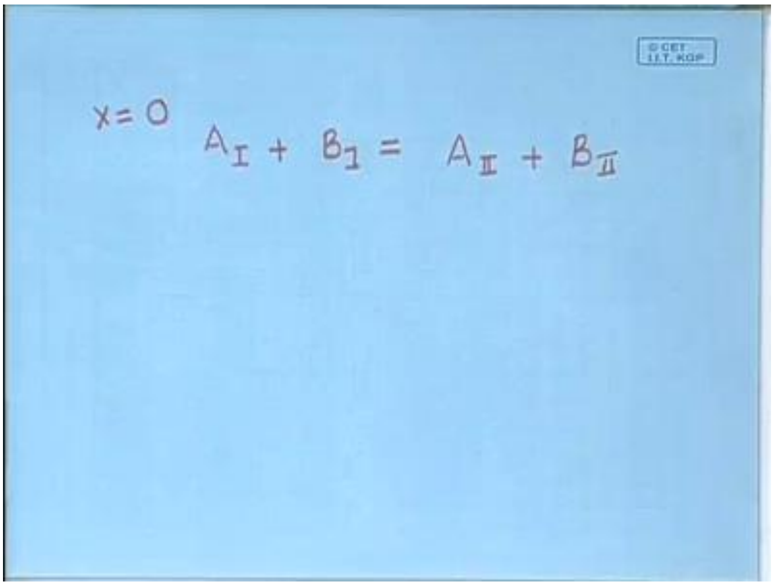

On the right hand side, right hand side is region two. We are considering the boundary between region one and region two. So, on the right hand side at x equal to 0 set x equal to 0 here and here. This term becomes 1, this term also becomes 1, so I have A 2 plus B 2. So, this should be equal to A 2 plus B 2.

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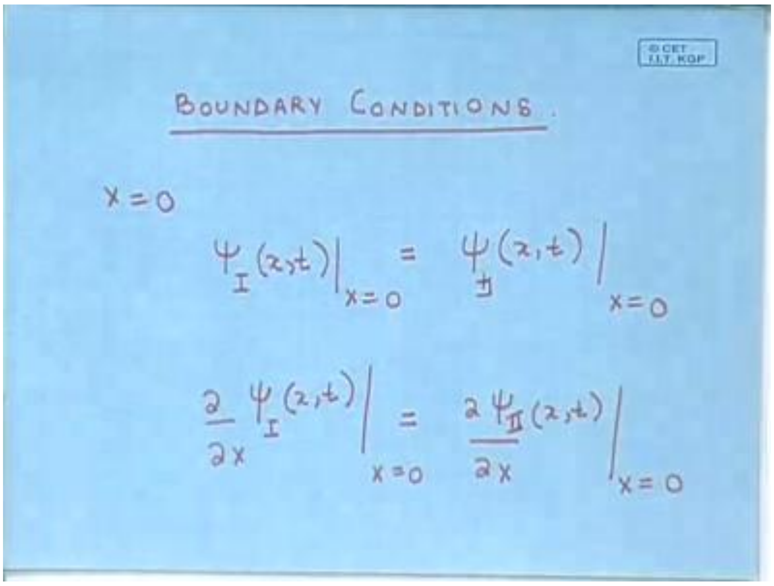

So, this is the first boundary condition, where we have matched the value of the wave function on the two sides. Now let us match the spatial derivatives of the wave function on the two sides.

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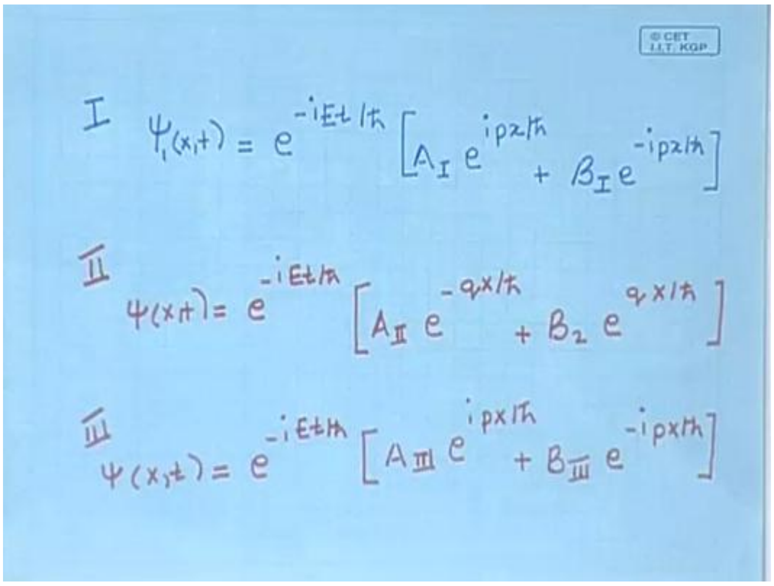

So, we have to take the wave function in region one. Calculate it is spatial derivative, and then set x equal to 0. So, the spatial derivative of this is the same function multiplied by i p divided by h cross. If the spatial derivative of this is the same thing multiplied by minus i p divided by h cross. So, I can say that the spatial derivative of the total, is going to be at x equal to 0. It is going to be i p by h cross. You see I can ignore h cross, because it will occur here also when I differentiate it, so I am not going to bother about the h cross.

