Physics I: Oscillations & Waves
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Lecture - 04
Oscillator with External Forcing-I

In today's talk we shall be considering what happens to an oscillator under the influence of an external force.

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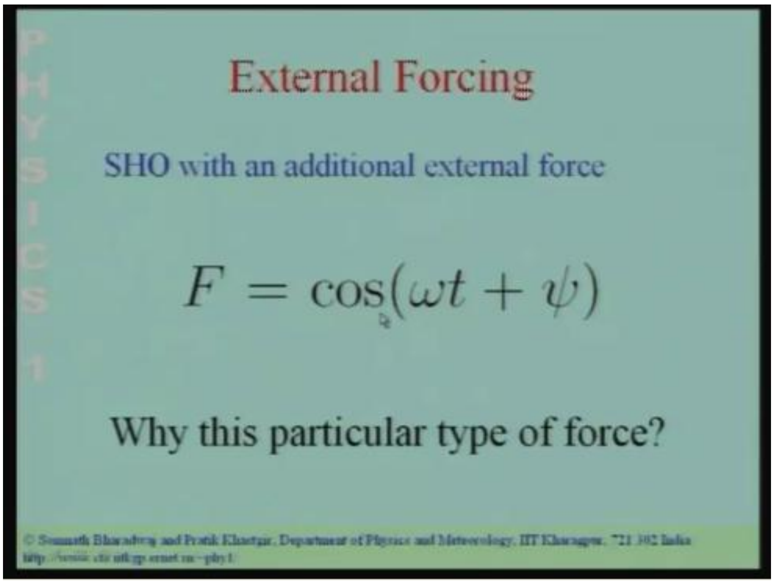

We shall be considering a force, which is of the type F is equal to cos omega t plus psi. So, the first question that arises is why this particular form, why this particular type of force, what. so, special about the this particular type of force, where the force itself is like the oscillation of a simple harmonic oscillator difference being that omega now could be arbitrary frequency.

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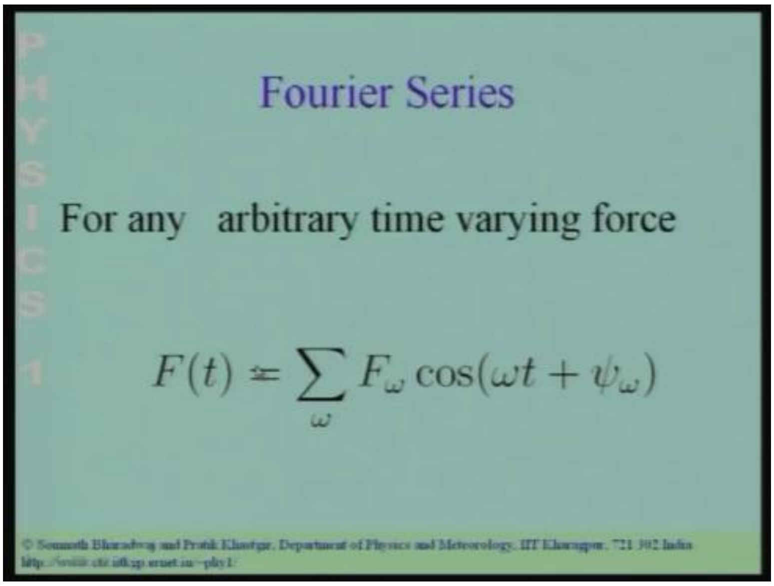

The reason why we have chosen this particular type of a force for why we have decided to study the effect of this particular type of a force on a simple harmonic oscillator is very interesting. And it has to do with a theorem which was proved by a French engineer called Fourier. The essence of Fourier's claim was that any arbitrary time varying function of time any arbitrary time varying force, which you apply which you could possibly apply to a simple harmonic oscillator could be expanded into a some of cosines with different frequencies. And each frequency component would have a different amplitude and different phase.

So, Fourier showed that any arbitrary function of time many arbitrary force. So, you could have in general you have a simple harmonic oscillator with any arbitrary force. And Fourier showed that this arbitrary time dependent force could be decomposed into a some of cosines of different frequencies, sometimes you may be required to take an infinite sum and take the continuous limits. So, you have a Fourier integral instead of a Fourier series.

But for our purposes we shall not be going into the details in all those details we should we shall be content with the statement that any arbitrary time dependent force can be decomposed into a sum of cosines of different frequencies. And depending on the nature of the force you would have different set of amplitudes F omega and a different set of phases psi omega. So, in general you could decompose any particular force into this form of sum of cosine omega t plus psi with different values of omega.

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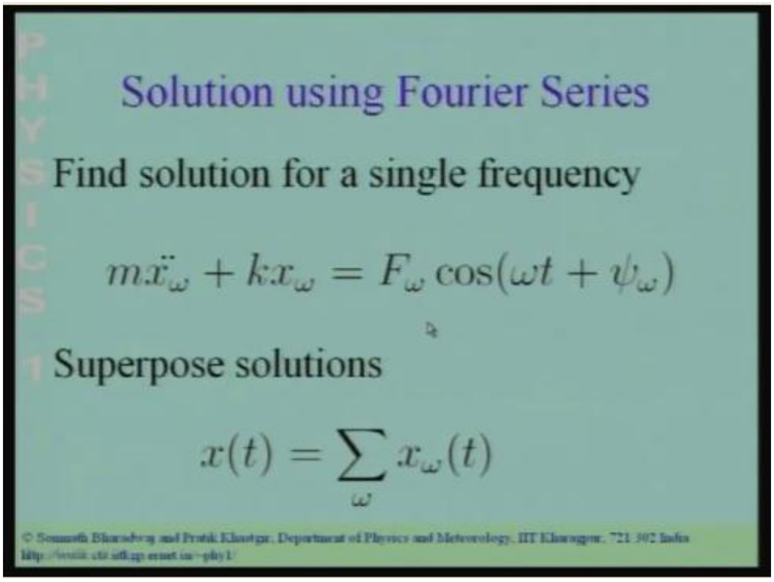

So, this is why this is the main reason, this is the reason why we have decided to focus our attention on a particular of a force of this particular type. Now, once you know the solution for a particular frequency omega you can determine the solution. For a super position of different frequencies by superposing the solutions the for these different frequencies. In this course we shall be focusing our attention to the situation, where that external driving force has only a single frequency omega.

And we shall not show frequency of the external force as a subscript for the amplitudes and the phase anymore.

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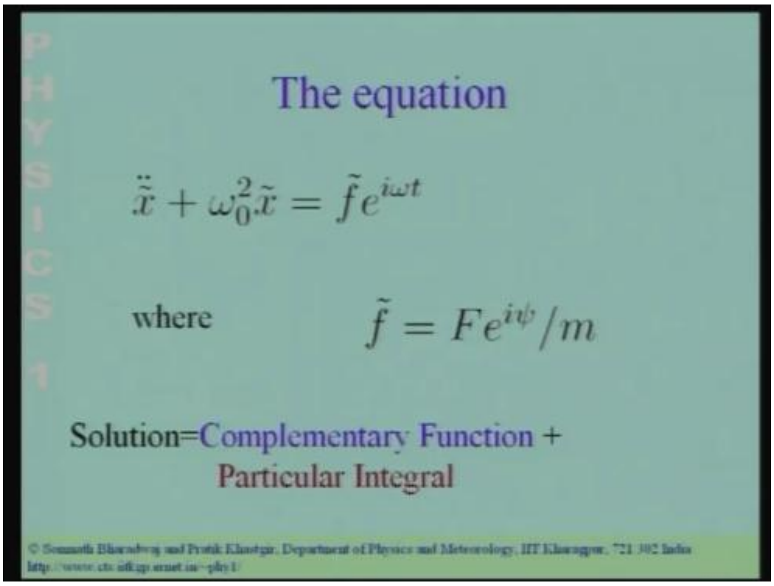

So, the problem, which we are dealing with is as follows let me again let me, let us again go back to the problem. The problem which we are dealing with this is as follows. We have the simple harmonic oscillator.

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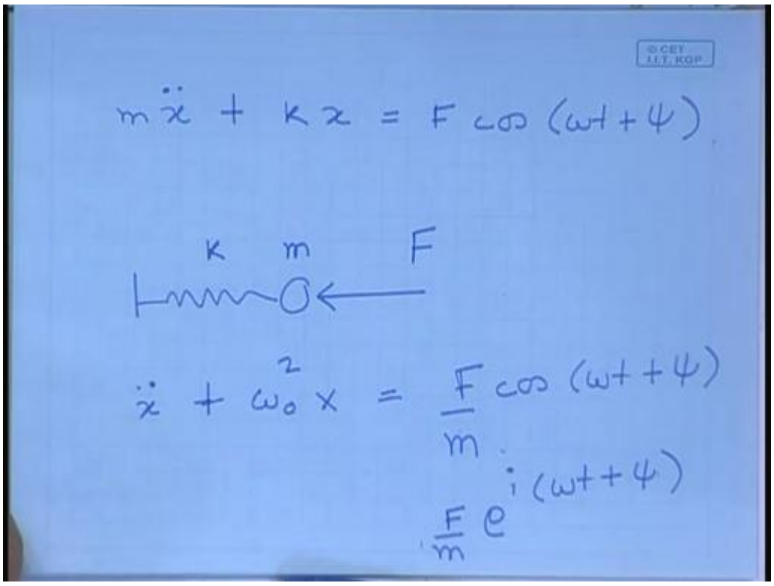

So, let us write down the equation for the simple harmonic oscillator Mx double dot minus Kx and now we have a force and the force is of the type F cos omega t plus psi. So, this is the situation which we are dealing with. So, we have a spring and a mass and there is external force acting on the mass. This is the problem which we are dealing with and the external force has is a sinusoidal force, it has a frequency omega the amplitude of the external force is F and it has a phase psi. Now, following the notation we which we had introduced earlier, we can divide this whole equation by m and if I divide his whole equation by m I get x double dot minus K by m and K by m we had called omega nought square x is equal to F cos omega t plus psi by m.

Now, recollect that omega nought square is the natural frequency of the simple harmonic oscillator and if there was no external force, the simple harmonic oscillator would be oscillating at the frequency omega nought. And we have studied this in considerable detail in the last class. So, this is the equation governing the simple harmonic oscillator in the presence of an external force. The external force is the frequency angular frequency omega.

Now, it is convenient to use the complex notation over here So, the same equation written in complex notation is: what I have shown over here. So, the same equation written sorry this should be a plus sign here that; the force is opposing the motion it should be a plus sign here. And the same equation written in complex notation is what I have shown on the screen over here. So, we have x tilde double dot plus omega not square x tilde where x tilde are now, complex is a complex variable the real part of which is the displacement x.

Now, let us look at the force. The force is now written, as this small f tilde e to the power i omega t. Recollect that the force which we had was term on the right hand side arising from the force which we had was the amplitude of the force F divided by m and then we had cos omega t plus psi. Now, in the complex notation you could write this as F by m e to the power i omega t plus psi and the real part of this term over here is the force which we have.

Now, in our in the notation that we are going to use we are going to take the phase of the force e to the power i psi into the amplitude of the force. So, into the complex amplitude of the force.

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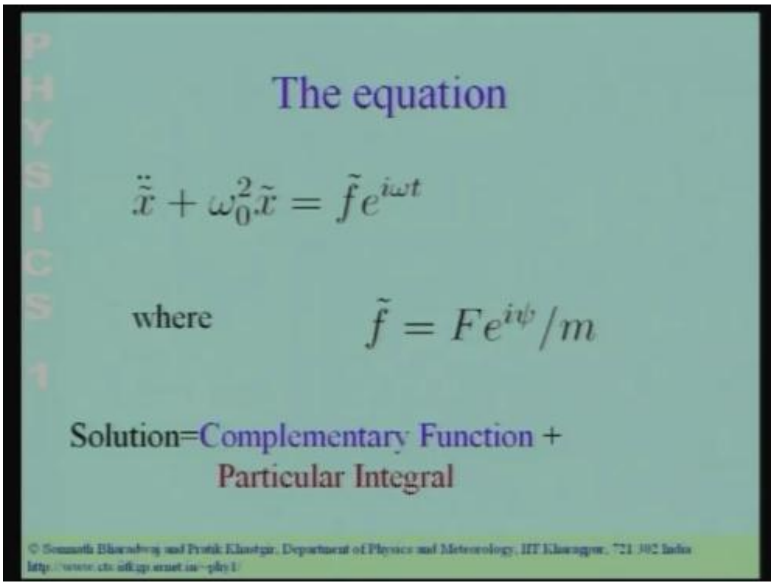

So, we have here, the complex amplitude of the force small f tilde which is capital F the force the magnitude of the force divided by the mass into the phase factor e to the power i psi. So, the forcing term in the complex notation is f tilde remember f tilde has both the magnitude of the force divided by the mass it also has the phase e to the power i psi. So, the external force is now, f tilde into e to the power i omega t it oscillates with the frequency e to the power i omega t.

Now, the question is we have to solve this; what is the solution to this equation we have to find the solution to this equation. Now, it is well known that differential equations of this type have 2 kinds of solutions. The first kind of solution is called the complementary function. Let us just recollect what we mean by the complementary function.

