Physics I : Oscillations and Waves
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Lecture - 41
Potential Well (Contd.)

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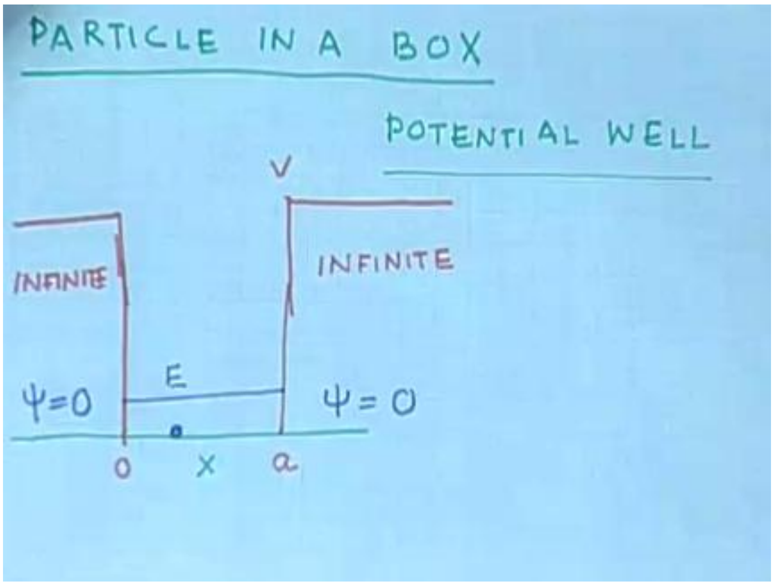

Good morning in the last lecture, we were discussing a situation where we have a particle which is free to move along the x axis, and the particle is constrained in the region zero x equal to 0 to x equal to a by two potentials, which are infinitely high. So, these potentials on the two sides of this region x equal to a to x equal to 0. The potential at x equal to 0, the potential to the left of x equal to 0, the potential rises to infinity. I have not been able to show that, but I have written here infinite. So, this potential is infinitely high, and again beyond x equal to a, the potential is infinitely high. As a consequence the particle is constrained in the region x equal to 0 to x equal to a, and this is often referred to as a particle in a box. The infinite potential represents the walls of the box, which do not allow the particle to go outside. It is also referred to as a particle in a potential well, and this is an infinitely deep potential well. One could also consider a finite potential well. The mathematics is a little more complicated. We shall not be considering it here. We shall discuss only the infinite potential well.

Now we have seen that the main. Let me just also remind you what the main difference is, if the potential well were finite if the potential well were finite. We are considering a situation where the energy of the particle is less than the height of the potential. So, the energy is here, the potential is much larger, it is here. And we have seen that if I had a finite potential here. The wave function of this particle would penetrate, to some extent into the region, where the potential is larger than the energy. But if the potential is infinitely high, if the difference between V and E is infinitely large, is infinite. Then the penetration is, penetration depth essentially goes to zero, or what we can say is, that the wave function goes to zero at the two boundaries, which is what I have shown over here also. So, psi is zero at this boundary, psi is also zero at this boundary. We have to solve for psi in the region, between zero to a, where the particle is essentially free. So, in this region there is no external potential, and the potential is suddenly infinite at these two boundaries.

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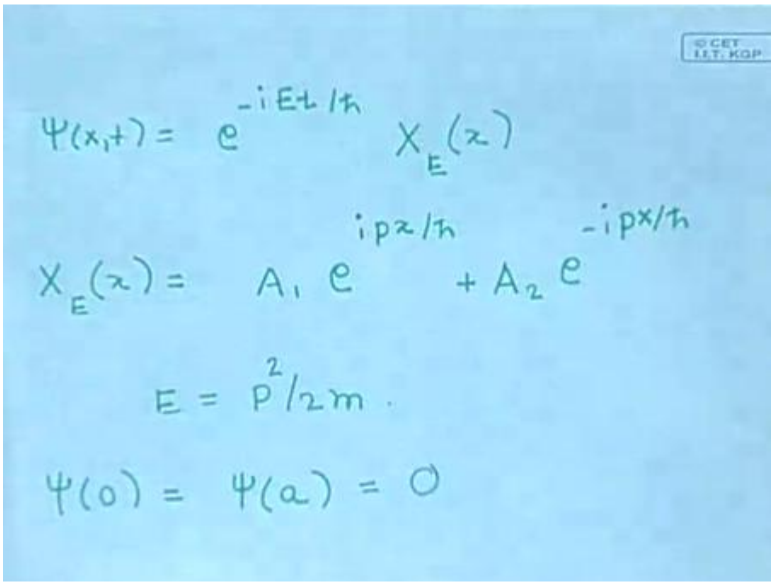

$$\psi(x,t) = e^{-iEt/\hbar} X_E(x)$$

$$X_E(x) = A_1 e^{ipx/\hbar} + A_2 e^{-ipx/\hbar}$$

$$E = p^2/2m .$$

$$\psi(0) = \psi(a) = 0$$

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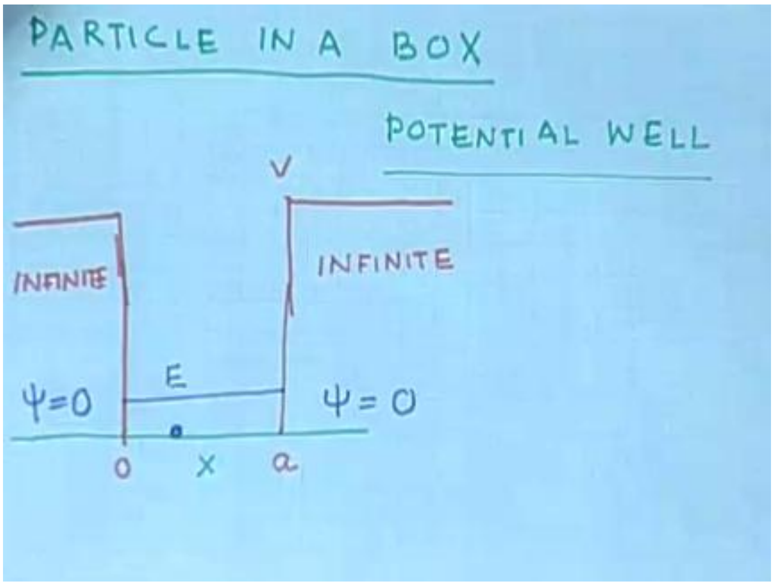

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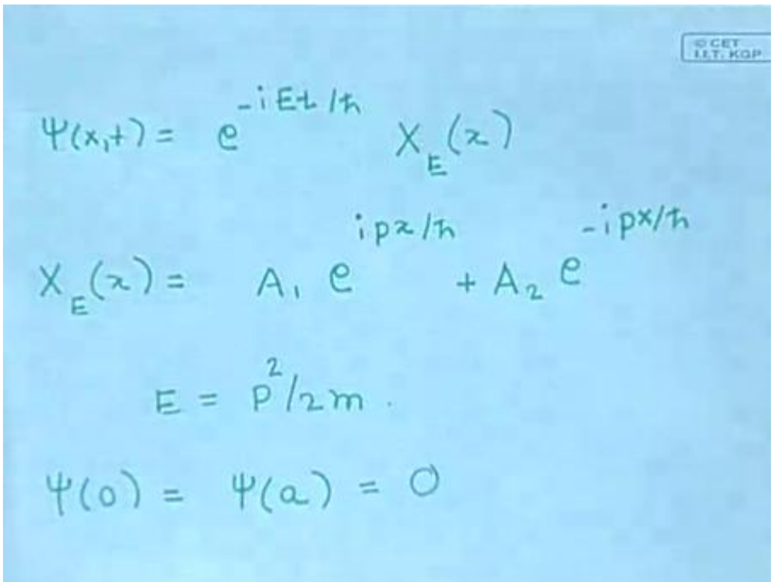

$$\psi(x,t) = e^{-iEt/\hbar} \chi_E(x)$$

$$\chi_E(x) = A_1 e^{ipx/\hbar} + A_2 e^{-ipx/\hbar}$$

$$E = p^2/2m.$$

$$\psi(0) = \psi(a) = 0$$

Now, we wrote down the wave function, and we have. As I have told you many times that in a situation like this, where I have a static potential which does not depend on time. The solution can be written as a product of a function of time, and the time dependence is e to the power minus i Et by h cross. And then we have this special part, which depends on the value of the energy. So, the time dependent part is fixed, and for any static potential you will have a time dependent part like this. And one can now superpose such different solutions with different values of E, to get a general solution. We are looking at a particular value of the energy. And the spatial part has to now be worked out.

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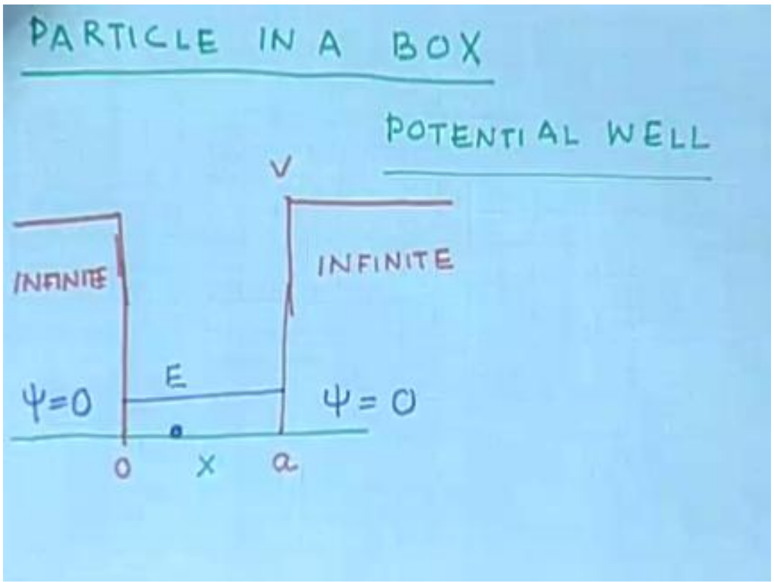

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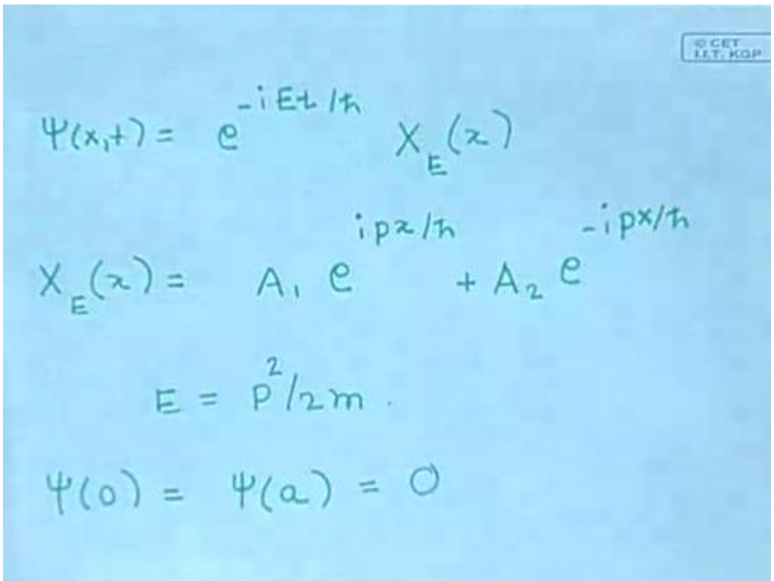

$$\psi(x,t) = e^{-iEt/\hbar} \, X_E(x)$$

$$X_E(x) = A_1 e^{ipx/\hbar} + A_2 e^{-ipx/\hbar}$$

$$E = p^2/2m .$$

$$\psi(0) = \psi(a) = 0$$

Now in the region where we are trying to solve for the wave function, the particle be is essentially free, and we have already worked out what a free particles, wave function is, and there are two possibilities; one is e to the power i p x by h cross, the other is e to the power minus i p x by h cross. And these two possibilities can have different amplitudes A 1 A 2. And this constant p is related to the energy of the particle E, as E is equal to p square by 2 m, this constant e is the momentum of the particle. So, there it is a superposition of two momentum states; one with plus p and with minus p. So, this is the spatial part of the wave function inside the potential well. And the wave function the product of these two, or essentially the x depend the spatial dependence, has is such, that it has to go to zero at both the left boundary and the right boundary.

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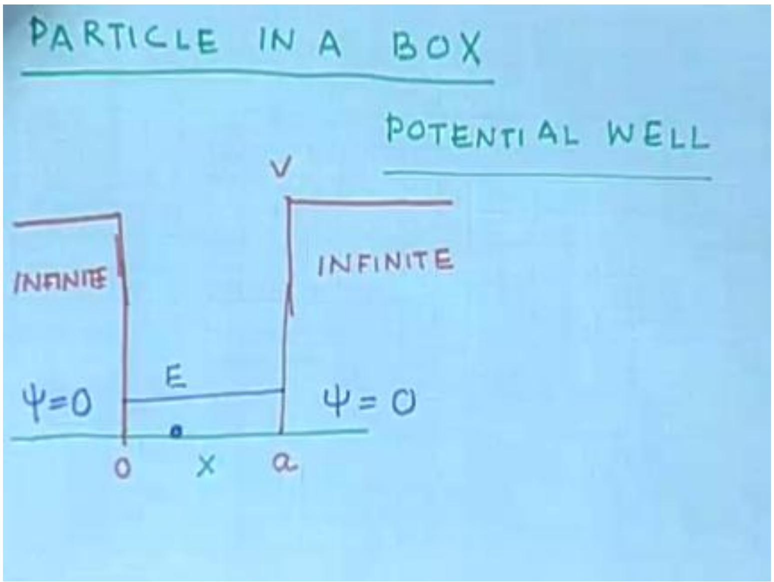

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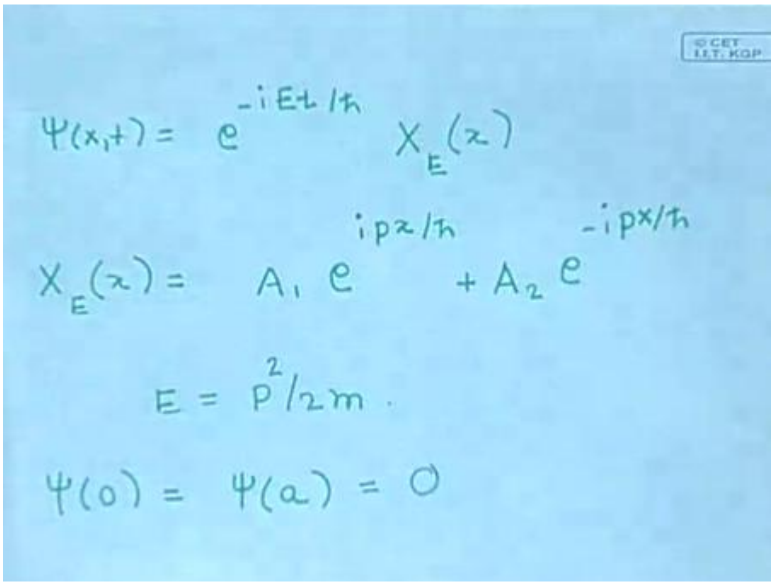

$$\psi(x,t) = e^{-iEt/\hbar} \, X_E(x)$$

$$X_E(x) = A_1 e^{ipx/\hbar} + A_2 e^{-ipx/\hbar}$$

$$E = p^2/2m.$$

$$\psi(0) = \psi(a) = 0$$

So, we have to find super positions of two such oscillating exponentials, which vanish at x equal to zero and x equal to a. we next applied the boundary condition, that this has to vanish at x equal to 0. And the requirement that this wave function has to vanish at x equal to 0. So, we just put x equal to 0 here and here. It tells us that A 1 plus A 2 must be equal to 0 which is what I have written here that A 1 plus A 2 is the value of this spatial part at x equal to 0.

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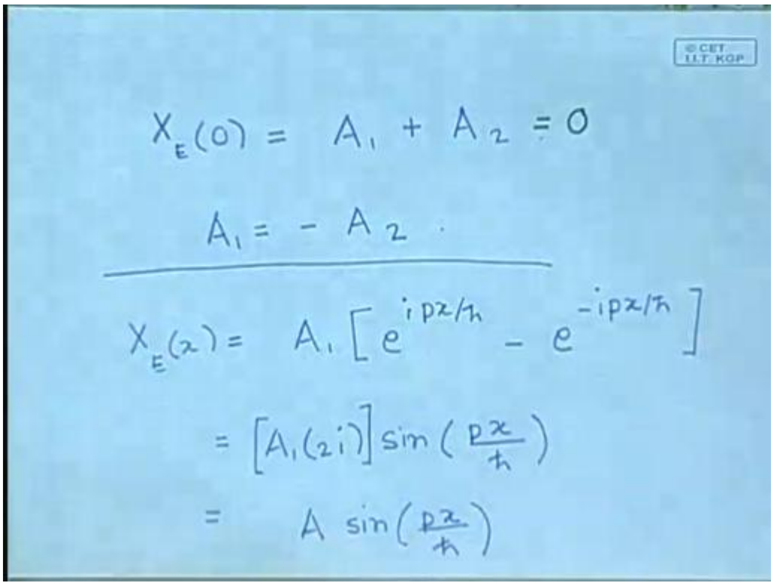

$$X_E(0) = A_1 + A_2 = 0$$

$$A_1 = -A_2 .$$

$$X_E(x) = A_1 \left[e^{ipx/\hbar} - e^{-ipx/\hbar} \right]$$

$$= \left[A_1 (2i) \right] \sin\left(\frac{px}{\hbar} \right)$$

$$= A \sin\left(\frac{px}{\hbar} \right)$$

And this has to be equal to 0; that is the boundary condition. On the left boundary x equal to 0 and it tells us that these two constants A 1 and A 2 should be equal and opposite, equal in magnitude and opposite in signs, so A 1 is minus A 2. So, I can now write these spatial dependence as A 1 e to the power i p x by h cross minus e to the power minus i p x by h cross. And these two functions, when I subtract them I get 2 i into sin p x by h cross. And what I could do is, I could now identify another constant A as 2 i into A 1. So, this is a different constant these constants A 1 and A etcetera can be real or imaginary complex in general. So, we can redefine the constant and call it A. So, A is the earlier constant A 1 into 2 i. So, having applied one of the boundary conditions, having applied the boundary conditions at this end.