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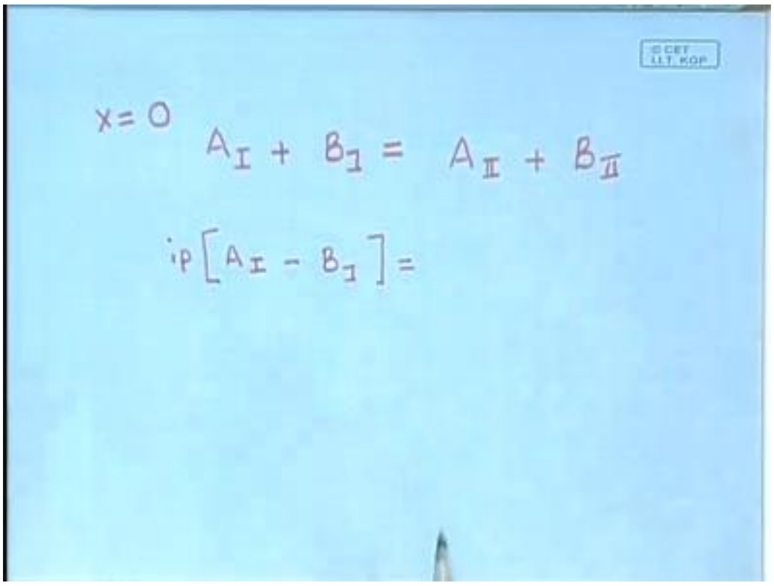

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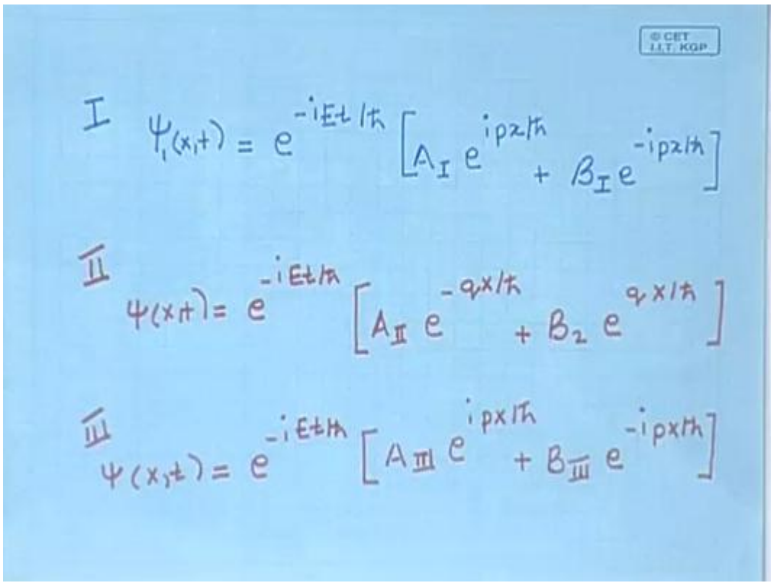

So, the spatial derivative of the left hand side at x equal to 0 is going to be i p A 1 minus B 1 is equal to. The spatial derivative of the right hand side, is going to be. The spatial derivative of this is going to be minus q by h cross into this, and here it is going to be plus q by h cross. So, if I take the minus q common outside I will get A 2 minus B 2; h cross is being canceled out from both here and here.

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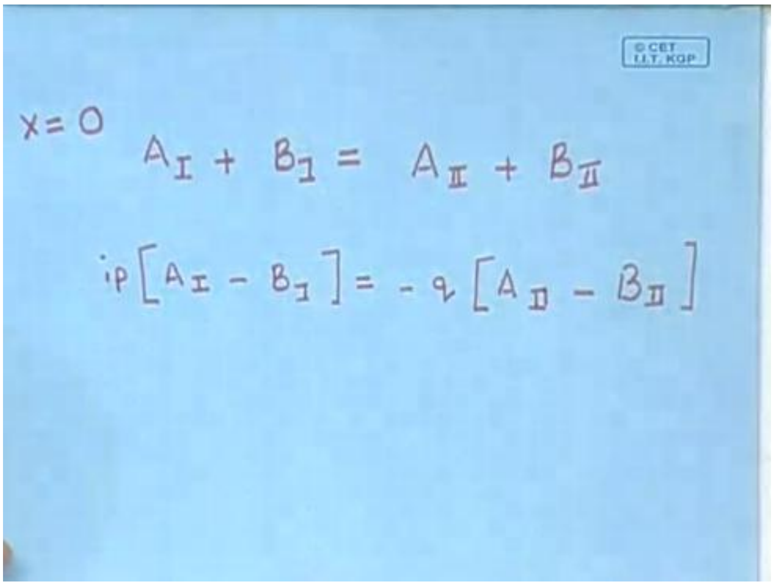

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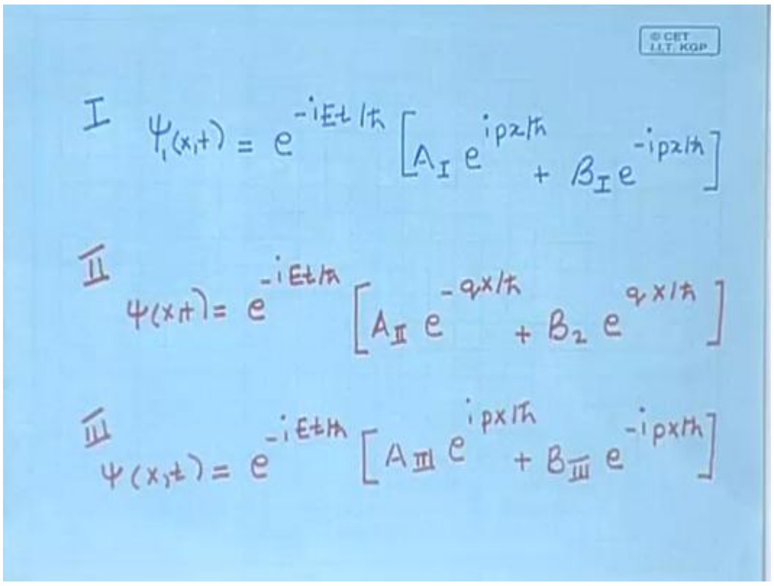