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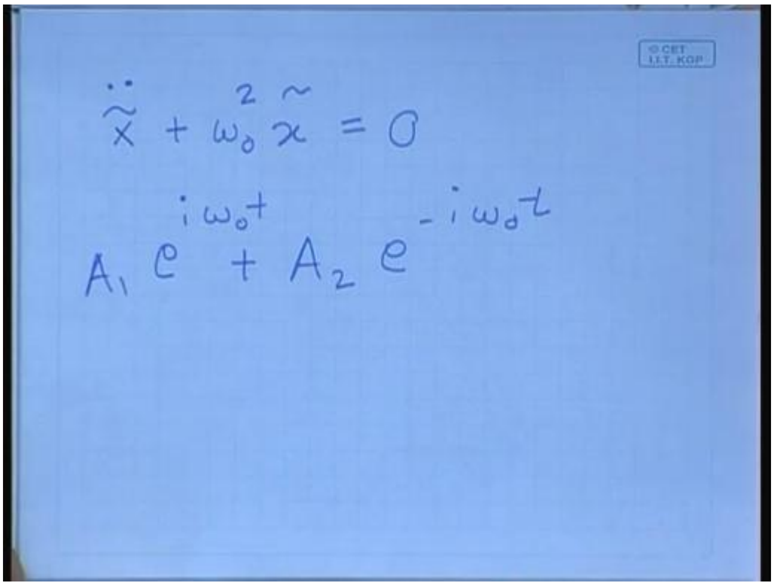

The complementary function is a solution, to only this part of the differential equation. So, the complementary function is a solution to only this part of the differential equation complementary function is a solution to only this part of the differential equation; where the external force has been set to 0. And we have already studied this solution in great detail there are 2 solutions and these are e to the power i omega nought t and e to the power minus i omega nought t.

And we have linear superposition's of these and this together constitutes the complementary function, it gives you oscillations at the frequency omega nought. These are the oscillations of the simple harmonic oscillator, if it is left free it is disturbed and left free to oscillate. We are not interested in this particular solution in today's lecture, in today's lecture we are interested in the particular solution.

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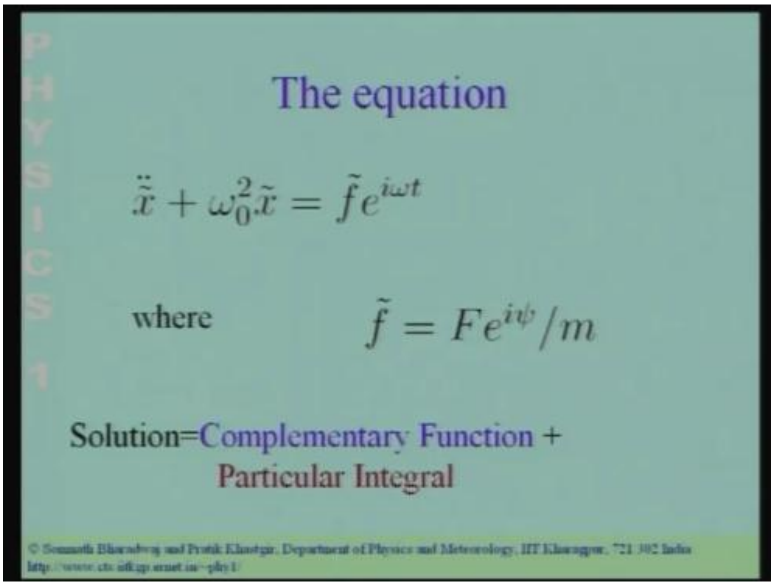

The particular integral, the particular integral is the solution is the part of the solution which also sacrifice takes into account the external force. The total solution is a sum of the complementary function and the particular integral.

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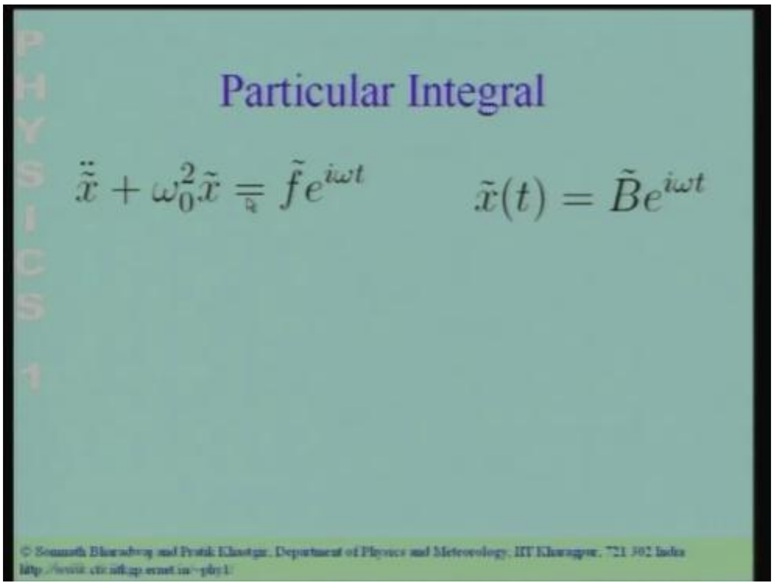

So, let us, look at the particular integral, the particular integral is a part of the solution which takes into account the external force also f tilde e to the power i omega t. So, we are now looking for a solution to this equation, where the equation has x derivatives of x the second derivative of x and the 0 derivative that is no derivative of x on the left hand side. And it has a function of time e to the power i omega t on the right hand side.

The question is we have to find the function of time x as a function of time which will satisfy this differential equation. Now, if x, if you have to find a x, the function of time x as the function of time which will satisfy this equation. You can see that x should have the same dependence as the right hand side time dependence as a right hand side. So, x should depend on e to the power i omega t. So, we take the trial solution x tilde t is equal to some constant B tilde into e to the power i omega t.

So, we take this trial solution and plug it in to this equation. So, let me do this little bit of algebra over here.

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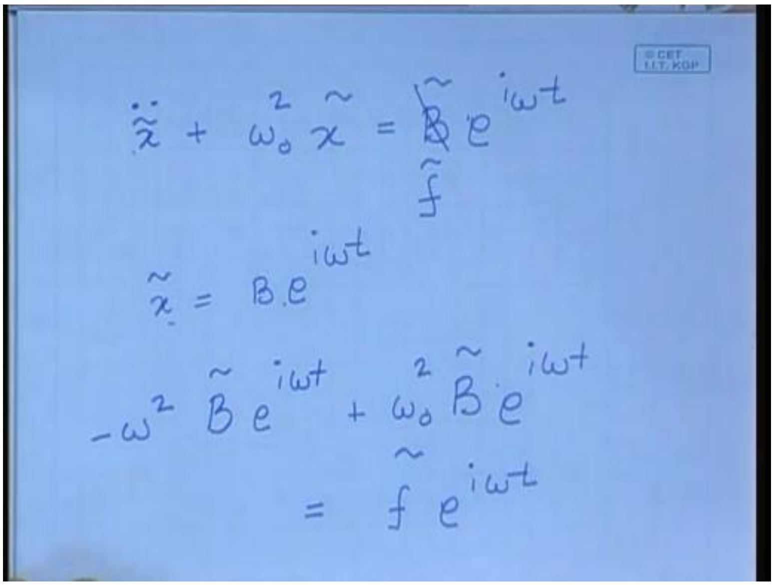

So, we have the equation x double dot this is the equation, which we would like to solve and we put in the trial solution, where x tilde is equal to B e to the power i omega t putting this into this equation. So, differentiating this twice and then combining it with this term gives me omega square with a minus sign if I differentiate this twice, I get minus omega square.

So, from here I get a term which is minus omega square into B e to the power i omega t from here, I will get a term plus omega nought square B this is B tilde e is a complex number i omega t this is equal to f tilde e to the power i omega t. So, notice that e to the power i omega t cancels out from both the left hand side and the right hand side it is there on both the sides. So, it cancels out. And we are left with an algebraic equation, the algebraic equation essentially gives us the value of B tilde and if you work out the value of B tilde from this if you work out the value of B tilde from this you take B tilde common over here.

If you take B tilde common you will have omega nought square minus omega square is equal to f tilde and then you can divide f tilde by the factor omega nought square minus omega square and what you get is.

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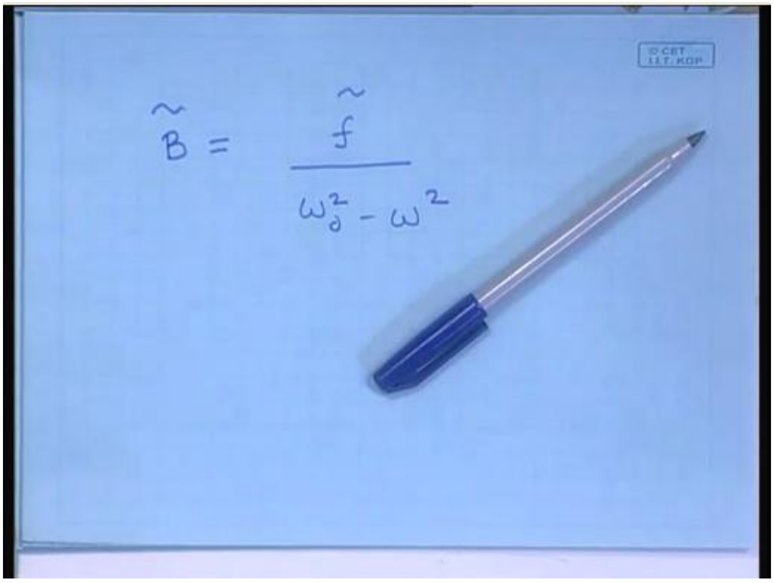

B tilde is equal to f tilde.

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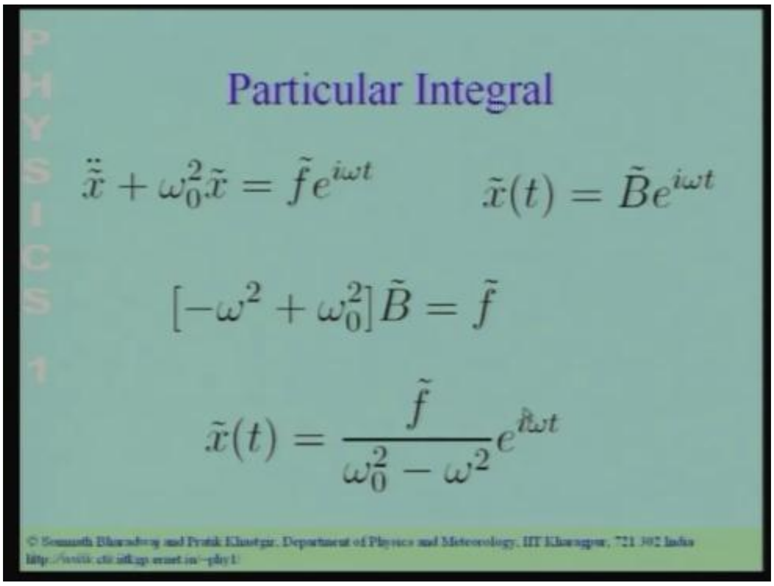

So, this is the let we go through the steps again. So, we have putting the trial solution into this equation and it gives us this relation between B tilde and f tilde I showed you just now how we get this. And then if you put this back into the expression over here, if you put in the value of B tilde back into the expression over here. It gives you the displacement x tilde the complex variable x tilde in terms of the external force f tilde e to the power i omega t. And you see that, the complex variable x the complex displacement x tilde is equal to the force external force divided by omega nought square minus omega square.

So, we have worked out the oscillation the displacement of the oscillator has a function of the external force. So, this is the particular integral remember that, we also have the complementary function and the total solution now you see has 2 parts: 1 part oscillates at the frequency omega nought that is, the solution even if the external force were not there we are not really interested in that particular solution. If the particular integral we see the part of the solution that arises specifically due to the external force that oscillates with exactly the same frequency omega as the external force that is the first feature.

So, whenever you have a simple harmonic oscillator and if you drive it with an external force, which is also oscillating a sinusoidal external force oscillating at a different frequency omega. Then the particular integral there is a part of the solution that oscillates at the frequency the same frequency as the external force. So, there is a part of the solution there is a part of a solution that oscillates at the same frequency as the external force at the frequency omega.

So, in general what you see is that; if I have a simple harmonic oscillator then and I have an external force acting on it. The simple harmonic oscillator under the influence of the external force will have 2 oscillation frequencies: 1 oscillation frequency is the frequency omega nought, which is the natural frequency of the simple harmonic oscillator that is the frequency at which the simple harmonic oscillator oscillates, even if there is no external force.

There is another frequency omega, the frequency at which the external force is acting and the simple harmonic oscillator also does sinusoidal oscillations at the frequency omega. So, it does a superposition of 2 kinds of oscillations 1 at the frequency omega nought another at the frequency omega.

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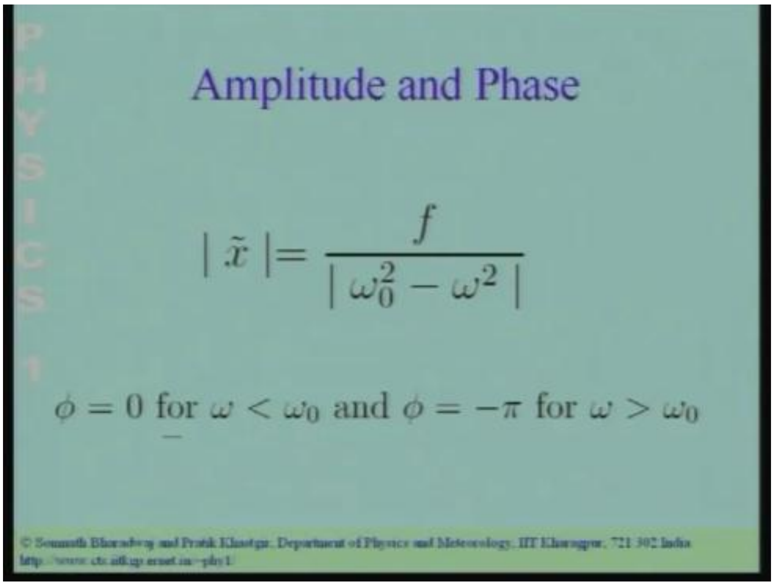

Now, let us look at the amplitude and the phase of the oscillations and its relation to the amplitude and phase of the external force; just remember that the external force also has an amplitude and an oscillation. The amplitude and the oscillation of the external force are both inside this variable f tilde which we have defined over here.