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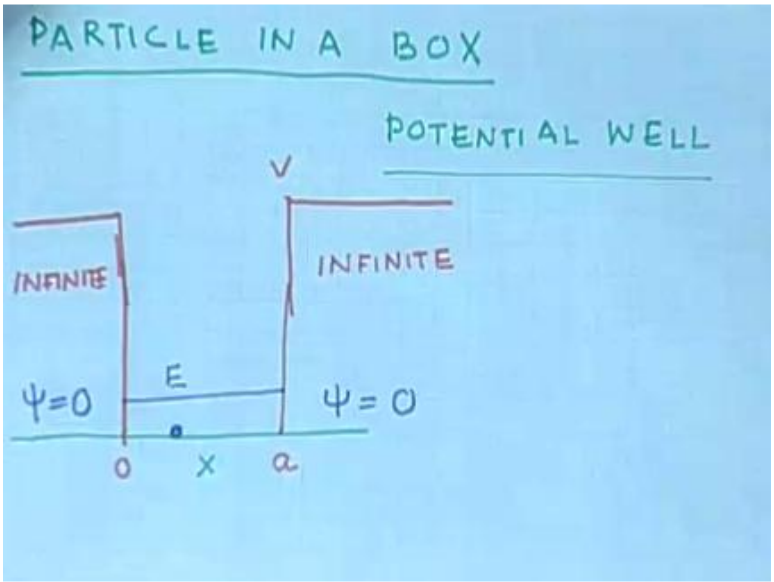

PARTICLE IN A BOX

POTENTIAL WELL

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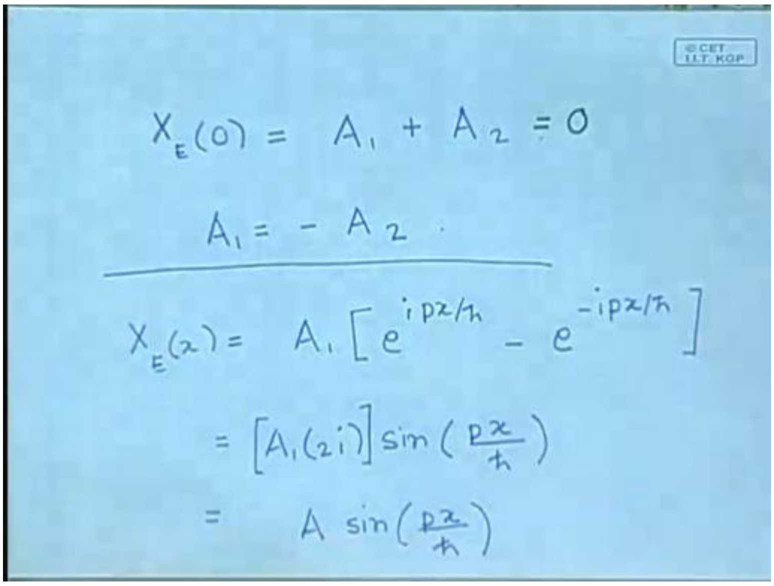

$$X_E(0) = A_1 + A_2 = 0$$

$$A_1 = -A_2$$

$$X_E(x) = A_1\left[e^{ipx/\hbar} - e^{-ipx/\hbar}\right]$$

$$= \left[A_1(2i)\right]\sin\left(\frac{px}{\hbar}\right)$$

$$= A\sin\left(\frac{px}{\hbar}\right)$$

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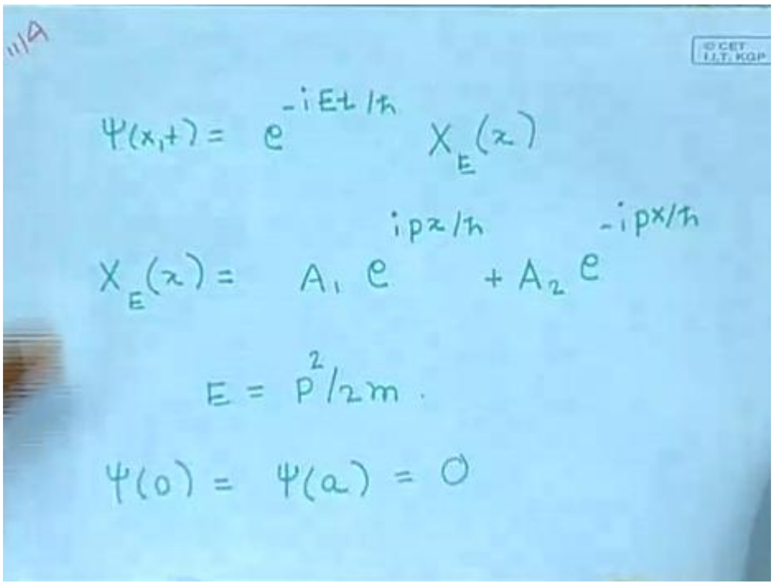

$$\Psi(x,t) = e^{-iEt/\hbar} X_E(x)$$

$$X_E(x) = A_1 e^{ipx/\hbar} + A_2 e^{-ipx/\hbar}$$

$$E = p^2/2m .$$

$$\Psi(0) = \Psi(a) = 0$$

We see that the wave function inside the potential, inside that potential, well infinitely deep potential well or inside the box, is of this kind where this function is sin p x by h cross. Let us now apply the second boundary condition.

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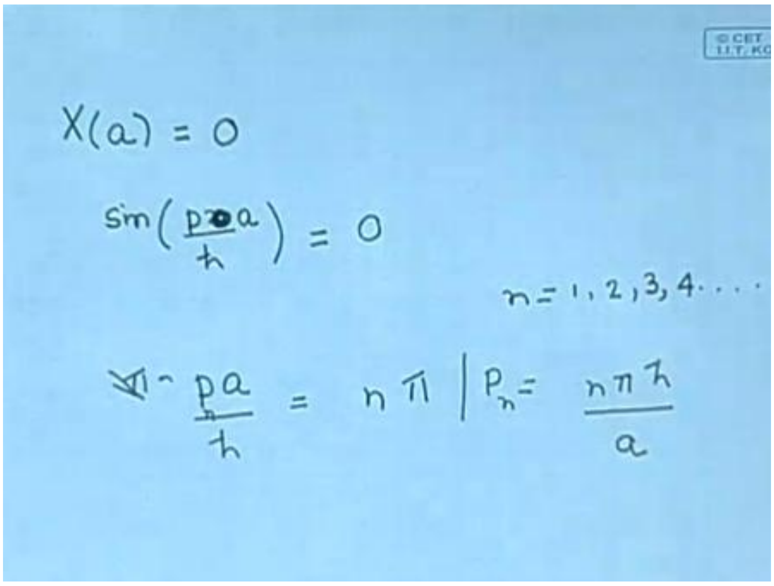

$$X(a) = 0$$

$$\sin\left(\frac{pa}{\hbar}\right) = 0$$

$$n = 1, 2, 3, 4 \ldots$$

$$\frac{pa}{\hbar} = n\pi \quad \Big| \quad p_n = \frac{n\pi\hbar}{a}$$

The second boundary condition is, that the wave function, the spatial dependence of the wave function has to be such, that it has to vanish at x equal to a. So, we have already seen that this has to be sin p x by h cross. So, the condition this boundary condition tells us that sin p a by h cross at the x equal to a, this has to vanish. Now, if you want sin p a by h cross to vanish at. So, if this has to vanish. It essentially tells us that, we know that the sin function vanishes whenever the argument, is an integer multiple of pi. So, this function is going to vanish, whenever this condition that p a by h cross is equal to an integer multiple n into pi. If this is condition is satisfied, this wave function is going to vanish at the boundary x is equal to a. Or what we can say is that corresponding to every integer there is a different value of p which I can denote by p n, such that p n is equal to n pi h cross by a. So, corresponding to every integer n cannot be 0. If n were 0, then the constant p would also be 0, the wave function would also be 0, which is a trivial solution which does not have any physical significance. So, we have n taking values the 1 2 3 4 etcetera. And we have a different solution for each such integer value.

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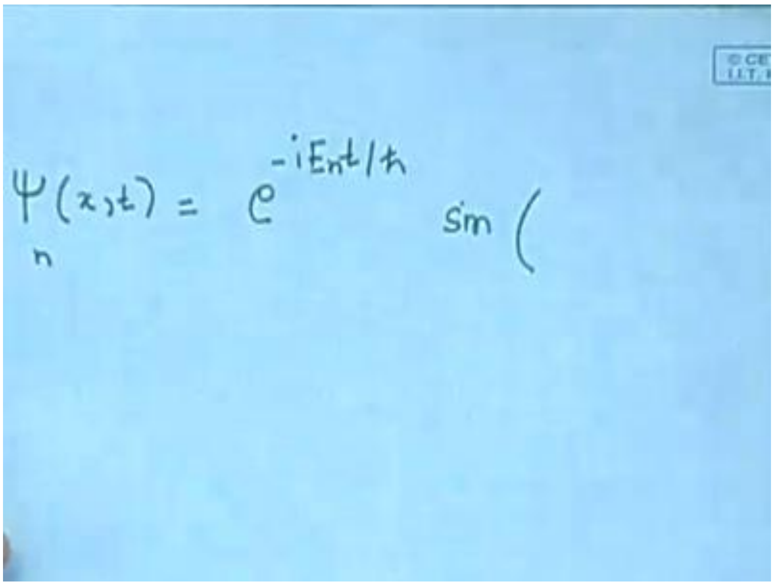

So, let me write down the general solution first. So, the general solution, which satisfies both the boundary conditions that it has to disappear at x equal to zero and x equal to a, is of this form x psi x t, these corresponding to every integer n we have a different solution, so psi n x t. This is going to have a energy time dependence minus i En t by h cross.

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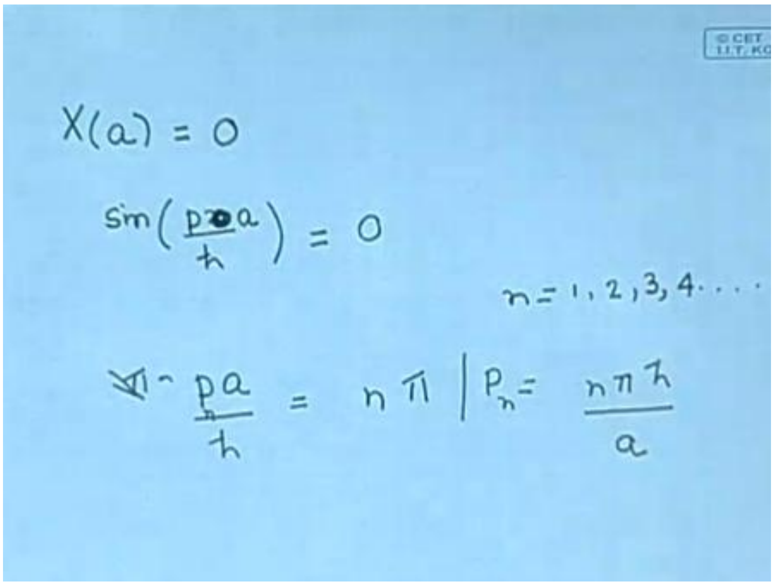

$$X(a) = 0$$

$$\sin\left(\frac{pa}{\hbar}\right) = 0$$

$n = 1, 2, 3, 4 \ldots$

$$\frac{pa}{\hbar} = n\pi \quad \Big| \quad P_n = \frac{n\pi\hbar}{a}$$

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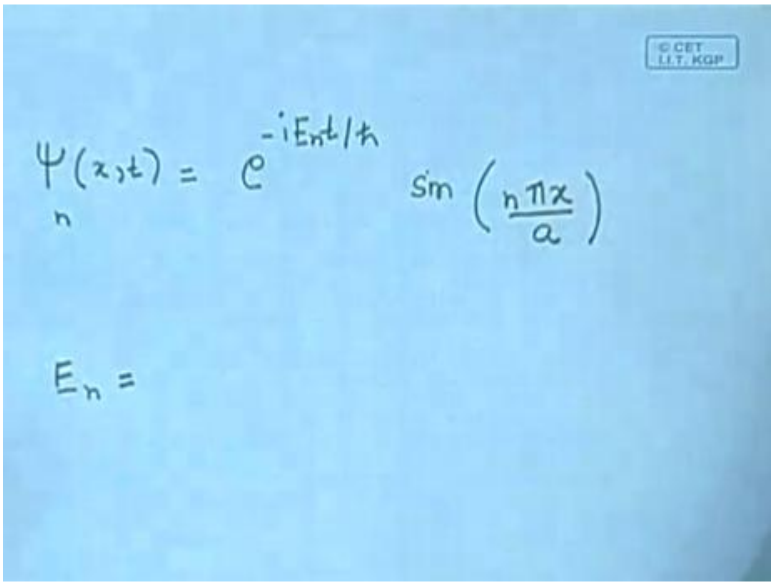

$$\psi_n(x,t) = e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n =$$

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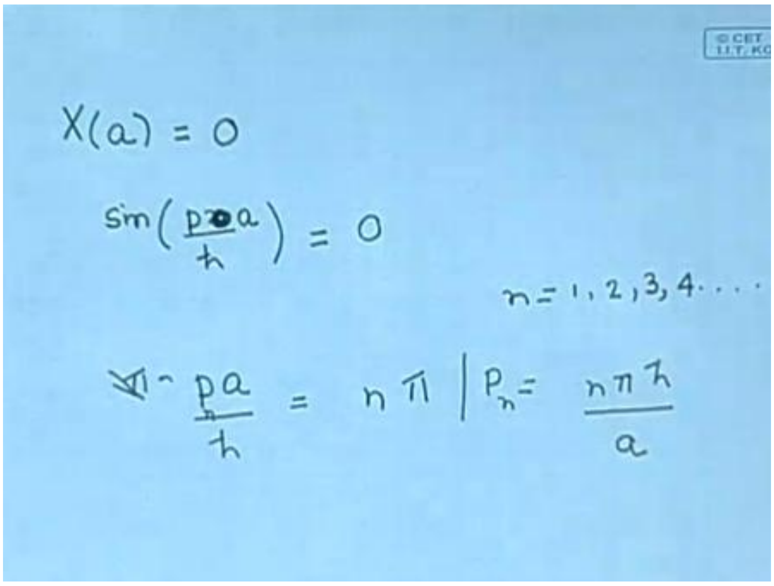

$$X(a) = 0$$

$$\sin\left(\frac{pa}{\hbar}\right) = 0$$

$$n = 1, 2, 3, 4 \ldots$$

$$\frac{p_n a}{\hbar} = n\pi \quad \Big| \quad P_n = \frac{n\pi\hbar}{a}$$

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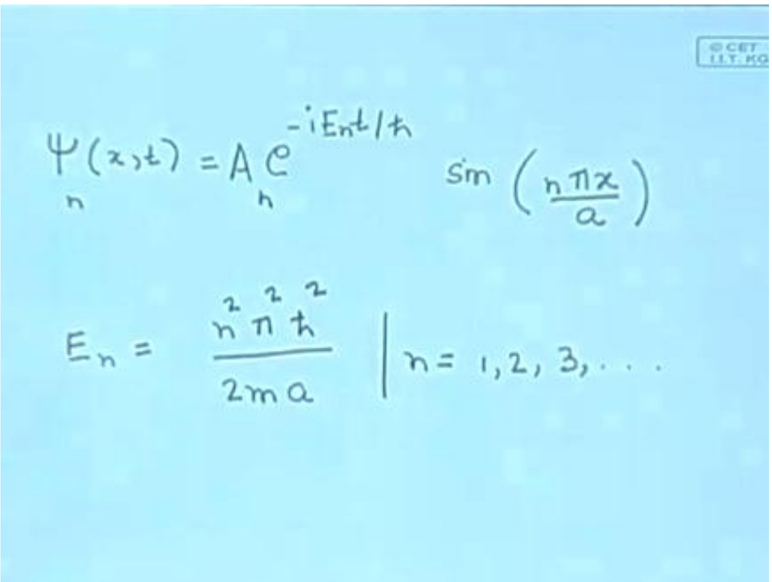

$$\Psi_n(x,t) = A_n e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma} \quad \Big| \quad n = 1, 2, 3, \ldots$$

En is the energy corresponding to the solution, corresponding to an integer n, and then we have sin p x by h cross, and p can have values n pi h cross by a, so sin n pi x by a and the corresponding energies En. Remember the energy is p square by 2 m, and the possible allowed values of P are shown over here. So, what we have is pi square h cross square n square by a square 2 m, n square pi square h cross square by 2 m a. And let me write this again n can have any value 1 2 3 4 etcetera. There could also be an arbitrary constant A n over here.