So, this is equal to minus q A 1 minus A 2 minus B 2. So, this is the spatial derivative of the wave function on the right hand side. This is the spatial derivative of the wave function on the left hand side of the boundary between one and two. There will be a difference of minus sign between these two, difference of minus sign between these two. I have taken minus q common outside, after differentiating, so this becomes A 2 minus B

2. I have taken i p common outside after differentiating this. So, this will be A 1 minus B 1, and this is the condition that I get.

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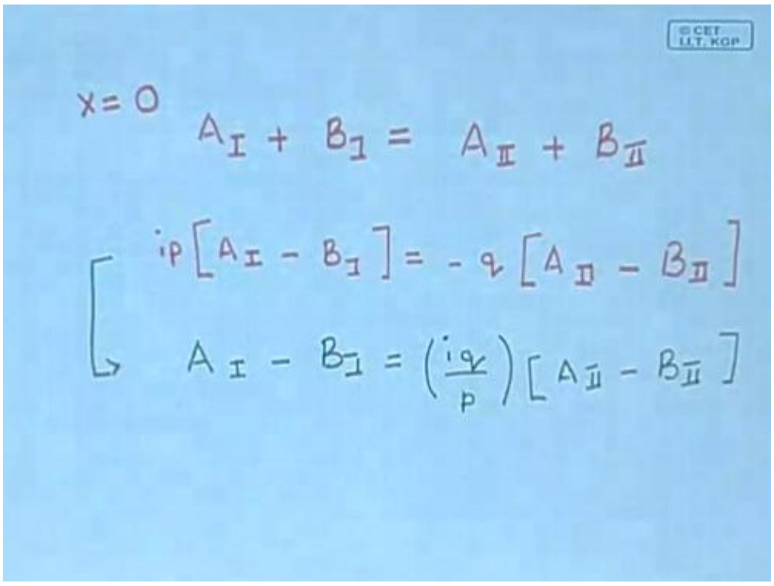

Now we can simplify this expression slightly, and write it as A 1 minus B 1 is equal to. I will divide this by i p minus 1 by i is i. So, this will become i q by p into A 2 minus B 2. So, this gives us relations between A 2 and B 2 with A 1 and B 1, and we could. It is quite obvious that we can solve both of these and get A 2 in terms of A 1 B 1. We could also get B 2 in terms of A 1 B 1, which I had told you we could do, but we will not do this right now. Let us look at the other boundary that we have in this problem.

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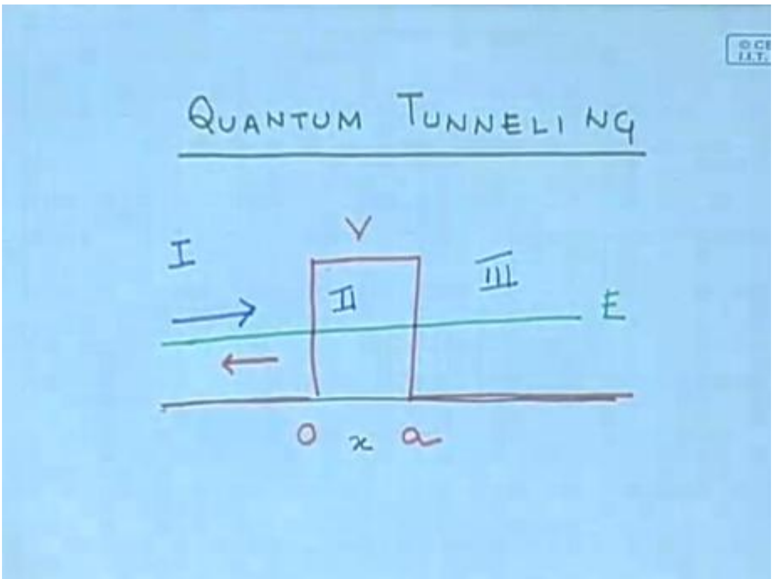

In this problem we have two boundaries one over here, and the other over here. So, let us look at now the boundary between region two and region three. This boundary occurs at x equal to a, the left and side is region two, the right side is region three.