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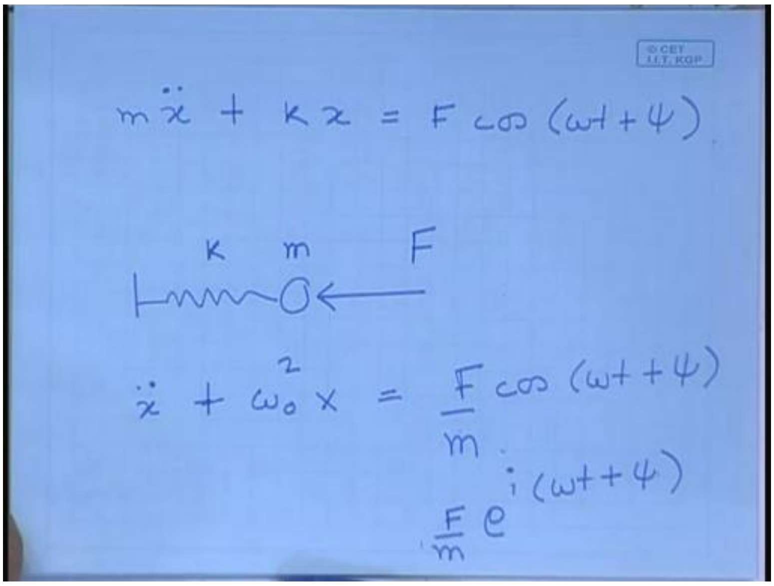

So, f tilde recollect that the external force which we give has both an amplitude and a phase the amplitude is F by m which is the magnitude of the small f tilde, which we have defined and it has a phase psi which is the phase of the f tilde variable which we have defined.

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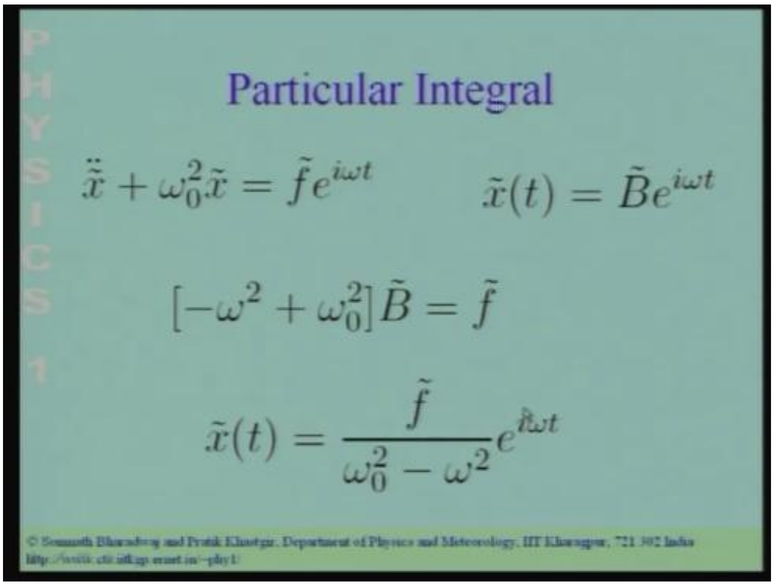

So, the variable f tilde f tilde which we have defined has got both the variable f tilde over here has got.

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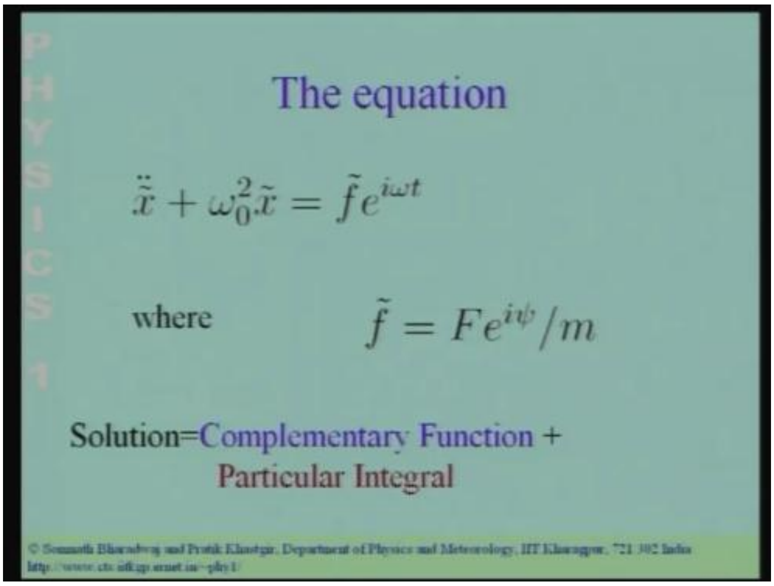

Both the amplitude of the external force, it also has the phase of the external force.

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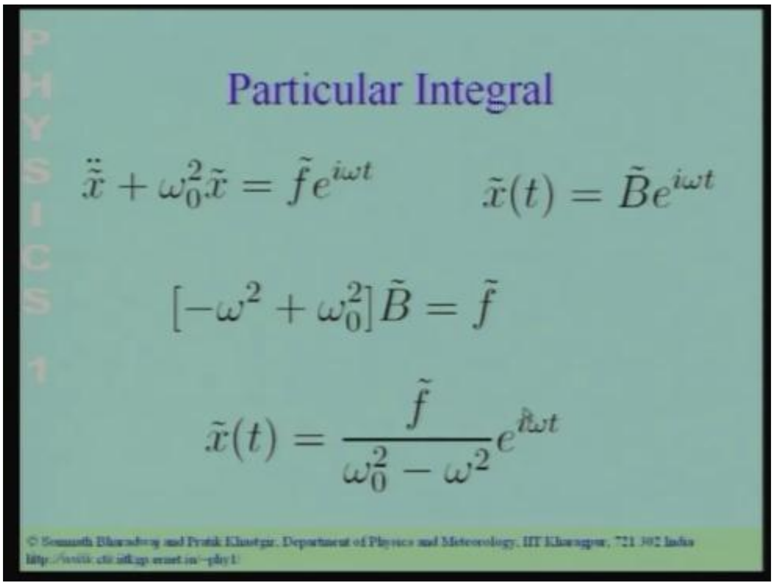

And this relation gives us the amplitude and phase of the oscillations relative to the amplitude and phase of the external force.

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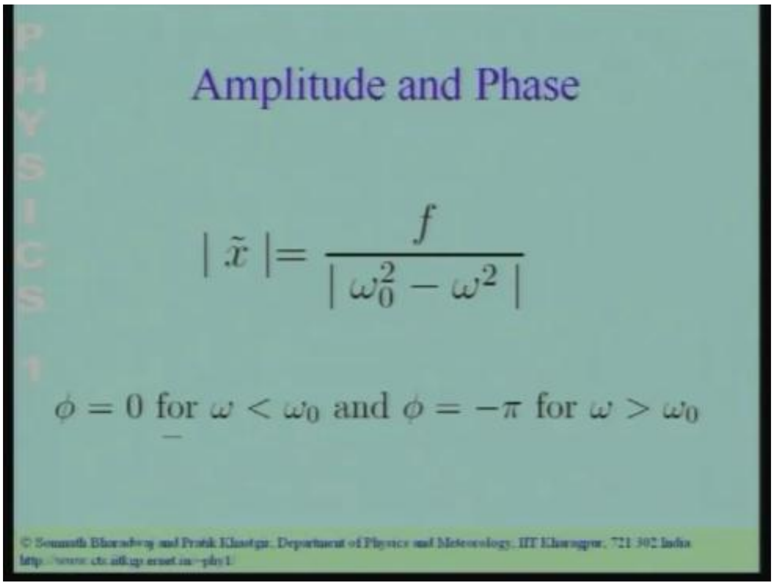

So, now let us study how the amplitude and phase of the oscillation relative to that of the external force behaves, if I change the frequency of the oscillation. So, amplitude of the oscillation is related to the amplitude of the external force f through this relation over here. So, you have to divide the amplitude of the external force by the modulus of omega nought square minus omega square to get the amplitude of the oscillation.

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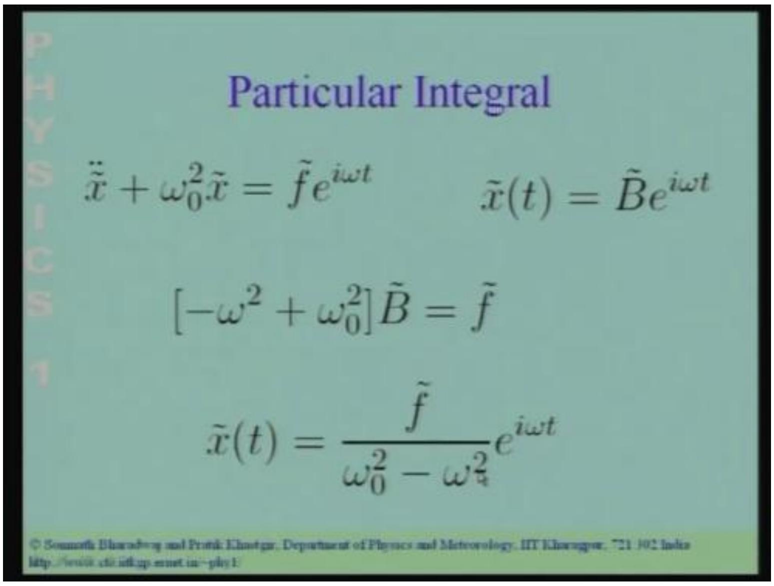

Let us next, look at the phase of the amplitude relative to the phase of the oscillations relative to the phase of the external force. Now, when omega is less than omega nought, if where omega is less than omega nought notice that the denominator over here, is positive. So, the amplitude of the displacement is related to the amplitude of the external force through a positive number multiplying a complex number with a positive real number does not change the phase of the complex number.

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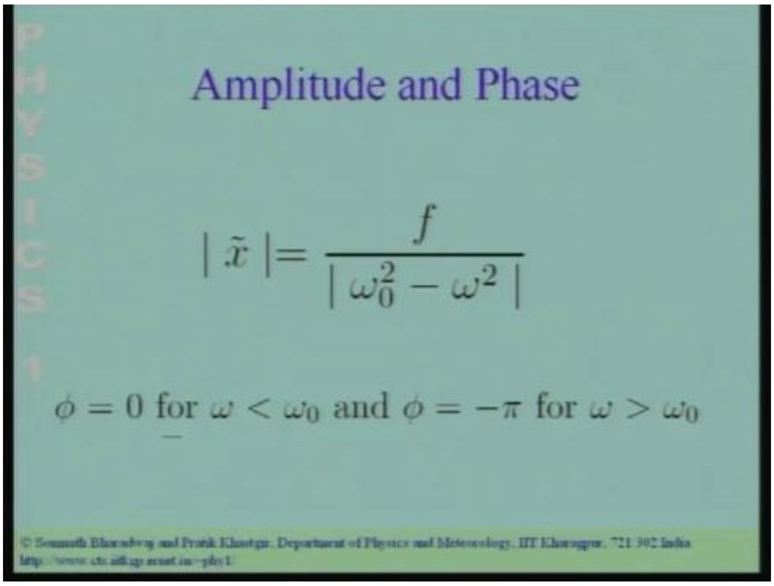

So, you are lead to the conclusion that when omega is less than omega nought the oscillation and the external force both have the same phase the relative phase given by phi between the oscillation and the external force has a value 0. So, when the angular frequency is less than the natural frequency of the simple harmonic oscillator both the external force and the oscillation occur at exactly the same phase. So, if you think of this as the external force and this as the oscillator when the frequency of the external force is less than the natural frequency of this they both oscillate in the same phase.

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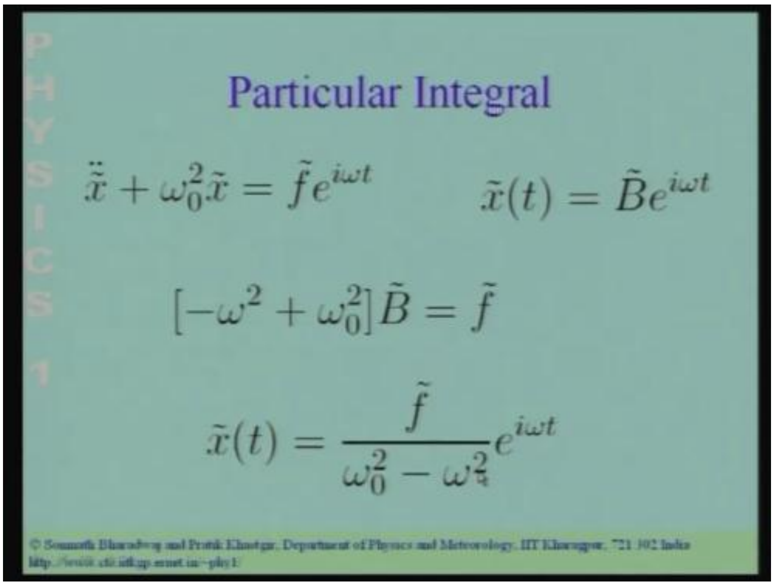

Now, let us look at the situation, where the external the frequency of the external force is more than the natural frequency of the oscillator omega is more than omega nought. In the situation where omega is more than omega nought notice that, the denominator becomes negative multiplying a complex number f tilde with a negative number.

