Physics I : Oscillations and Waves
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Lecture - 40
Potential Well

Good morning. We have been discussing a particle in a potential.

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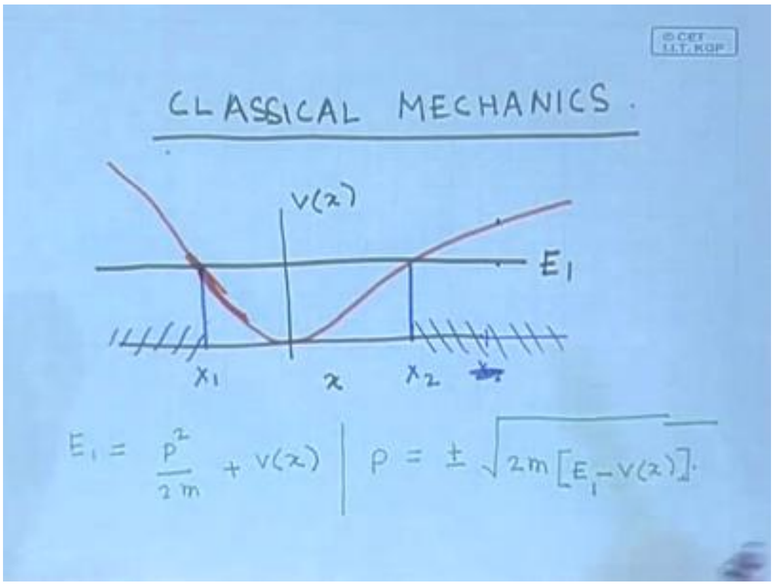

And in the last class I have told you that in classical mechanics if we have a particle in a potential which looks like this. We are dealing with the static potential let me remind you and the particle is free to move only along 1 direction the x axis and we have some potential which looks like this. And the particle has an, energy E 1 then we know that the energy of the particle is going to be conserved if the potential is static the energy of the particle is going to be conserved. And the energy is a some of the kinetic energy P square by 2 m plus the potential energy v x from which we can determine the momentum of the particle. Which is plus minus square root of 2 m into the difference of the energy and the potential E minus v x.

Now, what we see from this analysis is that you will not the, you will not find the particle in regions where the potential is more than the energy. Because if the particle goes into such a region where v x is more than E this number becomes negative. And the square root of a negative number is imaginary momentum is a real quantity which you can measure so, it cannot be imaginary so, that is ruled out. So, it essentially tells us that if the particle has energy E 1 and if it is moving in a potential like this. The motion of the particle is going to be restricted in the range x 1 to x 2 where the potential is less than the energy in this range.

The particle will not venture into the region where the potential is more than the energy of the particle total energy of the particle. So, it will never venture into x larger than x 2 or x smaller than x 1. So, that is the behaviour of the particle as predicted by classical mechanics the particle is going to oscillate back and forth between x 1 and x 2. Now, we are interested in studying what happens when you do a quantum analysis of this problem let me briefly recapitulate how this has to be done. And what we have already done regarding this and from I shall go ahead from there so, in quantum mechanics. We have to think of the particle as a wave and the wave is governed by the wave equation.

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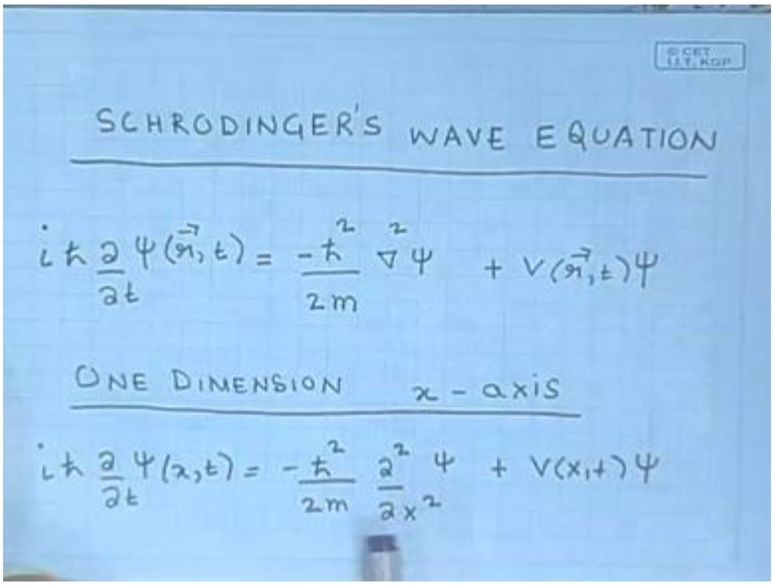

The Schrodinger wave equation and in 1 dimension this is the wave equation the Laplacian gets replaced by partial derivative with respect to x psi is a function of x and the.

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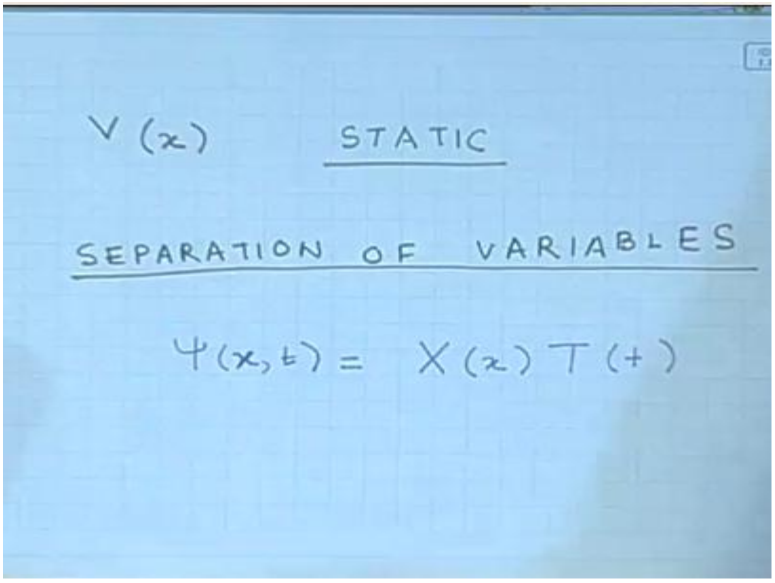

Further we had assume that the potential is static it has no time dependence where in we can use the method of separation of variables where we wrote psi as a function of x capital X a function of x alone and capital T a function of time alone.

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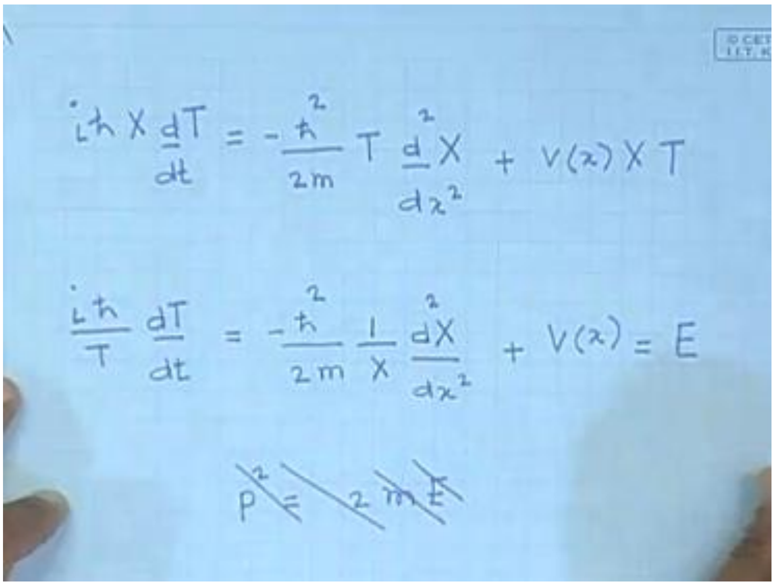

And we substituted this into the Schrodinger equation and this, what it gave us and then we divided by this psi. And we had this equation where this we had this equation when we divided by psi now this left hand term is a function of time alone. This right hand term is a function of x alone if these 2 are to be equal then they must be equal to a constant which is what I have written here. So, we have to now solve 2 separate equations 1 way this a constant another way this is a constant.

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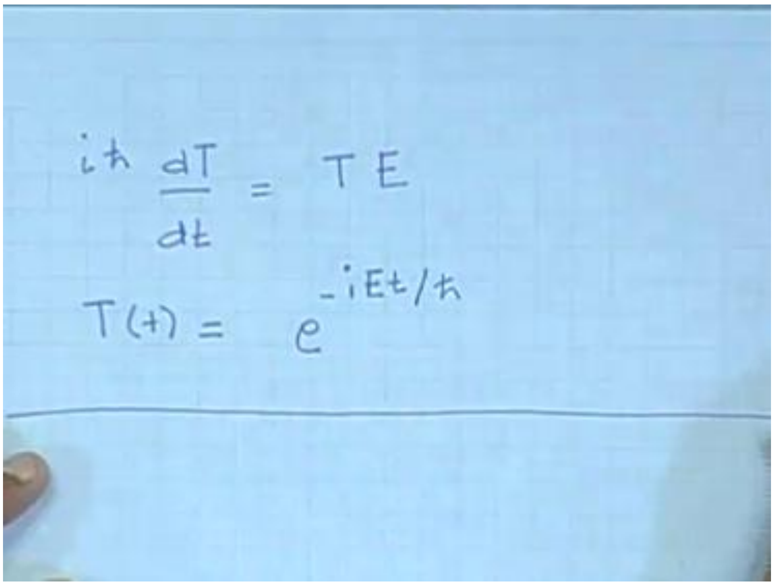

The time part is very easy to solve it is an exponential of minus i E t by h cross where E is the constant which we had introduced over here.

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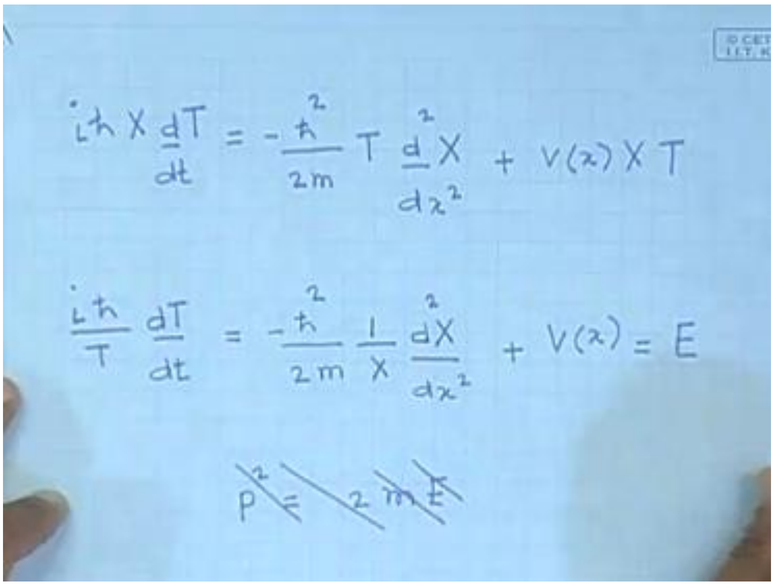

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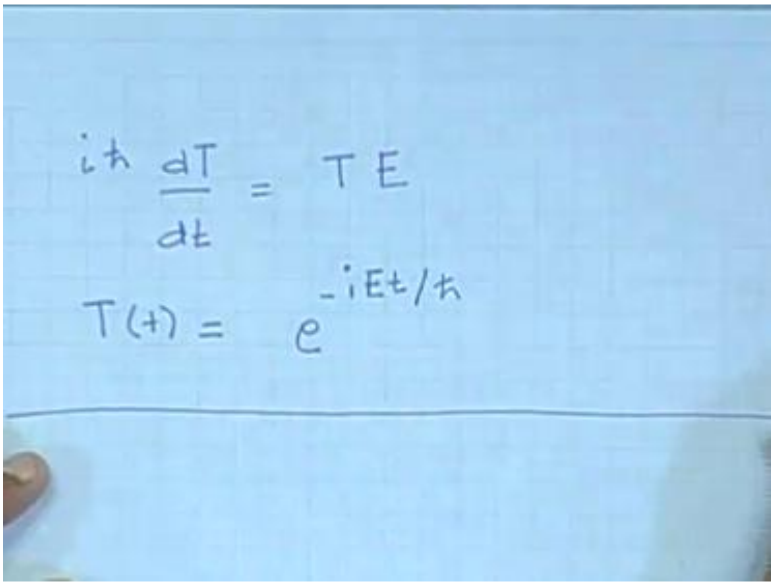

So, that is the time part of the wave function.

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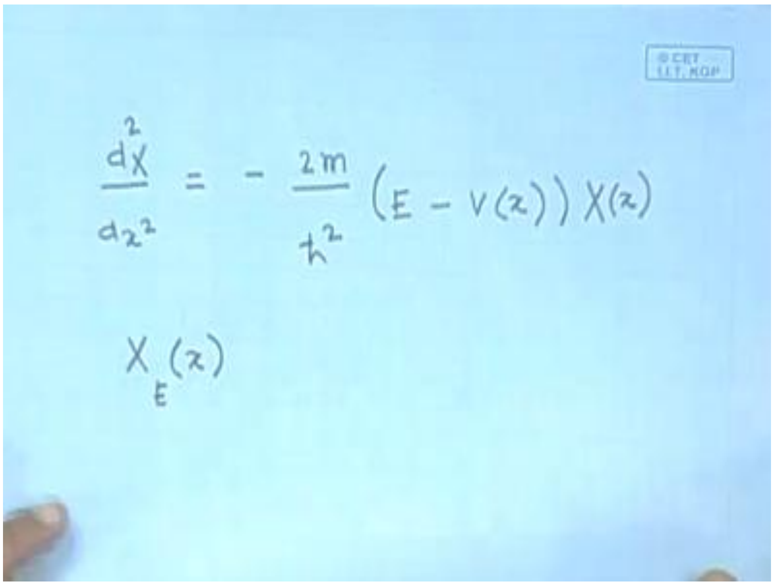

The space part of the wave function the spatial dependence of the wave function has to be obtained by solving this equation where E is the constant which we had introduced this this solution will depend on the form of v x. But there will be a solution which will depend on the value of the constant which I denote by XE as the function of the position.

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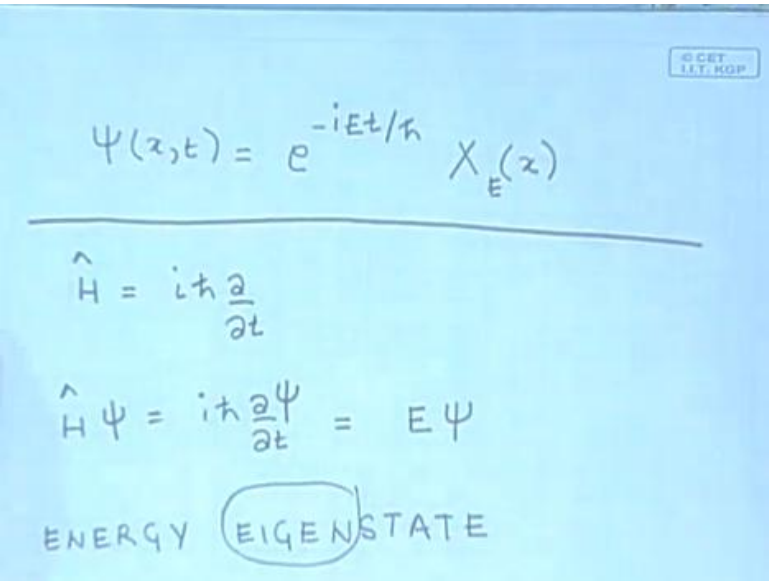

And the resultant is going to be something like this so; it is going to be the product of the time dependence and the spatial dependence. We have put it back together and this is the wave function so, in general for a static potential in 1 dimension the wave function is going look like this. It is going to be decided the form of the wave function is going to have is depends on the constant E arbitrary constant E for different values of E I will get different wave functions. And then I also told you about the of the physical significance of this constant E. If you act with this wave function with Hamiltonian operator which whose Eigen value is correspond to the energy. Then you will see that this wave function is an Eigen function of the Hamiltonian operator which tells us that if I make a measurement of the energy .That if I make a measurement of the energy I will get this value E as my result so, this is a energy this is an energy Eigen state whenever I have a particle in this state I. If I measure its energy I will always get the value E that is the constant which appears in the wave function we can now, say that that corresponds to the energy of the particle.

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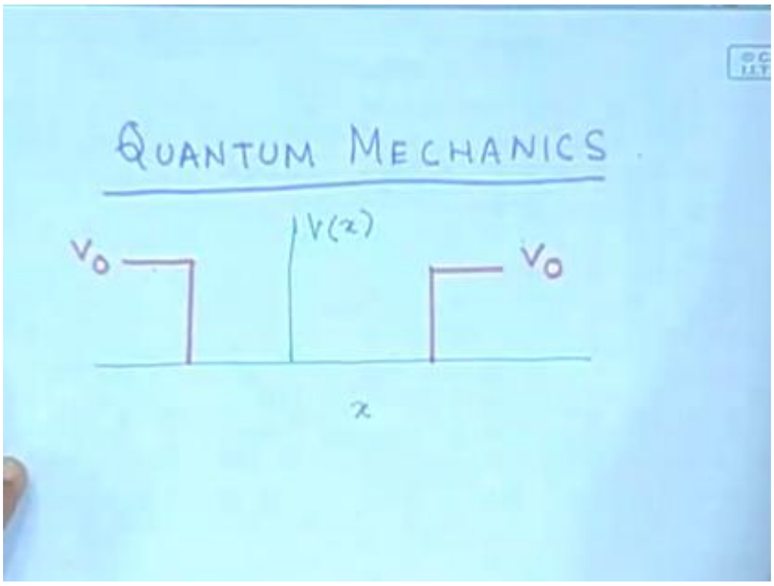

So, we for solving this spatial dependence.

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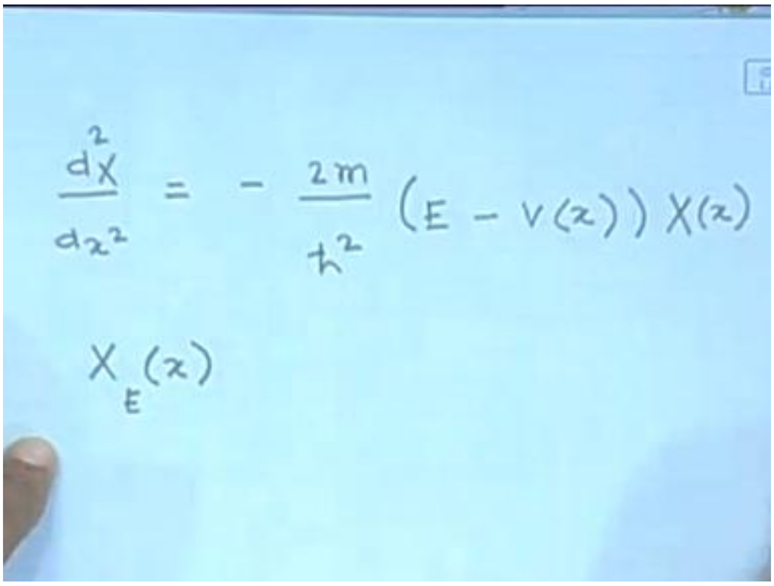

The wave the equation governing the spatial dependence depends on the form of the wave function a form of the potential the external potential. And depending on the external potential you have to this could be a mathematically challenging problem. And you may not be able to get an analytic solution if this v x is arbitrary. There are only a few potentials for which analytic solutions exits you can get the solution numerically for any arbitrary potential. Whatever, it is the solution the wave function does exists you can always determine this either analytically or numerically.

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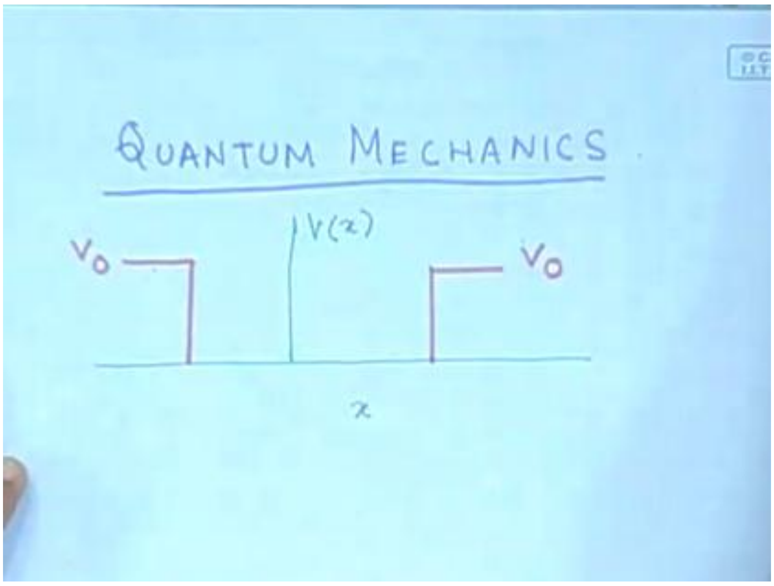

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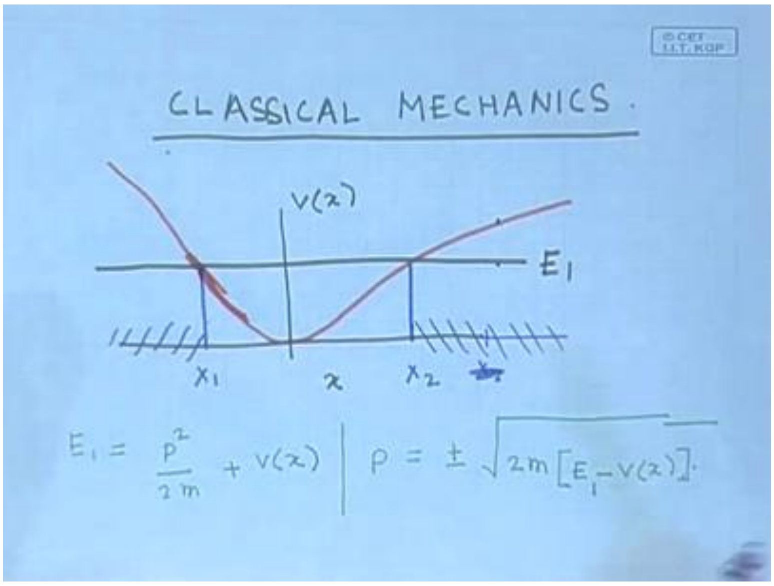

Now, we are going to consider a simple situation where the potential instead of having this kind of a variation where it varies with x in some continuous passion. We will be considering a simple situation where the potential varies in steps.