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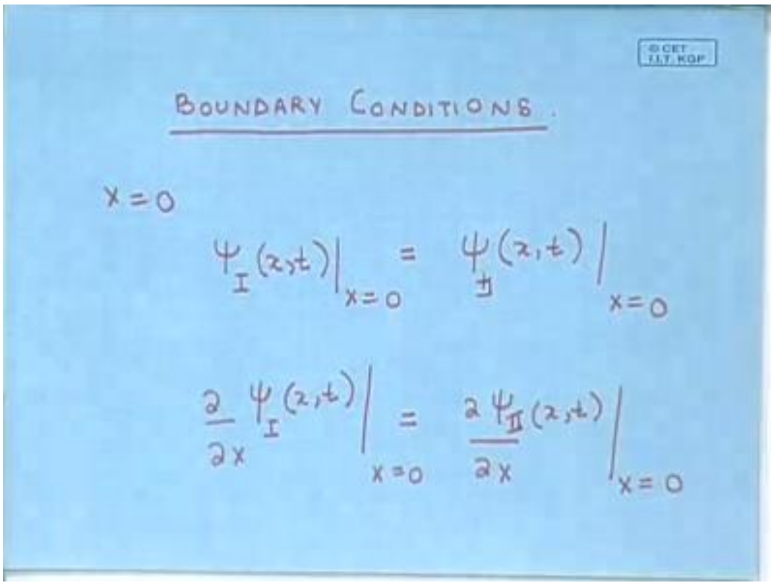

So, at this boundary we first, we will now apply the first boundary condition, the fact that the wave function should be continuous at x equal to A psi 2.

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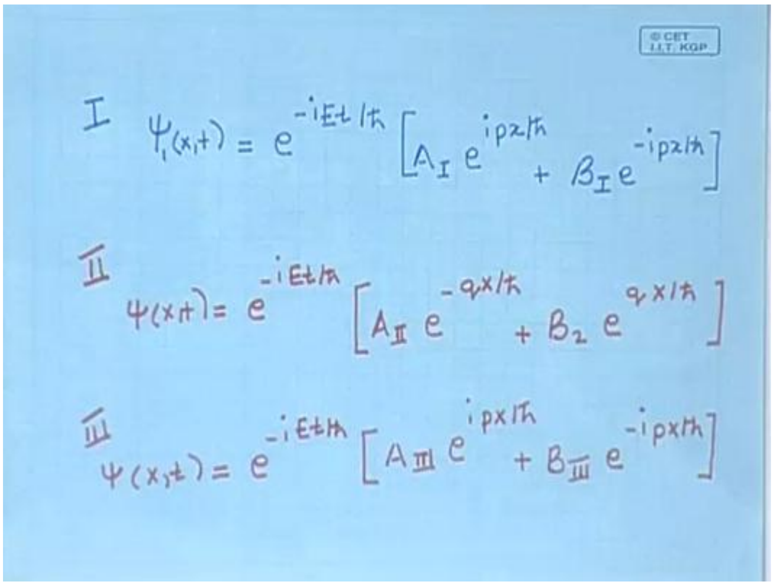

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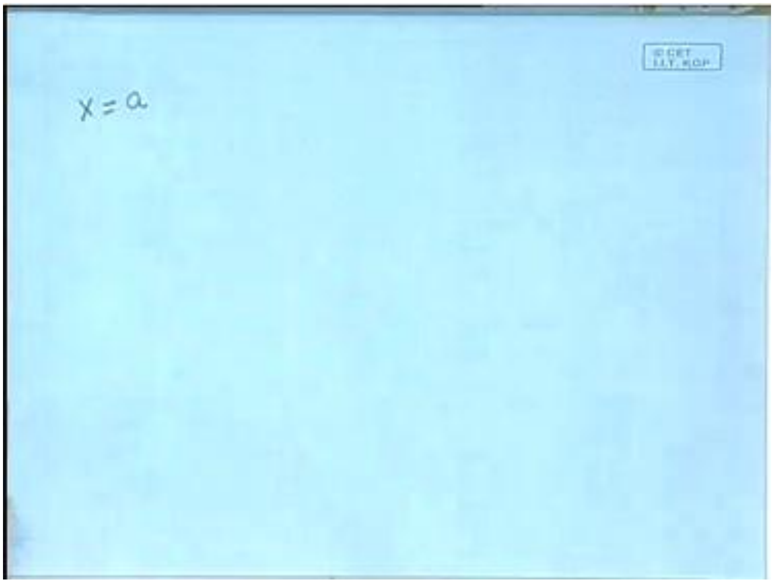

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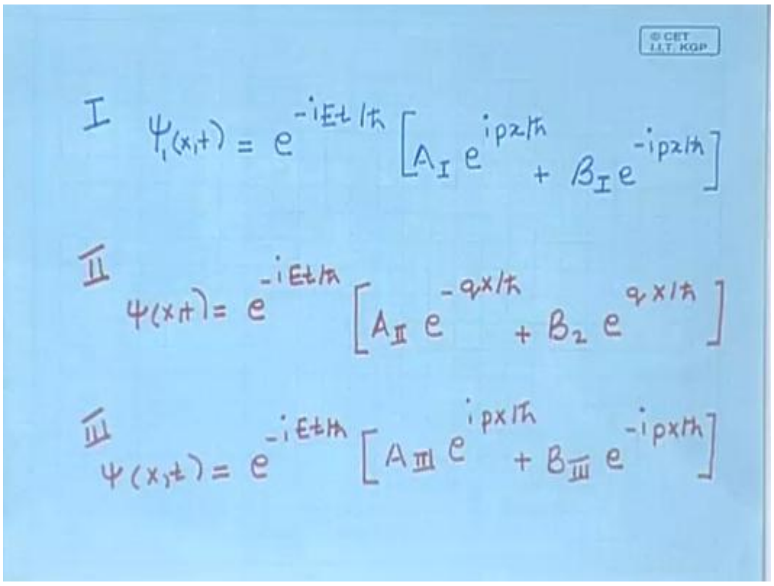