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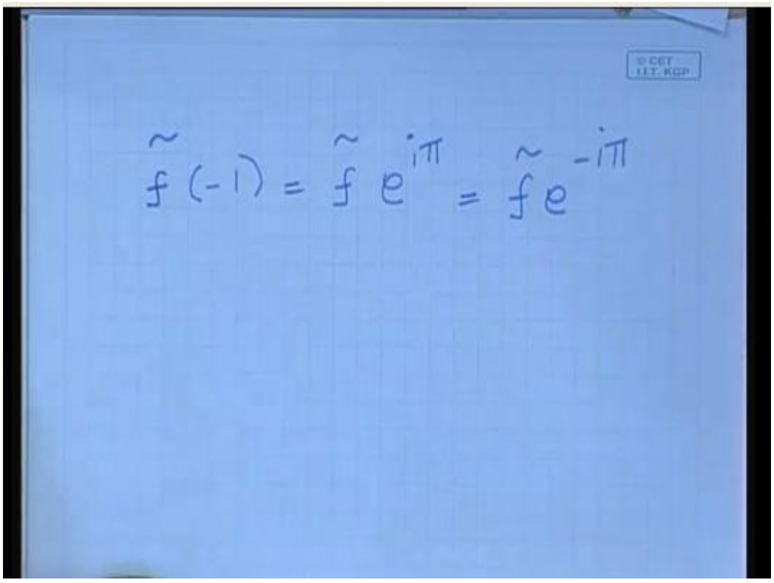

So, you are essentially multiplying f tilde with a negative number. So, let us just consider f tilde being multiplied by minus 1 now, minus 1 can be written as e to the power i pi recollect that e to the power i pi the real part of it is cos pi which is minus 1 the imaginary part is sin pi which is 0. So, e to the power i pi is essentially minus 1. So, multiplying f tilde with a negative number you can think of it as multiplying f tilde with e to the power i pi into the amplitude of the negative number which is a positive number.

So, you see that multiplying it minus 1 introduces a phase of pi a phase of pi.

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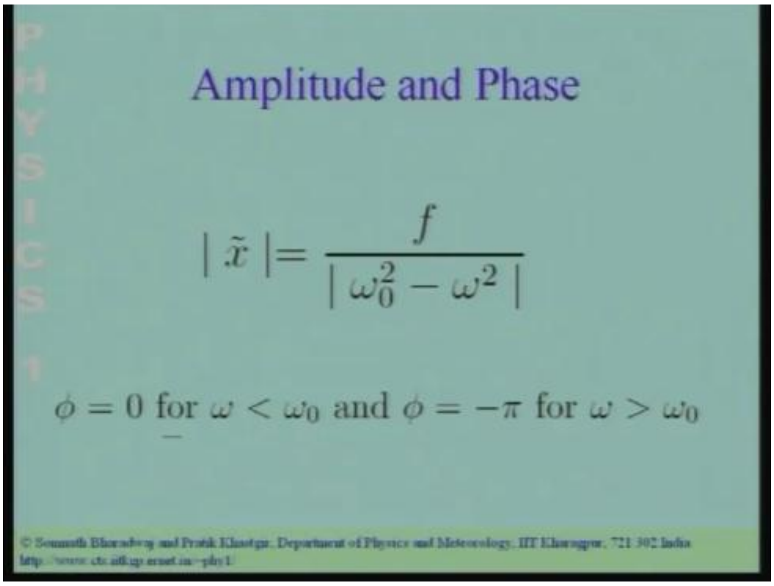

So, when.

So, when omega is greater than omega nought you have introduce there is a extra phase of pi between the amplitude and the force.

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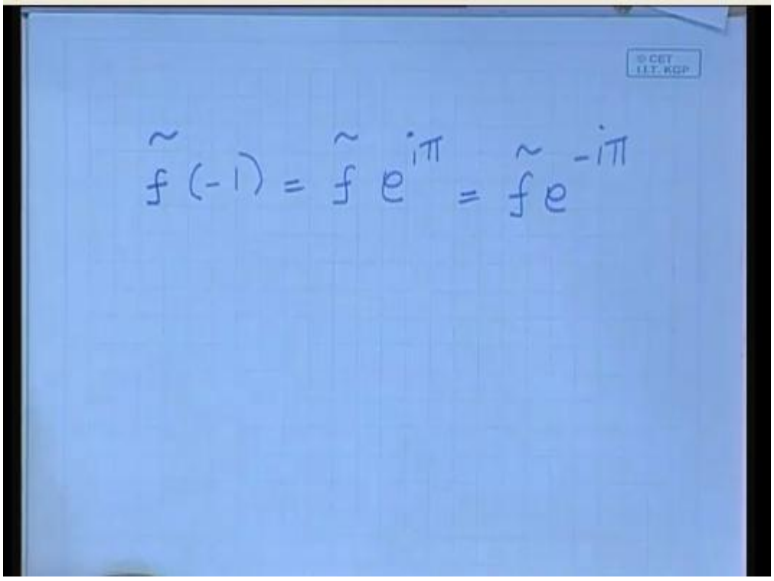

Now, the phase could be either plus pi or minus pi both plus or minus pi both these numbers represent minus 1. So, e to the power i pi is minus 1 e to the power minus i pi is also minus 1. So, there is an ambiguity if is the phase plus pi or minus pi. And we shall see shortly as we go long that, it is convenient to interpret it as minus pi.

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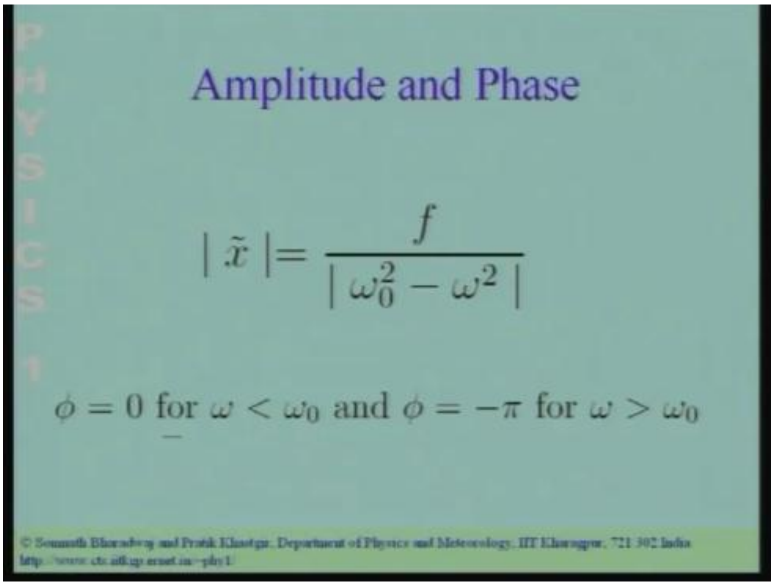

So, if the

So, if the angular frequency of the external force is more than the natural frequency then the oscillations occur at a phase difference of minus pi relative to the force. So, if this is the force then the oscillations, this is the motion of the oscillator they will occur at exactly minus pi outer phase. So, they will go last the motion will occur like this. This is the force, this is the motion and the both occur exactly minus pi outer phase. This is what happens if the external force is has a frequency, which is higher than the natural frequency of the oscillator.

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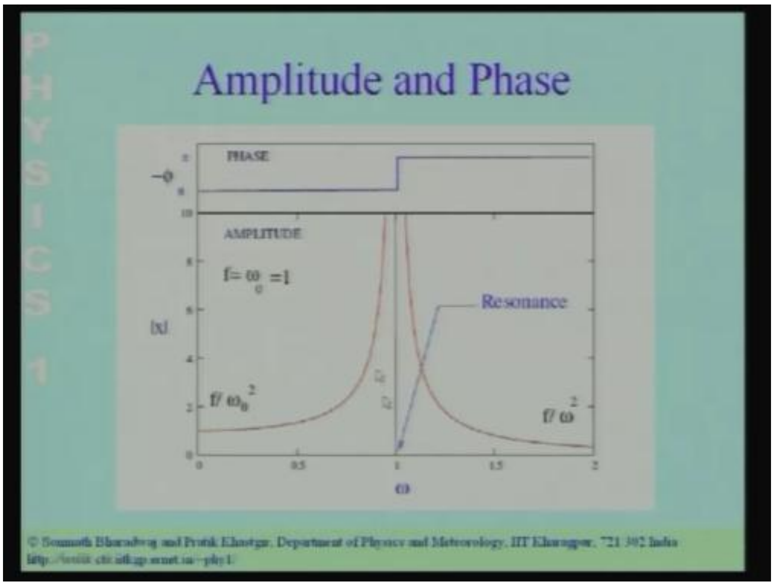

So, this is what is shown graphically over here when the angular. So, we have considered a here in this graph I have shown you both the behavior of the phase and the amplitude of an oscillator, as you vary the angular frequency of the external force. So, we have chosen a simple harmonic oscillator such that, it has a natural frequency omega nought equal to 1. So, we have a simple harmonic oscillator whose natural frequency is such that omega nought is equal to 1.

For this simple harmonic oscillator, we have applied an external force whose amplitude has also been chosen equal to 1. Now, we ask the question what is the relative phase between the external force and the oscillations of the oscillator. So, this oscillator has a natural frequency omega nought equal to 1. At frequencies at values of omega where the external force has an angular frequency less than omega nought which is less than 1 in this case.

The oscillations occur at exactly the same phase as the as the external force. So, the phase difference, if the phase difference is phi the quantity that I have plotted here is minus phi. So, the phase difference phi has a value 0 for angular frequency omega less than omega nought. Now, the moment omega crosses omega nought, the phase difference does a jump and it jumps to a value of pi minus pi. So, here I have plotted minus phi so, minus phi jumps minus the phase difference.

So, minus phi jumps from a value 0 at omega nought omega less than omega nought to a value pi when omega is more than omega nought. So, at angular frequencies more than 1 the phase difference between the oscillations and the force is minus pi the oscillations lag by pi relative to the force external force. This shows you what happens to the amplitude. So, let us now look at what happens to the amplitude.

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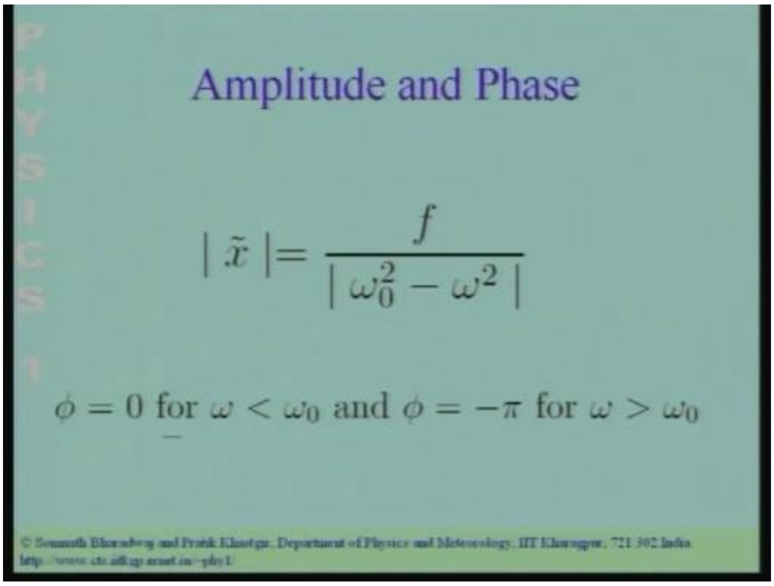

So, this shows you what happens to the amplitude. The most interesting thing occurs when omega is equal to omega nought. So, let us just see, what happens when omega is equal to omega nought that when omega is equal to omega nought notice that the denominator of this expression becomes 0. If the denominator becomes 0 this ratio f by omega nought square minus omega square this ratio becomes infinite.

So, it essentially tells us that, the amplitude of the oscillations the amplitude of oscillations blow up when the external force has the same angular frequency as the natural frequency of the oscillation.

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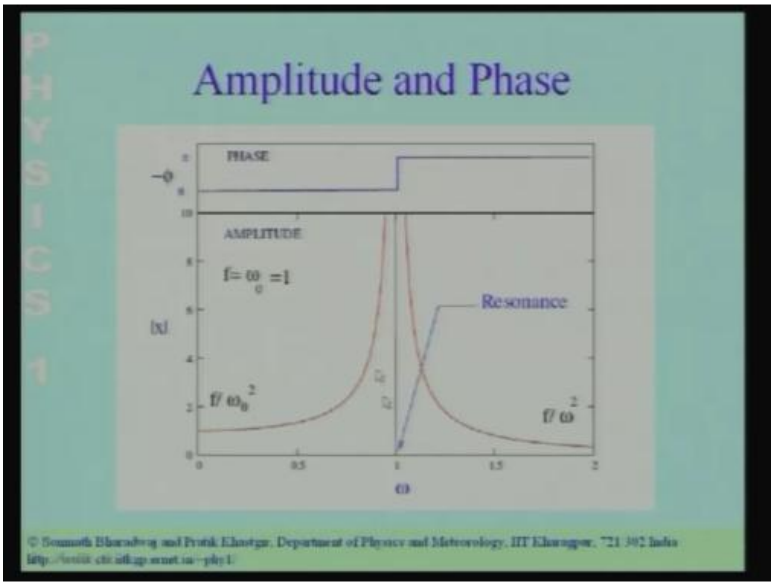

So, this is what you see over here, the amplitude of the oscillations blow up they become infinite, when the external force has the same angular frequency, as the natural frequency of the oscillator. This is the phenomena, which is referred to as resonance the amplitude of the oscillations become extremely large as omega.