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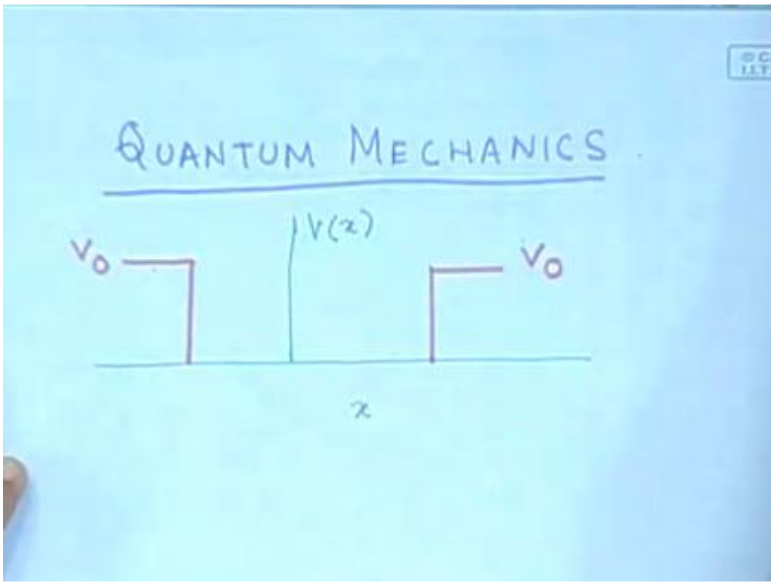

So, it has a value v naught for all x less than this value it has a value v naught for all x more than. This value in between the potential is 0 the reason why we are considering this step like potential where it is constant and constant these constants have different values. It is v naught here v naught here it is 0 here the reason why we are considering this instead of considering a situation like this is because it is relatively easier to solve.

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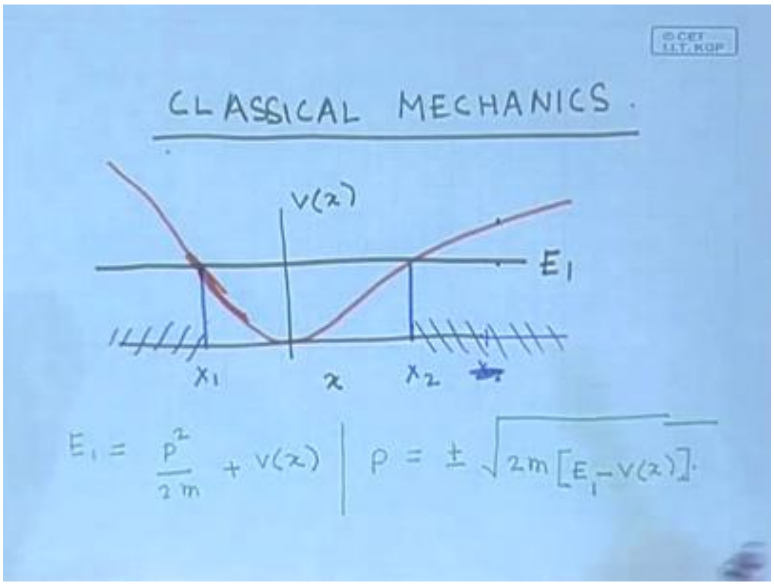

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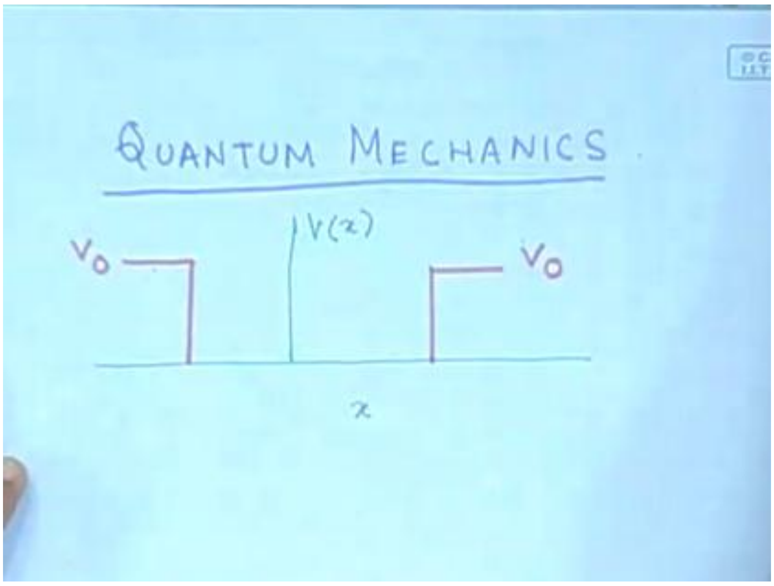

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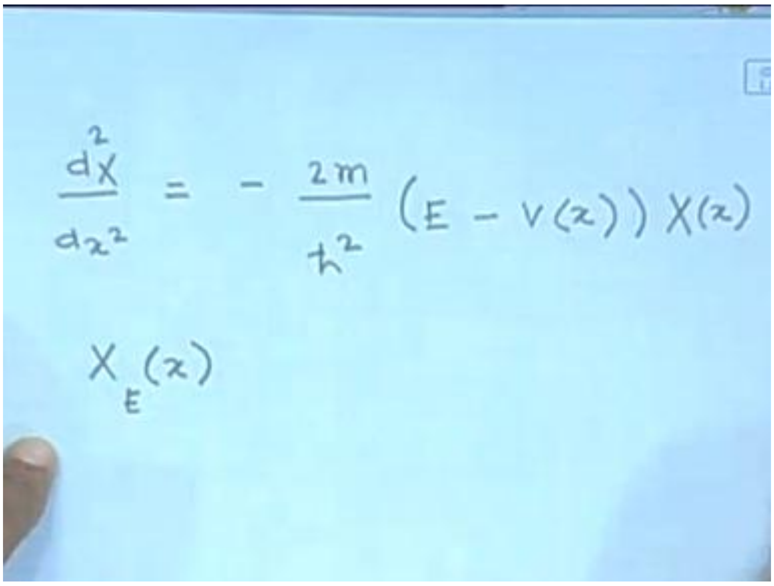

For the spatial dependence it is easier to solve this equation in a situation where the potential is a constant.

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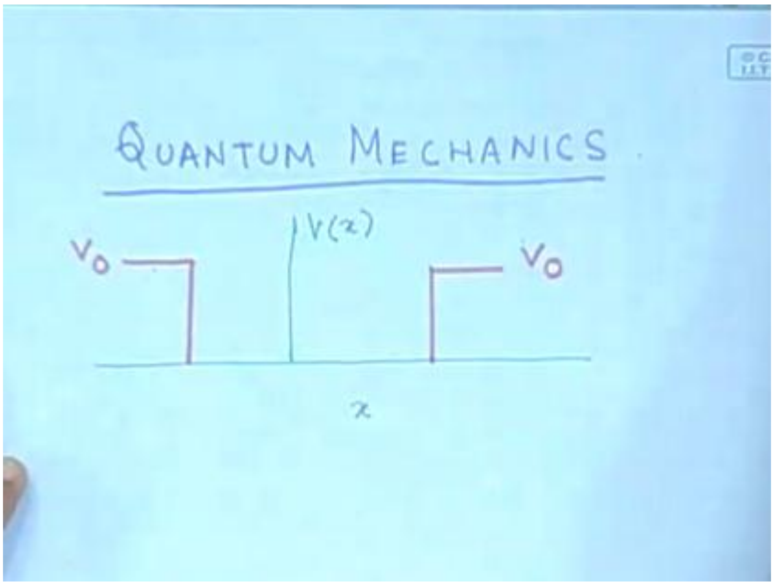

And by solving for a situation like this where the potential changes in steps.

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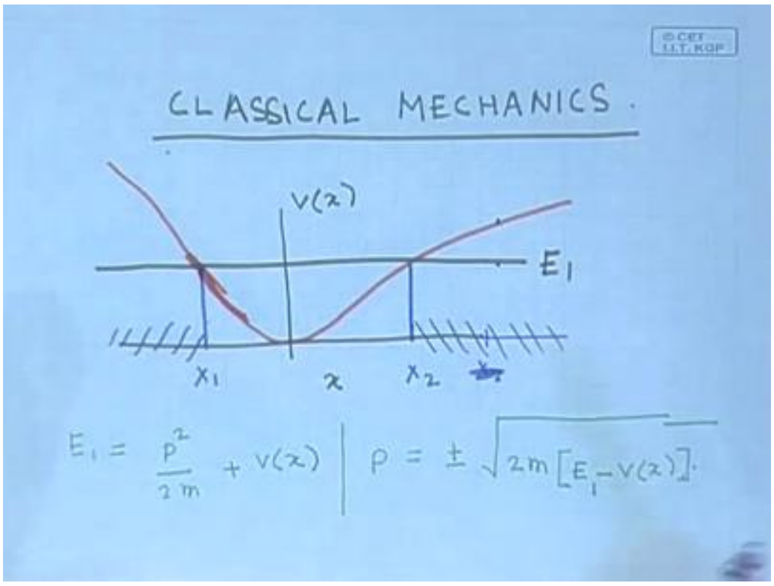

We will get some idea of what will happen in a situation where the potential the more general situation where the potential changes gradually.

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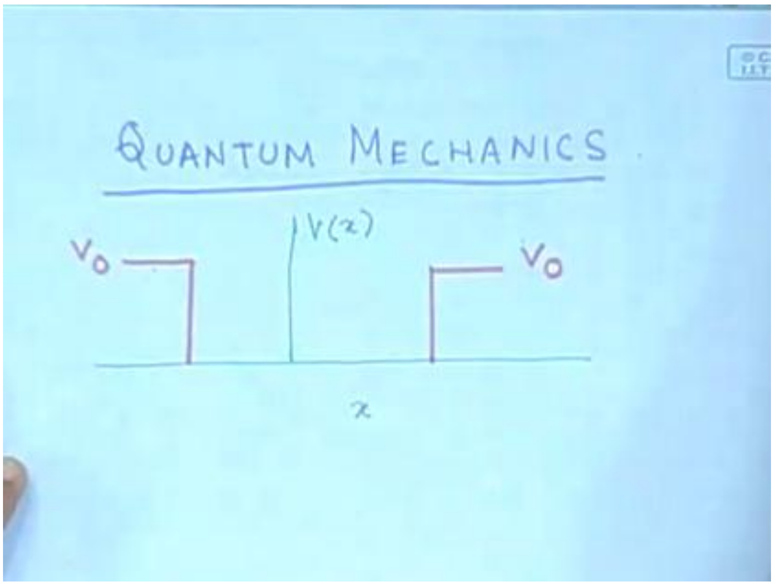

So, it is with this aim the fact that this is mathematically much simpler, but it still gives us a good idea of what would happen in a more general situation that we are considering such a step like potential. Now, we have already solved this, the wave function in the region in between where there is where the potential is 0.

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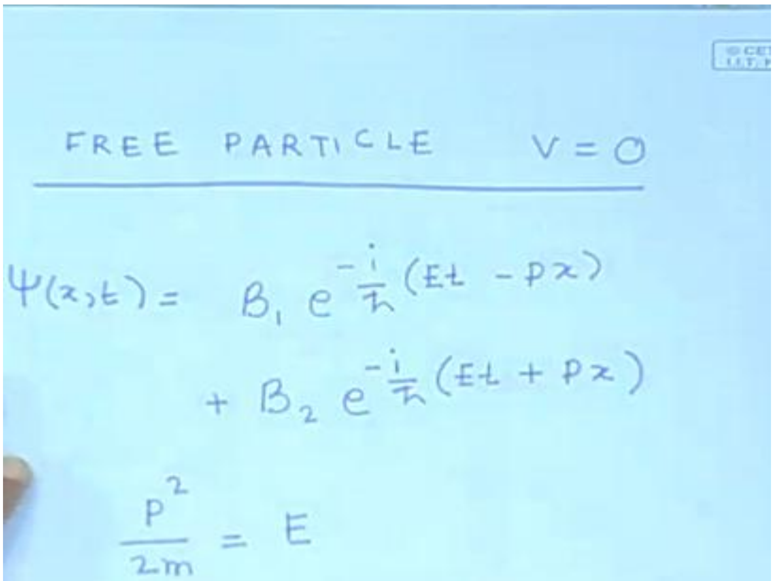

This is the free particle potential is 0 we have already worked out the solution much earlier and the solution the time dependence is e to the power minus i E t by h cross. We have this is in general whatever be the value of potential. The spatial part is just a plane wave and there could be a left travelling wave and a right traveling wave. If the super position of these 2 with arbitrary coefficients B 1 B 2 the constant P and the energy E are related as follows.

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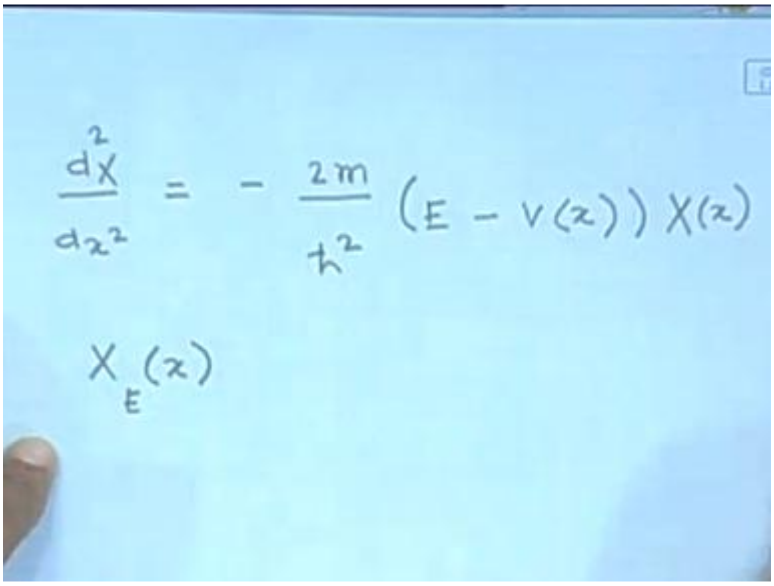

And so, like this you get this when you solve this equation they are related like this which tells you the dispersion relation for this wave.

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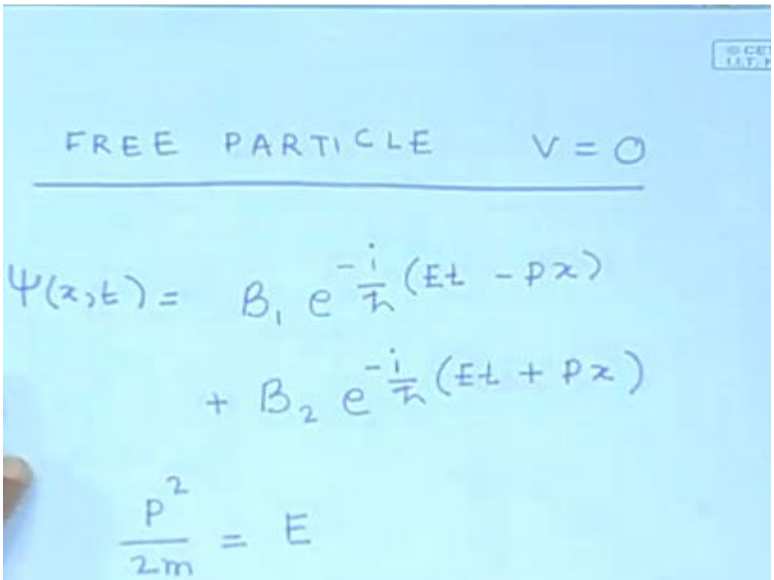

So, this, the free particle the wave function in the part of the wave where the potential is 0 we now have to solve the wave for the wave in the region where the potential has a value v naught.

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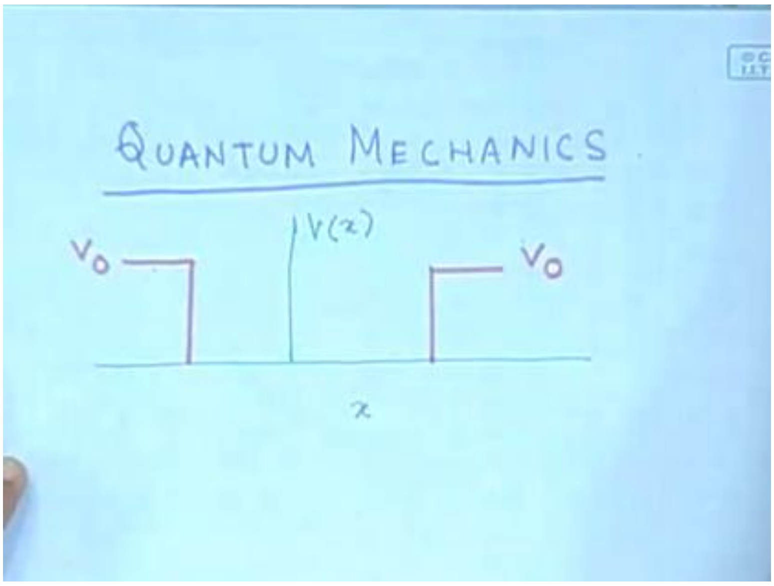

Now, there could be two possible situations which we shall consider separately the first. So, we shall we shall focus on only one of these steps the analysis at the other step is exactly identical there is no difference so, we shall restrict our attention to only this particular steps. So, we shall consider the wave function across this step and there are 2 possible situation. So, let us consider them separately the first situation is where the energy of the particle is more than the value of the potential.

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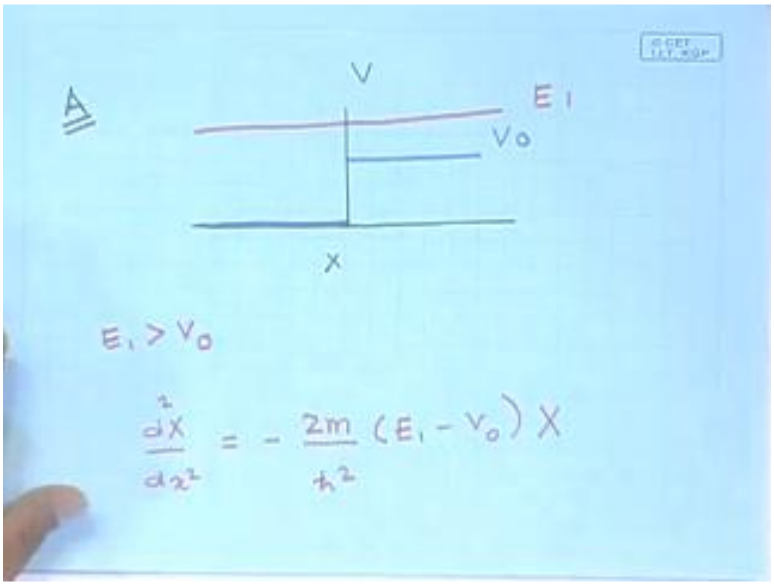

So, let we draw this. So, here is the region where the potential is 0 this is the x axis and this is v and there is a region where the potential is 0. So, this is the region where the potential is 0 and this is the region where the potential has a value v naught. And we will consider the situation where the energy of the particle E 1 is greater than v naught. So, we have already obtained the solution on this side where the particle is a free particle.