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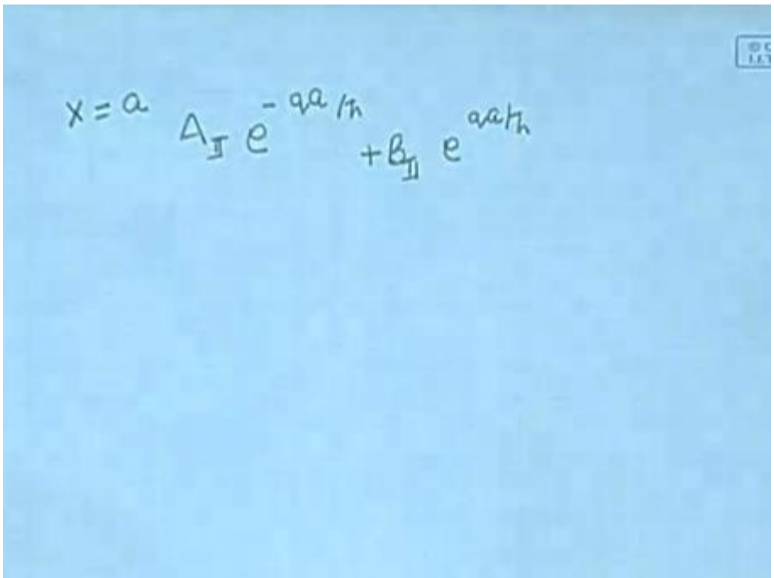

What is the value of psi 2 at x equal to A psi 2 at x equal to A. So, we have to set x equal to A here x equal to A here. So, let me write down, we are looking at the boundary at the interface at x equal to a and matching the boundary conditions, the left hand side. On the left hand side we have A 2 e to the power minus q a by h cross plus B 2 e to the power q a by h cross.

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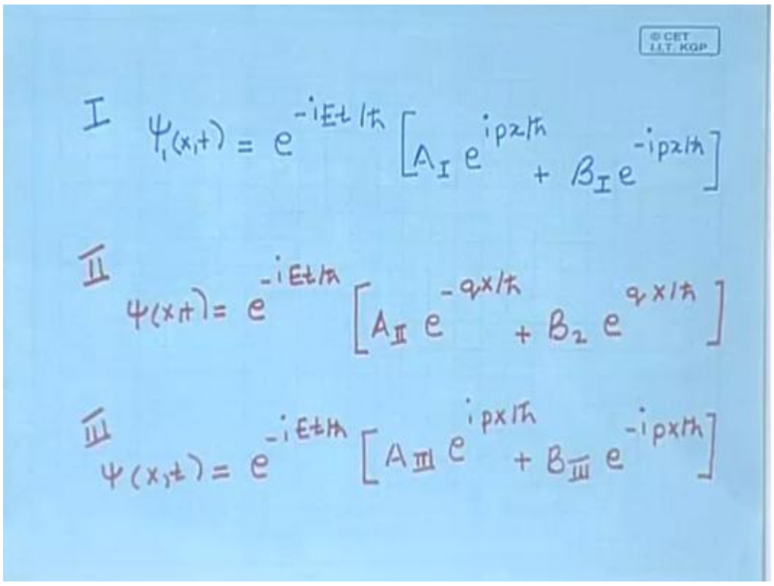

These are the two terms at x equal to A. And this should be equal to the wave function at x equal to A in region three. So, in region three we will have A 3 e to the power i p a by h cross plus this term is 0. So, we really do not have to bother about it.

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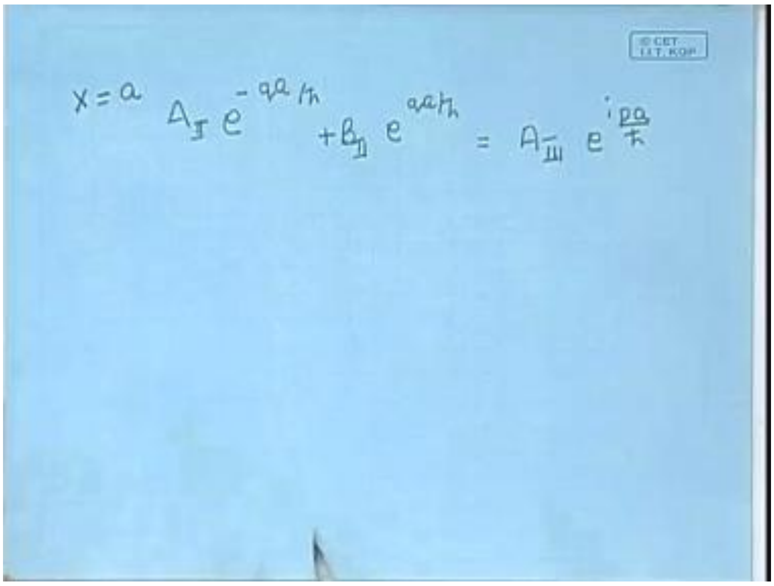