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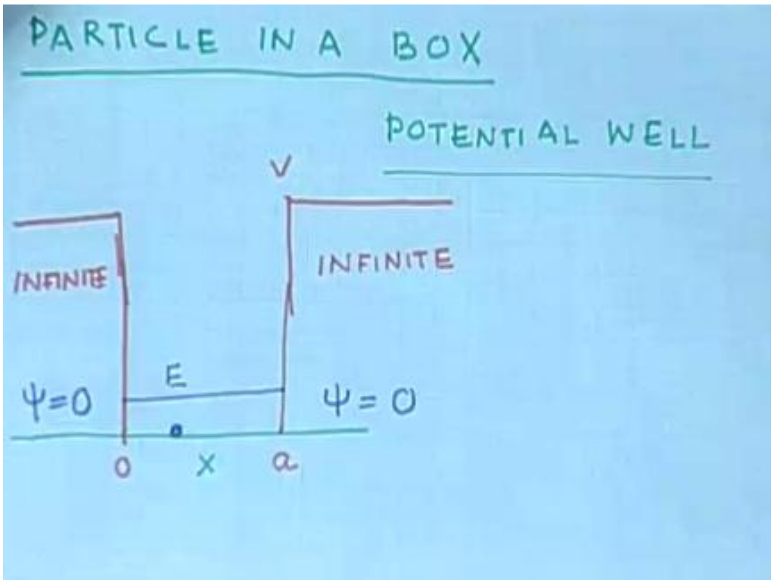
PARTICLE IN A BOX

POTENTIAL WELL

So, this is the general form for wave functions, for possible wave functions of a particle inside a potential well, of extend a, the potential well of extends from 0 to x equal to 0 to x equal to a. It is bounded by infinitely high potential to infinitely high potentials.

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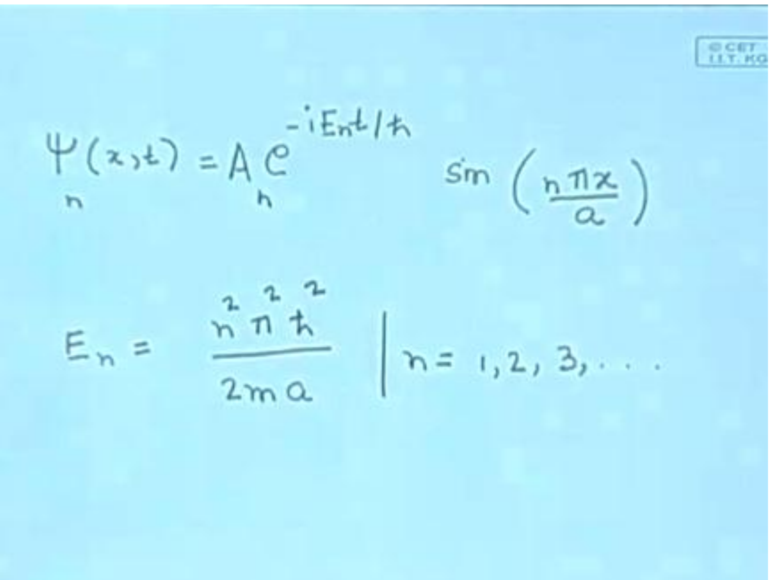

$$\psi_n(x,t) = A_n e^{-iE_nt/\hbar} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma} \quad \Big| \quad n = 1, 2, 3, \ldots$$

And the solution, allowed solutions are as or of this type; they are psi x t. So, there are solutions corresponding to different values of integer n. possible values are 1 2 3 4 etcetera. And the solutions are of this type, they are some arbitrary amplitude whose value is to be determined e to the power minus i En t by h cross into sin n pi x by a.

Where En is the energy of the state, which corresponds to some value of the integer, so n equal 1 will give you a different energy, n equal to 2 will give you different energy.

So, each of these state has a different energy each of these wave function has a different energy, and this is the expression for the energy corresponding to any integer value of n. So, essentially if I fix the value of n, I will get a wave function, I will get the corresponding energy. And we have any possible value of n gives you a valid solution. Let me now discuss what these solutions look like. So, we have n. Let we first take up n equal to 1; n equal 1 is referred to as the ground state, and a particle in this energy state in this particular state corresponding to n equal 1, has the lowest energy that you can see here, the energy levels are proportional to n square. So, a particle with n equal 1 in a state labeled by n equal to 1 has the lowest possible energy. So, it is called the ground state.

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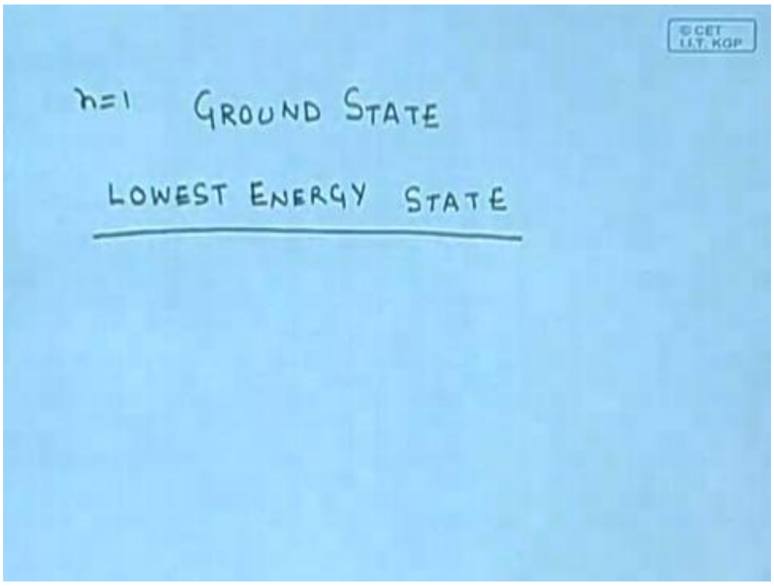

$n=1$ GROUND STATE

LOWEST ENERGY STATE

So, n equal to 1 is what we are going to discuss, this is called the ground state of the particle, or the lowest energy state level. And this is the lowest energy level which the particle can have, so that is why it is also called the ground state. What is the wave function, let me also write that down.

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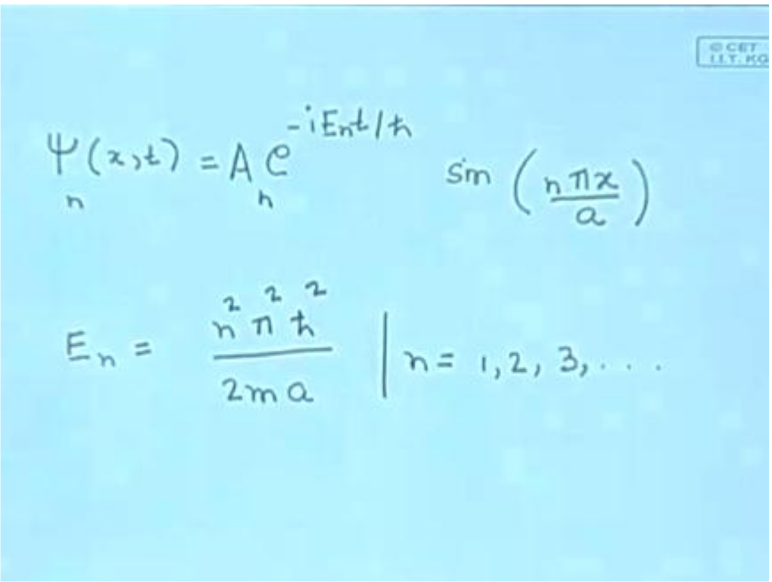

$$\Psi_n(x,t) = A_n e^{-iE_nt/\hbar} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma} \quad \Big| \; n = 1, 2, 3, \ldots$$

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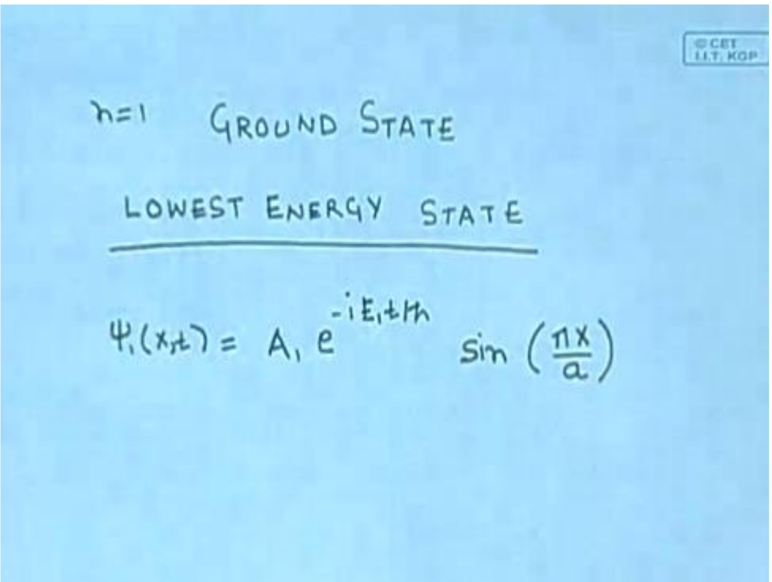

$n=1$ GROUND STATE

LOWEST ENERGY STATE

$$\Psi_1(x,t) = A_1 e^{-iE_1t/\hbar} \sin\left(\frac{\pi x}{a}\right)$$

So n equal to 1, the wave function is, Psi 1 x t is equal to A 1 e to the power minus i E one t by h cross sin pi x by a.

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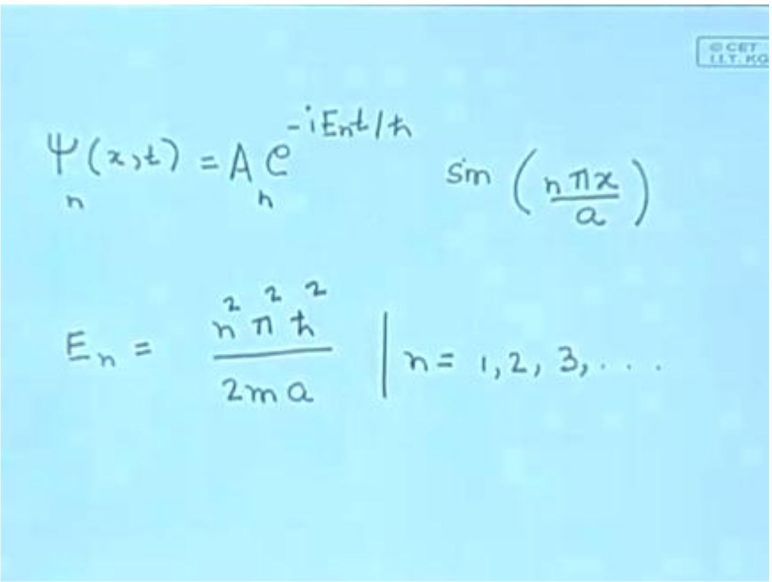

$$\psi_n(x,t) = A_n e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma} \quad \Big| \; n = 1, 2, 3, \ldots$$

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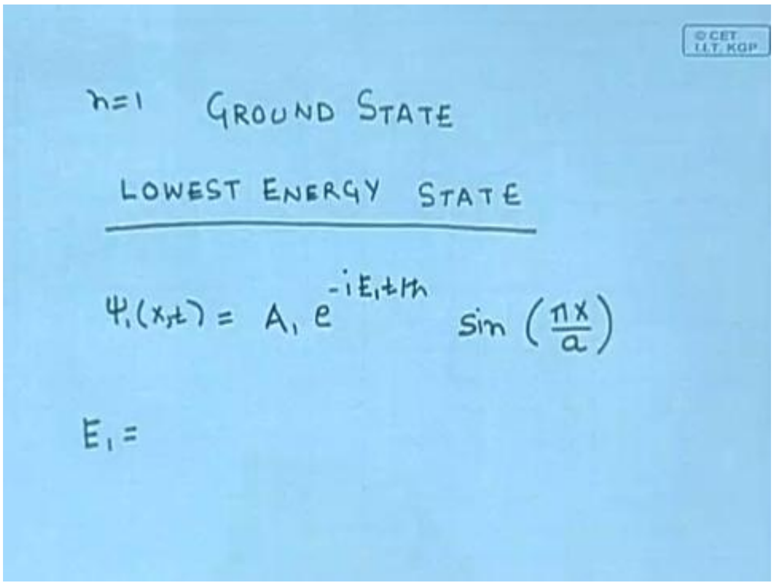

$n=1$ GROUND STATE

LOWEST ENERGY STATE

$$\psi_1(x,t) = A_1 e^{-iE_1 t/\hbar} \sin\left(\frac{\pi x}{a}\right)$$

$E_1 =$

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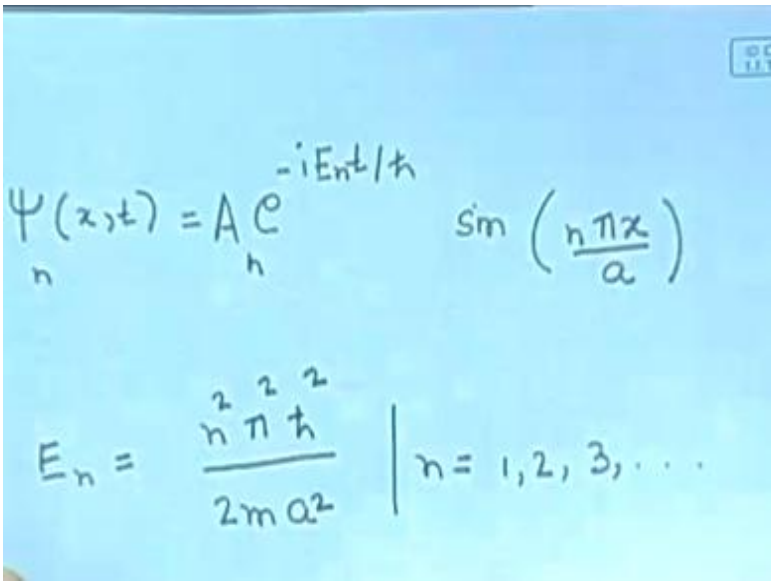

$$\psi_n(x,t) = A_n e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad \Big| \; n = 1, 2, 3, \ldots$$

So, we are looking at n equal to 1. So, n equal 1, this becomes sin pi x pi a. And the energy corresponding to this state E 1 is. Let me show you the expression for the energy again n square pi square h cross square by 2 m. This should be a square 2 m a square.

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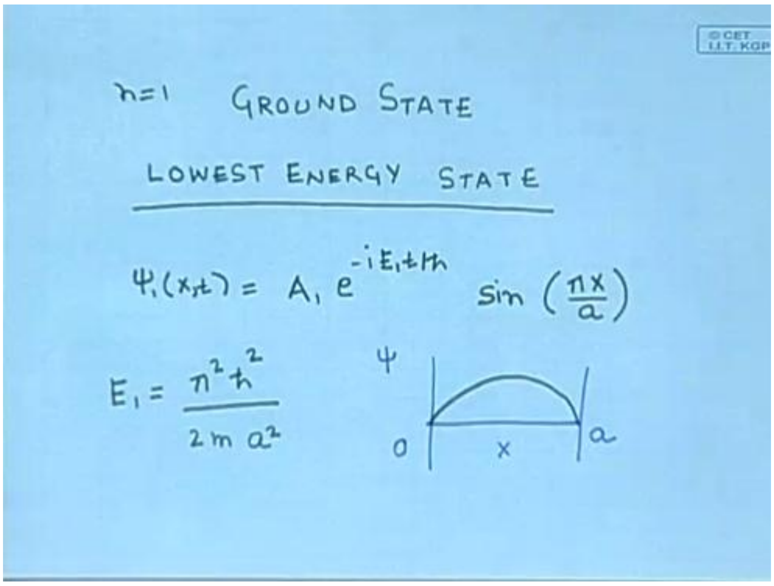

$n=1$ GROUND STATE

LOWEST ENERGY STATE

$$\psi_1(x,t) = A_1 e^{-iE_1 t/\hbar} \sin\left(\frac{\pi x}{a}\right)$$

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2}$$

So, the energy corresponding to the lowest, the n equal 1. The ground state energy that is the lowest energy state, it has energy Pi square h cross square by 2 m into a square. What does this wave function look like. This wave function looks like this. Let me draw the wave function. So, I am not going to draw this time dependent part. This time part is an oscillation, with angular frequency e one by h cross. So, I am not going to show this. We have discussed this kind of an oscillation in great detail through much of this course. I am going to show just this part. This is the part which you have to remember this part, also we have similar things we have also discussed, but this is the part that of interest over here. So, I have shown x over here, it goes from 0 to a. let me show psi here, psi. If I plot only this part it is real, and we have sin pi x by a, so the function looks like this. The wave function for the ground state looks like this; sin pi x by a. So, it goes from 0 and it rises to a maximum at pi by 2 and then again it goes to 0 at x equal to a. We still have to determine this constant A 1, and the value of this constant A 1 is determined by the condition that the wave function should be normalized. So, let me remind you what we mean by the fact that the wave function should be normalized. By normalization we mean, the condition that the total probability of finding the particle somewhere should be 1. So, the total probability of finding the particle somewhere in the range 0 to a should be 1.