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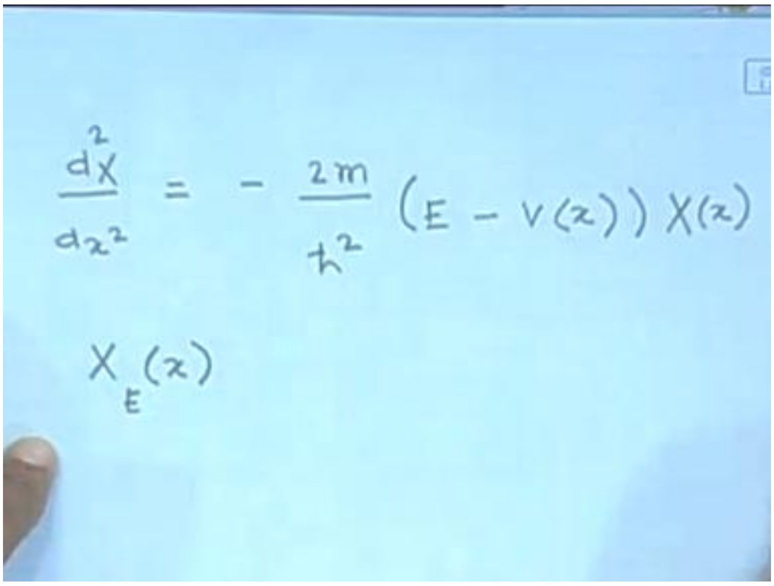

Now, we want to solve for the spatial dependence on the right hand side inside the potential with potential is a constant over there. And the constant is less than E E 1 the energy of the particle is E 1 so, this constant the potential is less than E 1 so, the differential equation governing the spatial dependence is.

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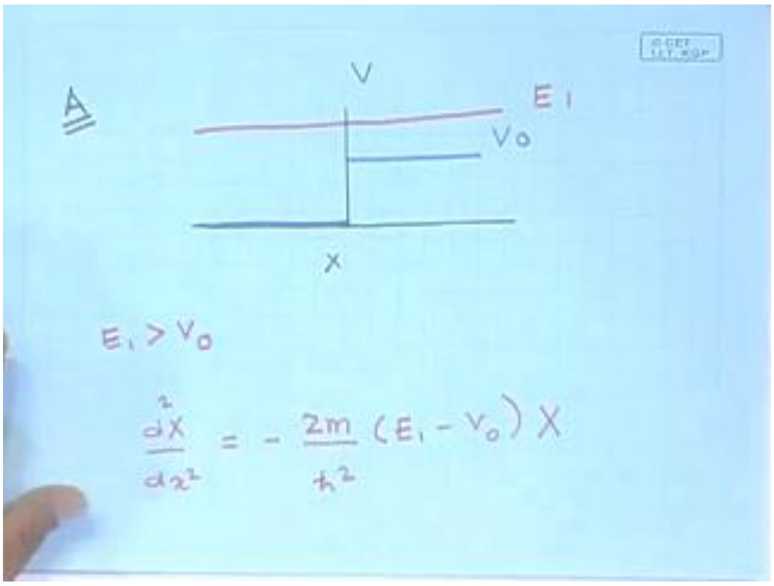

D square X dx square is equal to minus 2 m by h cross square E 1 minus v naught into x we want to solve this equation. Now, it is we will introduce another variable let we just go back to the free particle before proceeding further.

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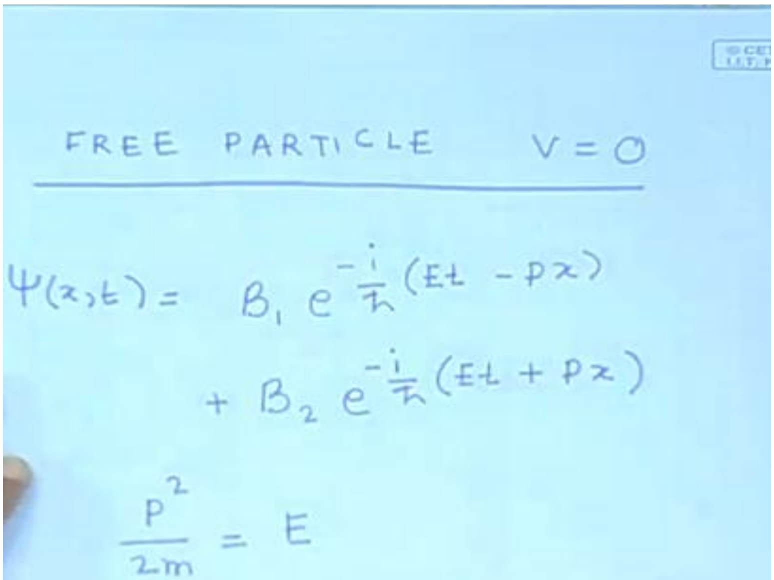

So, this was the free particle solution we had assumed that the energy of the particle was E and the momentum turned to be plus P and minus p so, we will follow the same notation here.

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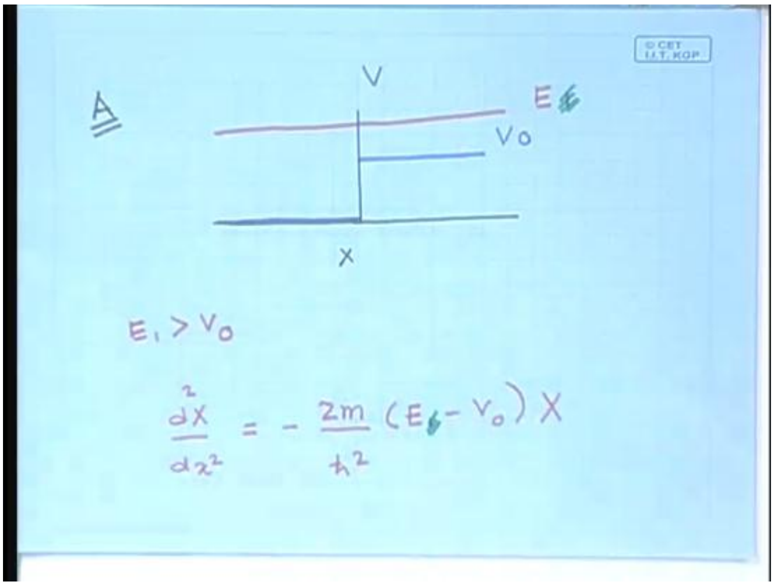

We will not use E 1 we will just use say that the energy of the particle is E and proceed okay. So, the particle has energy E and this is the equation that we wish to solve so, what we will do is we will define a new variable P. So, that the numerator over here is P squared we will define a new variable P prime.

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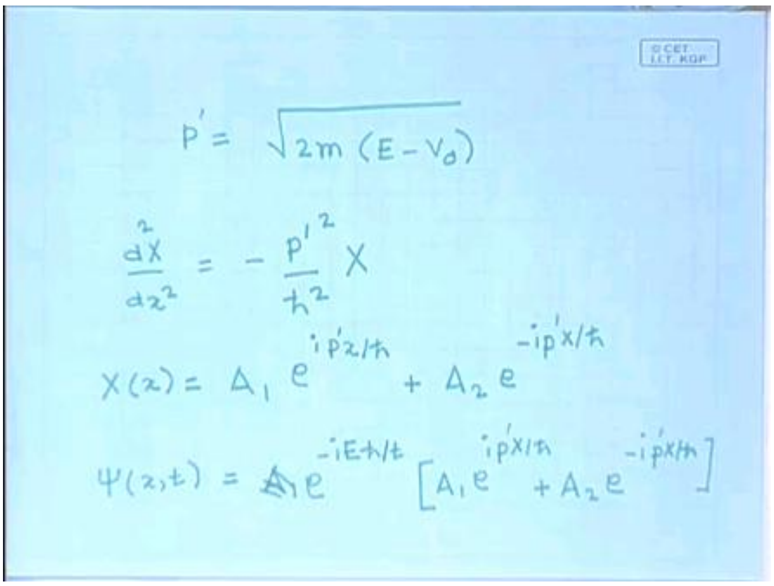

The interpretation of P prime is quite straight forward you can see.

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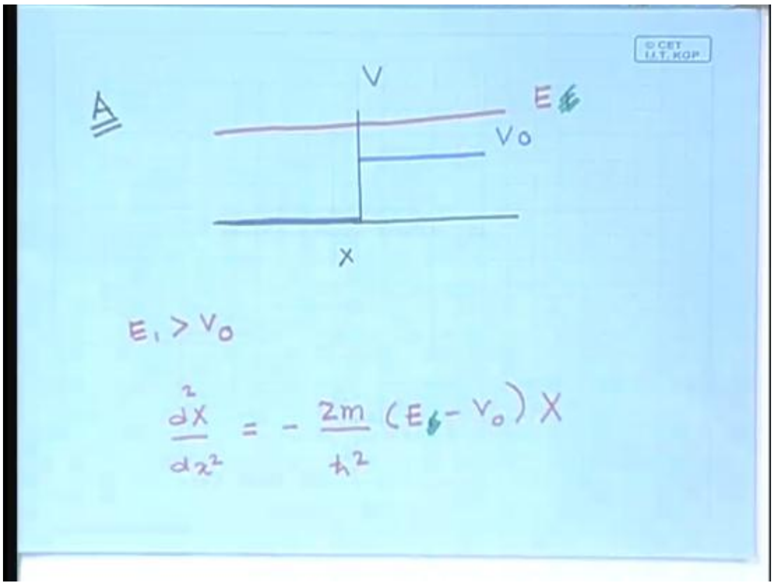

That it is the momentum of the particle in this region inside the potential and with this definition with this new variable P prime.

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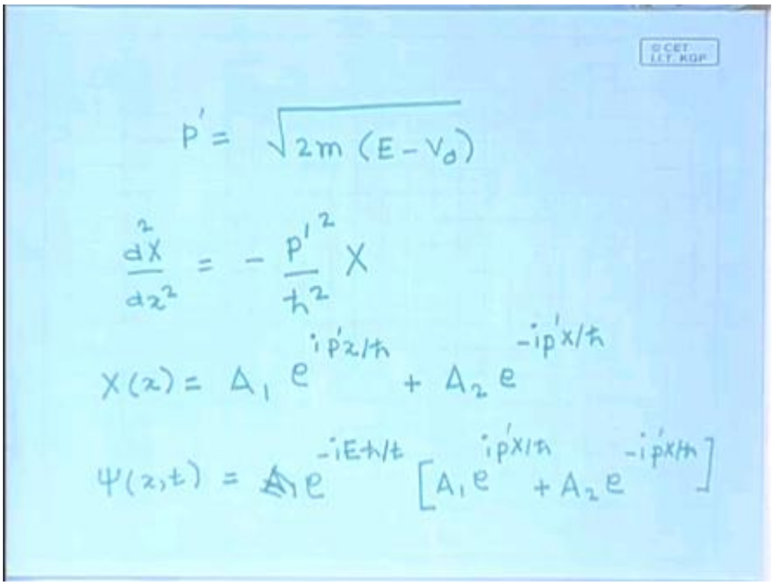

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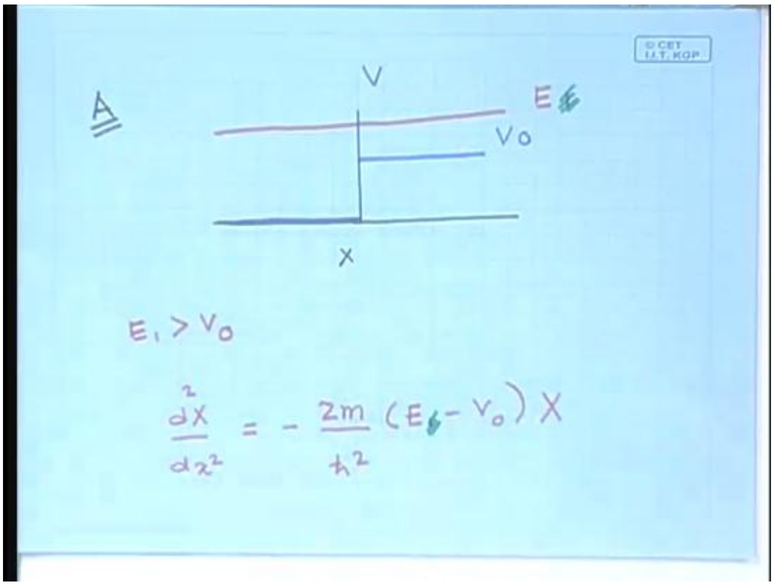

We can write this equation so, this will become P prime square and we can write this equation.

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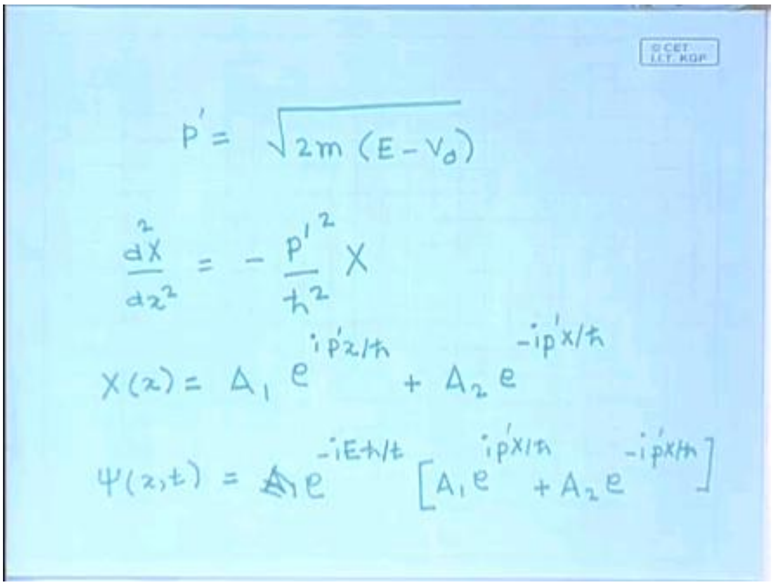

As d square X dx square is equal to minus P prime square by h cross, square X the solution to this equation is very simple it is the familiar simple harmonic oscillator equation. The solution to this is quite straight forward there are 2 possible solutions A 1 e to the power i P prime x by h cross plus A 2 e to the power minus i P prime x by h cross these are the 2 possible solutions. And we can now put in the time dependence also and we will get the total wave function psi rather it will be convenient to write it in this way e to the power minus i the energy is the same E h cross by the. The spatial difference is different inside the potential and it will be A 1 E to the power i P prime x by h cross plus A 2 e to the power minus i P prime x by h cross.

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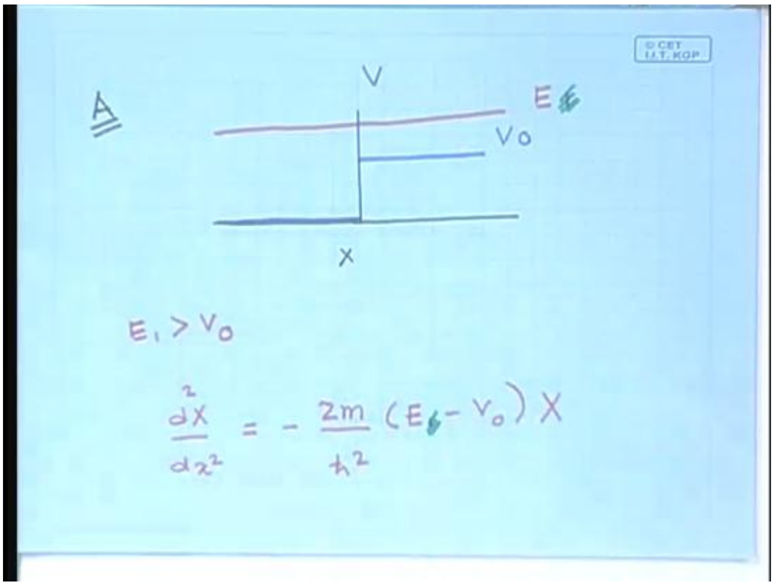

So, we have obtain the the wave function both inside on the right hand side as well as on the left hand side so, on the left hand side this is the wave function and it behaves like a free particle.

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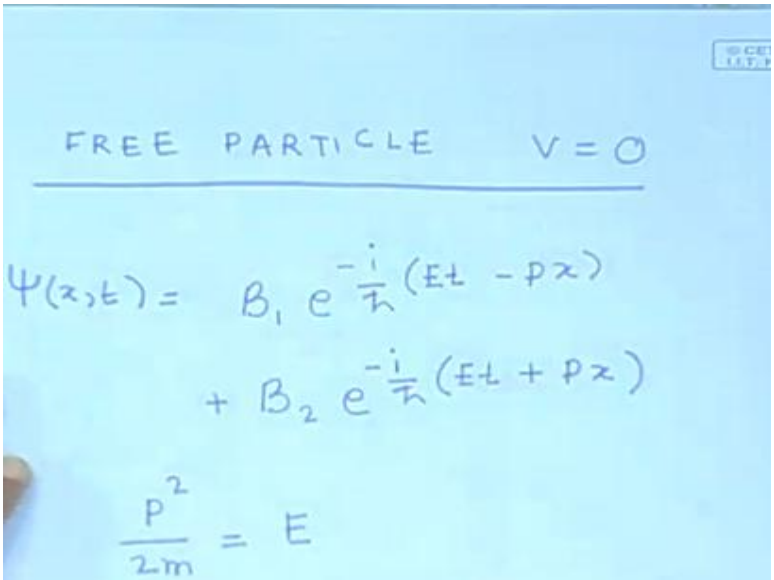

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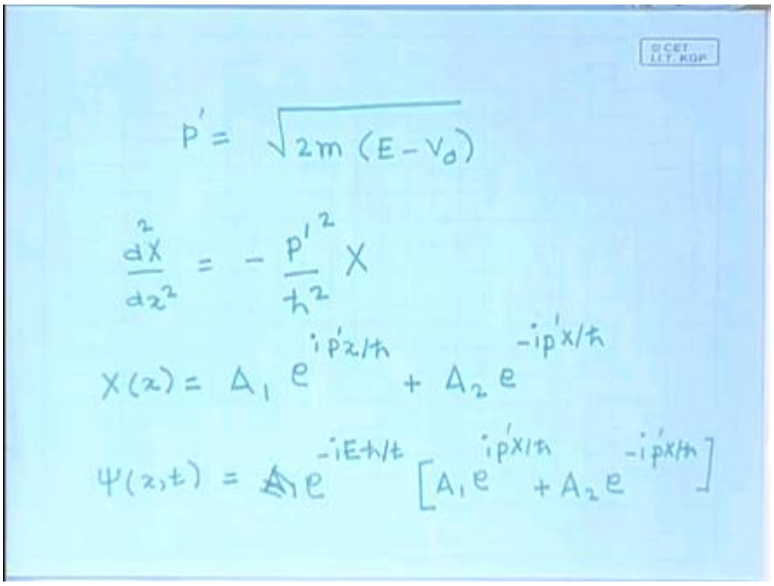

On the right hand side this is the wave function it is also again a plane wave, but with the different value of the momentum P prime which is the momentum we can calculate in classical mechanics for a particle inside the potential.

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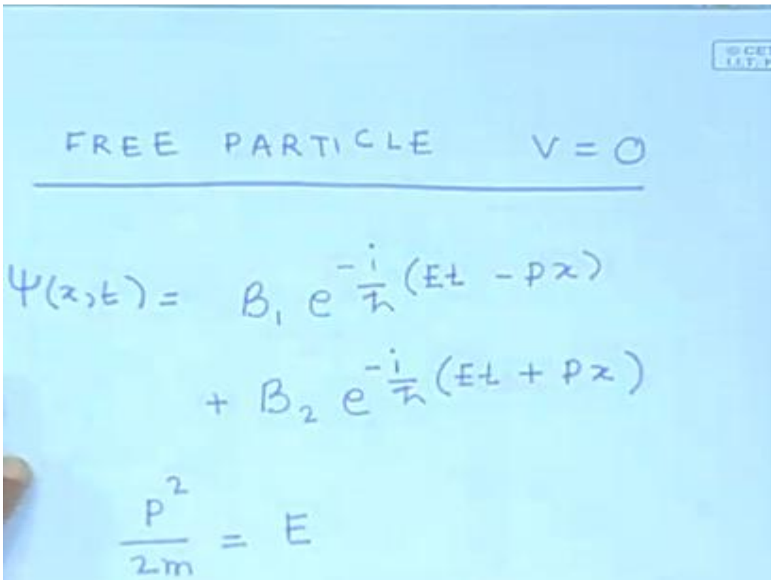

Now, note that you have a plane wave for the particle on the left hand side where it is free.

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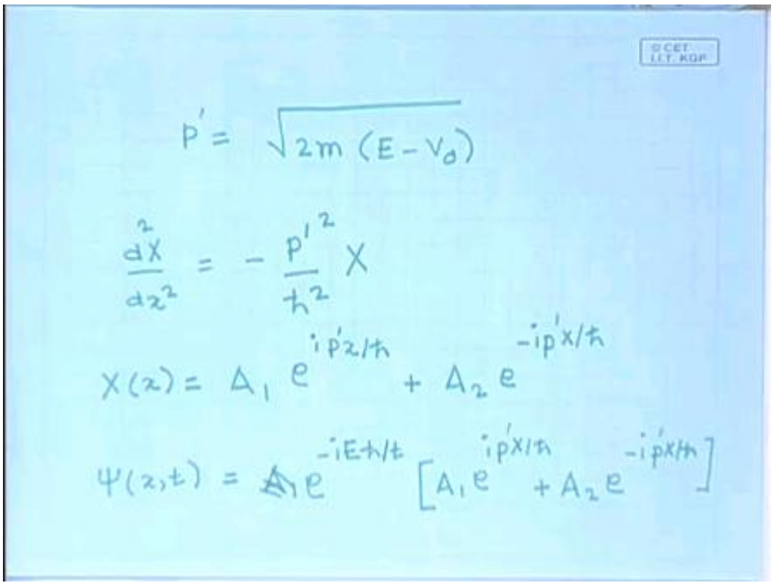

You also have a plane wave for the particle on the right hand side inside the potential.