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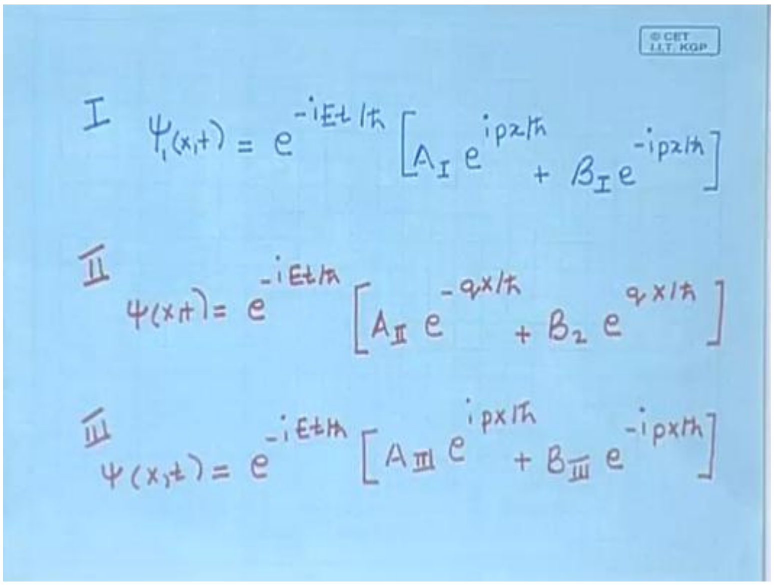

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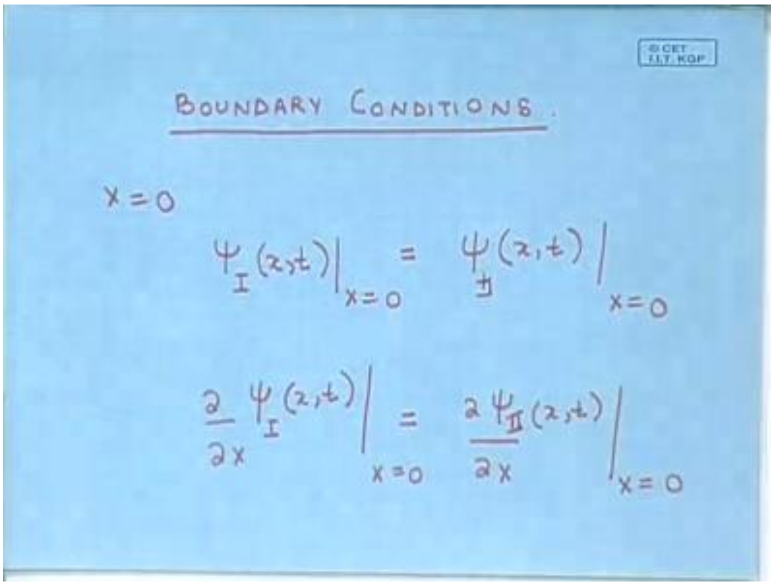

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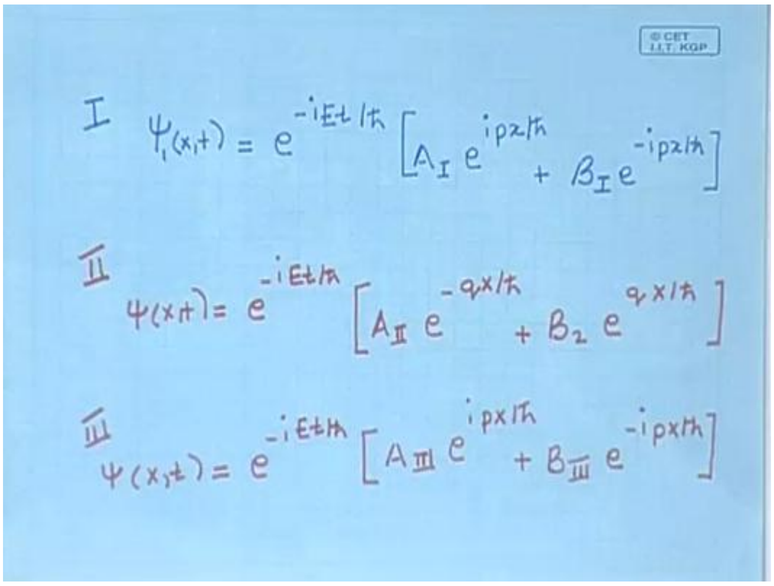

So, this term, this should be equal to A 3 e to the power i p a by h cross. So, we have got applied the first boundary condition at the boundary x equal to a. This is the interface between region two and region three. Let us now apply the second boundary condition, the second boundary condition is the requirement, that the derivatives should be same on the left and the right hand side, so we have to differentiate this and set x equal to A. If I differentiate this I will get a minus q here, I will get plus q here h cross will be there both here and here, so I can cancel it out.

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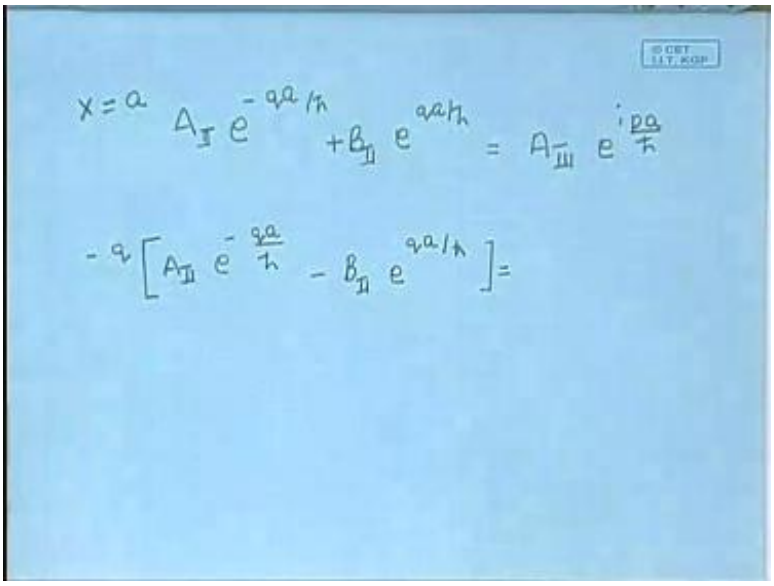

So, let me take this minus q common outside, ad what I get when differentiate is, minus q A 2 e to the power minus qa by h cross minus B 2 e to the power q a by h cross is equal to.