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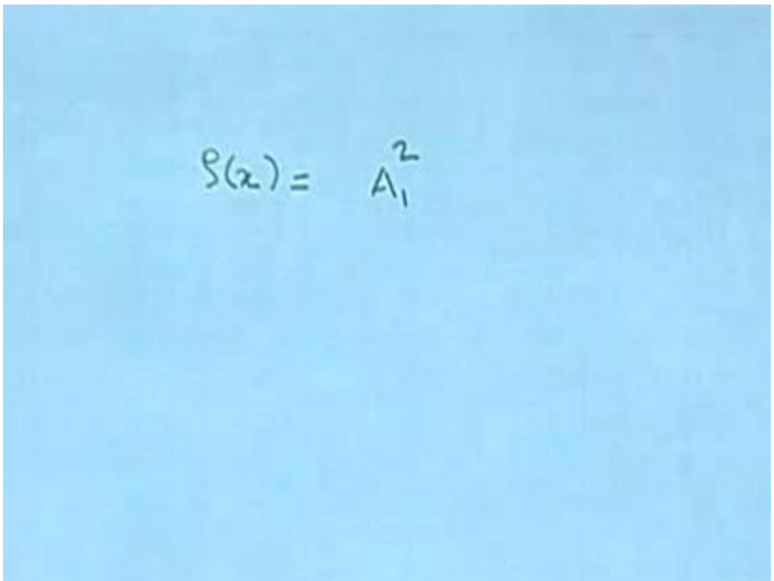

$\rho(x) = A_1^2$

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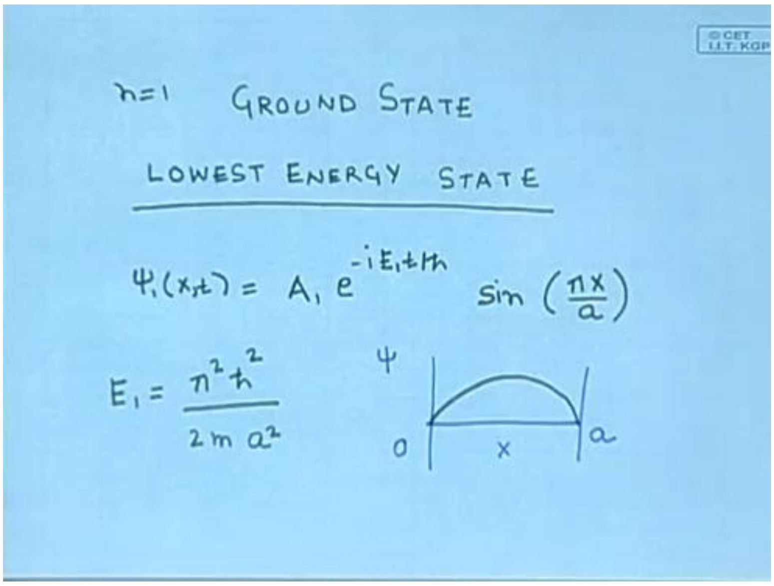

n=1 GROUND STATE

LOWEST ENERGY STATE

$$\psi_1(x,t) = A_1 e^{-iE_1 t/\hbar} \sin\left(\frac{\pi x}{a}\right)$$

$$E_1 = \frac{\pi^2 \hbar^2}{2 m a^2}$$

And how do we calculate the total probability of finding the particle. The probability density remember rho x is A 1 square. Rho x is psi into psi star for this particular state, the ground state rho x is psi. For any state it is psi into psi star we are calculating it for the state n equal to 1, the ground state. So, I have to take this wave function and multiply it with its complex conjugate.

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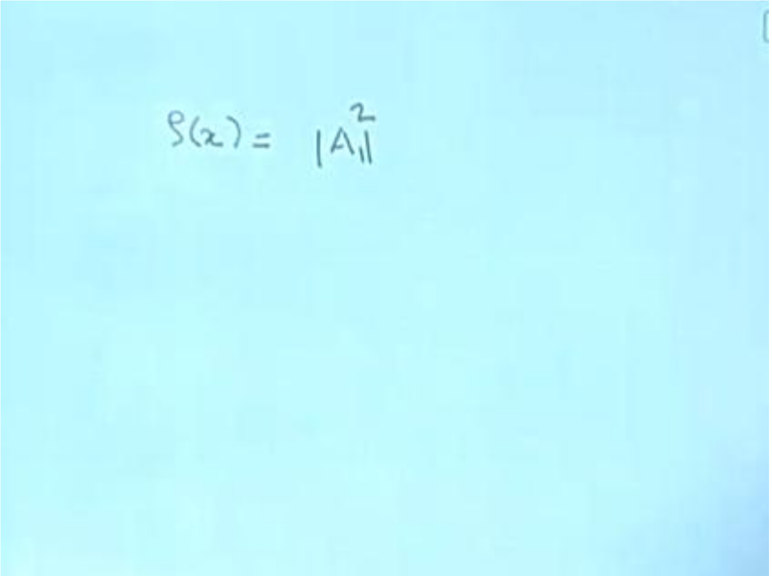

$$\rho(x) = |A_1|^2$$

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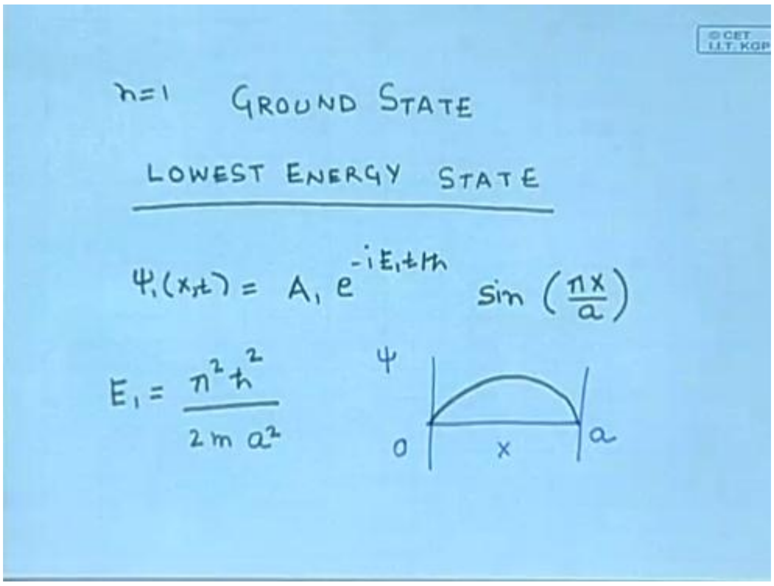

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$n=1$ GROUND STATE

LOWEST ENERGY STATE

$$\psi_1(x,t) = A_1 e^{-iE_1t/\hbar} \sin\left(\frac{\pi x}{a}\right)$$

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2}$$

So, I will have the mod of A 1 square. It should be the mod of A 1 the square of this. Multiplying this with it is complex conjugate gives me 1 e to the power minus i into any real number multiplied with it is complex conjugate, this will be gone, and I will have 1.

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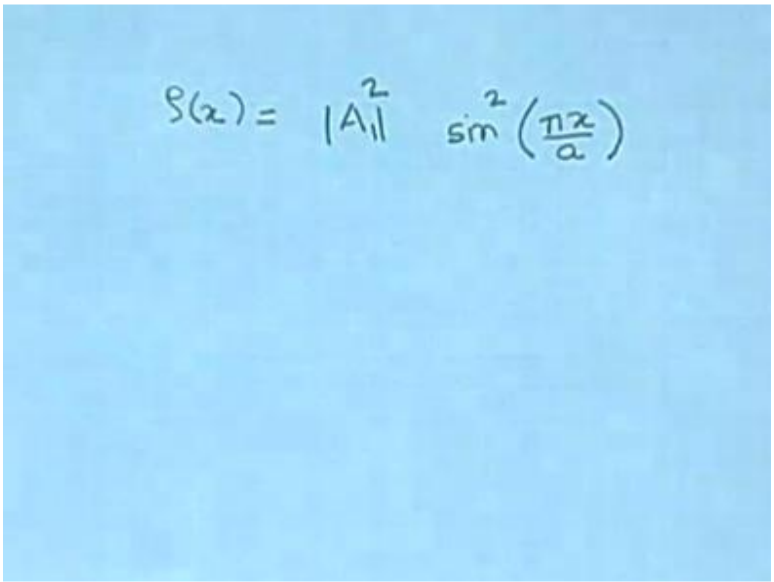

$$\rho(x) = |A_1|^2 \sin^2\left(\frac{\pi x}{a}\right)$$

So, I have this, and I have the sin square term sin square pi x by a. So, let me just spend a minute discussing the implication of this.

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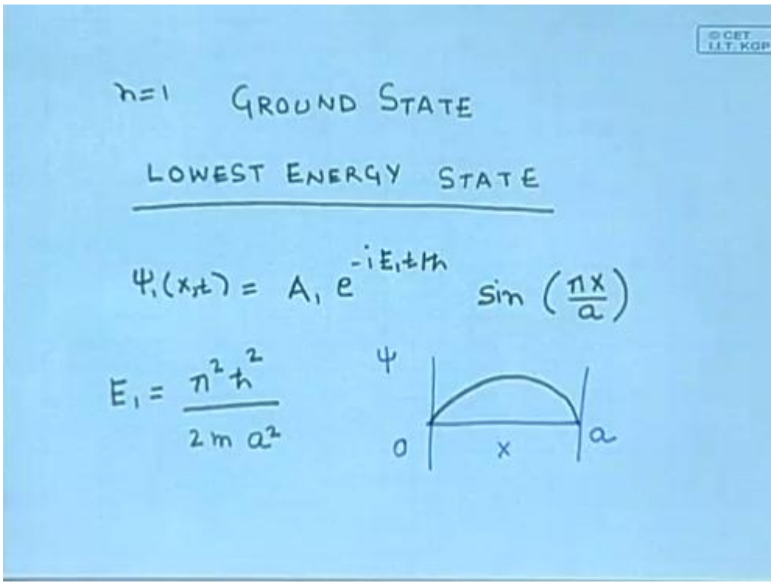

n=1 GROUND STATE

LOWEST ENERGY STATE

$$\psi_1(x,t) = A_1 e^{-iE_1t/\hbar} \sin\left(\frac{\pi x}{a}\right)$$

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2}$$

So, if I have a particle in this potential well. The probability of finding the particle outside this is 0, because the wave function vanishes.

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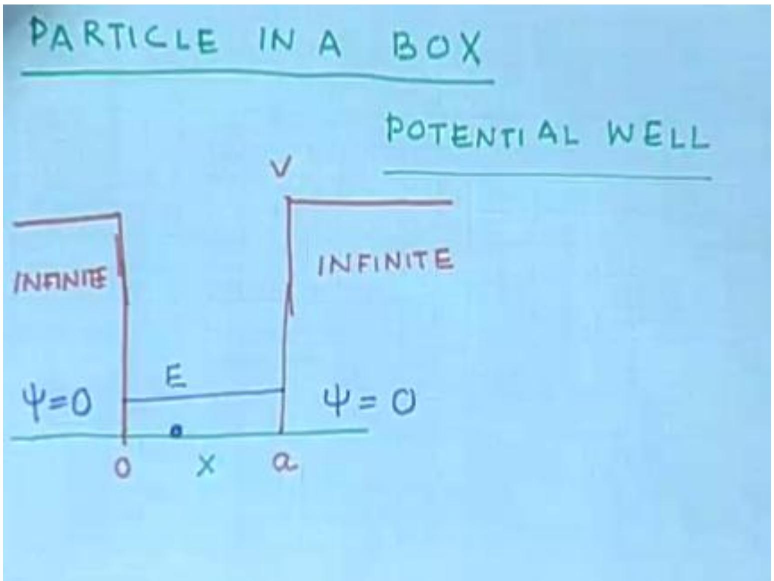

PARTICLE IN A BOX

POTENTIAL WELL

The probability of finding the particle here and here is 0, since the wave function vanishes.

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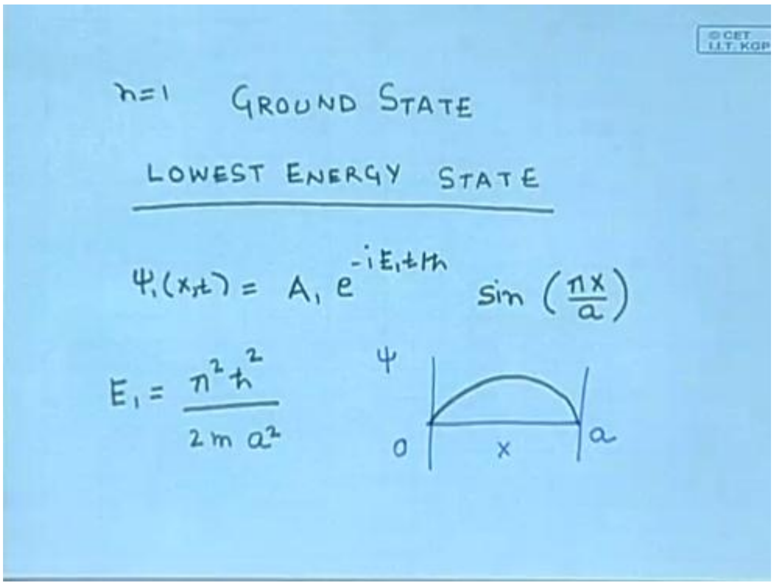

n=1 GROUND STATE

LOWEST ENERGY STATE

$$\psi_1(x,t) = A_1 e^{-iE_1 t/\hbar} \sin\left(\frac{\pi x}{a}\right)$$

$$E_1 = \frac{\pi^2 \hbar^2}{2 m a^2}$$

And the probability of finding the particle at any position, is the square of this wave function. So, you can see that the probability of finding the particle at any position provided the particle is in the ground state n equal to 1. The probability of finding the particle is the square of this. The square of this is a function which looks quite like this, it has a maxima at the center, and then it falls of at the two sides. So, the maximum probability of finding the particle, if I were to measure the position of the particle in this box, the largest probability would be to find the particle at the center, and then the probability of finding the particle falls off as you go to towards the edges of the box. So, that is the ground state.

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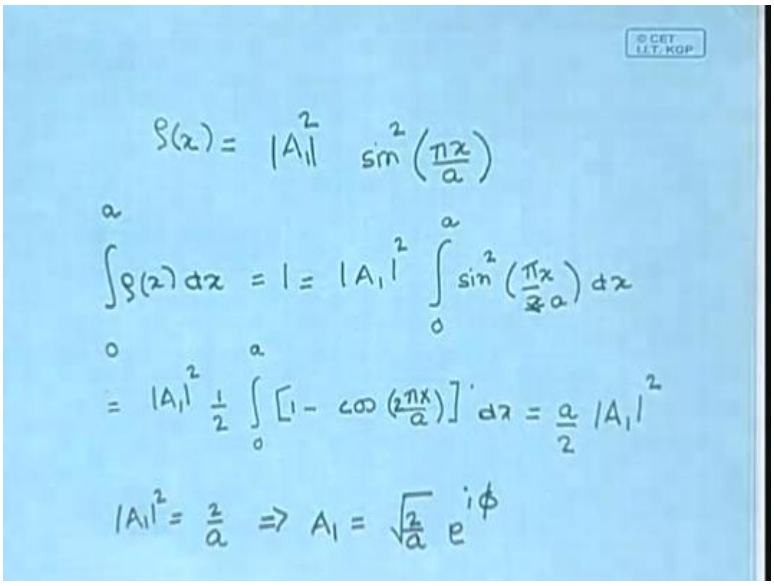

$$\rho(x) = |A_1|^2 \sin^2\left(\frac{\pi x}{a}\right)$$

$$\int_0^a \rho(x)\,dx = 1 = |A_1|^2 \int_0^a \sin^2\left(\frac{\pi x}{\cancel{2}a}\right) dx$$

$$= |A_1|^2 \frac{1}{2} \int_0^a \left[1 - \cos\left(\frac{2\pi x}{a}\right)\right] dx = \frac{a}{2} |A_1|^2$$

$$|A_1|^2 = \frac{2}{a} \Rightarrow A_1 = \sqrt{\frac{2}{a}}\, e^{i\phi}$$

In the ground state the probability falls off as sin square pi x by a go towards the. So, this function falls off towards the edges. And the total probability of finding the particle somewhere rho x inside that range is 0 to a rho x dx. This has to be equal to 1, or it A 1, the mod of A 1 square 0 to a sin square pi x pi x by a dx this has to be 1. Now, we know how to do this integral sin square pi x by a can be written as. So, this we can write as mod of A 1 square and then I have half 0 to A 1 minus cos pi x by a factor of 2 will come here now 2 pi x by a dx. So, there is a factory of two which comes in over here. So, this sin square pi x by a can be written as 1 minus cos 2 pi x by a dx. And the integral is from 0 to a. The first term gives me a, the second term gives me 0, because it is the full period, 0 to a is the full period for this function 2 pi x by a. So, this cos integrated over a period gives me 0. So, this is equal to a by 2, the lengthy by 2 into the mod of A 1 square. And this should be equal to 1. So, it tells us that the mod of A 1 square should be equal to 2 by a, or A 1 is the square root of 2 by a into e to the power i phi, some arbitrary phase where phi is some arbitrary real number, some arbitrary phase. So, A 1 the normalization coefficient is root 2 by a into e to the power i phi, where phi is some arbitrary phase. If you set phi equal to 0 it is just equal to this much. If you set pi equal to pi I pick up a minus sin. These are all possible solutions which give you this condition.