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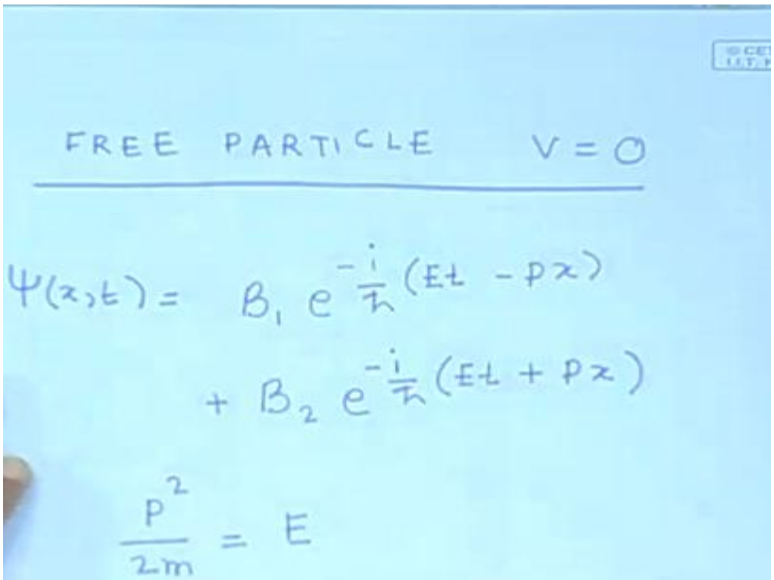

The angular frequency of the plane wave is the same the plane wave in this region.

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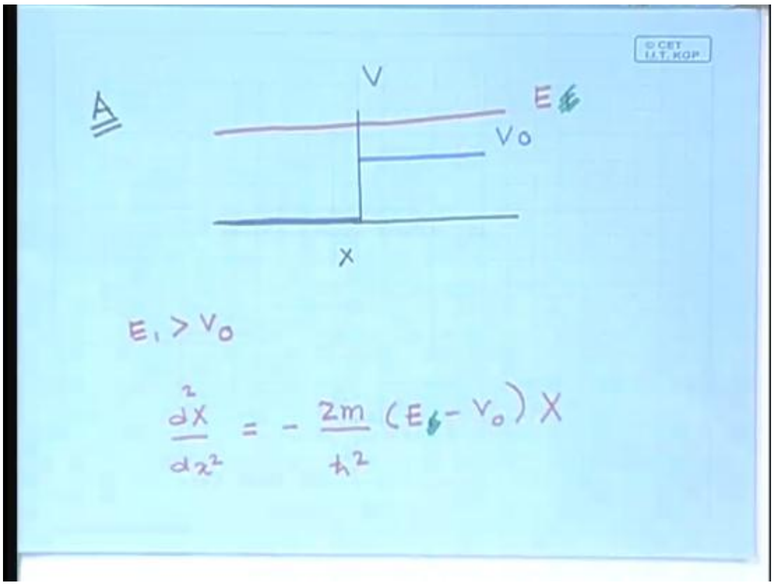

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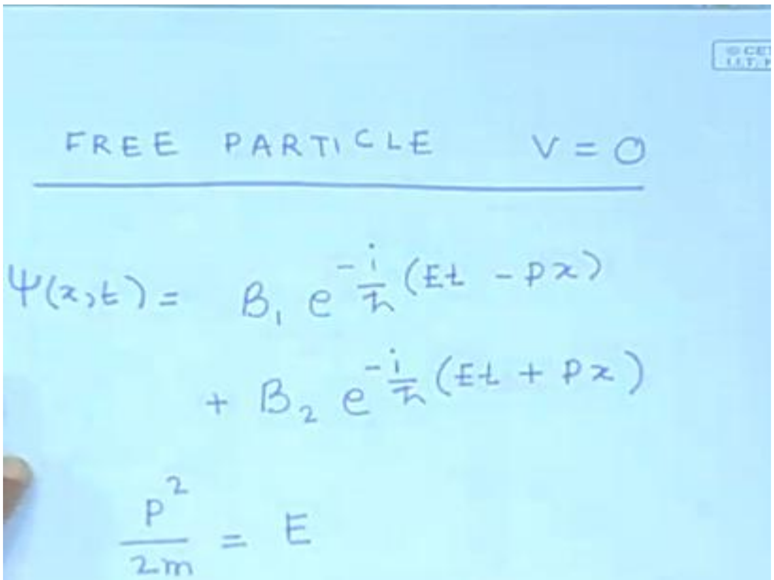

And the plane wave in this region have the same angular frequency the angular frequency is E by h cros it is also E by h cross it is also E by h cross it is also E by h cross in the region over here.

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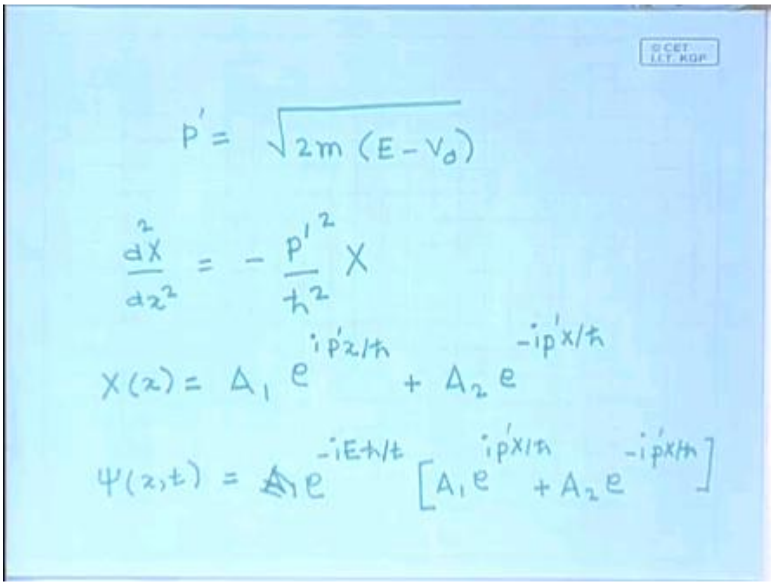

But the wave number has become different.

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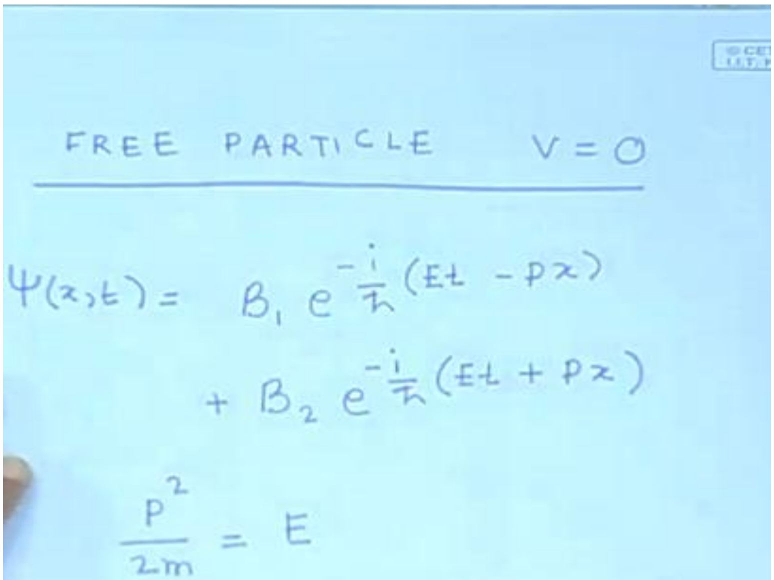

The wave number in the region where the particle is free is P by h cross the angular frequency was E by h cross the wave number is by h cross.

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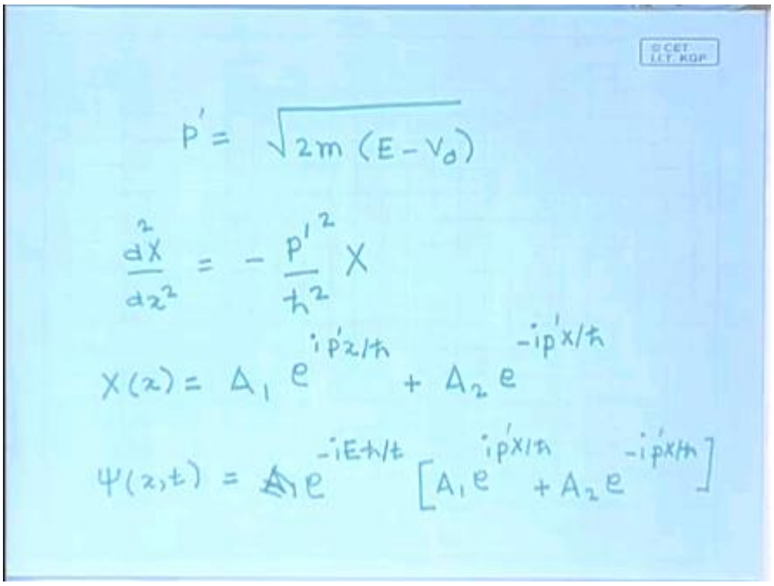

Inside the potential the wave number is P prime by h cross so, the wave number has changed. Now, let us ask the question what happens to the wavelength of the wave corresponding to the particle in the region to the left and in the region to the right. Now, the wave number inside the region where is there is a potential you can see is smaller because the wave number.

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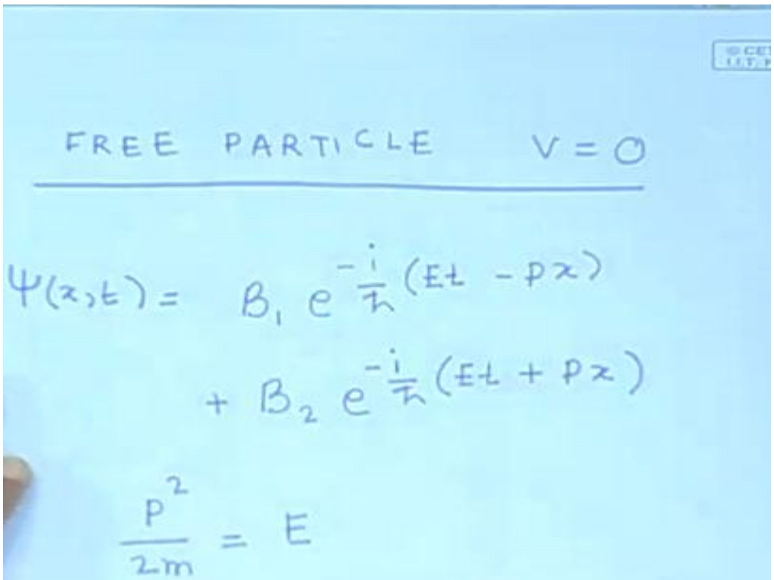

And where the particle is like a free particle is decided by E P is decided by E.

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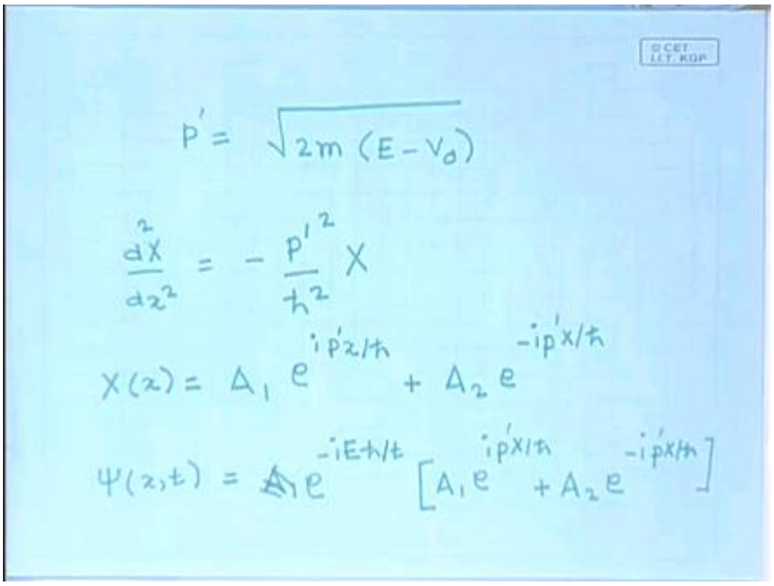

Whereas here it is decided by E minus v naught so, the wave number gets smaller the wave number is P by h cross where it is free it is P prime by h cross inside the potential. So, the wave number is smaller in the region where there is a potential and the wavelength of the particle is inversely proportional to the wave number the wavelength of the particle is inversely proportional to the wave number. So, what we can say is that the wavelength gets bigger insider the region where there is a potential the angular frequency or the frequency remains the same.

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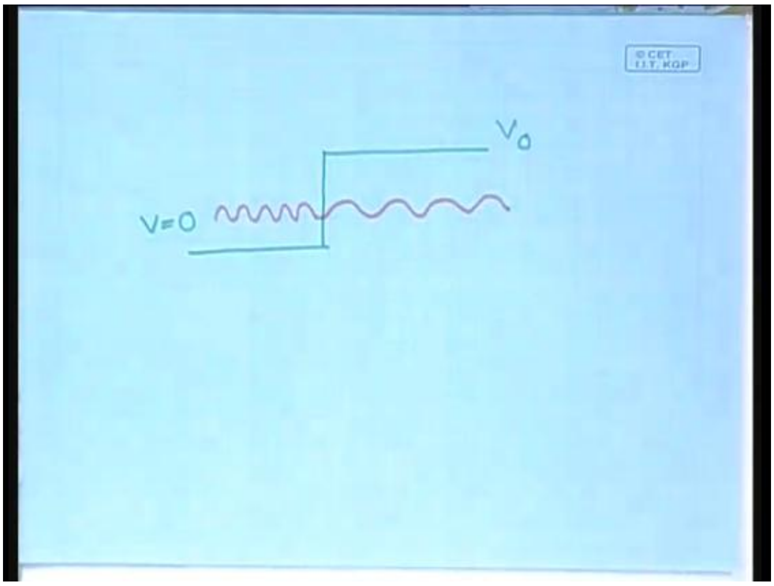

So, if draw the potential like this you have a the wave corresponding to the particle is a plane wave here as well as here. The plane wave the wavelength is smaller here the wavelength gets bigger inside the potential because the wave number gets smaller insider the potentials. So, the wave lengths gets bigger the frequency remains the same on both sides of the potential. So, this situation you see is exactly the same the situation where the energy of particle is more than. The value of the potential is exactly the same like the situation you have an, optics where light propagates from one refractive index medium to another. So, the potential can be thought of as a refractive change in a refractive index the frequency remains the same when light propagates from one refractive index material to another.

The frequency of the light wave remains the same the wavelength changes depending on the value of the refractive index exactly the same thing happens over here. And you see mathematically how it turns out that the frequency does not change, but the wavelength changes. So, you can think of the potential as a different refractive index for the wave corresponding to a particle. The particle, think of the particle corresponding to a wave the wave function propagating in a refracting medium and when it goes from one potential to another. You can think of this as a change in the refractive index the wavelength of the wave changes.

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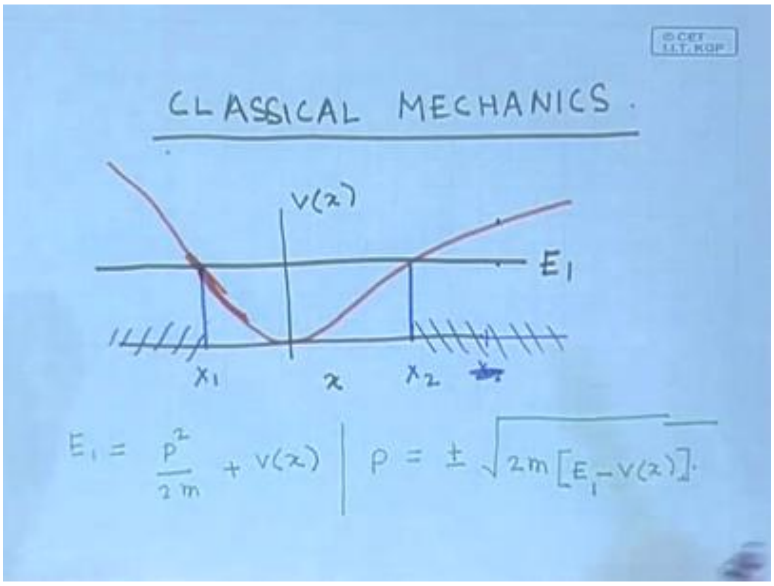

And a more general situation where the potential varies arbitrarily and it varies smoothly you can think of this as a medium where the refractive index changes continuously.

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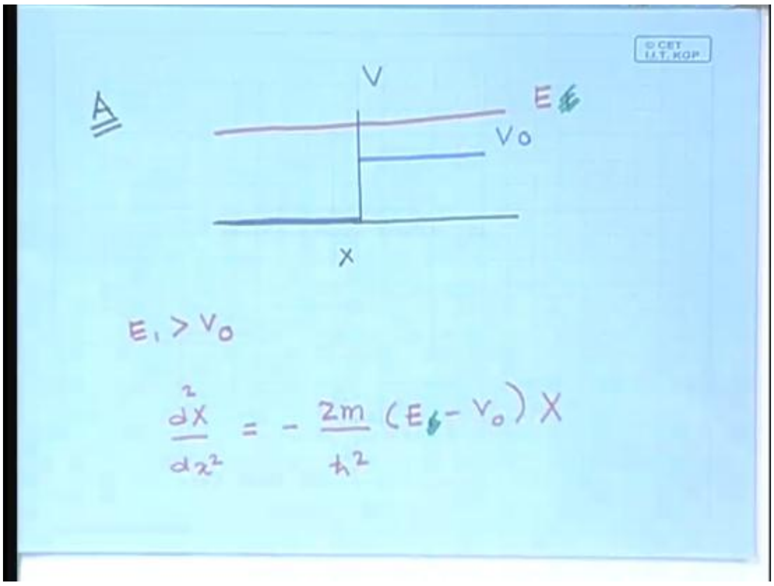

So, we have until now we have been discussing the situation where we have a particle which goes from a region where there is no potential to a region where there is potential. And the situation that we have been considering until now is where the energy of the particle is more than the value of the potential. Now, there is another possibility the other possibility is that the, that the value of the potential might be less than the value of the the more then the value of the energy. So, the value of the energy may be less than the value of the potential so, we can…

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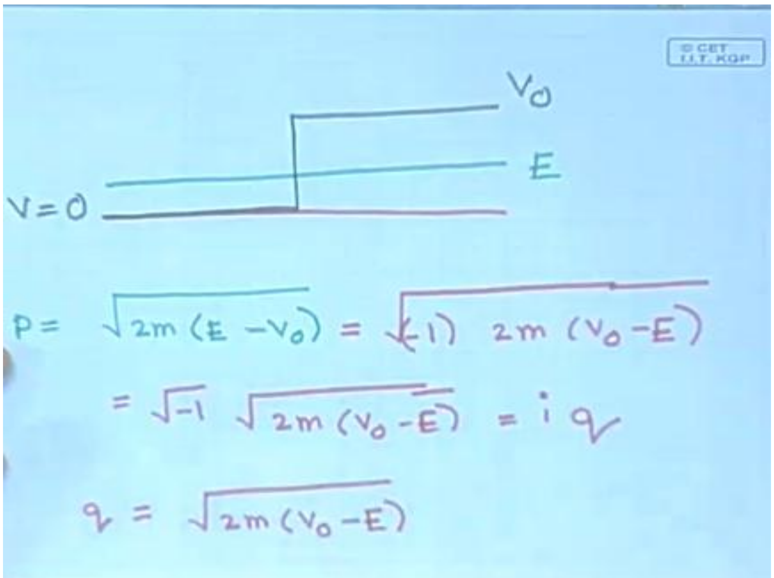

Let me draw the situation this is the x axis and this, the potential. So, it is v naught over here it is v equal to 0 over here. And the energy of the particle is now less than the value of the potential that is E so, until now the previous example that we had been considering.

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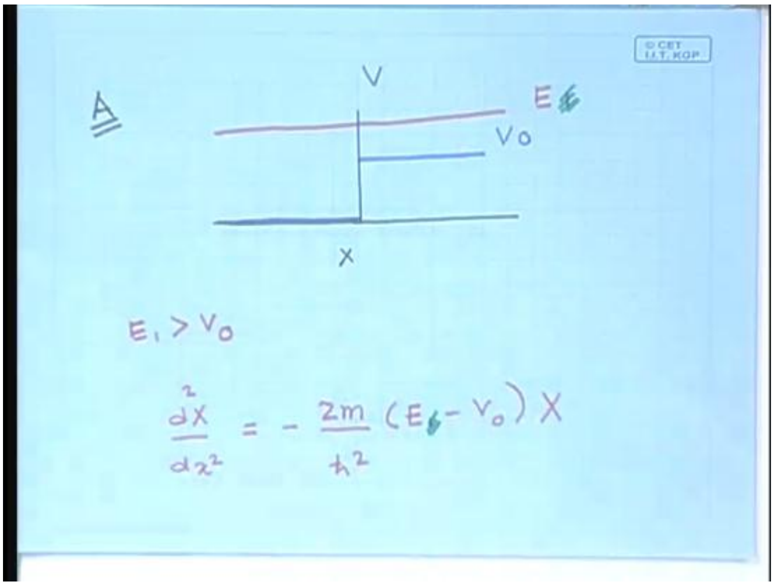

The energy was more than the value of the potential.