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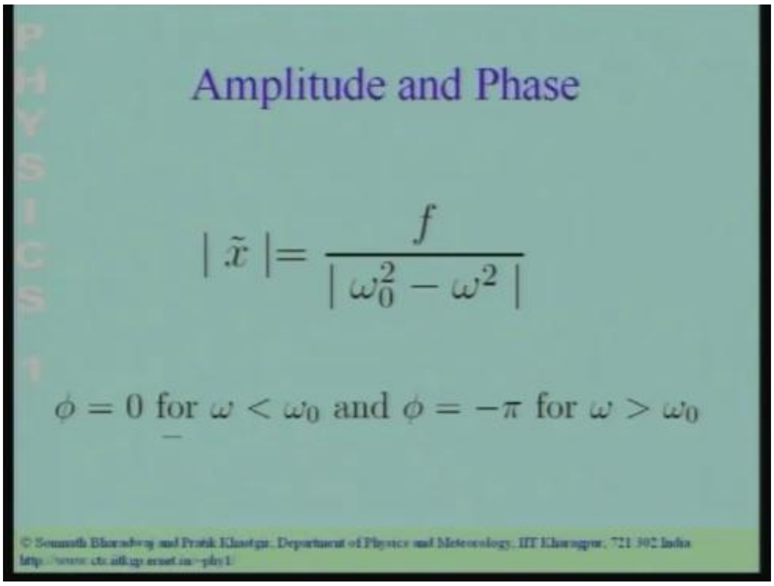

Approaches omega nought and it actually blows up when omega is equal to omega nought.

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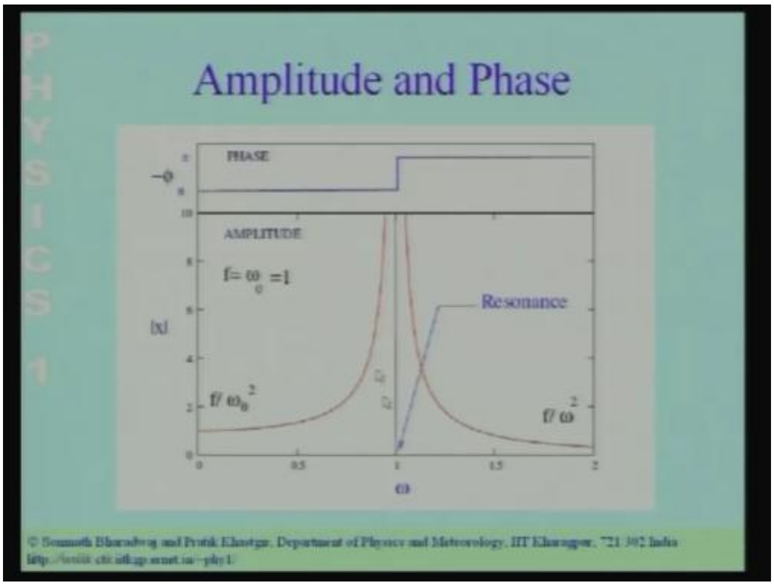

So, the fact that you have very large if you have a simple harmonic oscillator and you are driving it for the external force. If the external force has a frequency which is comparable to the frequency of the oscillator you get very large oscillations his phenomena is what is called the phenomena of resonance. So, if the driving force and oscillator both have the same frequencies then you get extremely large free oscillations and this is the phenomena, which is referred to as resonance.

And this is what I shown in this region of the graph the amplitude of the oscillations blow up as omega approaches omega nought as the frequency of the external force approaches the natural frequency which in this case is 1. The amplitude of the oscillations blow up and this is the phenomena of resonance. This phenomena is very important in nature and we shall be discussing it in some detail, in the next lectures.

Let us now, look at the behavior away from resonance. So, there are 2 regimes which are away from resonance: 1 regime is the region where omega is much smaller than the resonance frequency omega nought. This is the small omega limit and the other regime is where omega is much larger than omega nought very high frequency limit.

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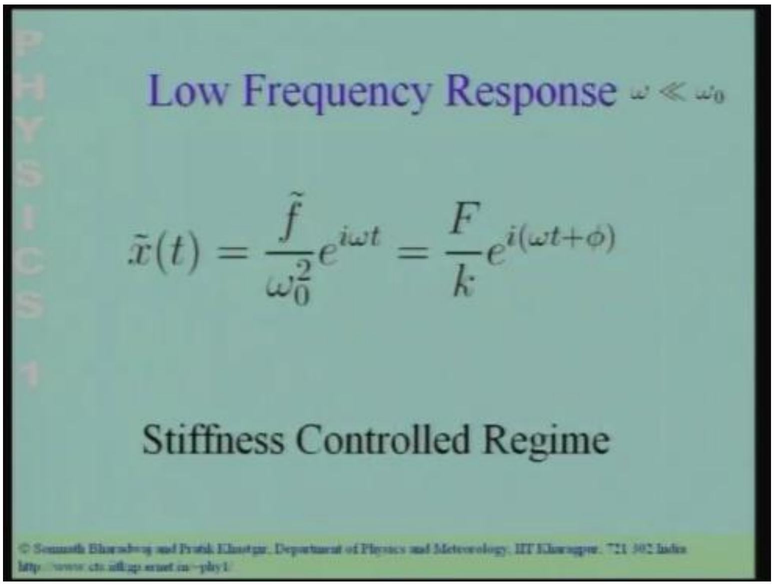

So, let us first look at, the low frequency response of the oscillator how does the oscillator behave if the driving force is much slower it has an angular frequency, which is much lower than the natural frequency of the oscillator. So, you could think of a situation for example, where I have let us say a building. Now, if you were to disturb the building and leave it for vibrate at some frequency. And this frequency would be the natural frequency of the oscillator.

Now consider a situation where there is an earthquake an earthquake is an external force you can think of the earthquake as being a periodic external force for our purposes. So, there is an earthquake which gives an external force. And you can think of it has been periodic if the force is not periodic you could at least decompose it into different periodic forces or sum of different periodic forces. So, you can think of a simple harmonic oscillator it has a natural frequency omega nought. Your forcing it whether different frequency which is slower than the natural frequency of the oscillator.

So, the question is what how does the oscillator respond, if it is forced by a force whose angular frequency is slower than the natural frequency of the oscillator. So, omega is much less than the resonant value omega nought.

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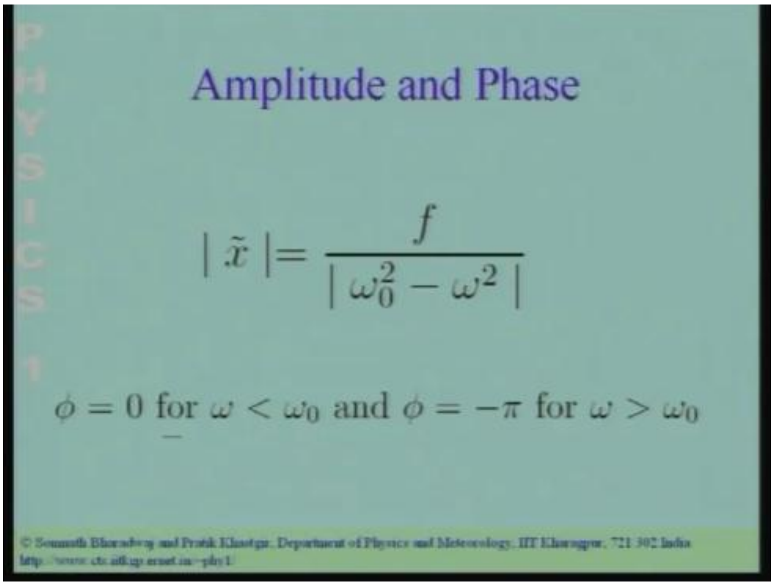

So, in this limit the 2 oscillations occur in phase that is the first thing. So, the oscillation occurs at the same phase as the external force that is the first feature.

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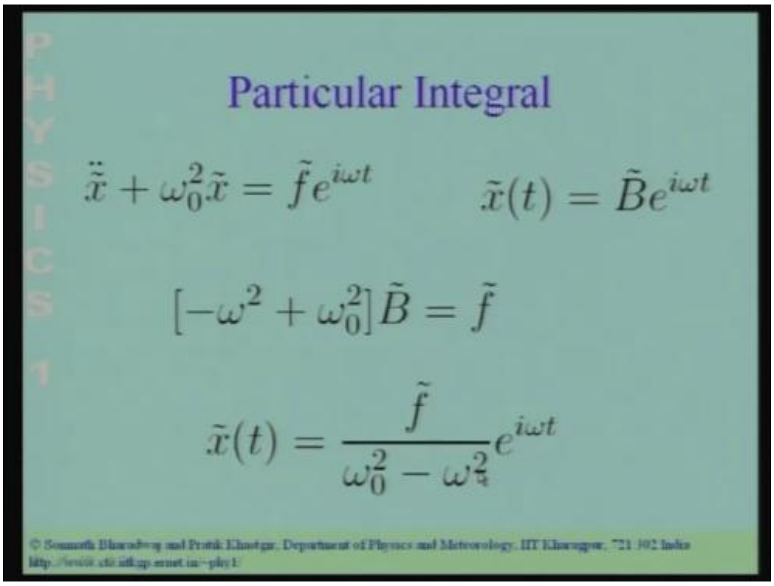

And if you take the limit of very small omega where omega is very small you can essentially neglect this term in the relation between the external force and the.

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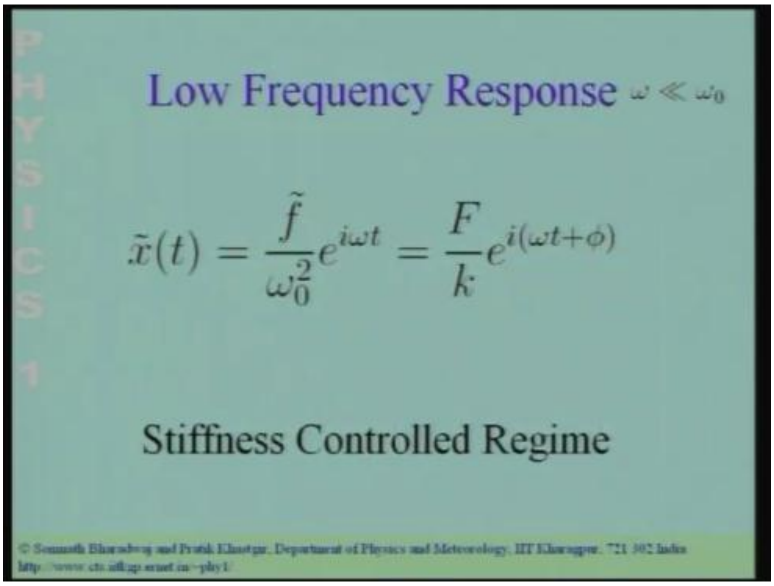

So, what you have is this relation that the displacement the complex variable corresponding to the displacement that is x tilde is equal to f tilde the complex amplitude of the force divided by omega nought square omega nought square is a natural frequency of the oscillator into e to the power i omega t e to the power i omega t is the is the cosine term of the external force. So, you get this relation. Now, recollect that f tilde was the force F the amplitude of the force F and you had the phase also in f tilde and if the whole thing was divided by the mass.

So, f tilde was F by m into this e to the power i phi e to the power i psi. And omega nought square is K by m. So, there is a 1 by m when you go from here to here and there is a 1 by m when you go from here to here these 1 by m factor cancel out. And you find that the displacement the complex displacement is equal to the amplitude of the force divided by the spring constant and you have this oscillating factor and the phase over here. So, this regime is referred to as a stiffness controlled regime.

So, let me give you and let me, try to give you some understanding of what happens in this regime of the oscillator. So, you let us get to get an understanding of what happens here let us consider, the limit where the frequency is extremely small. Now, if you set omega equal to 0 you have an external force which has no time dependence. So, you have an external force which is constant. So, let us study first the behavior of an oscillator under a constant external force. This I am sure is known to all of us.

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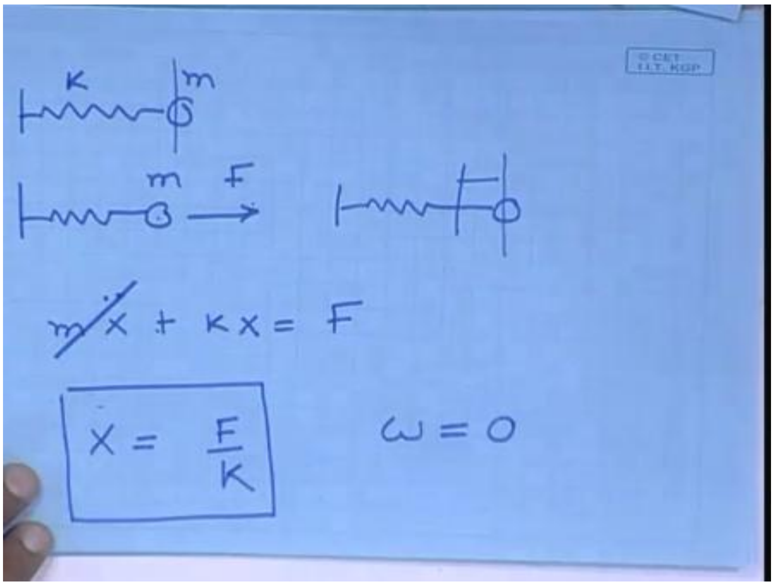

So, have a spring mass system this is the equilibrium position of the mass the spring constant is K a mass is m and this is the equilibrium position of the mass. Now, to this spring mass system if I apply a constant force F; if I apply a constant force F and ask what happens to the equilibrium situation, what happens to the spring mass system if there is a constant external force F. We can quickly analyze this system. So, we have m x double dot plus KX is equal to F.