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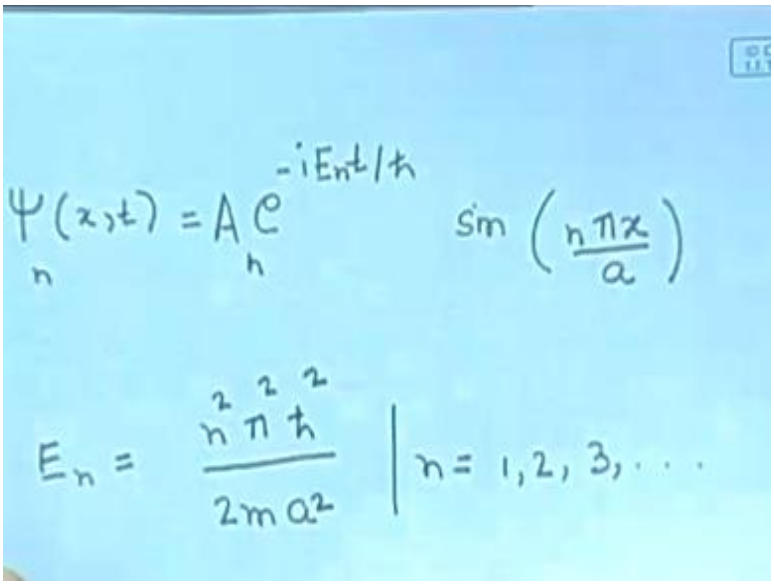

$$\psi_n(x,t) = A_n e^{-iE_nt/\hbar} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad \Big| \; n = 1, 2, 3, \ldots$$

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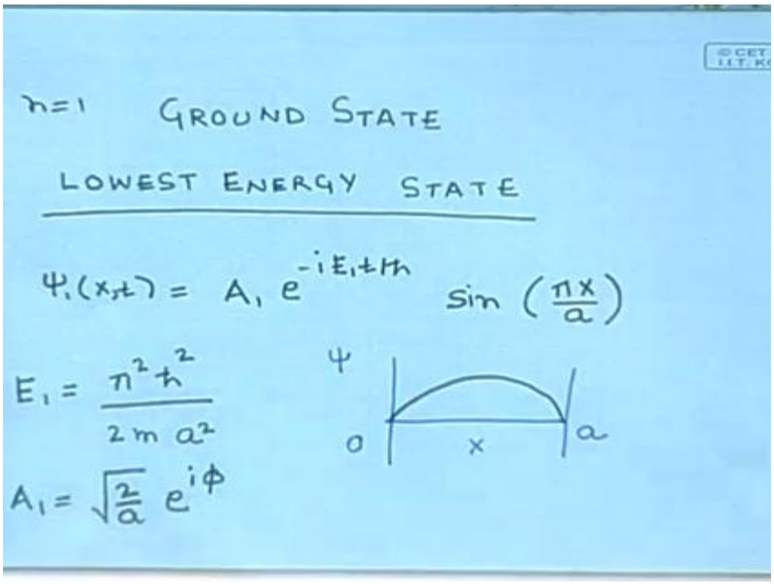

$n=1$ GROUND STATE

LOWEST ENERGY STATE

$$\psi_1(x,t) = A_1 e^{-iE_1t/\hbar} \sin\left(\frac{\pi x}{a}\right)$$

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2}$$

$$A_1 = \sqrt{\frac{2}{a}}\, e^{i\phi}$$

So, the normalization coefficient over here. Let me also write that, is for the ground state sorry this is the general expression. The normalization coefficient for the ground state is A 1 is equal to root 2 by a with some arbitrary phase e to the power i phi. And for any value of phi you put in the A 1 here it will satisfy the normalization condition. So, we have worked out what the ground state of a particle in a potential well is. Now let us look at the next possible state that is n equal to 2. So, we are going to now look at solution, to the wave function for n equal to 2, and this is the general expression.

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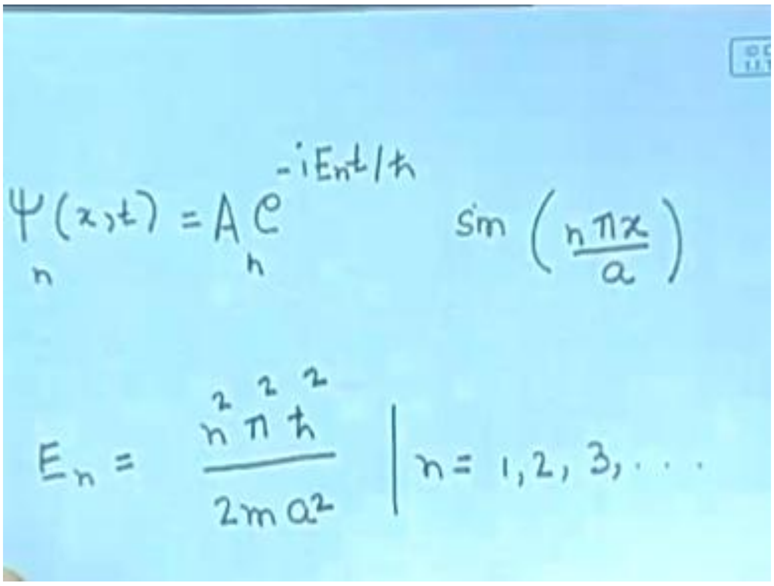

$$\psi_n(x,t) = A_n e^{-iE_nt/\hbar} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad \Big| \quad n = 1, 2, 3, \ldots$$

So, we have solutions corresponding to each value of the integer, starting from 1, then you can go to 2 3 4 etcetera. So, n equal to 2 is the next energy level. So, it is the energy level which is just one of from the ground state. So, this is called the first excited state, and here the energy levels increase proportional to n square. So, this is going to have an energy value 4 pi square h cross square by 2 pi a square.

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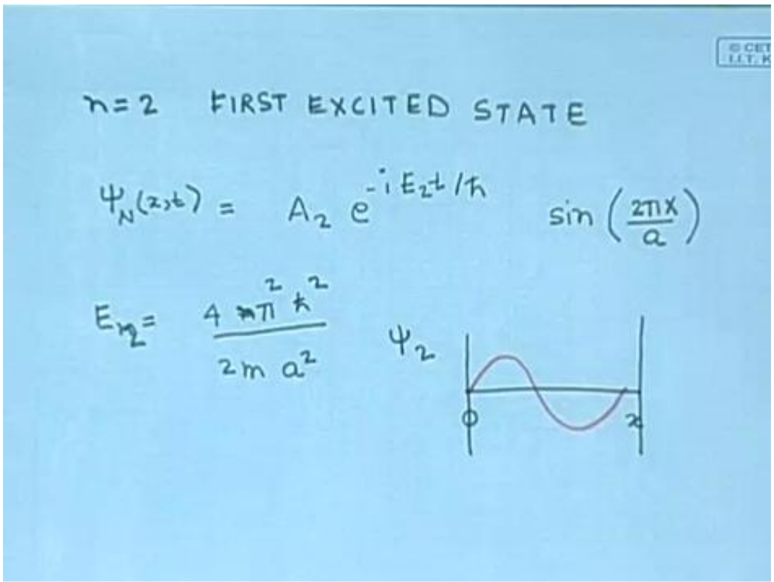

n = 2 FIRST EXCITED STATE

$$\psi_N(x,t) = A_2 e^{-iE_2t/\hbar} \sin\left(\frac{2\pi x}{a}\right)$$

$$E_{\cancel{n}2} = \frac{4\cancel{n}\pi^2\hbar^2}{2m\,a^2}$$

So, the n equal 2 state is called the first excited state. It is the second highest or rather the second lowest possible energy state, that the particle can have the lowest is the ground state, and then this is the one just above that. So, it is called the first excited state. If you give the particle some energy, it will jump to the state; that is a lowest energy state above the ground state; that is the first excited state n equal to 2. And the wave function now psi n x t is A 2, a normalization coefficient into e to the power minus i E 2 t by h cross into sin 2 pi x by a. And the energy level has a value E is equal to 4 pi square h cross square by 2 m a square. So, this is the energy level of the first excited state n equal to 2. And let me now plot this wave function again what I mean by plotting the wave function is, just plotting the spatial dependence which is a real function. We are not interested in this, and this is just a constant. So, when I plot psi 2 in the range 0 to x, I have to plot two pi x by a. So, this is a sin wave, sin with whose period is exactly equal a x equal to 0, the argument is 0, x equal to a the argument is 2 pi. So, the wave function, the argument of the wave function goes from 0 to 2 pi. So, the wave function does one full oscillation. So, this is n equal 2. And this normalization coefficient, again has to be found, by doing this, by calculating the probability density.

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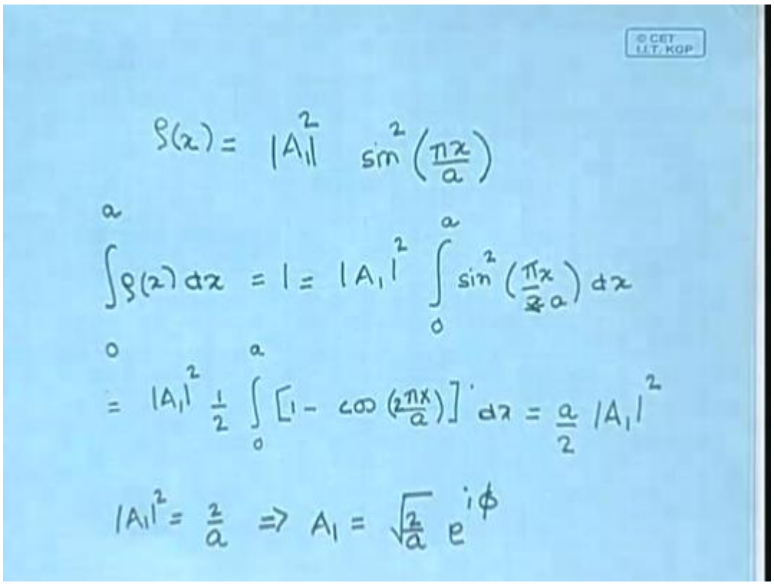

$$\rho(x) = |A_1|^2 \sin^2\left(\frac{\pi x}{a}\right)$$

$$\int_0^a \rho(x)\,dx = 1 = |A_1|^2 \int_0^a \sin^2\left(\frac{\pi x}{\cancel{2}a}\right) dx$$

$$= |A_1|^2 \frac{1}{2} \int_0^a \left[1 - \cos\left(\frac{2\pi x}{a}\right)\right] dx = \frac{a}{2} |A_1|^2$$

$$|A_1|^2 = \frac{2}{a} \Rightarrow A_1 = \sqrt{\frac{2}{a}}\, e^{i\phi}$$

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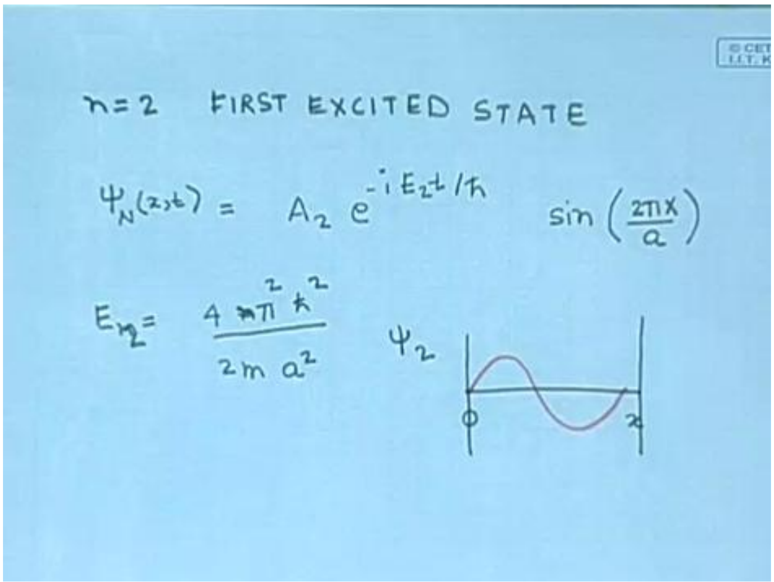

n = 2 FIRST EXCITED STATE

$$\psi_N(x,t) = A_2 \, e^{-iE_2 t/\hbar} \sin\left(\frac{2\pi x}{a}\right)$$

$$E_2 = \frac{4\pi^2\hbar^2}{2ma^2}$$

The probability density is now going to be mod of A 2 square into sin square 2 pi x by a. So, when i square this is the probability, to get the probability density. The probability density is going to have a peak over here, and the probability of finding the particle. If the particle is in the first excited state, the probability of the finding the particle is going to have one maxima, somewhere over here, another maxima somewhere over here. The probability of finding the particle is going to be 0 near the center, and is going to have two maxima; One at one fourth of the distance a by 4, and other one at 3 a by 4. So, the probability of finding the particle at different positions inside has changed, if I look at first excited state.

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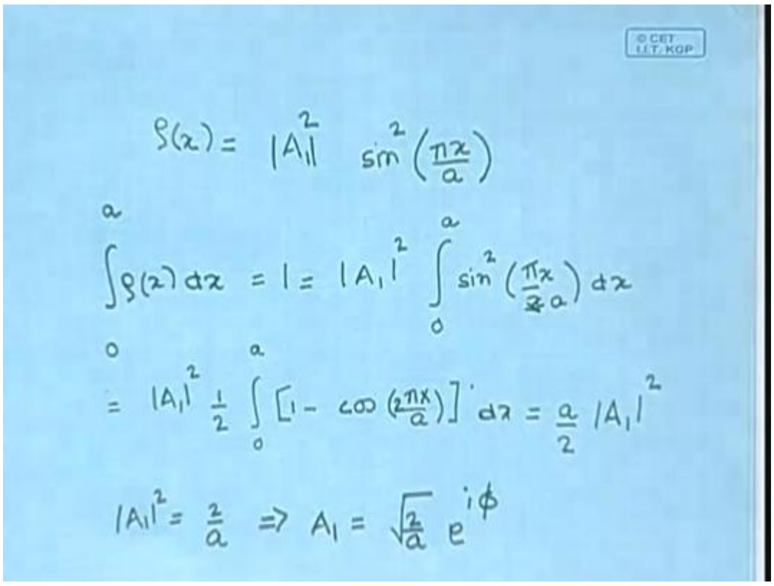

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$$|A_1|^2 = \frac{2}{a} \Rightarrow A_1 = \sqrt{\frac{2}{a}}\, e^{i\phi}$$

And we have to determine this coefficient A 2 by the requirement that the total probability of finding the particle somewhere inside should be 1, that is what we mean by normalization. And again we will have A 2 now instead of A 1 into sin square 2 pi x by a. now again the sin square can written as half 1 minus cos. Now, there is going to be another factor of 2 here, whether there is a factor of 2 or not. This integral is always going to be 0. So, we have a by 2, and the value of the amplitude is going to be the same.

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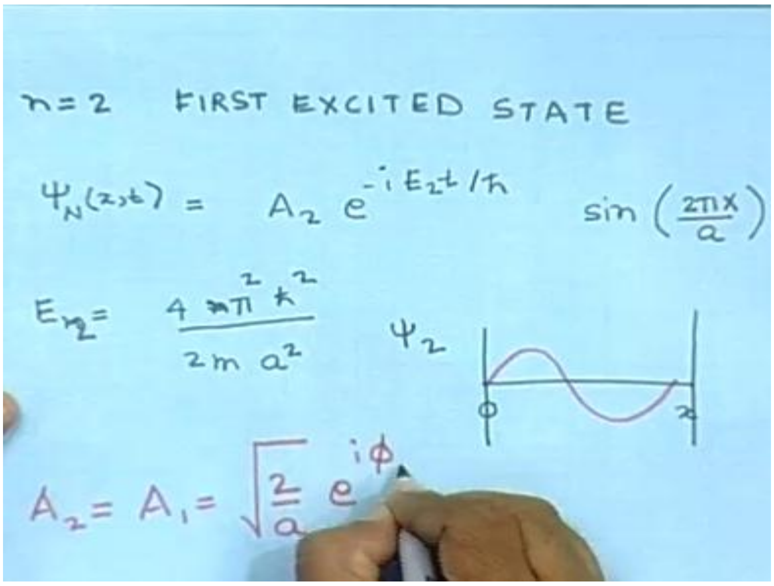

$n = 2$ FIRST EXCITED STATE

$$\psi_N(x,t) = A_2\, e^{-iE_2 t/\hbar} \sin\left(\frac{2\pi x}{a}\right)$$

$$E_{\cancel{n}2} = \frac{4\,\cancel{n}\pi^2 \hbar^2}{2m a^2}$$

$$A_2 = A_1 = \sqrt{\frac{2}{a}}\, e^{i\phi}$$

So, for all the wave functions, all the higher values of n. the value of the wave of the normalization coefficient, is going to be exactly the same. So, A 2 is going to be equal to A 1, is have the same value, but the phase could be different.