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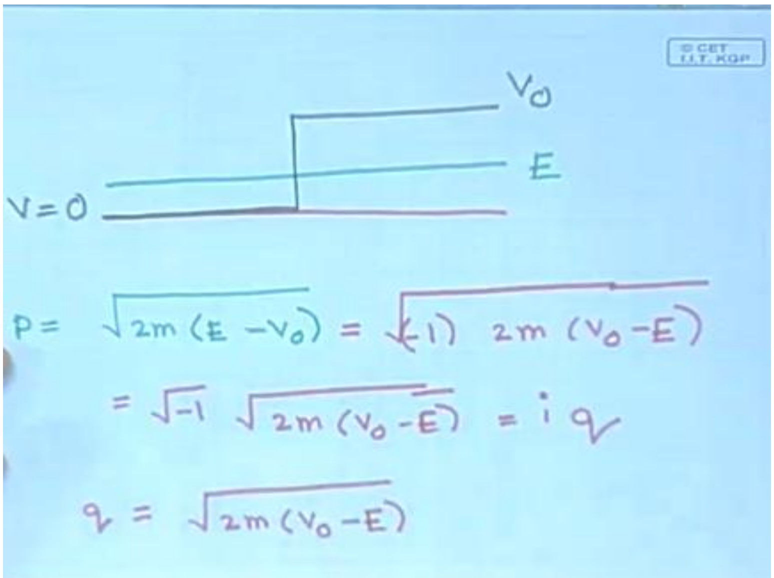

We will now consider a situation where the energy is less than the value of the potential now, in classical mechanics you would expect the particle to get reflected back. And the particle under no circumstances would enter a region where the potential is more than the energy. I have told you this right in the beginning of today's lecture we also discussed it in yesterday's lecture. So, in classical mechanics the particle would come like this and then get reflected back let us see what happens in quantum mechanics.

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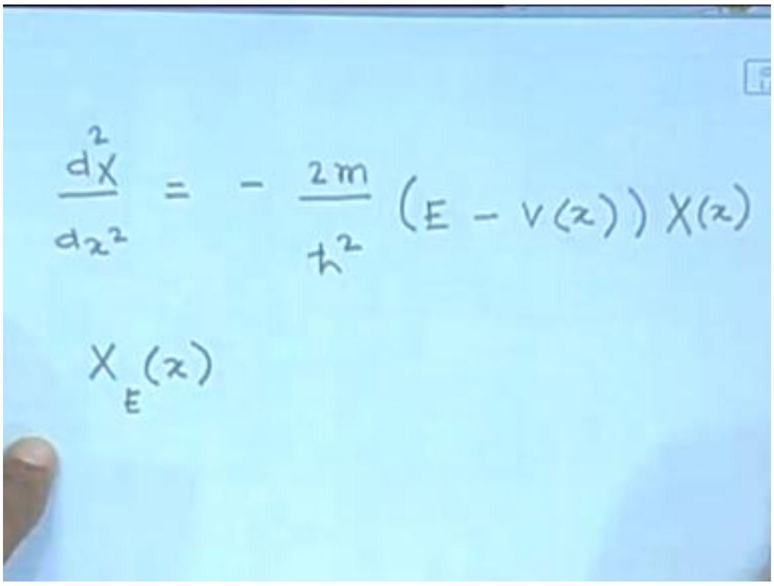

So, we have to now solve the spatial part of the wave equation in the region where the potential is more than the energy.

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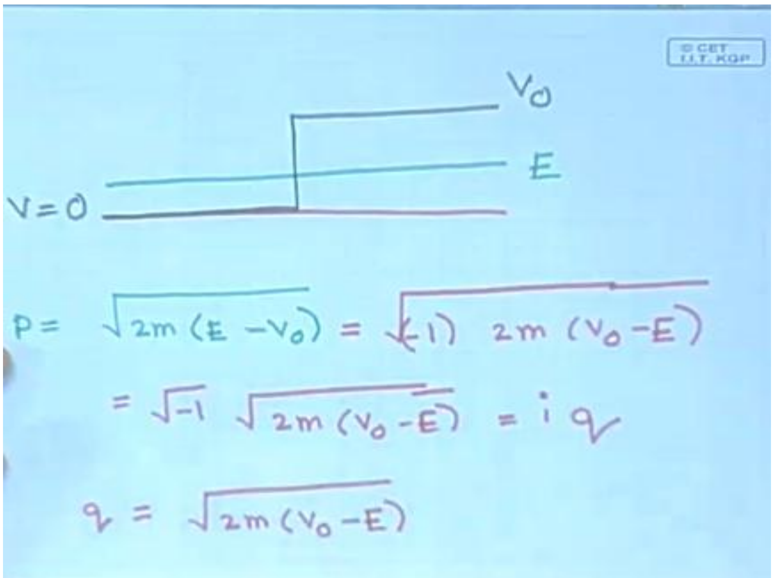

Now, the momentum of the particle in this region remembers when the energy was more we dealt with the momentum of the particle in this region. But in this case the momentum let us calculate what happens to the momentum in that region the momentum of the particle 2 m E minus v naught the momentum of the particle in this region. We see that the momentum becomes imaginary because the potential is larger than the energy.

So, it is now, convenient to define what we could do is we could write this as square root of minus 1 into 2 m v naught minus E. Now, this is a positive number so, the square root of 2 m v naught into E is a real numbe. So, we can write this as square root of minus 1 into square root of 2 m v not minus E square root of minus 1 is i and we will define this as q so, q is the square root of 2 m v not minus E. So, this imaginary momentum in this region we have written as i into some real number q where q is the square root of 2 m v naught minus E with this in terms of this variable q. So, let we write down here explicitly what q is q is the square root of 2 m v not minus E in terms of this variable q we can write.

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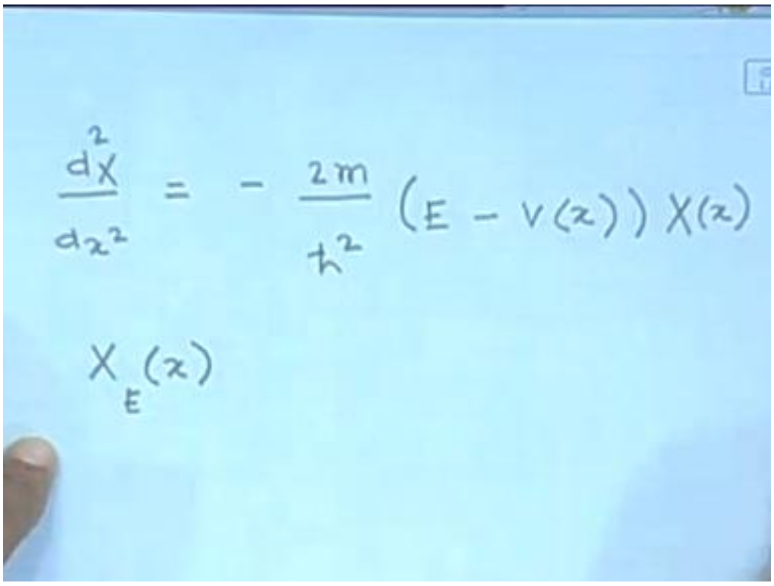

The differential equation governing the special part of the wave function so, v naught minus e if i write this as v not minus e this minus sign will be gone. And the numerator becomes q square by h cross square so, the differential equation governing the spatial part of the wave function that becomes.

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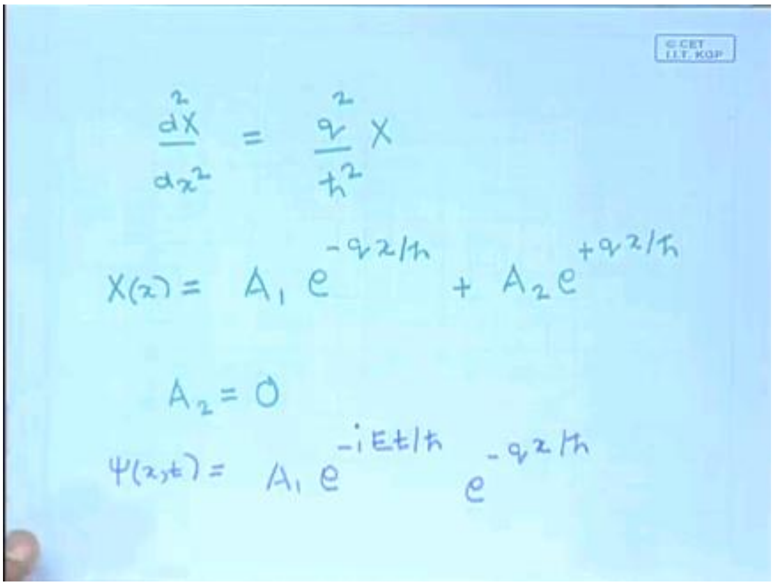

D square X dx square is equal to q square by h cross square X. So, we have to solve this differential equation for the wave inside the region where the energy is less than the value of the potential. Now, the solution to this equation is very straight forward we are all familiar with this the solution to this equation is some constant A 1 e to the power minus q x by h cross plus A 2 e to the power plus q plus x by h cross. So, this is the solution to the spatial part of the wave function in the region.

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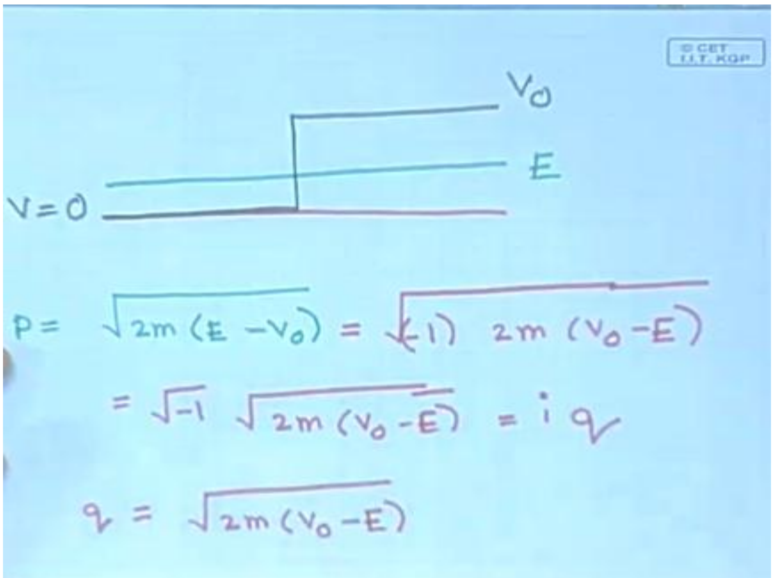

To the right over here where the energy is less than the value of the potential now, let us look at the behavior of these two solutions in this region.

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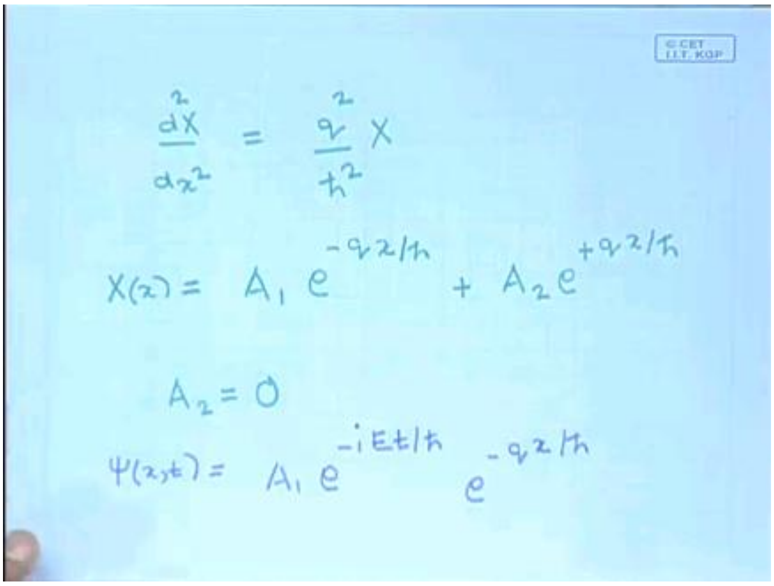

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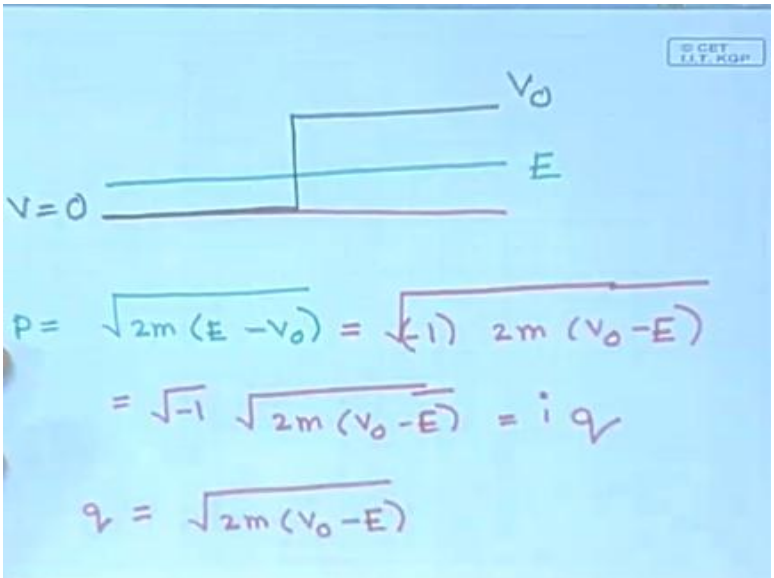

If you go deep inside this region as x becomes larger and larger this is the x axis for larger and larger values of x this term is going to decay.

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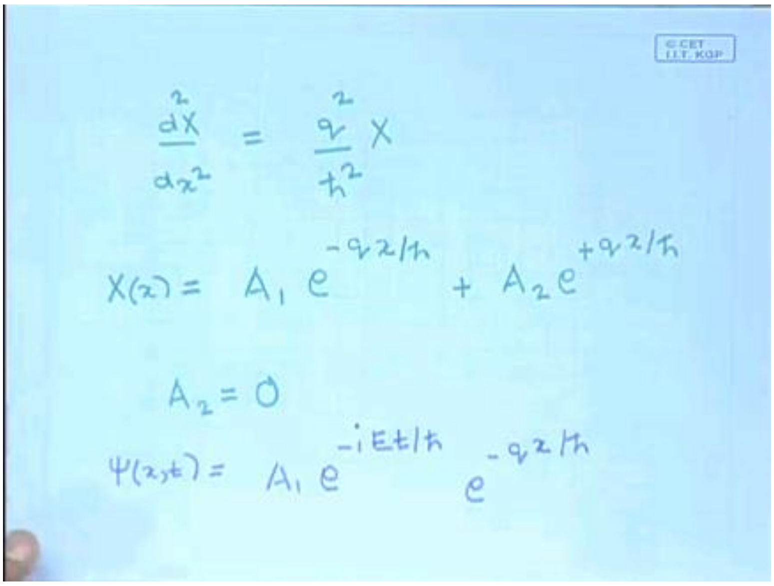

But this term is going to blow up and we do not want the wave function to blow up, because the mod square of the wave function gives the probability amplitude probability density and the probability density has to be normalized. So, and we do not want the wave function to become infinite, because it tells is that the probability of fining the particle becomes infinite it does not make sense. So, what this tells us is that we should set the constant A 2 to be equal to 0 so, on the right hand side over here.

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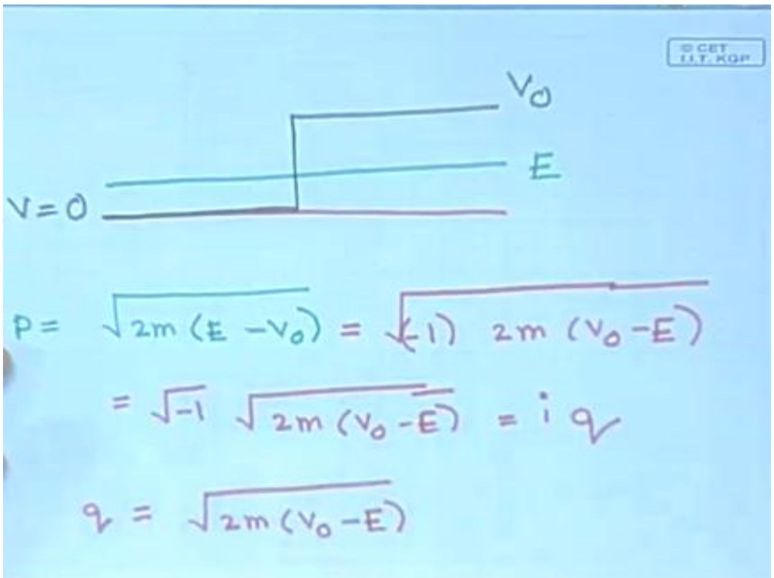

If we are dealing with the step like this and the energy of the particle is less than the potential then on the right hand side over here.

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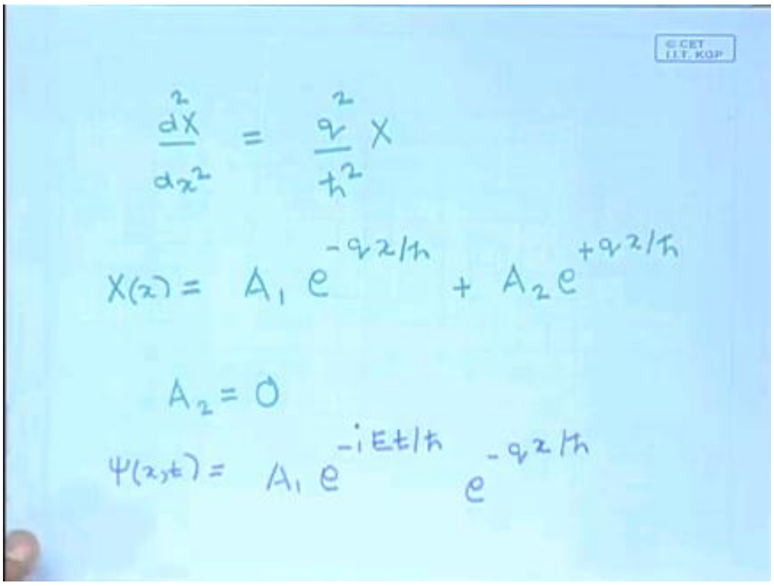

Then we have to set the coefficient of this positive exponential with the positive exponent we have to set this term to 0 the coefficient of this term to 0.

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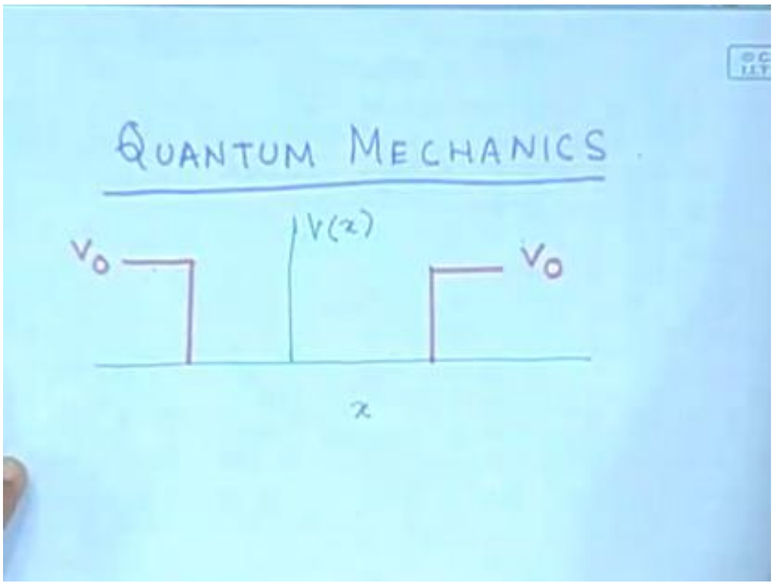

Similarly, if instead of dealing with this step we were dealing with this step then x would go towards minus infinity if we were dealing with this step.

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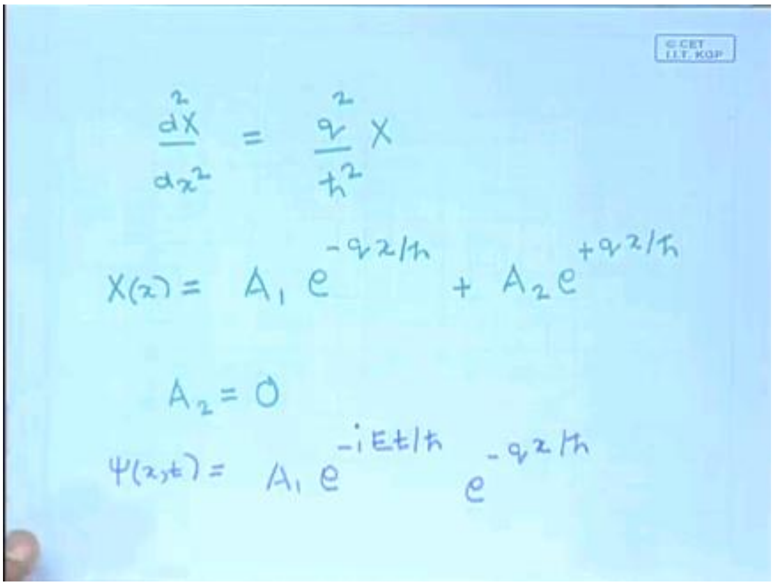

Then this term would tend to 0 and this term would blow up and for the other step we would have set this equal to 0. And keep this keep this term, but for this particular term kind of step that we are dealing with here which is what I have shown here.