So, it is the method to solve this is to basically that is no need to go into the mathematical solution of this equation sure all of us are familiar with the fact that the effect of a constant external force is essentially to shift the equilibrium position. So, if I put constant external force the spring mass the spring mass system on to this spring system essentially what happens is that this mass will move away from equilibrium to another from this original equilibrium position to another equilibrium position. And the new equilibrium position is such that the spring is extended by an amount.

So, that is exerts exactly the same force F and thus mass comes to equilibrium over there. So, if I have an external constant external force the net result F by K if the spring is extended by a constant amount F by K you can check that, it is a solution to this equation because X is a independent of time this cancels out and you see that this balance is this. So, if I put an external force F the spring is extended by an amount F by K and it remains at rest over there.

So, this is what happens to a spring if I apply a time independent force time independent means omega equal to 0. Now, let us consider a situation where the external force is not exactly omega equal to 0 it oscillates, but, the oscillations the external force is oscillating varying with time. But the oscillations of the external force are. So, slow that you can apply the this solution to that situation. The only difference is that this F itself now varies slowly with time.

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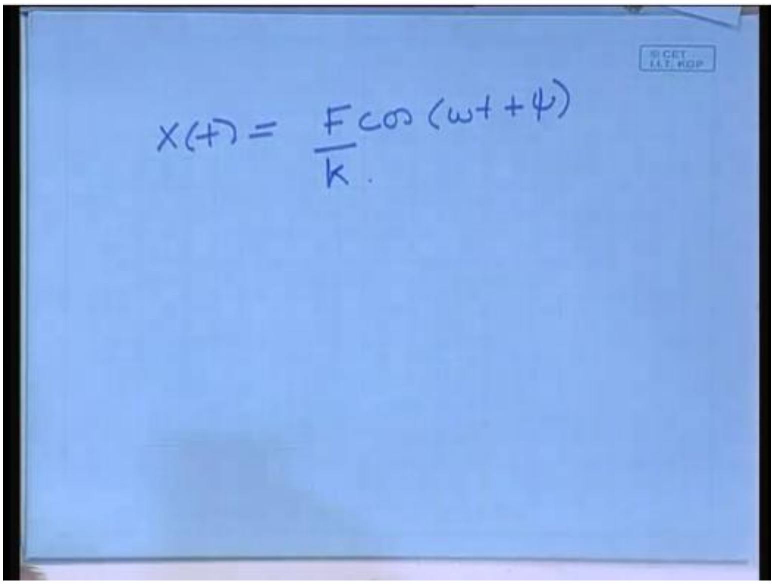

So, what you have is that, X of t is equal to F and F itself is now F cos omega t plus psi by K. So, as the force changes very slowly the equilibrium point also changes accordingly and the particle moves to the new equilibrium point. So, this is this kind of an intuition is applicable if the change in the external force occurs extremely slowly. So, you can think of the particle moving from 1 equilibrium position to another to another and to another.

So, it moves at exactly moves at exactly the same phase as the external force and the same frequency the only effect of the external force is that, it shifts the equilibrium position of the particle the particle now displaces to the new equilibrium position. So, this is what happens if the frequency of the external force is much smaller compared to the natural frequency of the oscillator.

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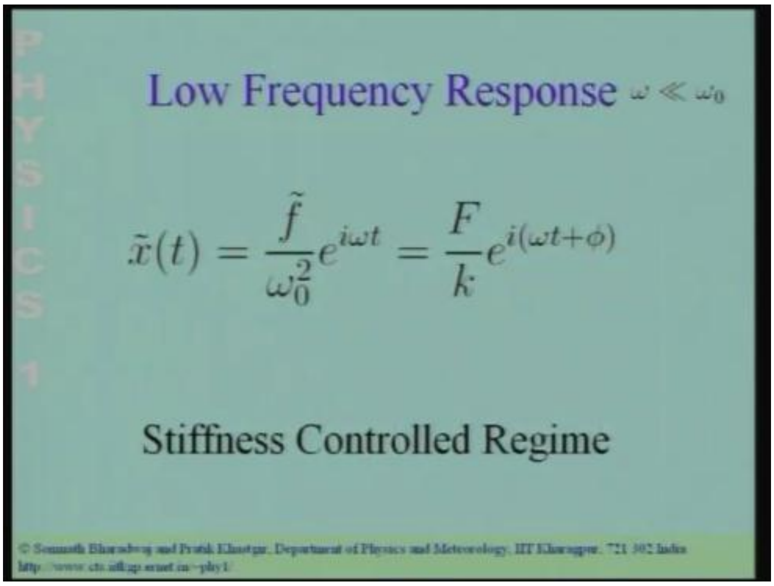

So, if omega is much less than omega nought this is the behavior that you get the particle goes to new equilibrium positions, the particle gets thus just string the spring at extended. And the particle goes to new equilibrium positions whose value is determined just by the force. This is called the stiffness controlled regime.

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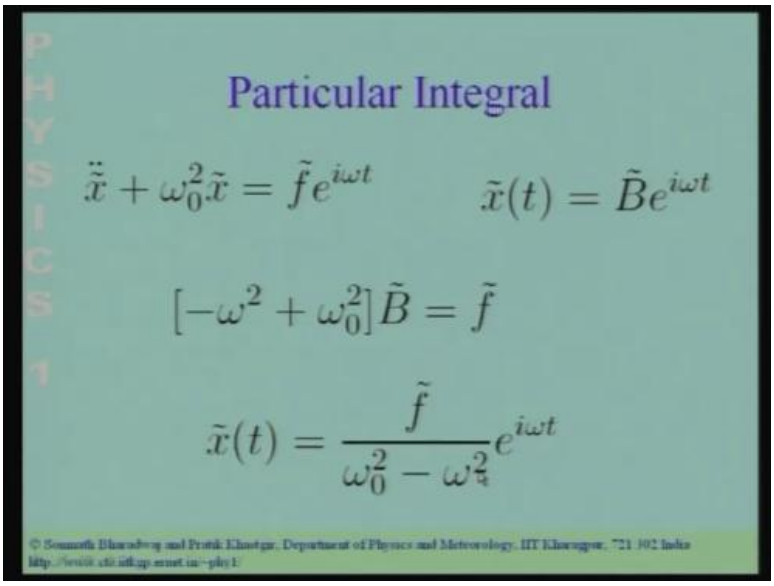

Let us now, look at the other extreme end of the behavior of the response which is the situation, where omega is much larger than omega nought what happens when omega is much larger than omega nought, as we have already discussed there is an extra phase of minus pi between the oscillations and the force. And in the limit where omega is much greater than omega nought you can essentially ignore this term.

So, what you have is the displacement the complex variable corresponding to the displacement x tilde is equal to.

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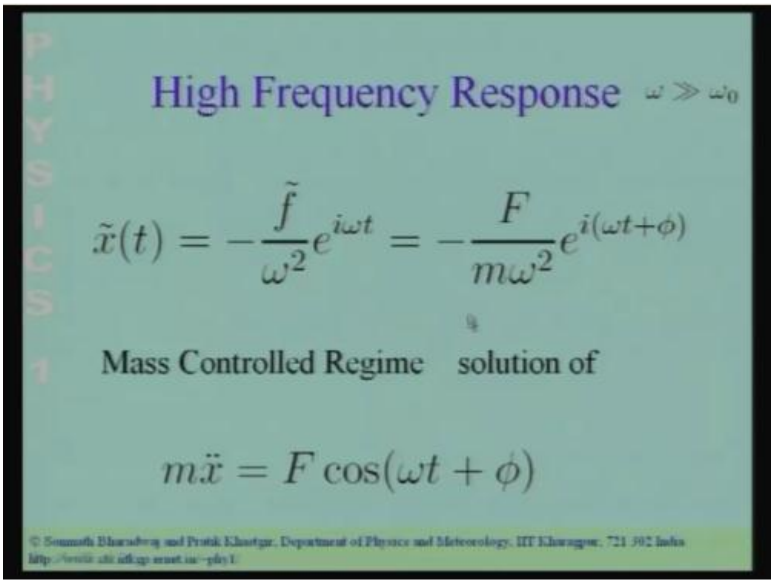

F tilde divided by omega square there is a minus sign that is the phase e to the power minus i pi and you have this e to the power i omega t. Again putting back the factor of the mass 1 by mass which occurs here and the phase what you find is that the variable x tilde is related to the amplitude of the force and the phase through the relation given over here. It only depends on 1 by omega square. So, as you keep on increasing the frequency of the external force the oscillations get smaller and smaller this regime is called the mass controlled regime.

Let us now take a look at this regime and try to get an understanding of what happens in this regime. So, this regime you can you get an understanding of what happens in this regime by looking at this particular equation over here.

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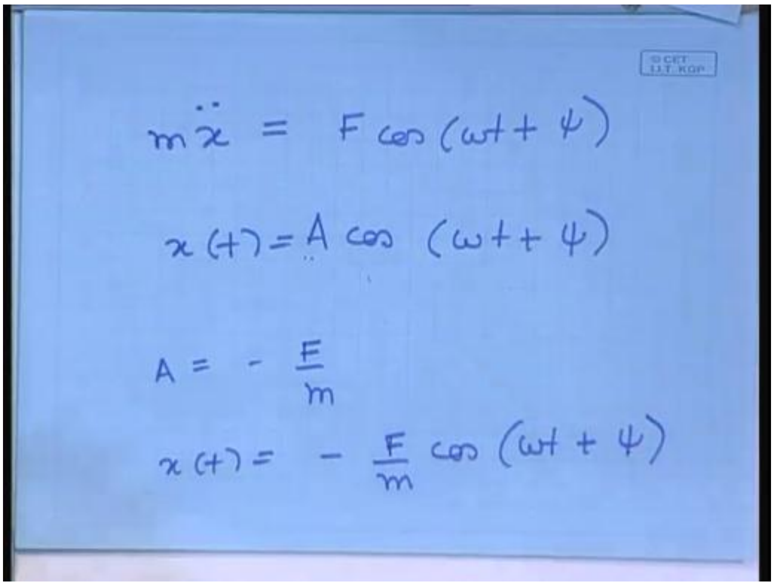

So, you need not bother about the spring at all in this regime. So, you have this equation F cos omega t plus psi. So, in this regime where omega is very large, the external force is oscillating very fast, the oscillation of the external force is. So, fast that it effectively boils down to the fact that you can ignore the spring. The spring it occurs much faster than the time scale on which the spring can react and essentially what happens is that the force due to the spring gets cancelled out averaged out over the oscillations of the external force you can essentially ignored the spring and you are left with the equation which you have over here.

So, the spring no longer effectively comes into the picture, you have the equation of motion of a free particle under the influence of an external force the spring is not there and the solution to this equation, if you put in the trial solution xt is equal to cos some amplitude A cos omega t plus psi. Then this gives you the relation that A is equal to minus F by m because if you differentia cosine twice you pick up a minus sign.

So, this gives you the relation that this amplitude A over here is equal to minus F by m which essentially gives you the solution that x of t is equal to minus F by m cos omega t plus psi. So, what we have seen till now, is that when you drive a simple harmonic oscillator with an external force, which is oscillating with the frequency omega there are 3 distinct regimes 1 regime is when the angular frequency of the external force corresponds to the angular frequency.

The natural angular frequency of the oscillator this gives this is where you see the phenomenon of resonance. You have very large oscillations this is a phenomena which we shall study in some detail as we go long then we have the regime, where the external force is much slower than the natural frequency of the oscillator. In this regime you can think of the whole thing, the whole system as moving to a new equilibrium under a constant force. And then that constant force vary slowly with time.

So, the equilibrium position shifts and the equilibrium position is determine just by the time external force. And the external force is vary. So, the equilibrium position also varies slowly with time. This is the situation where omega is much less than omega nought and then you have the other extreme, where the external force has an angular frequency which is much larger than omega nought much larger than omega frequency. In this regime you can forget about the spring this regime is totally governed by the mass of this system.

So, you can forget about the spring and you can think of at being a free particle under the influence of an external force, the spring can be ignored. So, there are 2 regimes when the external force is very slow the inertia, inertia the acceleration of the object of the mass can be ignored. And when the external force is very fast as time dependence you can forget about the spring. It is only the inertial term mass into the acceleration which really comes into the picture we have these 2 limits.

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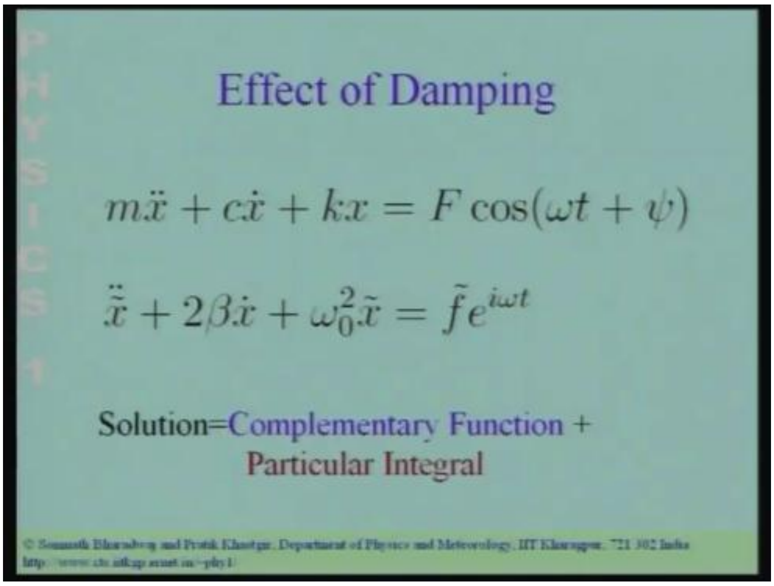

Let us now consider, what happens during resonance. So, you see there is no damping in the absence of damping we have a simple harmonic oscillator and if the external force has a same frequency as a natural frequency of the symple harmonic oscillator, you have an infinite Amplitude infinity the large oscillations.