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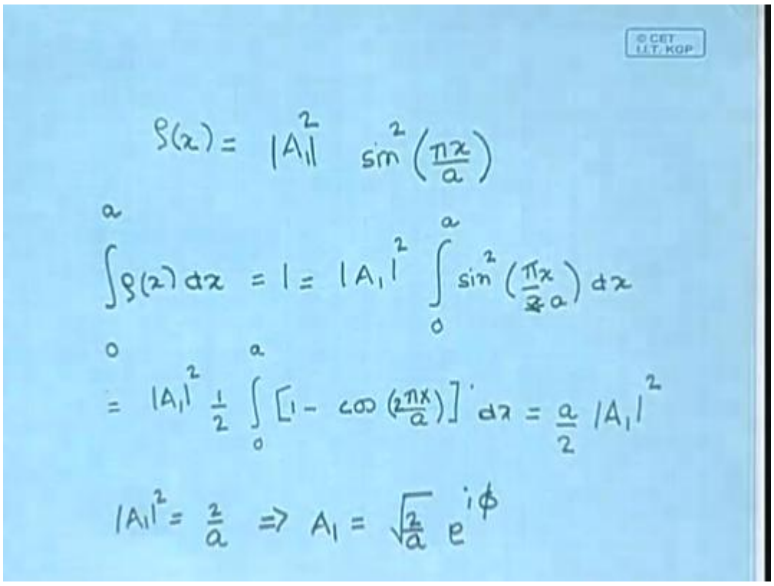

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$$|A_1|^2 = \frac{2}{a} \Rightarrow A_1 = \sqrt{\frac{2}{a}}\, e^{i\phi}$$

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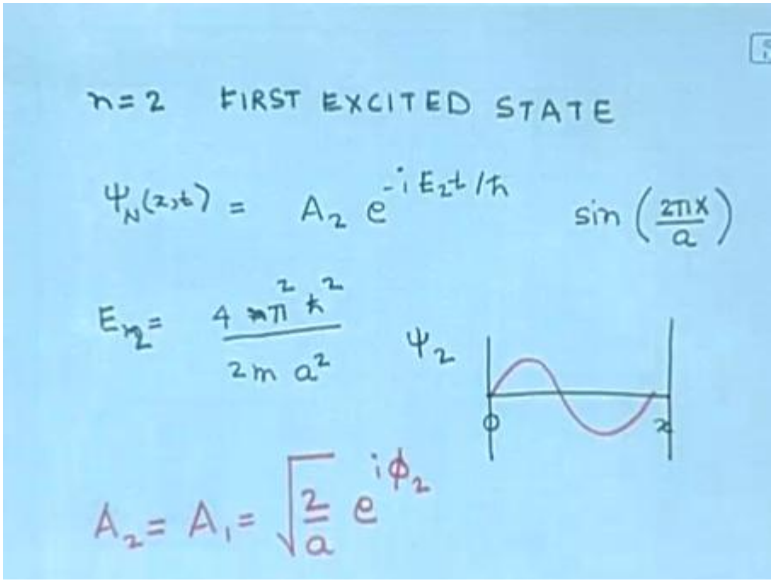

n = 2 FIRST EXCITED STATE

$$\psi_N(x,t) = A_2\, e^{-iE_2 t/\hbar} \sin\left(\frac{2\pi x}{a}\right)$$

$$E_{\cancel{n}2} = \frac{4\cancel{n}\pi^2\hbar^2}{2m a^2}$$

$$A_2 = A_1 = \sqrt{\frac{2}{a}}\, e^{i\phi_2}$$

So, I could have a different phase for A 1, and I could have different phase for A 2.The magnitude is exactly the same. We could similarly workout A 3 A 4 A 5 etcetera, and let me discuss some general features of the wave function for a particle in a potential well. So, the first thing that you notice is that, now once you have confined the particle to a finite region. Here you have confined the particle to a region 0 to a. You no longer have possible solutions for any value of the energy. There are only discrete values of the energy for which for which we have solutions.

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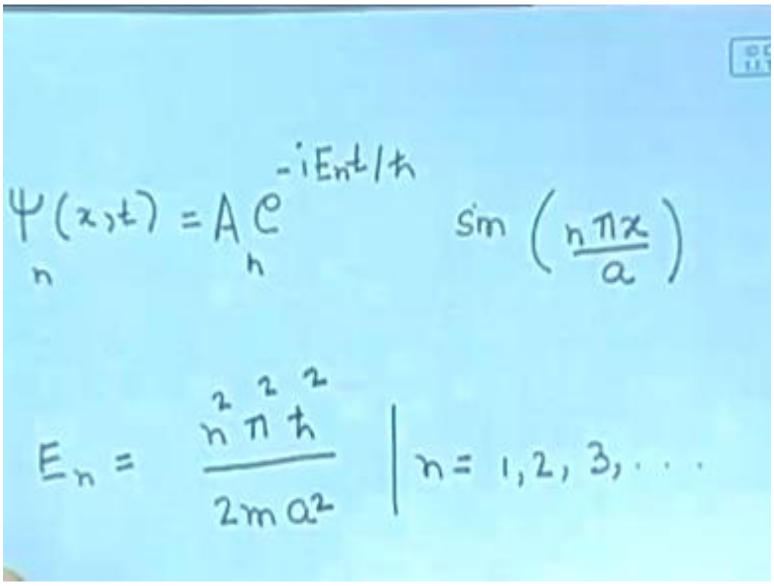

$$\Psi_n(x,t) = A_n e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad \Big| \quad n = 1, 2, 3, \ldots$$

And in this particular case, these energy levels correspond to different values of the integer, and you could have values of the n ranging from 1 all the way to infinity. There could any integer value will give me a different solution. So, in this particular case, we have discrete energy levels, and we have an energy level corresponding to every value of the integer, starting from 1 all the way to infinity. Now, this feature that we have discrete energy levels, is a very generic feature. It arises whenever we have a particle; that is constrained to a finite region. So, if you have a potential, such that it constrains the particle to a finite region, then you always have discrete energy levels. This is a very generic feature, we have studied. Here we have rigorously shown it, for a particular case where we have an infinite potential well, but such a situation also occurs if I had a finite potential well.

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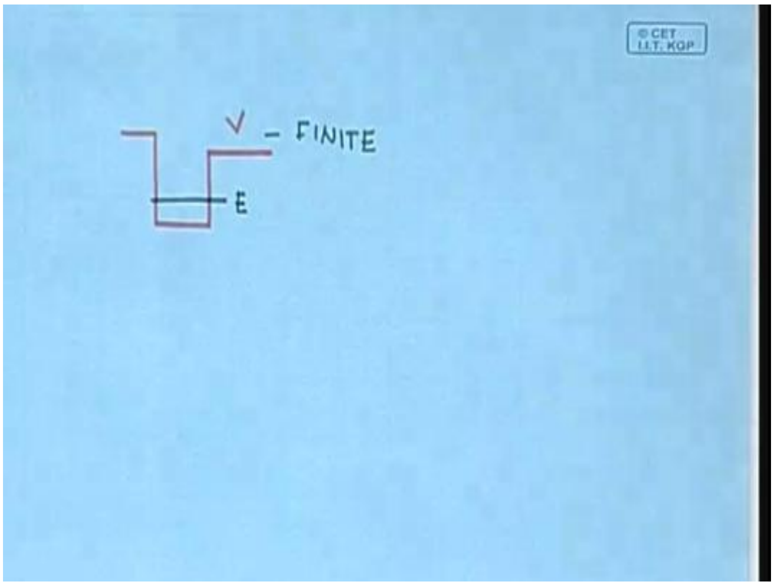

So, if I had a potential well which is finite, and we had a particle in this potential well whose energy was less than the height of the potential. So, this was the energy. So, v is finite not infinite, like the previous case. So, again we if you solve this problem you will find, that there will be only discrete values of the energy for which there will be solutions possible. And solutions will not be possible for any arbitrary value of the energy. For such a finite potential well, there may be the number of possible solutions will not be infinite.

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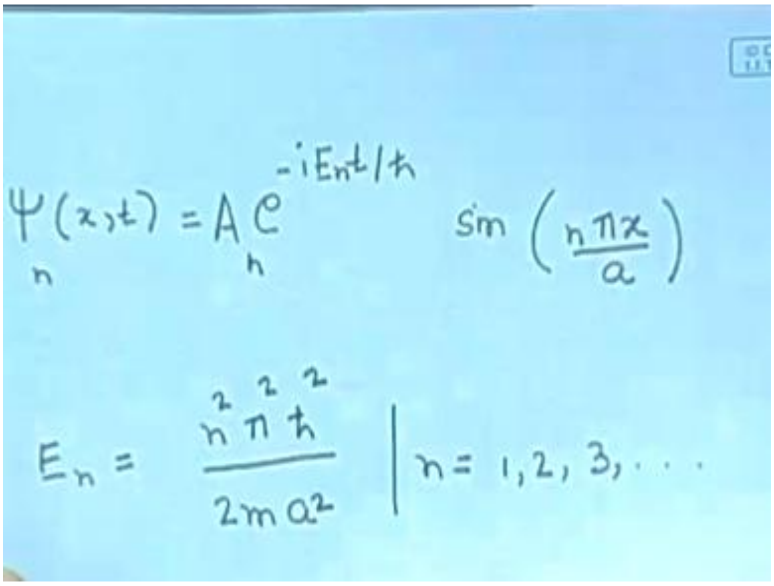

$$\psi_n(x,t) = A_n e^{-iE_nt/\hbar} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad \Big| \quad n = 1, 2, 3, \ldots$$

In this case you have infinite number of possible solutions, the integer. All values of the integer 1 to infinity will give. Each value in the range one to infinity will give a different solution.

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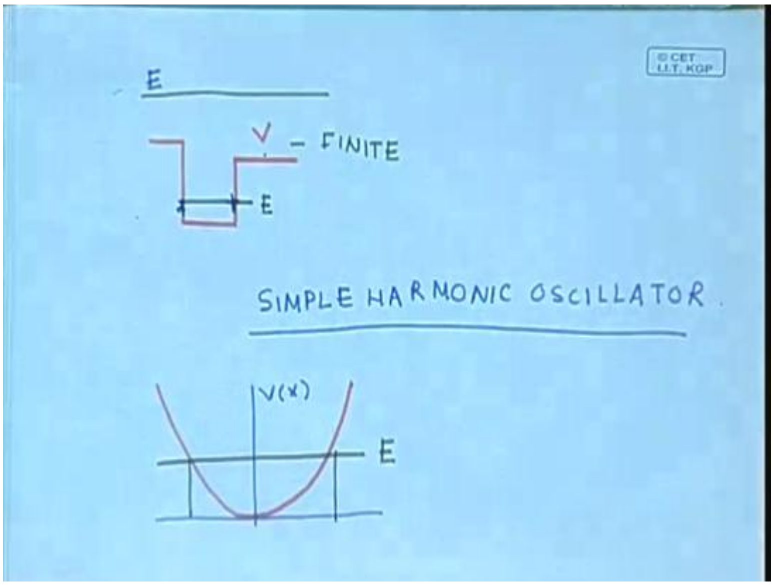

For this case there will be some restrictions, beyond which you will not have a bound solution. So, on the contrary if you have a situation where the energy is higher than the value of the potential, you do not have discrete energy states. The allowed values of energy are in general continuous. So, what we have seen here is a very generic feature, whenever the particle is confined. We notice that if the energy level is larger than the potential, the particle is in principle free to move over the entire range of x, whereas if the energy is less than the value of the potential outside, the particle is confined within the potential well. It is free to move here. If you apply classical mechanics the particle is constrained to move only in this region. In quantum mechanics there will be a amount of penetration, but still the particle is confined, and you always have discrete energy levels in such a situation. Whereas if the energy is more than the value of the potential, we have seen that you have this plane wave solutions, and you have solutions for any value of the energy. So, this is the very generic feature which occurs whenever you have a particle which is constrained within a certain region by a potential. So, there are other examples which I could give you.

For example, you have a simple harmonic oscillator. We have studied this extensively in classical mechanics. The simple harmonic oscillator potential is quadratic. So, the potential looks like this. And if I have a particle with some energy E, the particle is going to be constrained to oscillate in a finite region. And if I apply classical mechanics, it is going to be constrained to oscillate in this region. in quantum mechanics the particle may extend beyond this region, the wave function will extend beyond this region, but the particles wave functions is still going to be localized, because the potential goes to infinity at x equal to plus infinity and minus infinity. So, the particle is going to be bound somewhere in between. It is a bound particle, constrained to a finite region. And again in this case, you have finite energy level. So, in this case, the energy levels are. The spacing between the energy levels is proportional to end, whereas for a particle, in a square potential well the energies are proportional to n square. Another example is a hydrogen atom, if I have an electron bound, within the potential of the nucleus in a hydrogen atom, in any atom for that matter. If the electron is bound within the potential of the atom, so it is an electron bound to the atom, inside the bound in the atom. Then the allowed energy levels are discrete.

So, whenever in quantum mechanics, whenever a particle’s position is localized by a potential, it has only a set of energy discrete energy levels for which you can find solutions to the Schrodinger equation, and you will not have solutions for arbitrary value of the energy. Whereas if you had a free particle, or for the atom if I had a situation where I had an iron. So, singly ionized hydrogen atom let us say, or a singly ionized sodium atom or something like that, and I had an electron, so this together constitutes the atom. But the electron had more energy than the potential, the electron came from infinite it encounters the potential, it got scattered and then it went off to infinity. In such a situation the allowed energy levels are continuous energy. The electron could have any possible energy, as long as it is more than the value of the potential. So, I have a discrete, I do not have a discrete spectrum, I have a continuous range of allowed values for the energy, but the moment you have an electron which is confined constrained to a certain region, and which is bound, you have discrete energy levels. And we have seen this for. In this particular case, where I have an infinite potential well, and the particle is bound. The wave function is constrained, confined within a certain region, and we these discrete energy levels.

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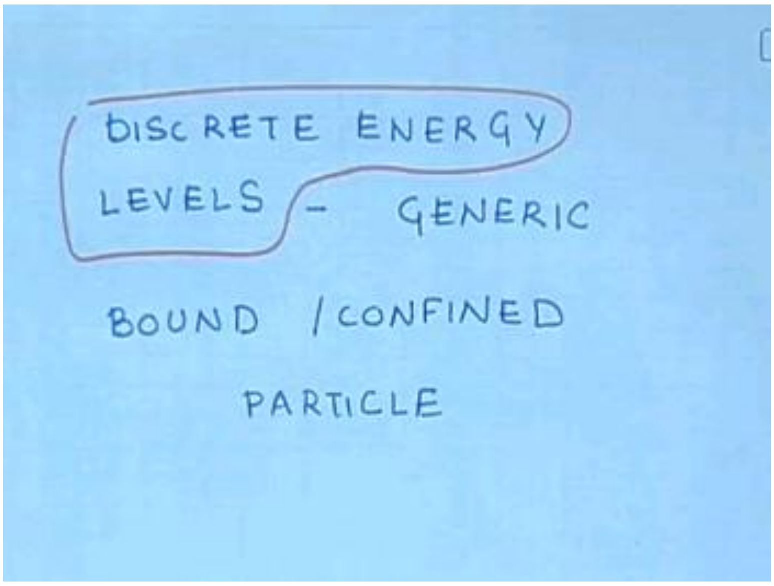

DISCRETE ENERGY LEVELS - GENERIC

BOUND / CONFINED

PARTICLE

So, the point I am trying to make over here, is that discrete energy levels are very generic. It occurs whenever we have a bound particle; bound or confined. So, whenever we have a particle confined to a certain region, we will always have discrete energy levels, as we have seen in this particular situation. Now, in this particular situation, the energy of these discrete energy levels is proportional to n squared. So, I can also draw that.