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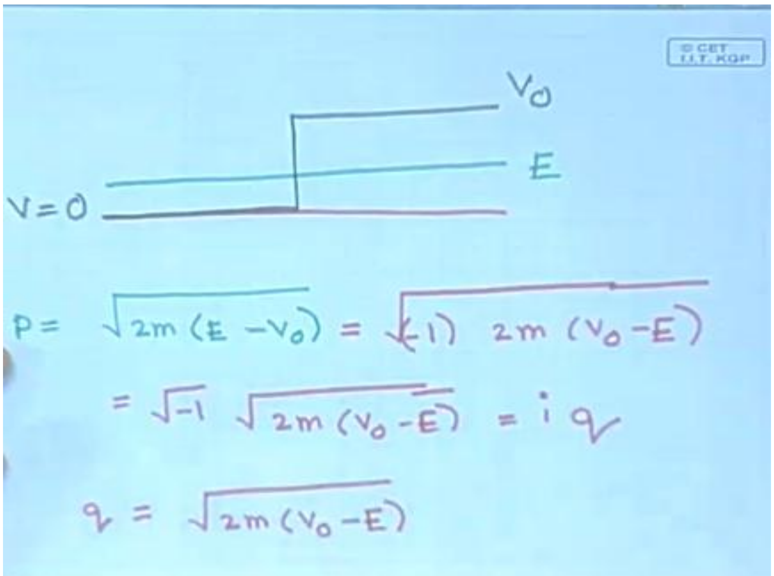

In this picture where the steps extends to the right?

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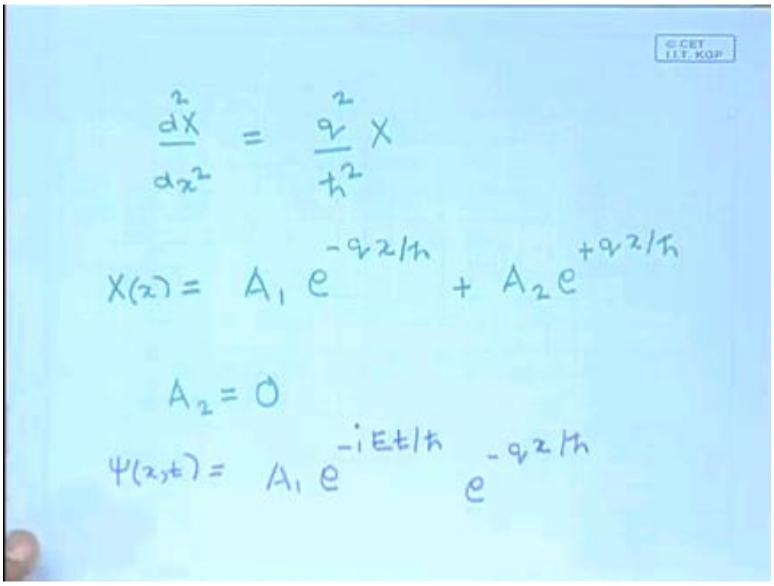

We have to send set this co efficient A 2 to be 0 so, with this let we write down the form of the, a wave function. The form of the wave function is psi x t is equal to some constant A 1 e to the power minus i E t by h cross e to the power minus q x by h cross. So, we have worked out the the solution for the wave function in both the possible situations where the energy is more than the value of the potential and also for the situation where the energy is less than the value of the potential. Now, for both these solutions we saw that there are these coefficients let me first discuss this there are these coefficients which are still undetermined.

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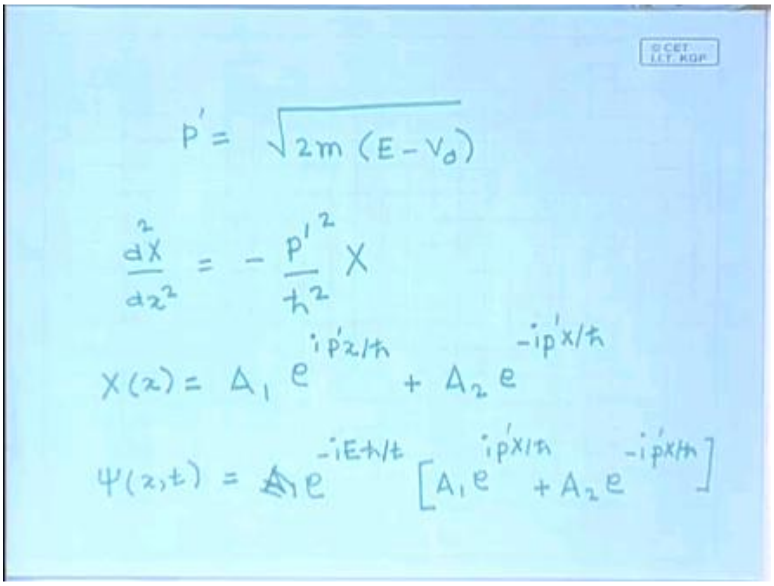

So, this is the solution when the energy is more than the potential we have a plane wave. The coefficients A 1 A 2 are undetermined.

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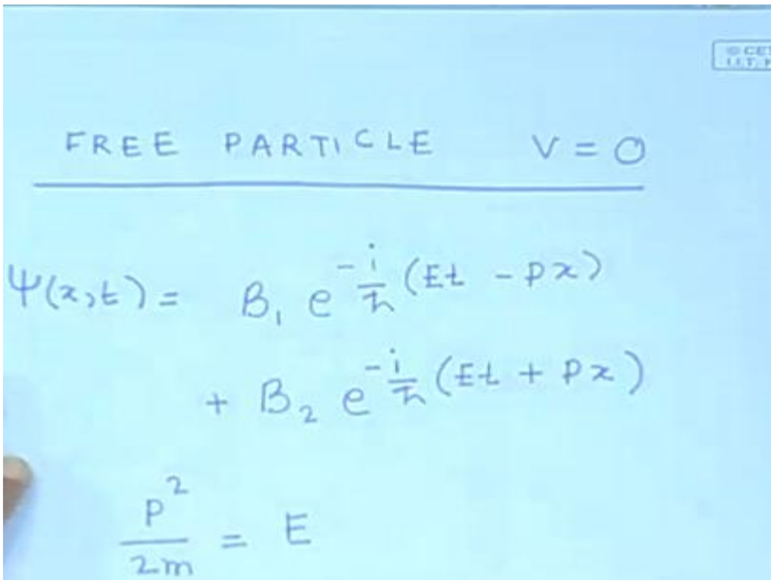

So, are the coefficients of the wave function in the part free particle part?

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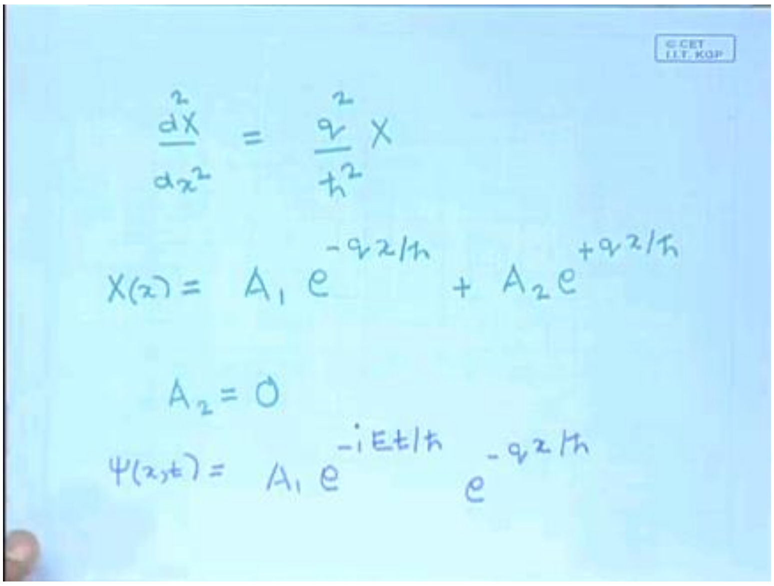

And in the situation where the energy is less the less than the value of the potential we have only 1 coefficient, but its value is still undetermined. The value of these coefficients have to be determined from by from the matching of boundary conditions and this is something which I will discuss in later lecture in the later lecture. So, we shall not discuss how to determine the value of these coefficients today let me discuss the interpretation the significance of this. Now, what does this tell us let we first what does this tell us let we first draw the form of the wave function. So, in the situation where the energy is less than the potential the step this is.

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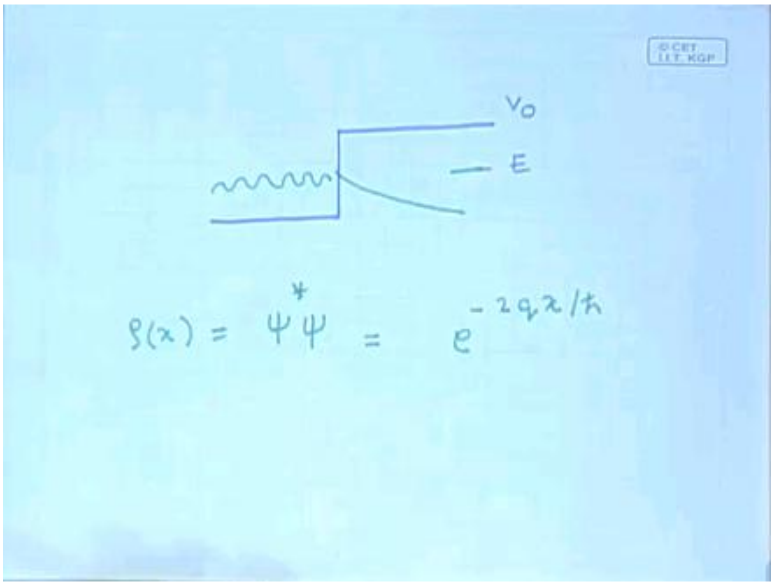

The potential step the energy is less than the value of the potential and in the region where the particle is like a free particle where there is no potential. The, we have a plane wave solution with the wavelength that looks like this. And the moment the wave enters the potential and the energy is less than the potential.

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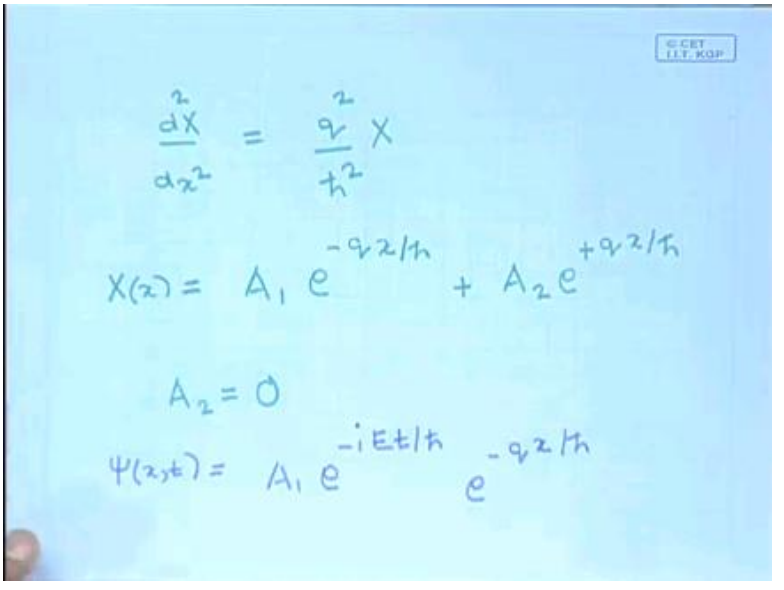

You only have an exponentially decaying solution.

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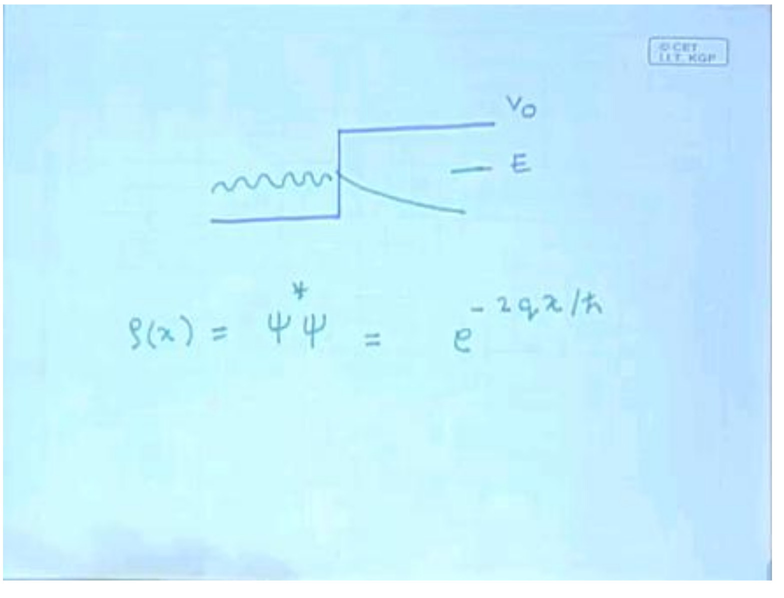

So, the solution inside this region is exponentially decaying it looks like this now, you see this is something which is quite remarkable, because if you calculate the probability density of finding the particle at different points. The probability density is not going to be a function time; because this is a stationary state it has a fixed value of the energy which in this case is here the energy level. This is going to be psi star psi and in the region inside the potential this is going to be e to the power minus 2 q x by h dot. Now, in classical mechanics recollect that in classical mechanics you will never find a particle in a region where it is energy is less than the value of the potential, right this is something we have discussed several times.

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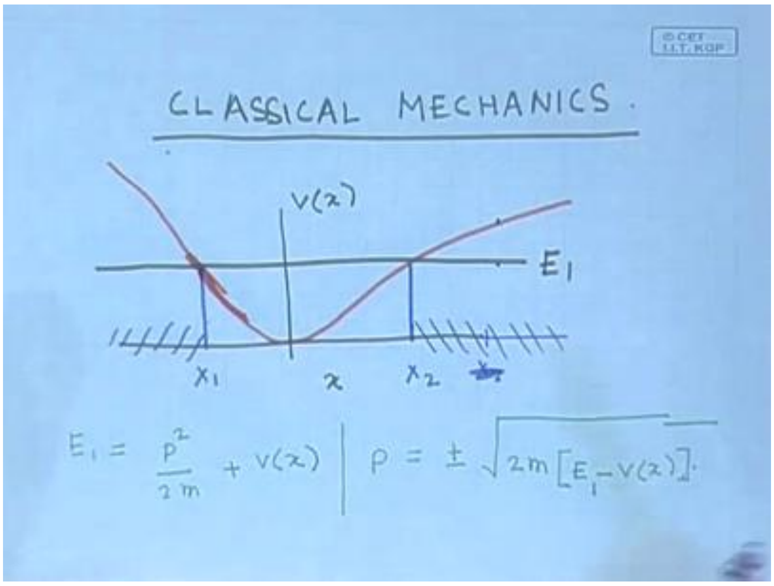

So, in classical mechanics you will never the particle in this region the particle is going to be over here.

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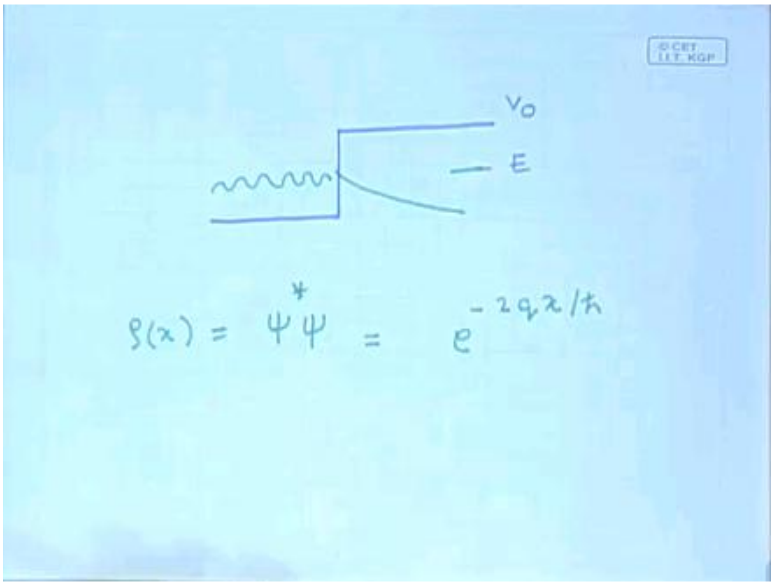

But when we do the quantum analysis what do we see when we do the quantum analysis we find that the wave function penetrates inside the potential to some extent. It does not go to 0 abruptly at the boundary even though the energy is less than the potential. The wave function penetrates inside this region it falls exponentially. It decays exponentially as we go inside this region so; the probability density of finding the particle is not non 0. There is a finite probability density of finding the particle in this region where the energy of the particle is more is less than the potential. The probability density the probability of finding the particle somewhere in this region falls exponentially as I go insider, but it is finite it is not 0. So, there is a finite probability of finding the particle in the region in quantum mechanics we find that. There is a finite probability of finding the particle in the region where it is energy is less than the potential.

So, this is something remarkably different which is predicted by quantum mechanics in classical mechanics you will never find the particle in the region where the energy is less than the potential. But in quantum mechanics we see that there is the finite probability of finding the particle here although the probability decays exponentially as we deeper. And deeper inside this region another point which you should note is the question, as to how does the probability of finding the particle in this region vary. If I change the height of this potential so, let us come back to the definition of q.

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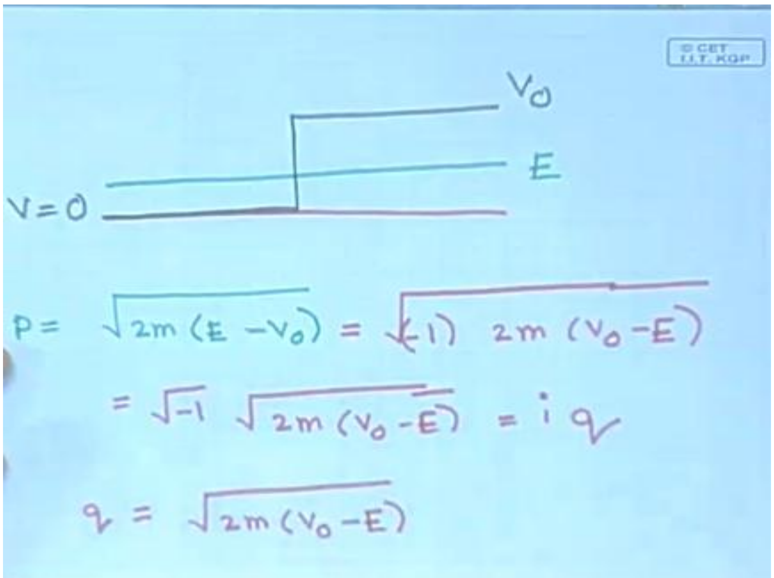

Q is defined as the square root of 2 m the difference of the potential and the energy of the particle. So, the higher the potential or the higher the difference the larger the difference between the potential and the energy the larger is the value of q. So, the higher you make the potential keeping the energy fix the larger becomes the value of q.

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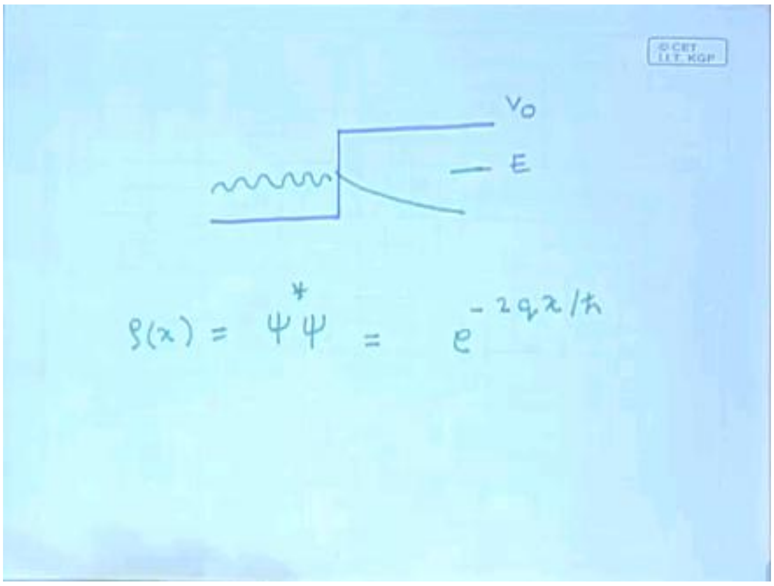

So, as you make the potential higher and higher maintaining the same energy the wave function is going to decay faster and faster insider this region and you can see that.