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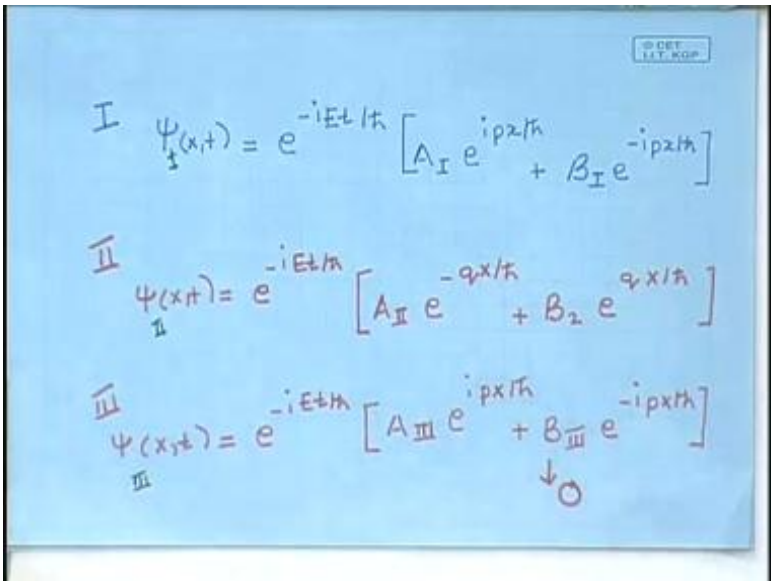

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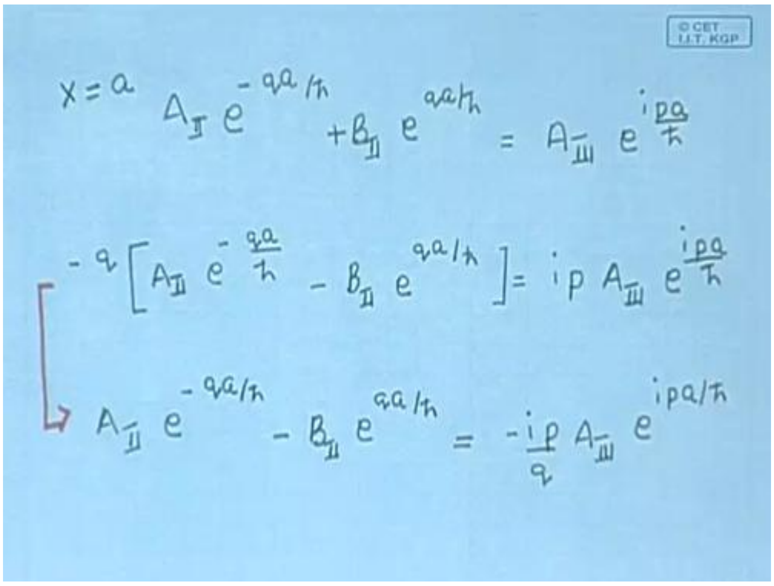

If I differentiate the right hand side. If I differentiate this wave function in region three. So, this I should denote two here. In region three if I differentiate the wave function, I have only this part. I will pick up a factor of i p. So, what this should be equal to i p. Then I have A 3 e to the power i p a by h cross. So, we have now got the two boundary conditions at the second boundary x equal to a, and we can simplify this one step. So, let me simplify it one step. If I simplify it then what I get is, A 2 e to the power minus q a by h cross minus B 2 e to the power q a by h cross should be equal to should be equal to minus i p by q A 3 e to the power i p a by h cross. So, this has been obtained from this. It is exactly the same just that we have taken this factor of minus q on to the other side. So, we now have two boundary conditions at the boundary x equal to a, and again we can eliminate the A 3 here and write it in terms of these.

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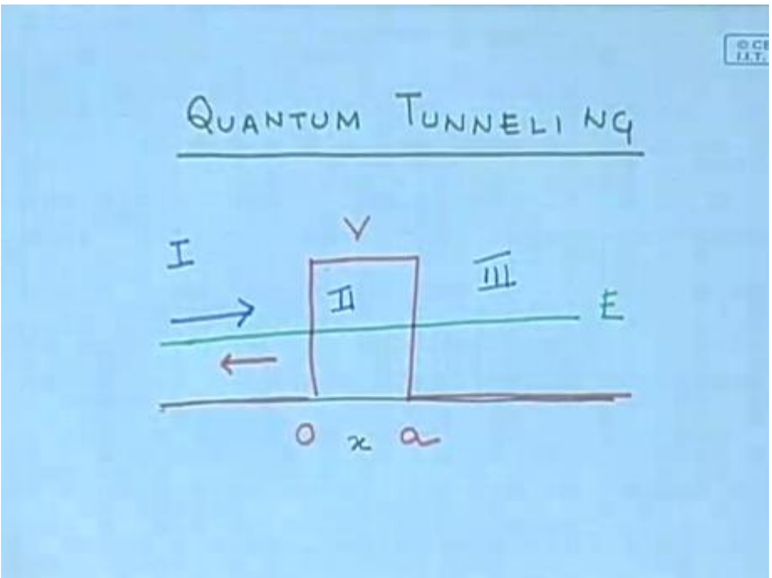