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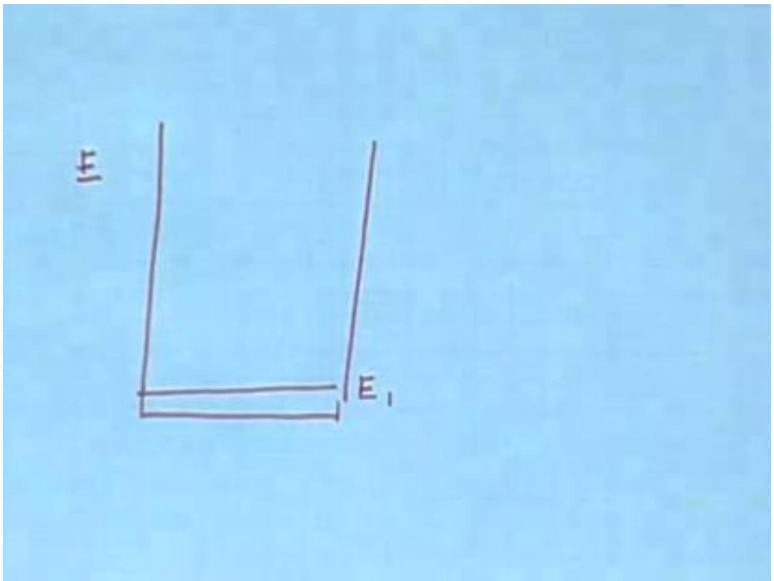

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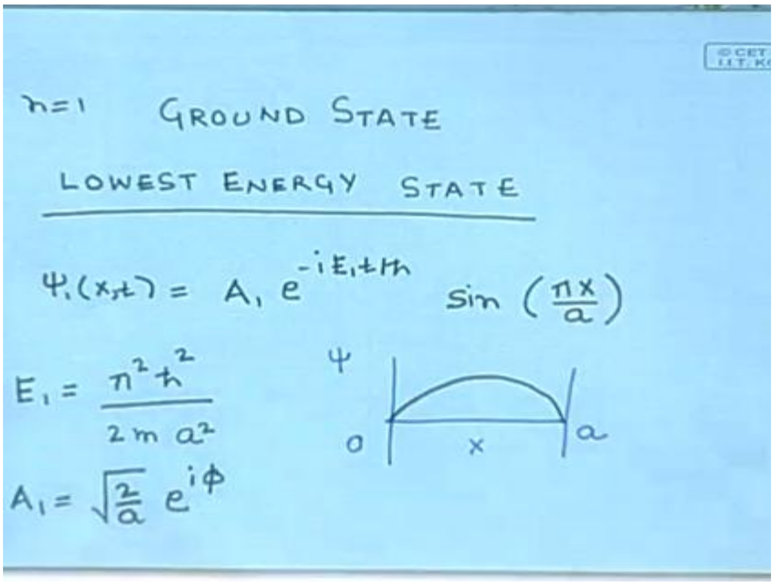
n=1 GROUND STATE

LOWEST ENERGY STATE

$$\psi_1(x,t) = A_1 e^{-iE_1t/\hbar} \sin\left(\frac{\pi x}{a}\right)$$

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2}$$

$$A_1 = \sqrt{\frac{2}{a}}\, e^{i\phi}$$

So, let me draw the energies along this axis. So, this is the potential well, and here I am going to show the energies. So, we have the lowest energy state E 1, which has energy h cross square into pi square divided by 2 m a squared.

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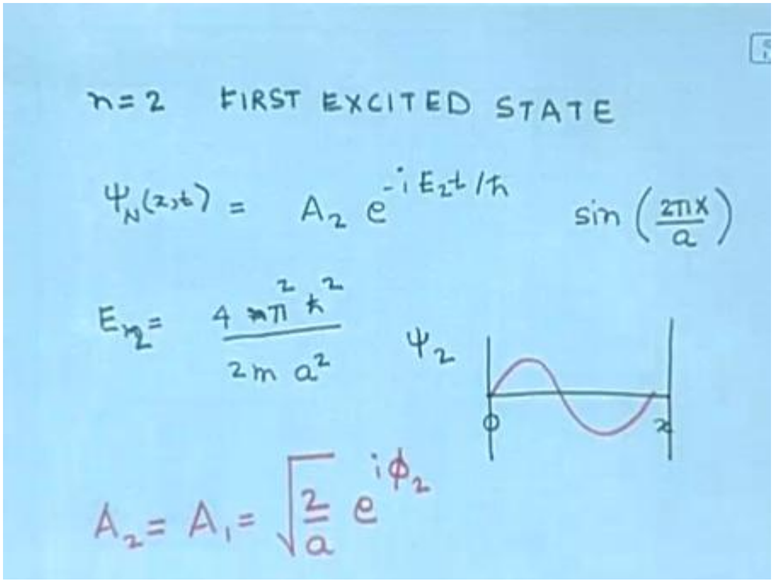
n=2 FIRST EXCITED STATE

$$\psi_N(x,t) = A_2 e^{-iE_2t/\hbar} \sin\left(\frac{2\pi x}{a}\right)$$

$$E_2 = \frac{4\pi^2\hbar^2}{2ma^2}$$

$$A_2 = A_1 = \sqrt{\frac{2}{a}}\, e^{i\phi_2}$$

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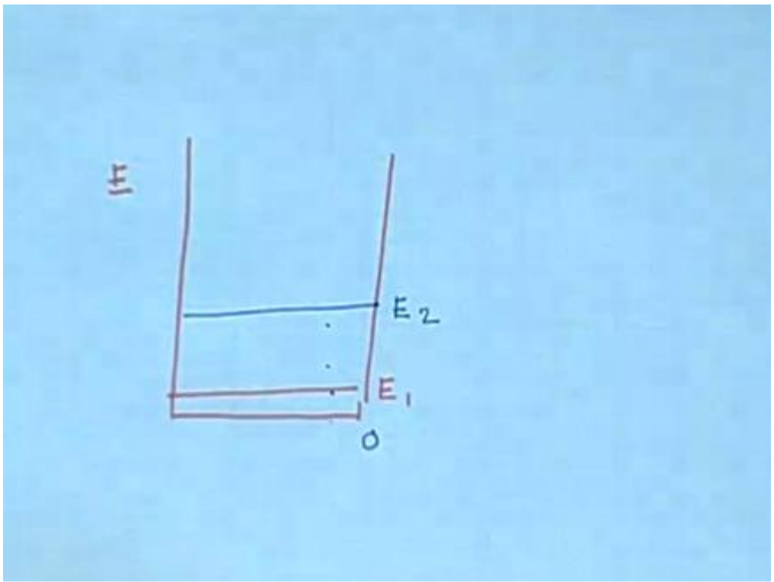

Now, the next energy level is n equal to 2, the energy level is 4 pi square x square by 2 m a square. So, this is E equal to 0 over here. So, the difference from here is going to be four times this much. So, 1,2,3,4, somewhere over here.

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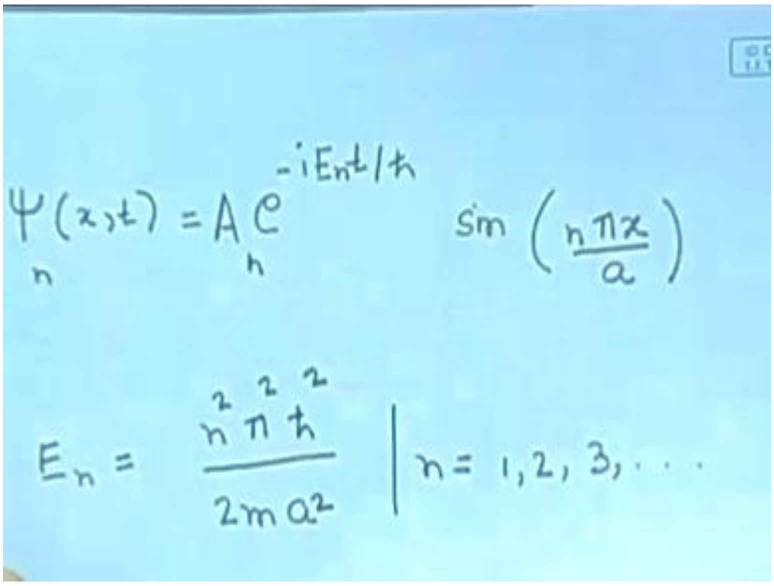

$$\psi_n(x,t) = A_n e^{-iE_nt/\hbar} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad \Big| \quad n = 1, 2, 3, \ldots$$

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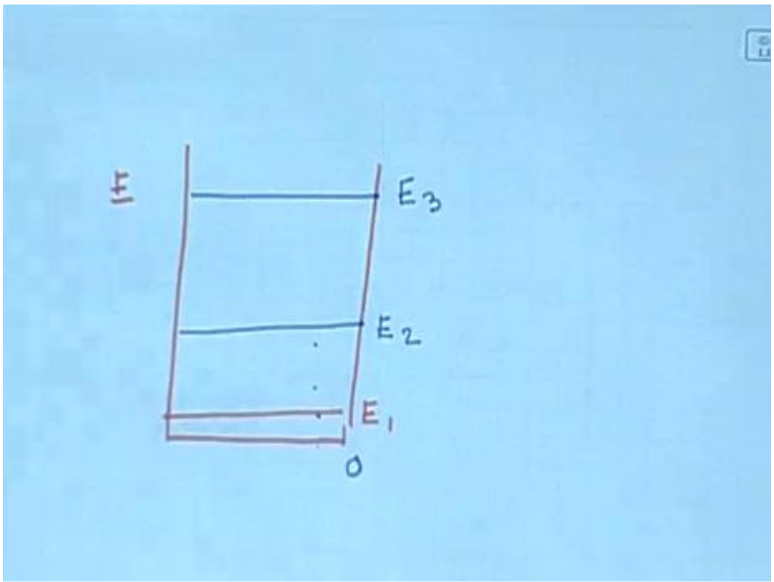

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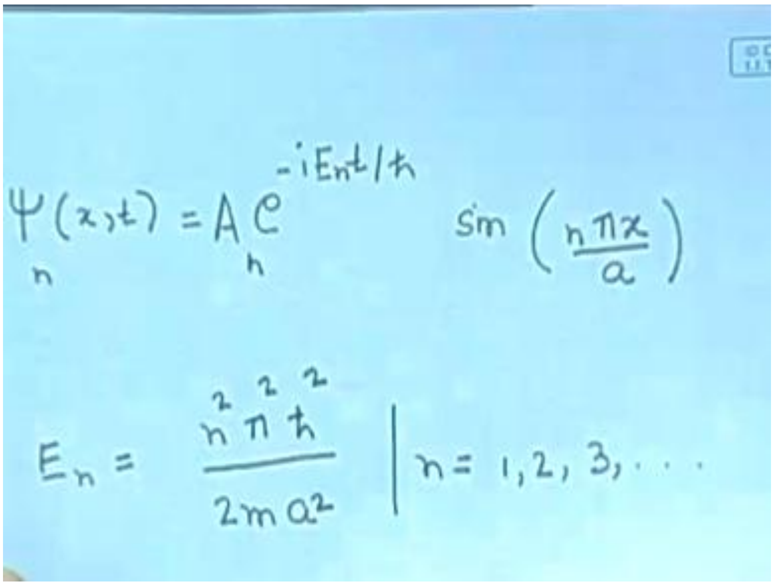

$$\psi_n(x,t) = A_n e^{-iE_nt/\hbar} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad \Big| \quad n = 1, 2, 3, \ldots$$

And the next possible energy level, is n equal to 3. So, the energy level is going to be 9 pi square h cross square by 2 m a square. So, the difference from here is going to be 9 times this difference. So, this is already 4. So, 4 and one more 5, so it is going to be somewhere here this is E 3 and then I have E 4 which is 16 pi square h cross square by 2 m a square.

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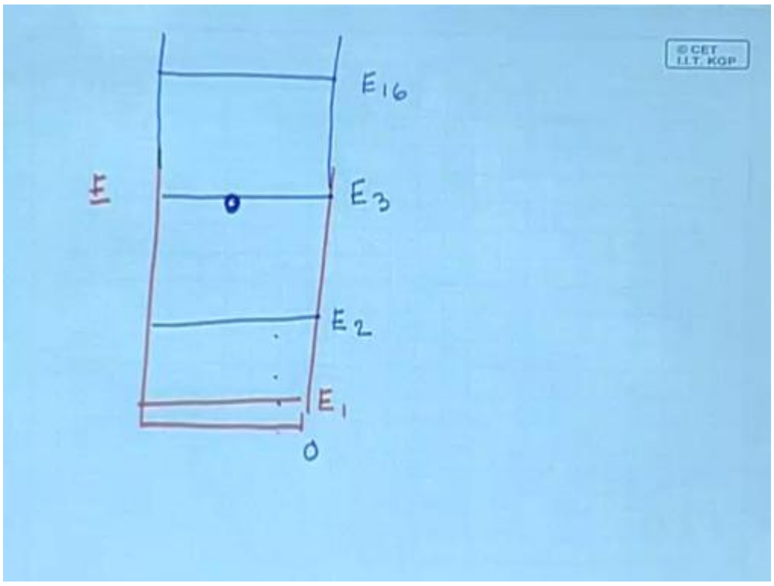

So, this is 9 and 16 is going to be somewhere over here. So, let me this is E 16, and the spacing is going to get larger and larger and the energy level are going to get further and further higher and higher. So, this is what the energy levels, discrete energy levels, the different energy levels, in this particular situation look like. And the spacing between the energy levels is going to be different for different forms of the potential. Also the number of allowed energy levels, is going to be different, for different forms of the potential. And if I have a finite potential, well the numbers are going to be in general few, they are not going to be infinite. Now, let us consider a situation where I have a say a particle over here. A charge particle in some kind of a potential well. Let me talk about an example, an example is in semiconductors. We know that in semiconductors, if you dope a certain part of the semiconductor.

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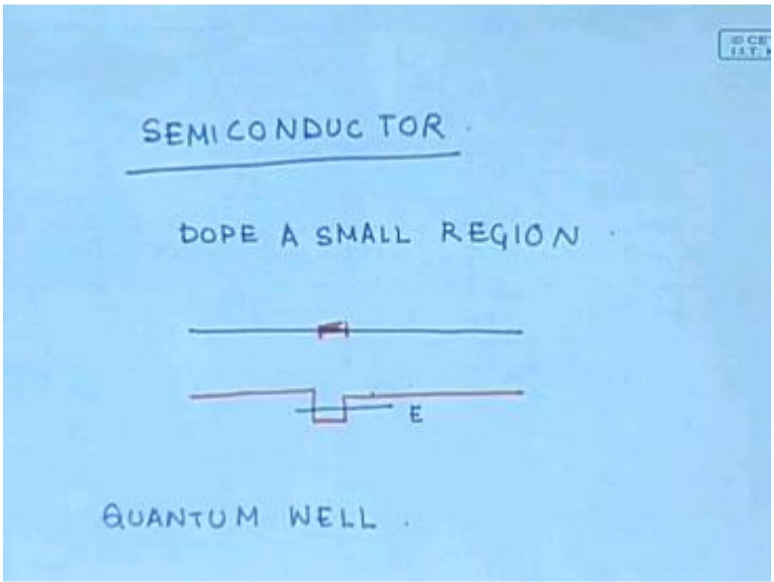

SEMICONDUCTOR

DOPE A SMALL REGION

QUANTUM WELL

So, if I have a semiconductor for example; an intrinsic semiconductor, say silicon or something like that, and i a dope a part of the semiconductor. I am sure we know what we mean by doping, we either make it n or p type. So, we have doped a certain part of the semiconductor. Inside this dope region we conduct the carriers, the charge carriers, so the electron or the hole, will experience a different potential. So, if you dope a small region, inside this region, the electron or the hole is going to experience a different potential. So, if this is, I am just going draw one dimensional representation. In general it is going to be three dimensional, but let me just draw a one dimensional representation. So, this is a semiconductor, and we have doped this part. So, an electron or a whole whichever is the carrier, charge carrier is going to have a different potential in this region. And if you draw the potential the potential is going to appear, it is going to be higher over here. So, if I have a particle, the electron or hole, whose energy is less than the energy outside, or a particle from outside could lose some energy, and fall into this, get trapped by this potential, so this is the particles energy. And there could be different such possible energy levels. So, this is what is called a quantum well. A quantum well can be made in semiconductors by doping very small regions, and it is a technologically challenge challenging task, because you have to dope uniformly dope a very small region, but it can be done.

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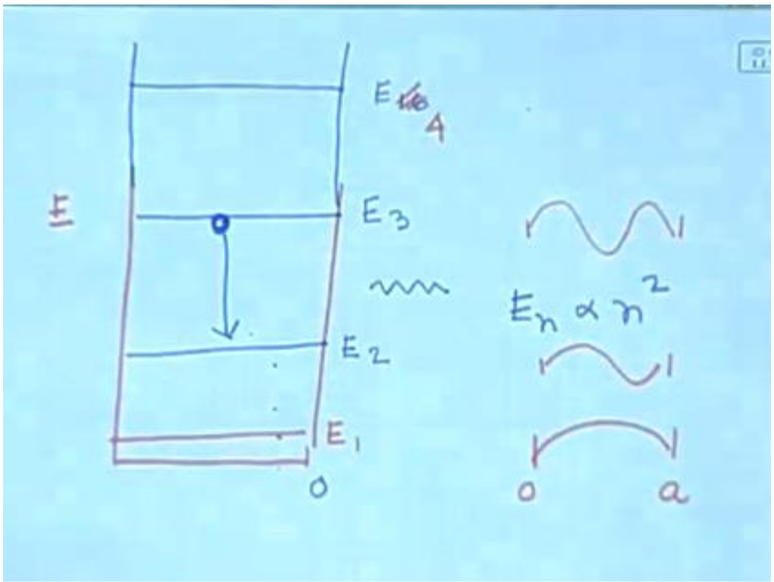

And if you do that then inside this region, the energy levels of the electron of the hole whichever is the case are roughly proportional to n square. So, if you can have trapped particles inside this small dope region. This small dope region is called a quantum well. And you can have particles trapped inside this energy levels of particles inside this, are approximately proportional to n square, as you expect in a potential well, the potential well which we have discussed. And if the electron or the hole goes from a higher energy level to a lower energy level inside this. We in this transition the electrons looses energy, and this energy is going to come out in the form of photons. So, you can have all kind of such transitions inside these quantum wells, and you can have lasers, you can use these things to make lasers, and it has got various other technological applications. So, the particle in box that we have studied, serves as a model to understand this, to understand the energy levels inside such a quantum well. And there are various other situations, where also you could use this as a model, to understand what is going on. And this gives you a rough idea of the kind of thing that go also goes on in an atom, where you find discrete energy levels.