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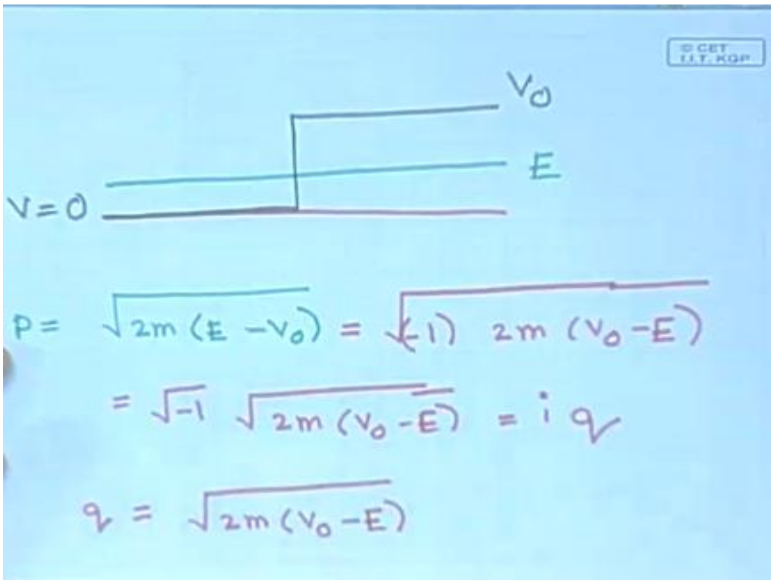

If I make the potential v naught infinitely large if I make the potential v naught infinitely large.

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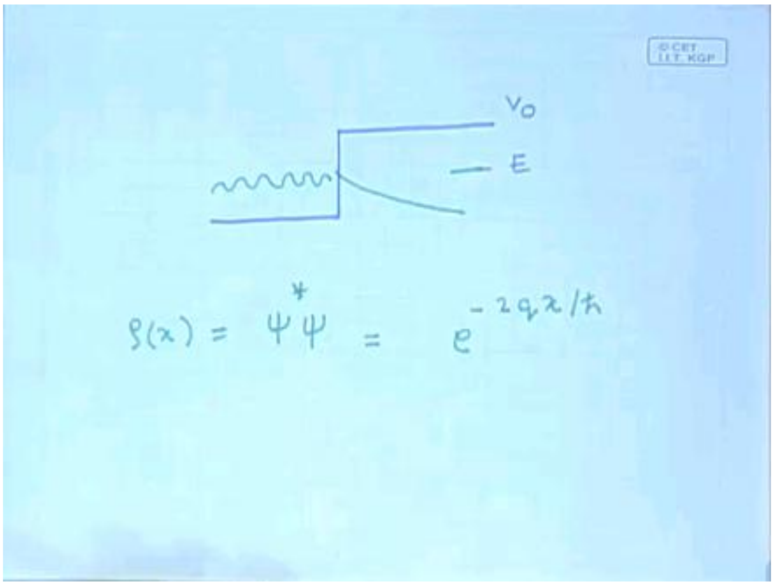

The wave function is going to decay faster and faster.

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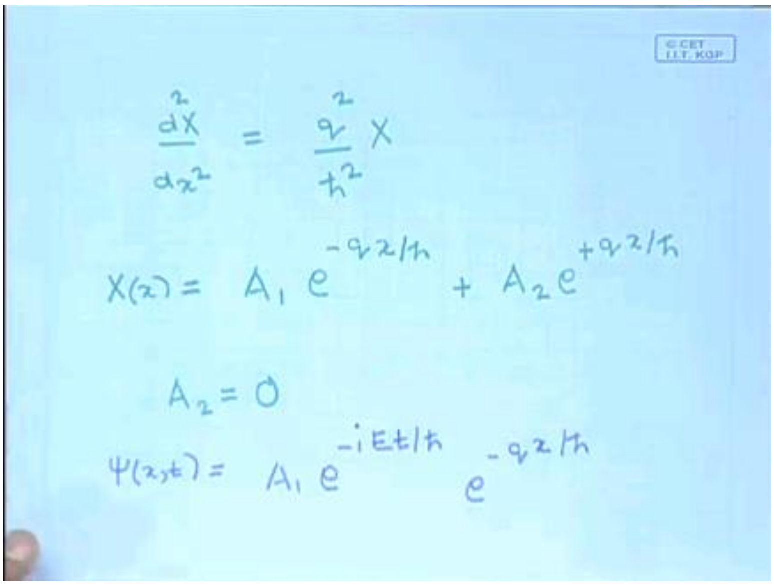

At the boundary and when I make this infinitely large the wave function is going to become 0 at the boundary.

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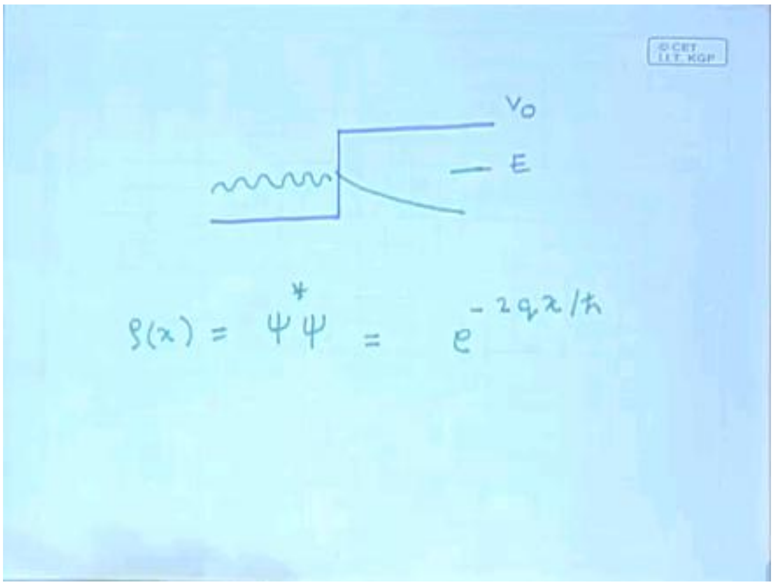

Because q becomes infinite.

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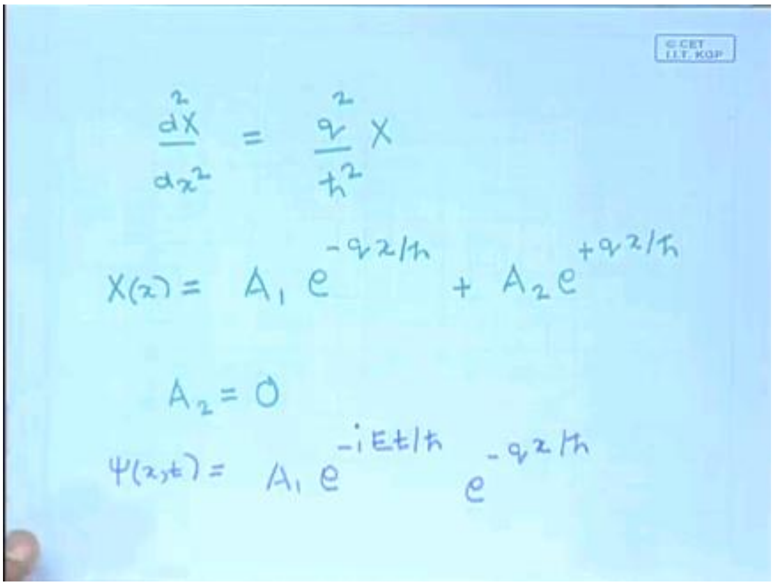

So, if I make this potential infinitely large.

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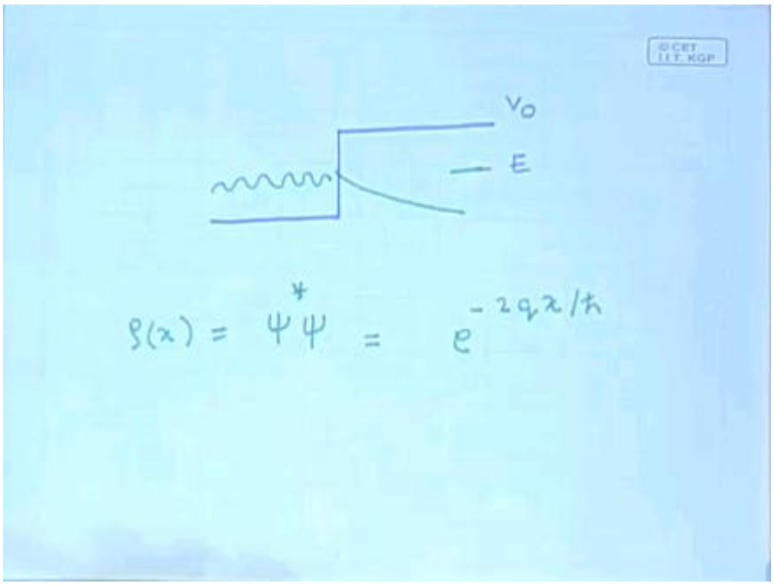

As I make it larger and larger the decay of the wave function is going to get faster and faster. And in the limit where the potential becomes infinite the wave function is going to go 0 at this boundary it is not going to penetrate inside this region. So, until now we have been discussing in general terms what happens when you have a particle moving in a potential in quantum mechanics. And what I have told you is that what we have seen is that you have these stationary states where the particle has a fixed angular frequency. The wavelength of the particle or the wave number corresponding to the particles wave or the wavelength of the wave changes depending on the value of the potential if the energy of the particle is more than the. If the energy of the particle is more than the potential then you can think of the propagation of the particle in this potential as like a like a wave moving in material of varying refractive index.

So, the potential can be thought of as variations in the refractive index what it does is it causes. The wavelength of the wave associated with the particle to change from place to place let me repeat this again if the energy of the particle is more than. The value of the potential then you can think of the potential as a refractive change in the refractive index. And the wavelength of the wave changes as the particle moves from position to position but if the energy of the particle is less than. The value of the potential in classical mechanics you would never expect the particle to go there. But what you find is that in quantum mechanics the particle does penetrate into that region where it is energy is less than the potential.

The wave function decays exponentially inside the region where the energy is less than the potential, but there still is a finite probability of finding the particle in that region. And finally, what I told you was that as you make the potential higher and higher this exponential decay gets faster and faster inside the region where the energy is less than the potential. And if you make the potential infinitely large this decay is quite abrupt and the wave function goes to 0 at the boundary where the potential where you have made the potential infinite. So, this has been the general discussion which we have been doing and I would just summarize it for you now, we are going to discuss a particular specific problem and this problem is as follows.

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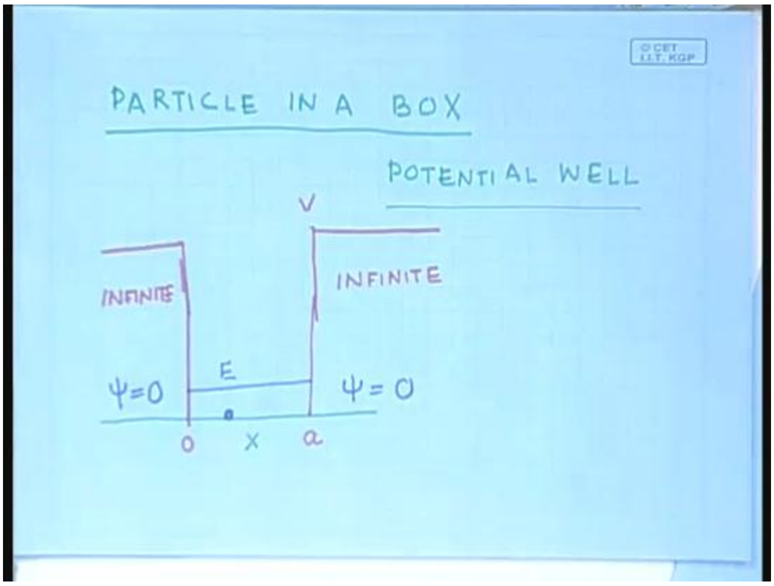

The problem is often referred to as a particle in a box or it is also refer to as a particle in the potential well the situation is as follows. So, we are dealing with the particle which is free to move along the x axis as usual. And the particle is confined to a range along the x axis a range of x along the range being 0 to a so, the particle. So, the particle is confined to the region x x inside the region x equal to 0 x equal to a, and it is confined by 2 potentials which are infinitely high. So, these are infinitely high so, in this axis I have drawn the potential v and there are on the 2 sides of this region. There are potentials which are infinitely large they are infinitely high I have not drawn it I cannot obliviously draw it has being infinitely high but the potentials on the 2 sides are infinite.

So, the particle if you think of it classically first the particle can never go outside this region the particle has an energy E. So, there is a particle inside here and the particle has some energy E and if you think of it classically the particle is never going to get outside this region because there are these 2 infinite potentials on 2 sides. So, as long as the particle has a finite energy it is going to remain inside and that is why we think call this a particle in a box. These infinite potentials are the walls of the box which restrain the particle to be inside this which is also why we refer to this as a particle in a potential well this represents a potential well. And remember why let me just remind you why it is called a well if you dig a hole in the ground or well in the ground. The potential energy if you are insider the well is going to be lower than if you are outside, because you have gone down.

The potential energy is mg h under the influence of the earth's gravitational field. So, if you are insider a well the potential is lower if you are outside the potential is higher if you climb Mount Everest the potential is highest that you can have on the surface of the earth. So, here insider here the potential is lower outside the potential is higher and we have made this infinitely large just for mathematical convenience. And which is why this is called a potential well now, the situation that we are going to analyze the well the the height of this potential is infinite. But it could have also been finite the situation where the potential is finite is mathematically a little more difficult. But many of the features which we get when we consider an infinite potential well will still hold. If you consider a finite the potential having a finite depth we are considering infinite because is it mathematically simpler.

So, we would like to solve for the particles wave function inside this potential inside this particle inside. This potential well or in you may say for this for the particle in this box and a particle in a box could be generalized to 3 dimensions we are dealing with just a 1 dimensional situation. So, we have just now studied one of these steps we have analyzed one of these steps and what we saw was what we saw was that if you make this potential higher and higher. So, that this is finally, infinite the as you make this potential higher the if we have a finite potential the wave function of the particle can penetrate into this region. But if you make this potential infinite the penetration disappears the wave function abruptly become 0 at the boundary. So, this is the mathematical convenience which we get if you take infinite potentials on the 2 sides if you have infinite potentials.

The wave function abruptly goes to 0 at the boundaries it does not penetrate inside the potential if I had a finite potential the wave function would penetrate inside. And I would also have to consider that, but with infinite potentials we have seen that. The wave function is not going to penetrate it is going to go to 0 abruptly at the 2 boundaries. So, we have to now solve the wave function inside this region we know that the wave function goes to 0 at the boundaries and is 0 everywhere over here. So, these 2 regions the wave function is 0 we have to solve for the wave function only in this region. Inside this region we have a free particle the there is no potentials inside this region. So, we could proceed either in either of one of the 2 waves we could take this equations.

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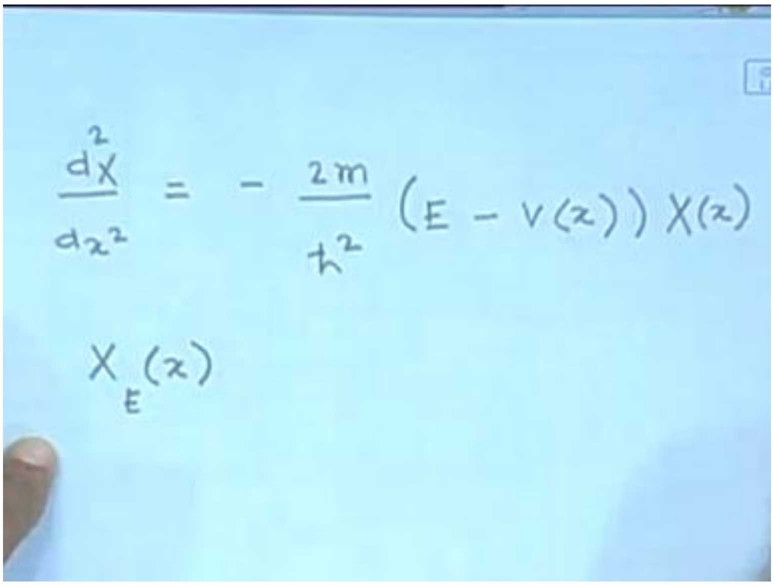

At potential v equal to 0 and solve it inside this with the appropriate boundary conditions.

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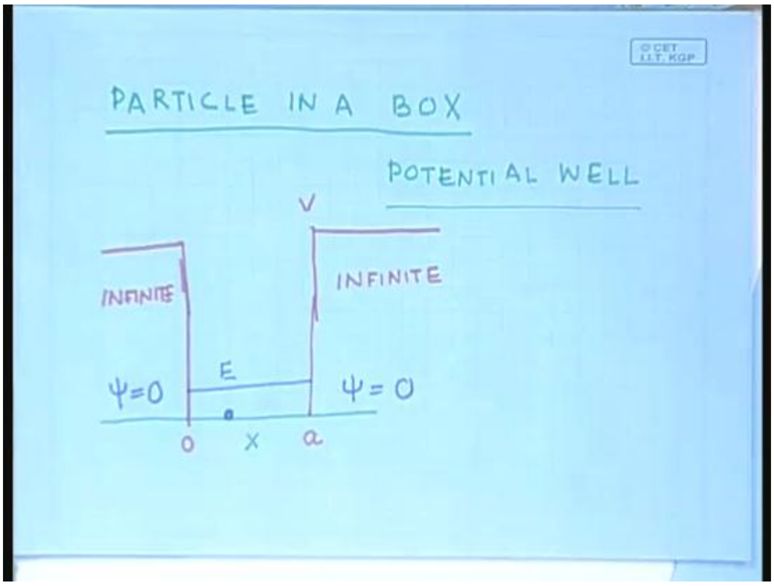

Or we could take the solution which we have already which has already been worked out let me just write it for you again over here the solution has already been worked out for a particle inside here. So, let me write it down again for you. The solution using the method of separation of variables is.

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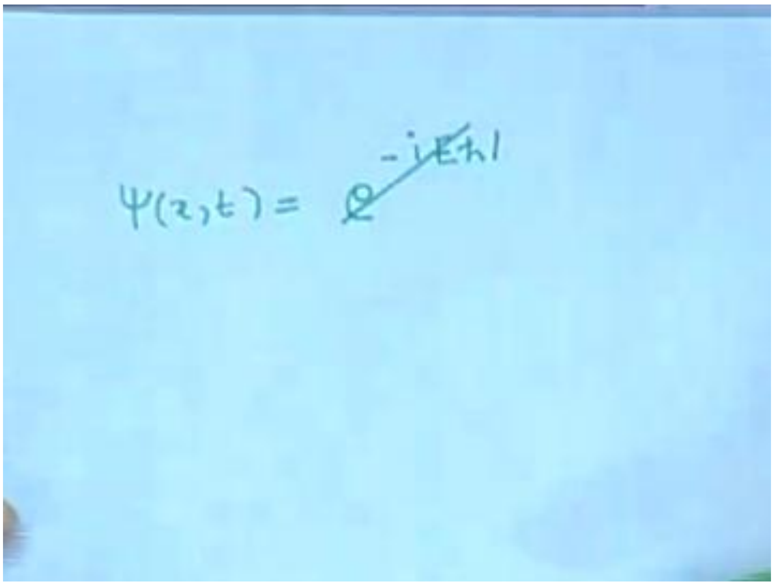

Psi x t is equal to e to the power minus i E h cross sorry the solution inside is.

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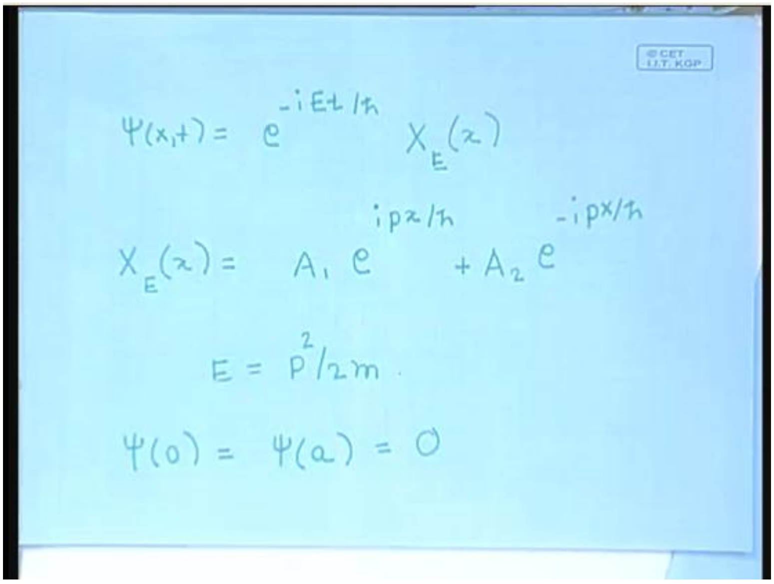

Psi x t is equal to e to the power minus i E t by h cross. And then we will have XE where this has the spatial dependence and this function XE is A 1 e to the power i Px by h cross plus A 2 e to the power minus i Px by h cross. And since it is a free particle E is equal to P square by 2 m that is a dispersion relation. Because it is a free particle the energy this E and P these 2 constants are related like this we have already discussed this quite a few times. This is the solution in the region where the potential is 0 which is the region in between these two potential wells.