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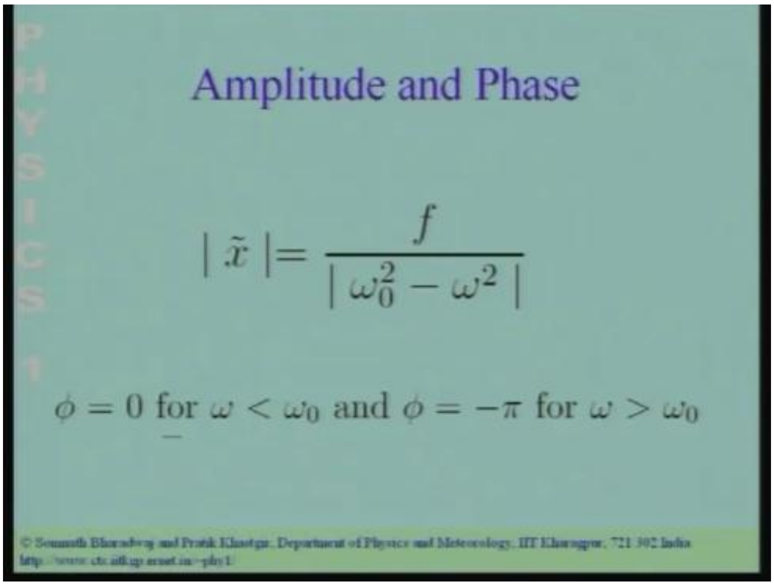

Now, in reality such a thing does not occurred in reality you always have

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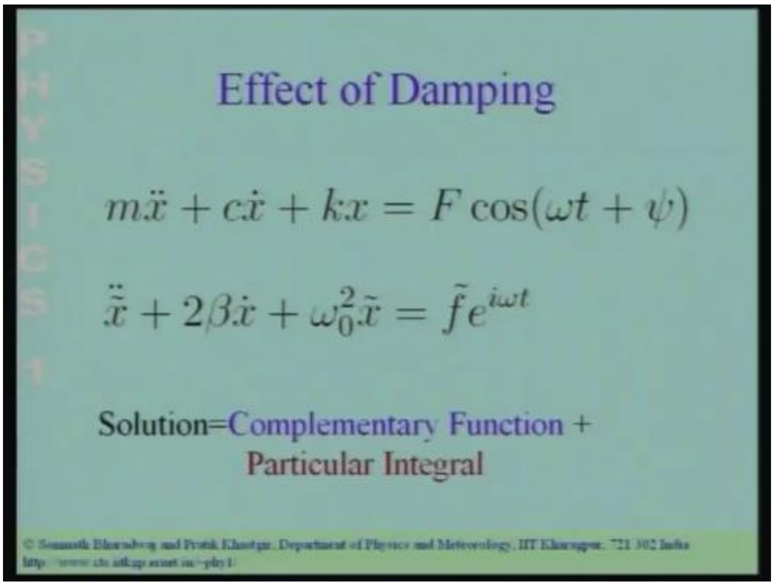

Damping and damping as we shall see reduces finite oscillations. Damping ensures that the oscillations do not become infinitely large it regulates, it maintains the oscillations at a finite value. So, let us now look at happens that resonance and to get finite answers for what happens at resonance, it is essential to consider damping. So, let us putting the damping term and now study what happens to the simple harmonic oscillator, if it is driven by an external force.

So, we have the good old equation, where I have a simple harmonic oscillator with damping this the damping term cx dot, this is the mass into acceleration, this is the damping term this is the effect of the spring. And here we have the external force, the external force has an amplitude F it is at a phase psi.

Now, we can divided this whole equation again by m dividing it by m we have x double dot over here c by m I have written as 2 beta that is in the notation we had introduced earlier k by m is omega nought square and F the amplitude of the force divided by m is f small f and I have absorbed the phase. So, we have f tilde over here. And the x over here remember is a complex variable. So, here again we have a second order differential equation, linear equation homogenous, second linear differential equation which has an external force on the right hand side.

So, as we have discussed earlier, such an equation has 2 solutions, the solution of this such a differential equation has 2 parts. The first part is the complementary function the complementary function is: the solution to this equation in the absence of this term over here in the absence of the external force. And we have seen that the compliment the that the complimentary function we have studied this in the last 2 lectures. And remember that the complementary function in the case of damped oscillator the complementary function is a time at a function of time which decays as time increases.

So, in all cases for a damped oscillator the complementary function decays with time the solution when there is no external force decays with time. So, if you have a damped simple harmonic damped oscillator and if you disturb it and there is no external force. So, you have disturbed it and left it the disturbance slowly decays with time. There we have studied there were 3 possible situations there was the critically damped the under damper and the over damped.

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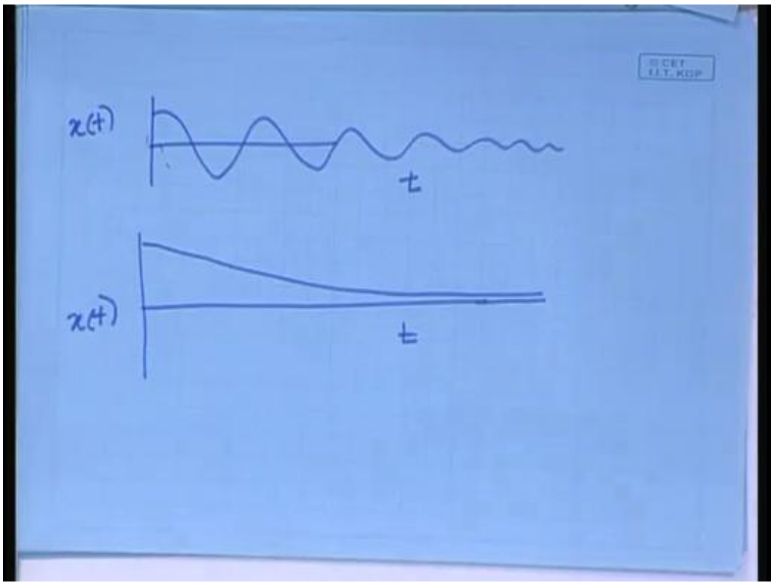

For the under damped oscillator, we had oscillations and the amplitude of the oscillation decayed with time for over damped and critically damped there were no oscillations. So, the oscillations decay with time with the displacements the deviations from the equilibrium decay with time the decays exponentially in both cases. For under damped you have oscillations along with the decay here you have no oscillations.

The crucial point is that, if you wished to study the large time behavior of the oscillator irrespective of whether the oscillator is over damped or under damped the oscillations die away as you go to late times. So, if you put a disturbance at t equal to 0 and observe what happens at late times these disturbances die away. So, the cracks of this whole thing is that.

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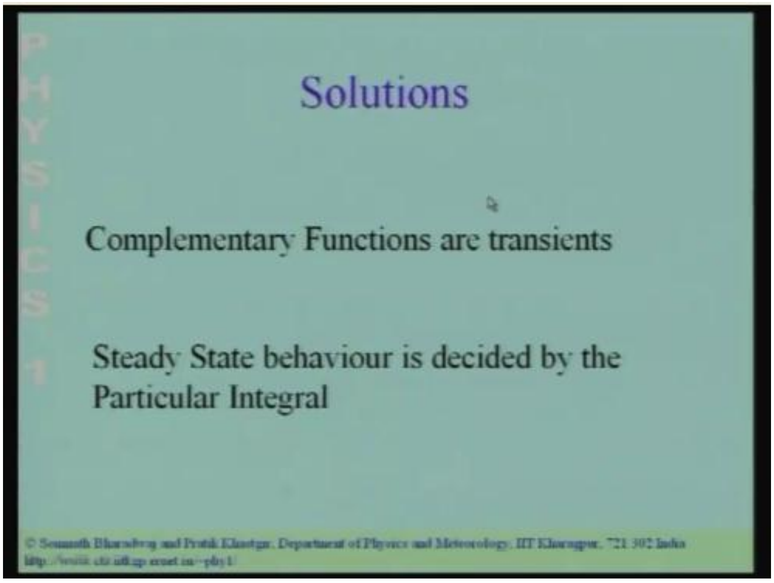

The complementary function.

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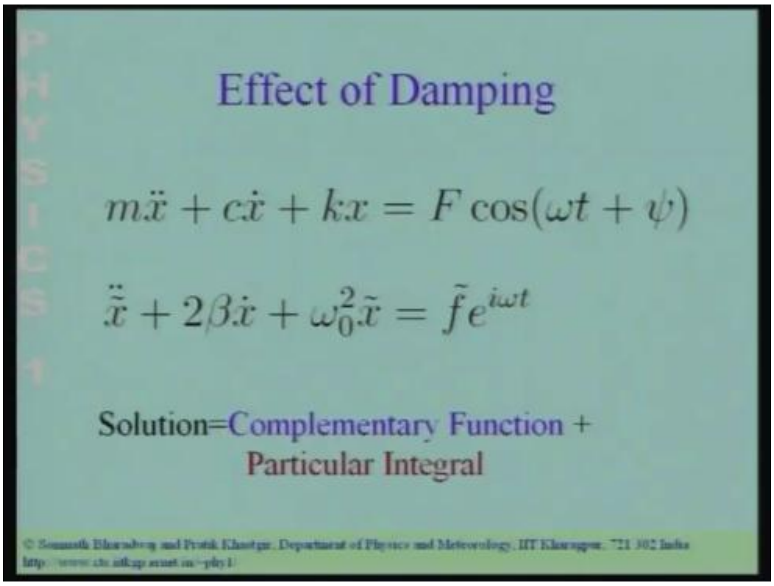

The function when this term over here is 0.

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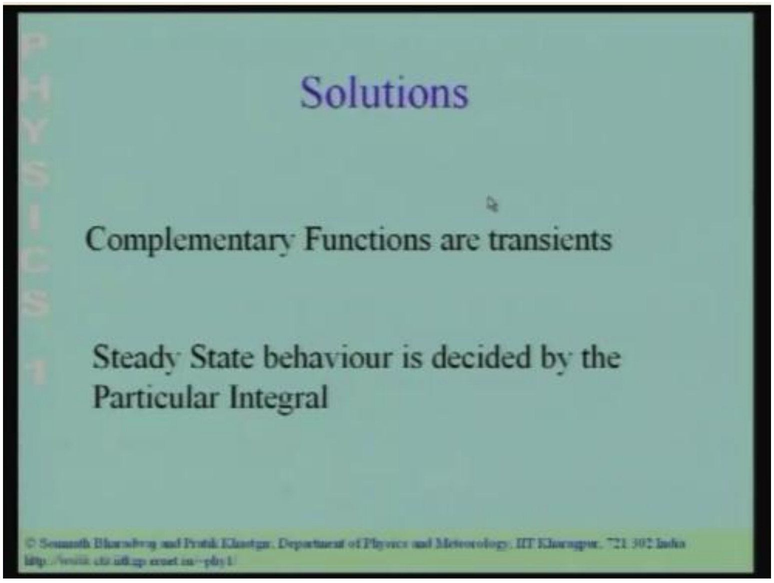

The complementary function are all transients these solutions there are always 2 complementary functions these solutions are always transients for a damped oscillator. And at late times these solutions become very small. So, if you wished to study the late time behavior you have to look at the particular integral alone there is no need to be concerned about the complementary function. These are transient functions by transients we mean things which are short lived which die away at large times.

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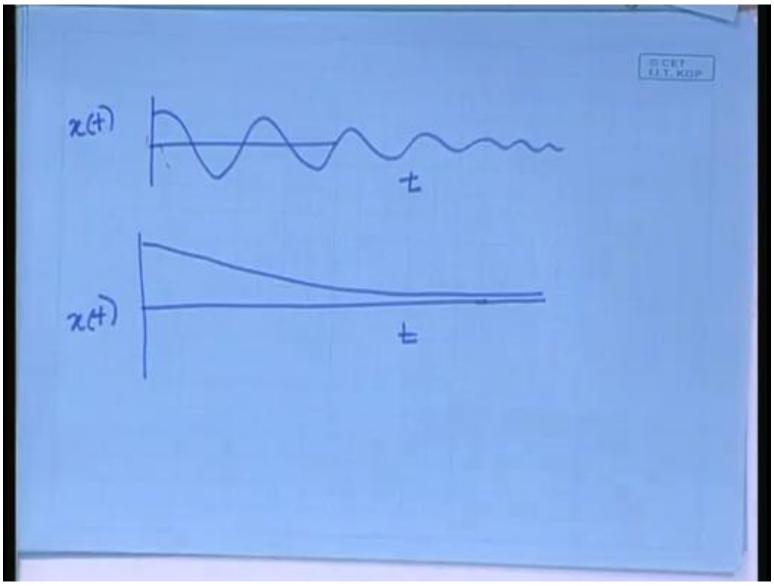

So, the complementary function.