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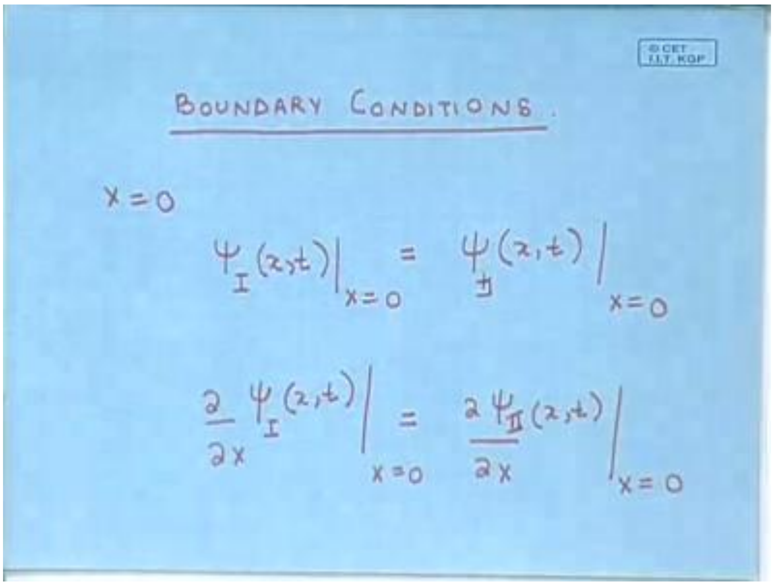

$$\psi_I(x,t)\Big|_{x=0} = \psi_{II}(x,t)\Big|_{x=0}$$

$$\frac{\partial}{\partial x}\psi_I(x,t)\Big|_{x=0} = \frac{\partial \psi_{II}(x,t)}{\partial x}\Big|_{x=0}$$

So, what we have done now, is that we have, until now is that we have a free particle wave function here. We have the solution in this region, and we have the solution in this region. We have to match the boundary conditions at this boundary and this boundary. The boundary conditions are two. The wave function should be continuous; its first derivative should be continuous.

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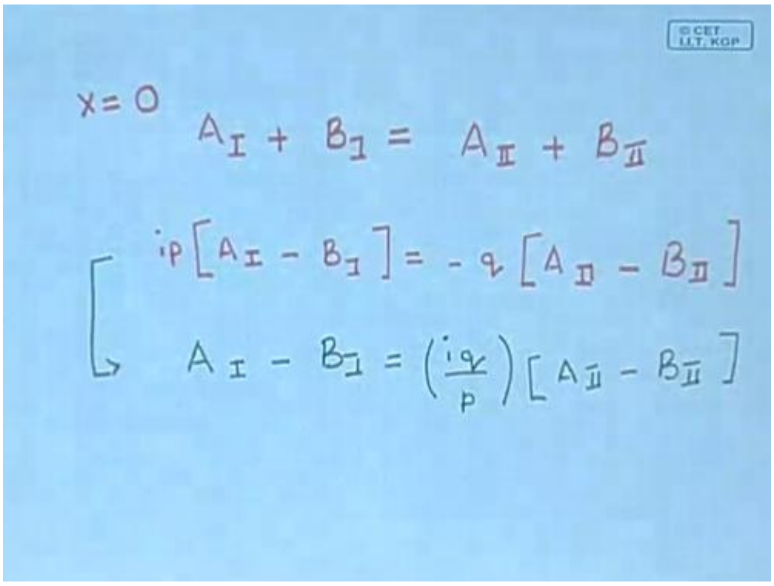

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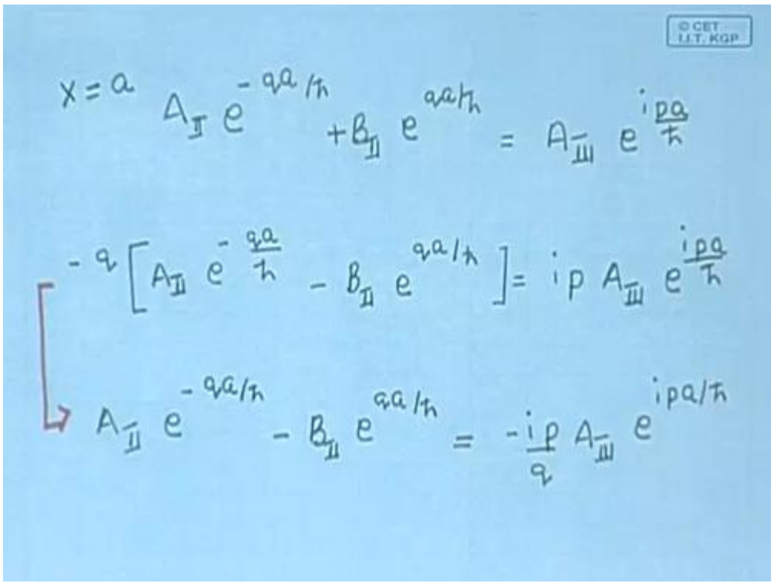

At the boundary x equal to 0 we have these two conditions. At the boundary x equal to a, we have these two conditions. In tomorrow’s lecture, I will take these boundary conditions and use them to calculate the probability, that the particle can tunnel through this potential barrier. That is what I am going to take up in tomorrow’s lecture.