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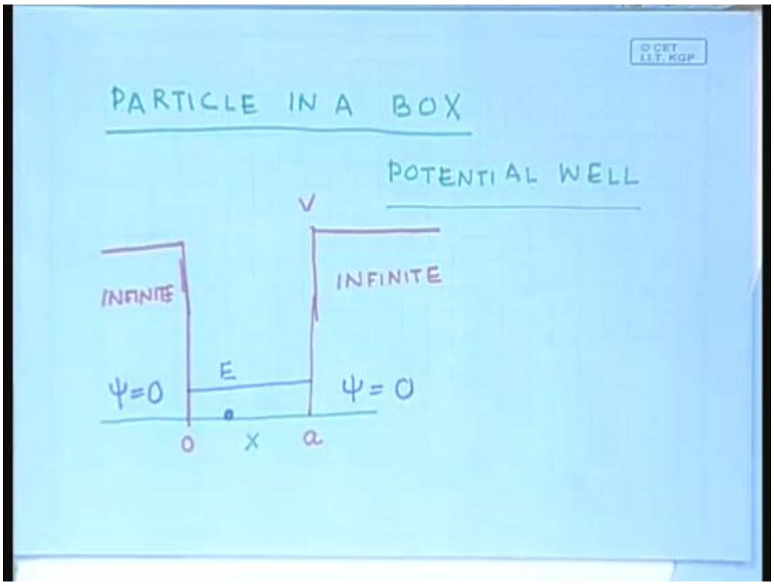

The wave function has to vanish at these two boundaries.

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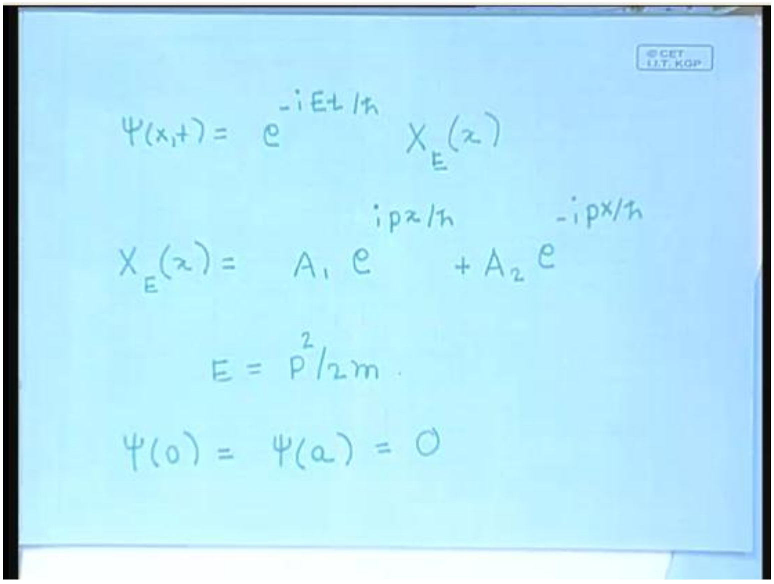

So, this is the solution it has a time part. So, which we can interpret that the particle has energy E and there is the spatial dependence which is of this form E and P are related like this. Now, the only difference that the extra feature that we have over here is that we have to apply some boundary conditions. And the boundary conditions that we have to apply are that the wave function has to vanish at this point.

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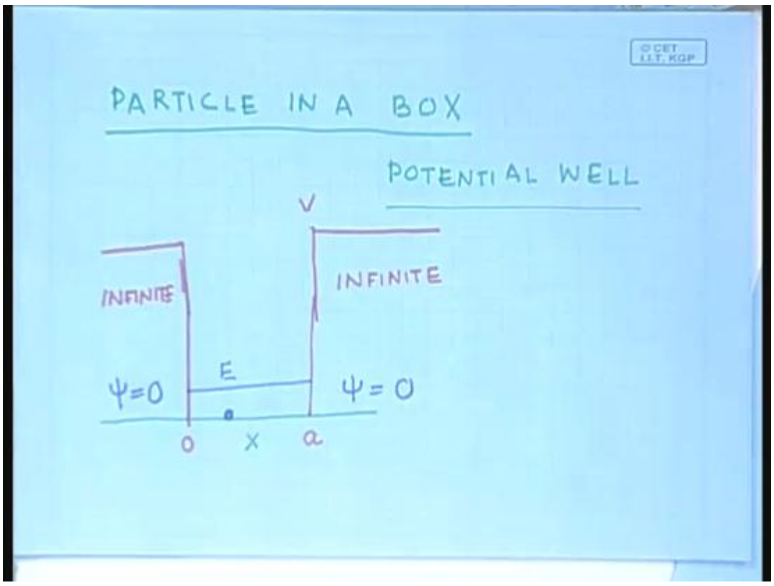

At this boundary and this boundary, because the wave function is 0 the momentum it encounters the infinite potential well it does not penetrate inside.

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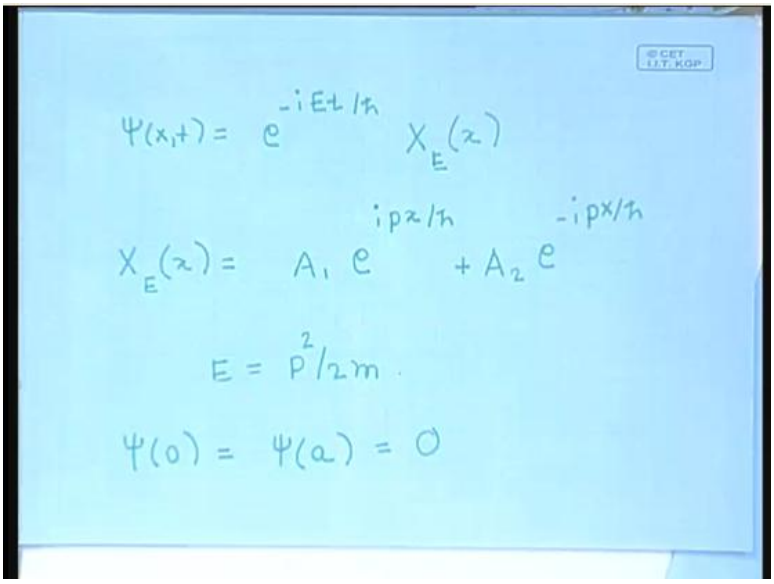

So, the boundary conditions that we have to apply are psi 0 is equal to psi a and both of these are equal to 0. Now, let us apply the first boundary condition that the wave function has to vanish at x equal to 0 and see what it tells us. So, what we are going do is we are going set x equal to 0 and this function should be 0 for at x equal to 0. At x equal to 0 this exponential is 0 this exponential also is 0.

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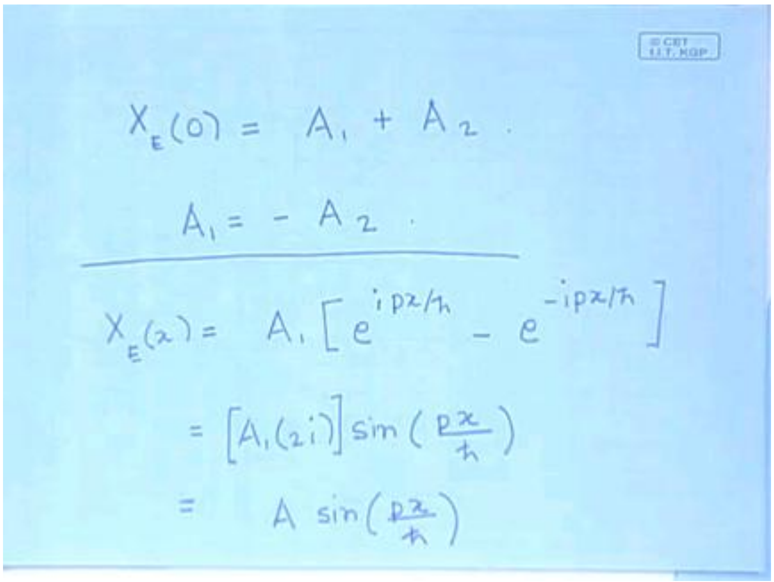

So, what it tell us is that 0 is equal to A 1 plus A 2 or it tells us that A 1 is equal to minus A 2.

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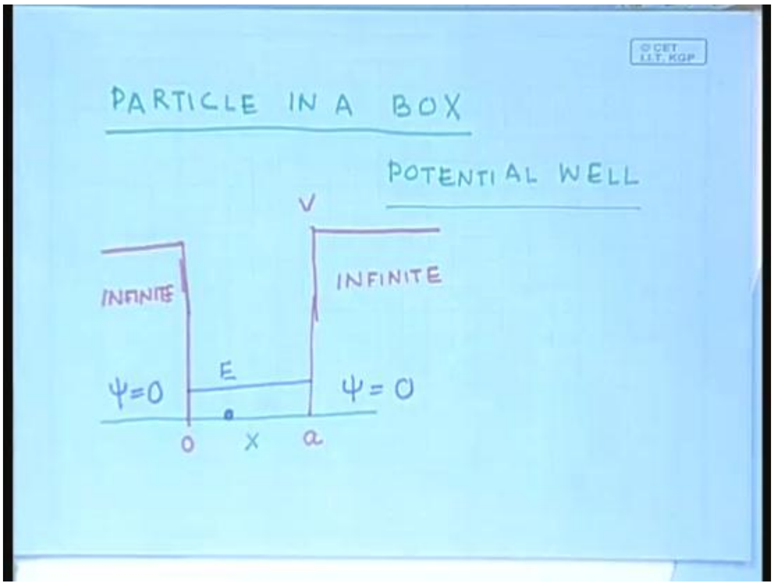

So, the first boundary condition that the wave function has to vanish at this boundary tells us that the 2 coefficients which occur 1 for the right travelling wave and 1 for the left travelling wave, they have to be opposite to each other.

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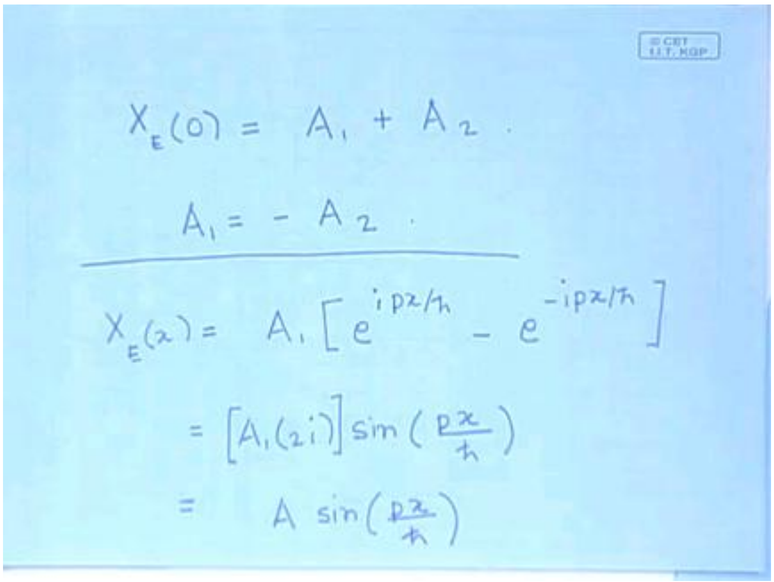

So, now we can put this back into the solution.

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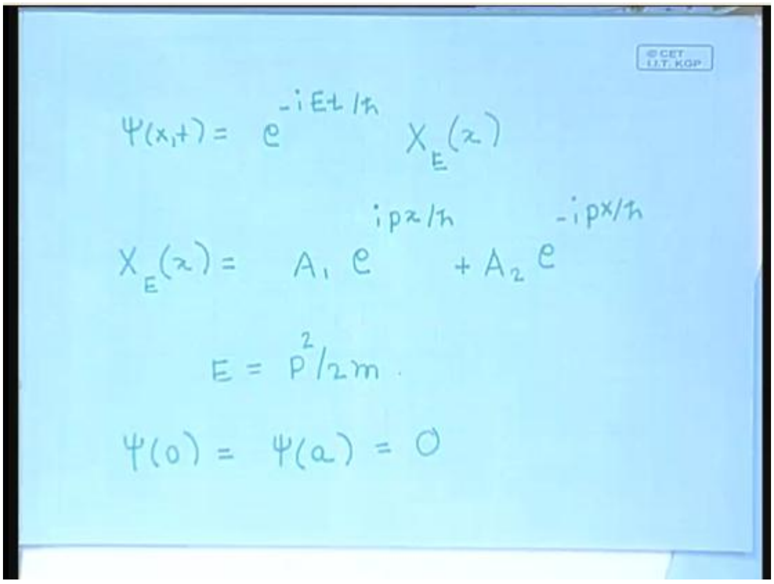

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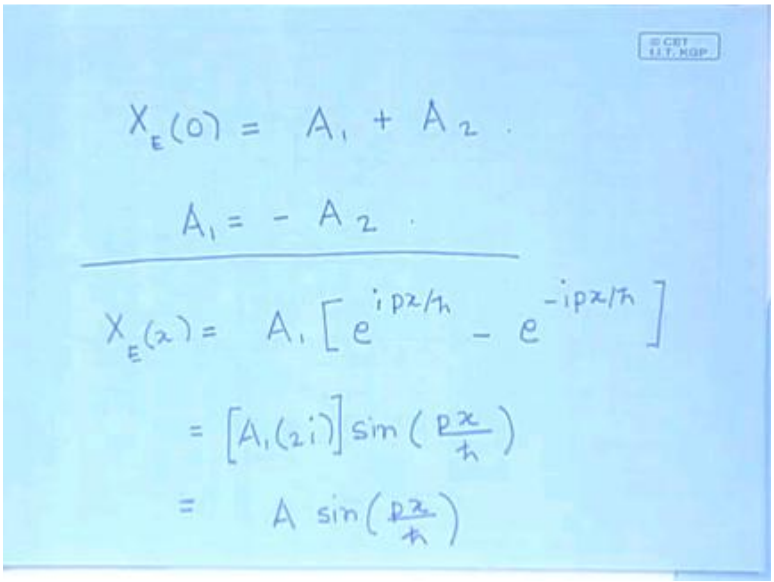

And what we have is that XE is equal to. So, i can write this as constant A 1 e to the power i Px by h cross minus e to the power minus i Px by h cross. Now, this we know is we can this is cosine Px by h cross plus i sign Px by h cross this also will be cosine and a minus i sin. So, this is going to be A 1 2 i into sin Px by h cross and this whole factor over here. I can write as another constant. So, this will be A sin Px by h cross. So, what we have let we stop over here and just remind you what we are doing we are solving for the wave function of a particle in this region with the boundary condition that the wave function has to vanish at the two end points and the wave function for a free particle over here.

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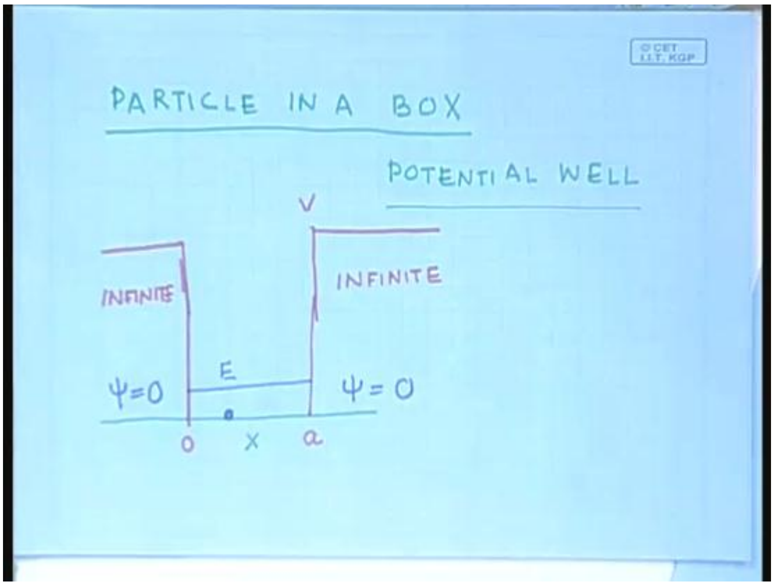

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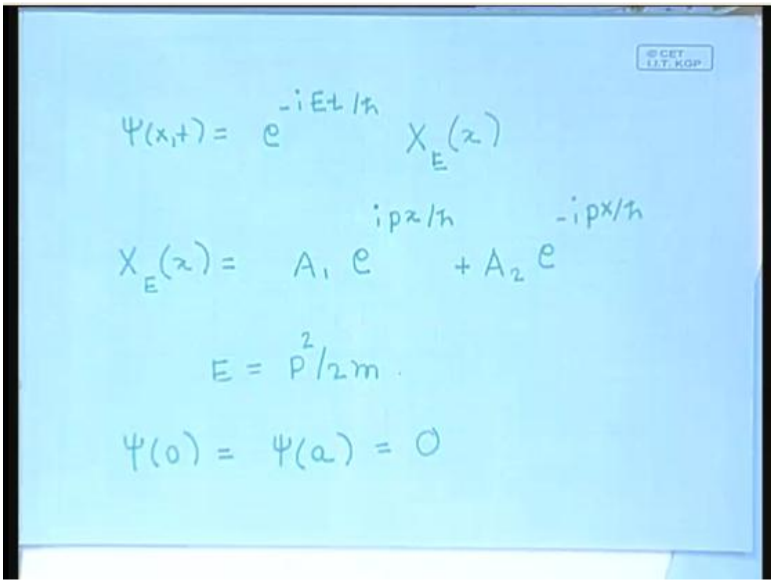

Is this familiar form where the spatial dependence is a plane wave with constant P over here which is and this constant P is related to the energy like this we can interpret this constant P as the momentum.

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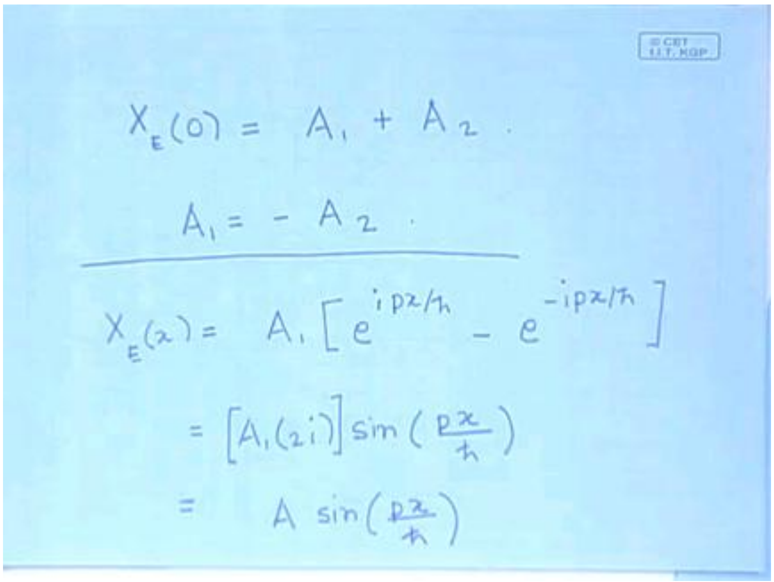

Now, applying the first boundary condition that the wave function has to vanish at X equal to 0 we find that the 2 constants must be exactly opposite. And it tells us that the wave function must be the spatial dependence of the wave function must be a constant into sin Px by h cross. So, in tomorrow's class I shall start of by considering.

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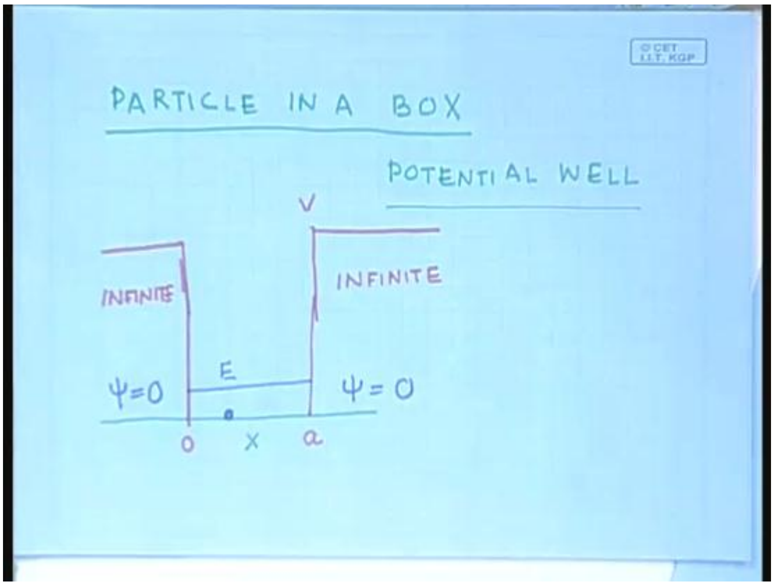

The next boundary and seeing what this boundary condition implies. Let me stop here for today.