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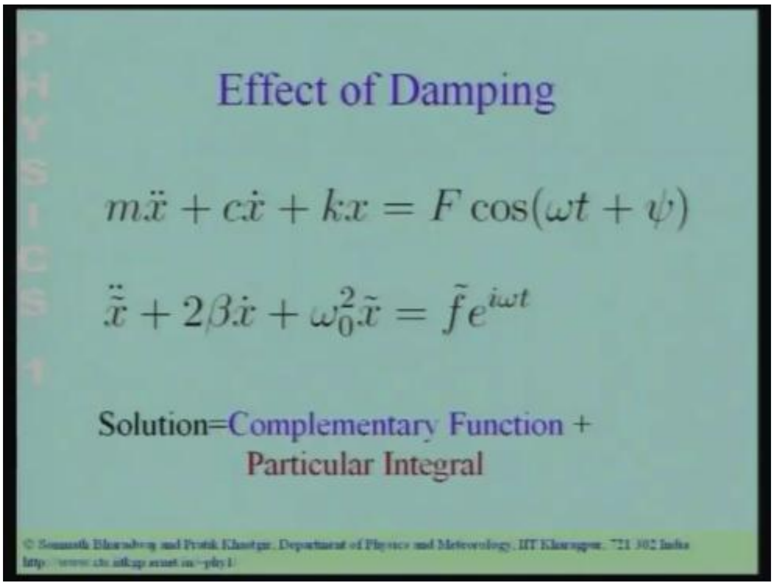

The total solution of this damped simple harmonic oscillator with an external force has 2 parts the complementary function and the particular integral the complementary function these are transients you do not have to bother about them if you are looking at the late time behavior. So, let us look at the particular integral. So, let we work out the particular integral for you over here.

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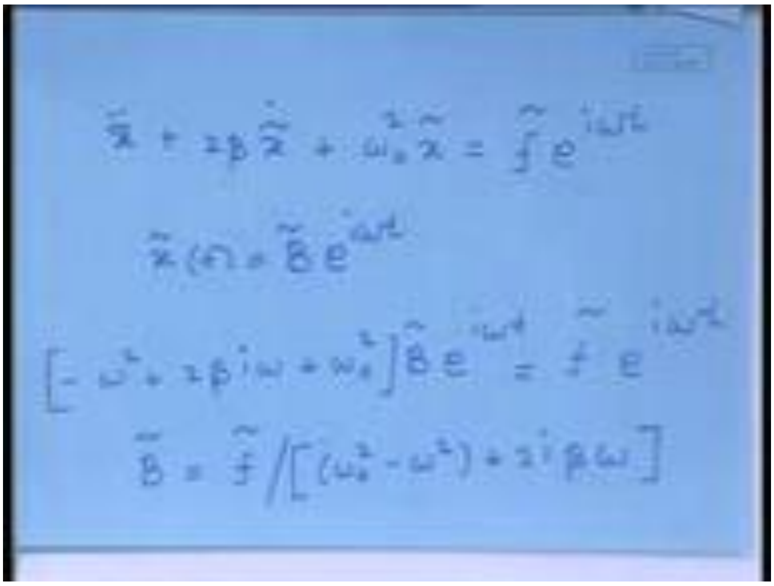

So, the equation at hand is X double dot plus x tilde double dot plus 2 beta x tilde dot this is the acceleration, this is the damping term and then we have the spring omega nought square x tilde this is equal to f tilde e to the power i omega the I shall not be referring to the tildes explicitly anymore assume, it is always there. Now, we need to find the particular integral of the his equation. So, we have to find a functional a function of time x as a function of time which will satisfy this equation.

The right hand side note is e to the power omega i omega t. So, if you wish to find a solution then you should chose x also to be some co-efficient B into e to the power the same time dependence as the right hand side, it is only then that these 2 can match. So, if you take this kind of a trial solution and put it into his equation the first term over here gives us minus omega square. The second term over here, gives us plus 2 beta and if I differentiate this once if I differentiate this once I will get a factor of i omega.

And here I do not have to differentiate it at all. So, I will have the factor of omega nought square this whole thing will multiply B e to the power i omega t and this is equal to B this should be a B tilde over here. So, we have taken a trial solution of this type and put it into the equation which we wished to solve we want the particular integral for this equation. And doing this gives us this relations. So, straight away you see that e to the power i omega t cancels out from both the sides. And you have B in terms of f.

So, what you get is B is equal to f divided by now, this can be written as omega nought square minus omega square plus 2 i beta into omega. Now, if you put this back into the into the trail solution you were lead to this expression for

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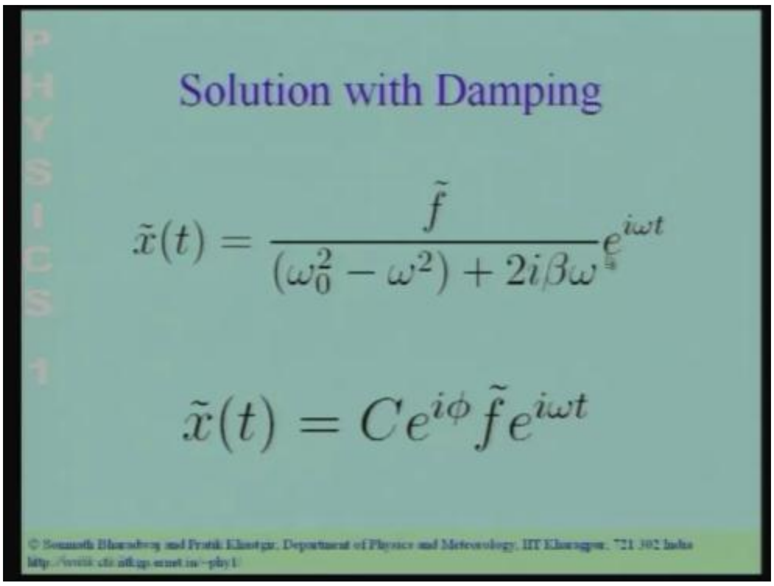

The displacement x as a function and the amplitude of the oscillation f and you have the solution for the displacement x as a function of time it is f divided by omega nought square minus omega square plus 2 beta i omega e to the power i omega t. So, the you have this extra term because of the damping this term does not derives when you do not have damping. Damping the role of damping is that you have this extra term which comes about.

Now, you can also write this relation between the external force between the external force f and the displacement x as in terms of an amplitude and a phase e to the power i phi. So, this amplitude is essentially the amplitude of this factor over here and this phase is the phase of this factor over here. So, let us now analyze the relation between the displacement and the external force in some more detail. The first point to note is if you consider the very low frequency regime, the low frequency regime that is omega is much smaller than omega nought much smaller than the angular the natural frequency of the oscillation.

So, you are away from resonance and you would like to study this

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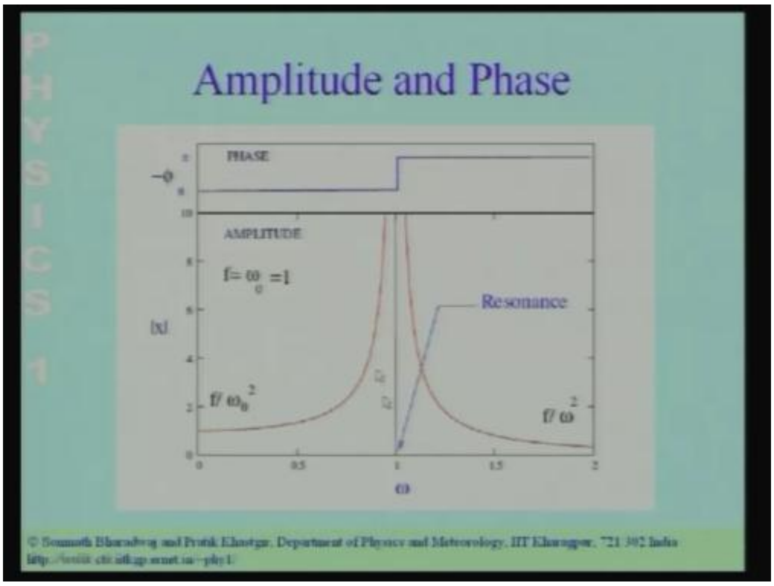

Regime of the oscillator in the presence of damping.

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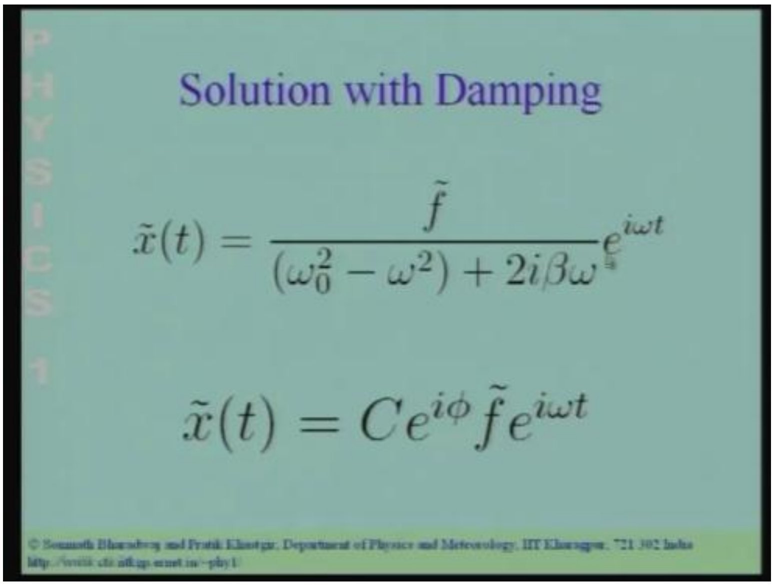

So, let us study the oscillator in the low frequency regime of damping which I just showed you. Now, when you take the limit of omega going to 0 the dominant term over here the term that the dominant term that remains is f tilde divided omega nought square and you have e to the power i omega t here which is exactly the same as when you have no damping. So, damping does not make any difference to the low frequency the low omega behavior.

The behavior at angular frequencies much smaller than the resonance value there is no difference which is caused by damping. So, the intuition which we had developed earlier in that regime still holds even if you introduce damping. Similarly, if you consider the other extreme where you have very large omega again note that for very large omega if you take, the limit of very large omega where omega is much larger than omega nought. Again note that the leading order term now this term is much more than this, it is also much larger than this omegas for large omega omega square is much larger than any term proportional omega.

So, in the limit when omega is much larger than the resonant frequency omega nought you have x is equal to minus f divided by omega square e to the power i omega t.

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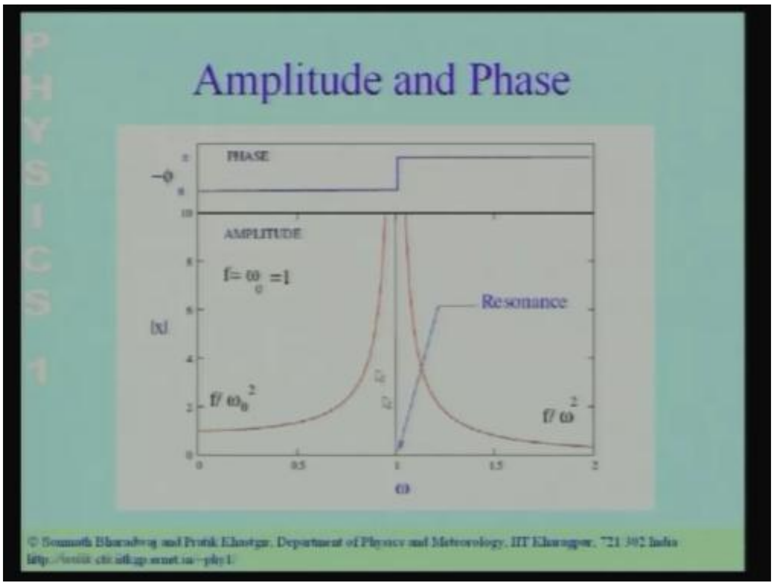

So, in the 2 extreme cases, in these 2 limits in the low angular frequency limit where omega is: much smaller than omega nought or in the very high angular frequency limit where omega is much large than omega nought. The role of damping is there is no there is no difference that is caused by damping these. So, this kind of a solution over here and over here remains unchanged whether you have damping or not.

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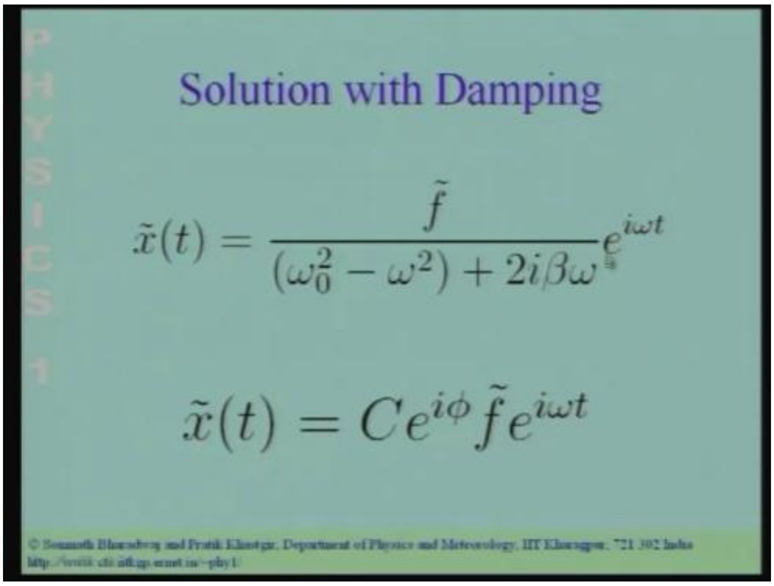

Damping makes a big difference around resonance at resonance you have omega equal to omega nought. So, at resonance in the absence of damping you had infinity in the denominator you had 0 at the denominator and the displacements become infinite. Now, because of damping you see that even at resonance even when omega is equal to omega nought the denominator does not become 0 and you have a finite amplitude for the oscillations at resonance.

So, this is what we are going to study, in the next class we are going to study the behavior of this expression in the presence of damping around the value around the resonant value, where omega is equal to omega nought. So, let me stop here and continue in the next class.