Although the mathematics there is a little more difficult, because the potential is not a step, but something else, but this gives you a rough idea of the fact that you expect to get discrete energy levels and so forth. So, this is our discussion of the energy levels. Now let me also discuss the nature of the wave function. So, we have looked at the wave function for the ground state, the first excited state. We also know what the wave function is going to look like for the second excited state, the third excited state, etcetera. So, we saw that the ground state wave function is going to look like this. It has no nodes, the only two nodes are at the end points, it has no nodes in between. So, this is what the ground state wave function looks like. The first excited state wave function is going to have one node in the middle.

The second excited state wave function, is going to have two nodes, is going to look like this. And similarly the third excited state, is going to have three nodes. This is not 16, sorry this is 4. So, the n equal to 4 is going to is a third excited state, it is going to have 3 nodes. So, you see that, you can relate the, you can predict the number of nodes that you expect in the wave function. The ground state will have no nodes, the first excited state will have a single node, the second excited state will have 2 nodes, and the third excited state will have 3 nodes etcetera. And this again is a very general feature. So, there are you find this feature again in all situations, where you have a particle trapped in a potential. If the particle is in the ground state, you will have no nodes in the wave function. If it is in the first excited state, you will have 1 node etcetera. This is a very generic feature, it is not true just for this, it is true for all situations where you have a particle trapped in a potential, where a particle is confined, is bound in a potential. Finally, let me also discuss the analogy of this problem, with the standing waves that we have discussed.

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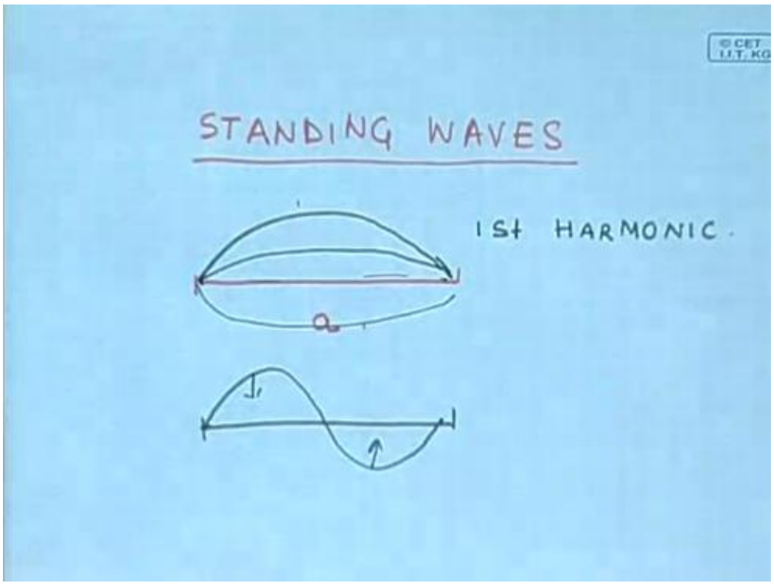

So, the relation of a particle in a box with the standing waves… So, remember we have discussed standing waves in a variety of situation; we could have standing waves in a variety of situations. So, for example, if you had a string of length a. There are certain solutions, standing wave solutions, and the spatial dependence of the solution does not move forward or backward. It just evolves in the whole wave, just the whole pattern just changes with time, and these are what are called standing waves. So, for standing waves we had these harmonics, we had the first harmonic. The first harmonic; For example, if you have a string, in the first harmonic the disturbance of the string looks like this. And with time, the string vibrations look like this, whole moves down, and then the vibration become 0, and then it goes like this, and it goes back and forth. So, this is the first harmonic for a standing wave; for example, a string. And then you have the second harmonic; in the second harmonic you have one node over here, and with time this goes down and this goes up, and the vibration this goes up and the whole thing moves up and down like this. So, in strings for electromagnetic waves, elastic waves, if you put boundary condition that that wave has to vanish at the two edges, you have these standing waves. And the wave the disturbance pattern of the standing waves looks exactly the same, it is the exactly the same, as the different wave functions that we have obtained for a particle in a box.

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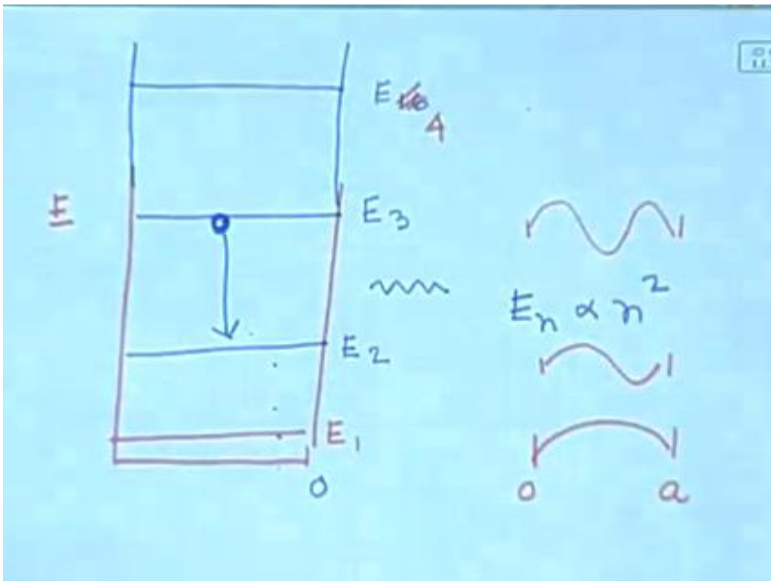

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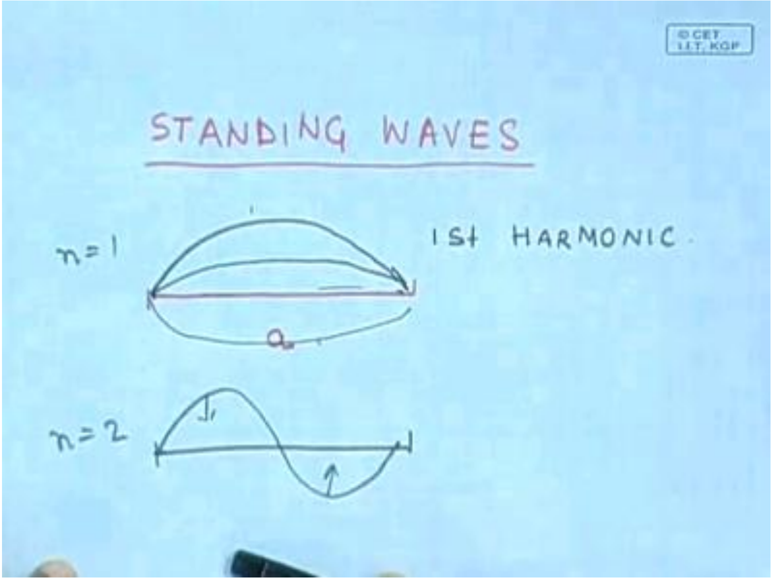

You have the ground state, the first excited state, the second excited state etcetera; n equal to 1 2 3 etcetera. These are also labeled by n equal to 1 n equal to 2 etcetera. So, the wavelength of the lowest energy state, is the twice the length of the box or the twice the length or the string. The wavelength of the first excited state, or the first second harmonic, is exactly equal to the size of the box or the size of the string etcetera. So, the spatial dependence is exactly the same, but there is a very. There is also big difference between these two.

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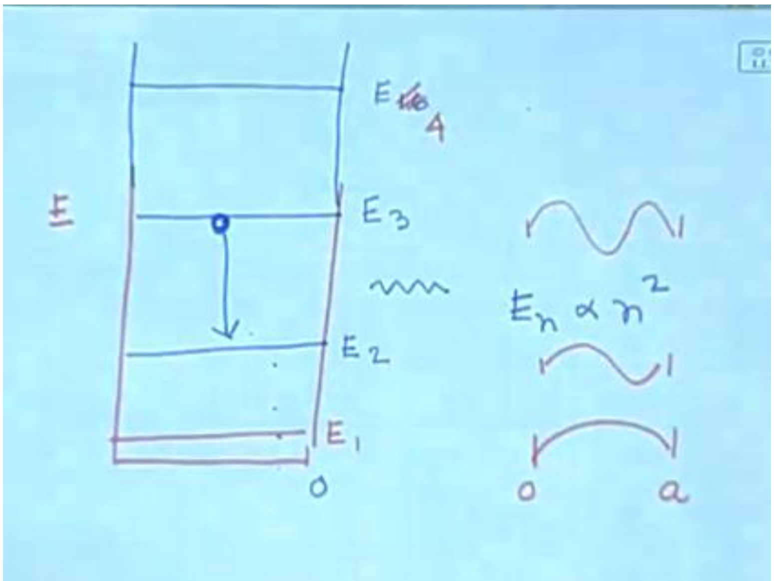

The difference is as follows, if you look at the energy levels of a particle in this box, the energy levels go proportional to n square. Now remember, how the energy levels, the energy is the angular frequency into h cross. So, you can say that the angular frequency of these different waves, also increases proportional to n square.

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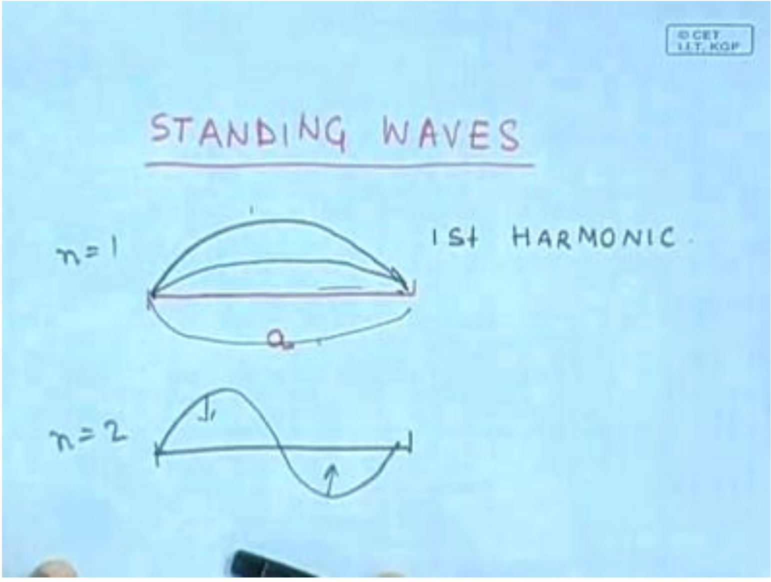

But if you ask the question how does the angular frequency of the standing waves increase. You will remember that the angular frequency increases proportional to n. For a standing wave the second harmonic has a frequency, which is exactly double the frequency of the first harmonic. The third harmonic has a frequency which is three times the frequency of the first harmonic so forth. So, the frequency increases proportional to n for the standing waves in elastic rod vibrating string, or electromagnetic waves. Whereas, the angular frequency goes proportional to n square for a particle in a box.

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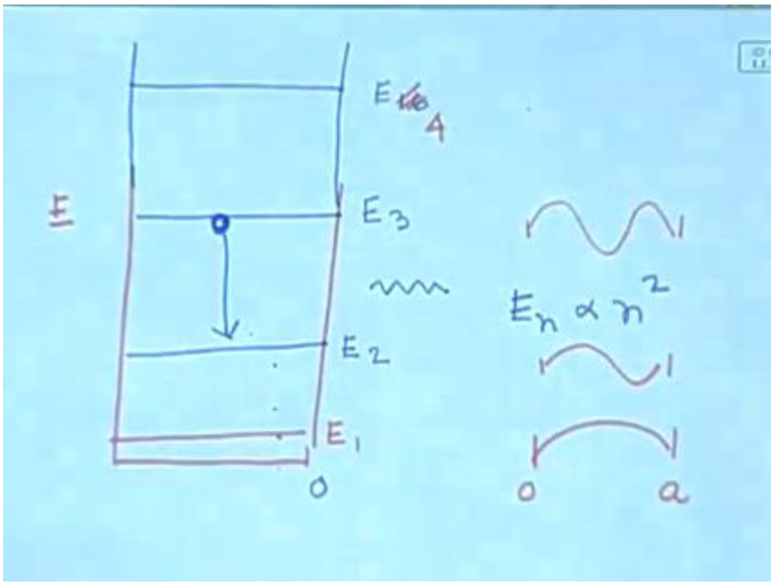

Now you may ask the question why, and the reason why this difference occurs, is essentially related to the dispersion relation. The wave function is governed by a Schrodinger equation, and the dispersion relation for this particular wave, the wave function, if you remember.

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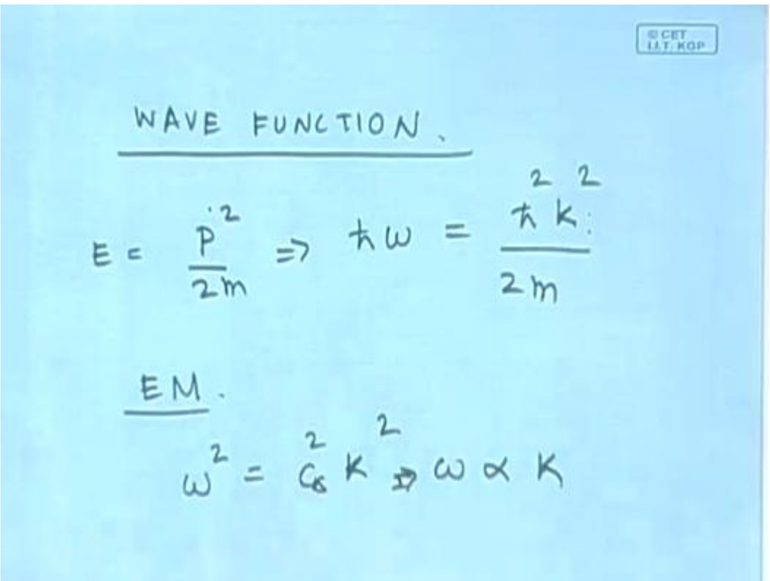

WAVE FUNCTION.

$$E = \frac{p^2}{2m} \Rightarrow \hbar\omega = \frac{\hbar^2 k^2}{2m}$$

EM.

$$\omega^2 = c_0^2 k^2 \Rightarrow \omega \propto k$$

So, for Schrodinger equation or the wave function, the dispersion relation comes from p square E is equal to p square by 2 m which gives us that h cross omega is h cross square k square by 2 m. So, we see that omega increases proportional to the wave number squared. Whereas, for electromagnetic waves or elastic waves or a vibrating string, we have a dispersion relation; omega square is equal to C s or C square k square. You see that C omega, is proportional to k, whereas here it is proportional to k square. And it is this difference which is the source of the difference in the behavior of the angular frequency with the different excitations. Finally, there is another point which I should make. We have seen that for a particle in a box. We have these different wave functions n equal to 1 is the ground state n equal 2 is the first excited state, n equal to 3 is the second excited state.

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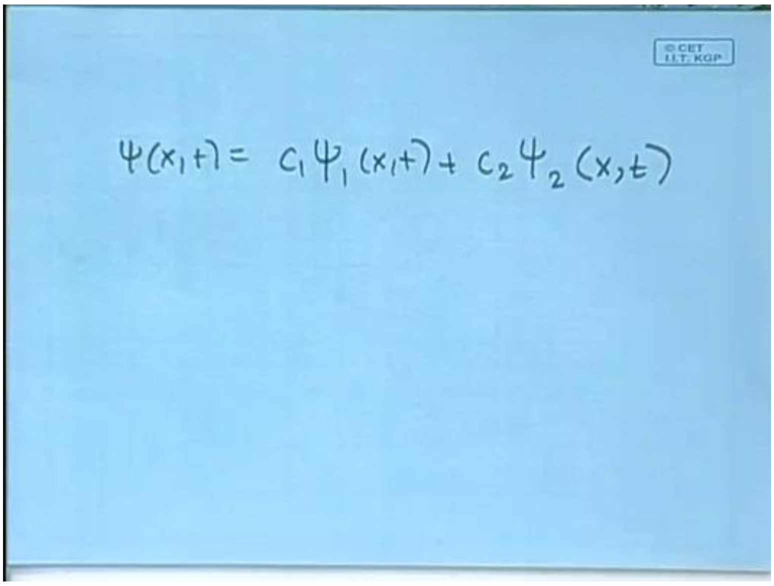

$$\psi(x,t) = C_1 \psi_1(x,t) + C_2 \psi_2(x,t)$$

Now, I should also tell you that any linear superposition of these different states, is also a possible solution of a particle in a box. So, here I have just superposed the ground state and the first excited state. In general I could superpose many of them. This is not a stationary state, but it is an allowed solution of the Schrodinger equation. And if you make a measurement of the particles energy, you will. For this particular state you will either get E 1 or E 2, and you can calculate the probability of getting E 1; that is the mod of C 1 square divided by the mod of C 1 square plus C 2 square etcetera, we have discussed this earlier. So, the point here is that any linear superposition of these different solutions, is also a possible solution. So, let me end today’s lecture of a particle in a box, a particle in a potential well over here. And the point which you should bear in mind, is that this the particle in a potential or the particle in a box, is the prototype for all situations where we have a particle which is bound.