Physics I: Oscillations and Waves
Professor S. Bharadwaj
Department of Physics and Meteorology
Indian Institute of Technology, Kharagapur

Lecture-27
Solving the Wave Equation

Good morning. In yesterday's lecture we were discussing the evaluation of a disturbance in an elastic medium, an elastic rod. Let us say, and the kind of disturbance that we were discussing we had deformation of the rod. And we consider the situation where the deformation we studied the evaluation of the deformation in the same direction as the deformation itself. And we had obtained an equation which governs the evaluation of disturbances in an elastic rod.

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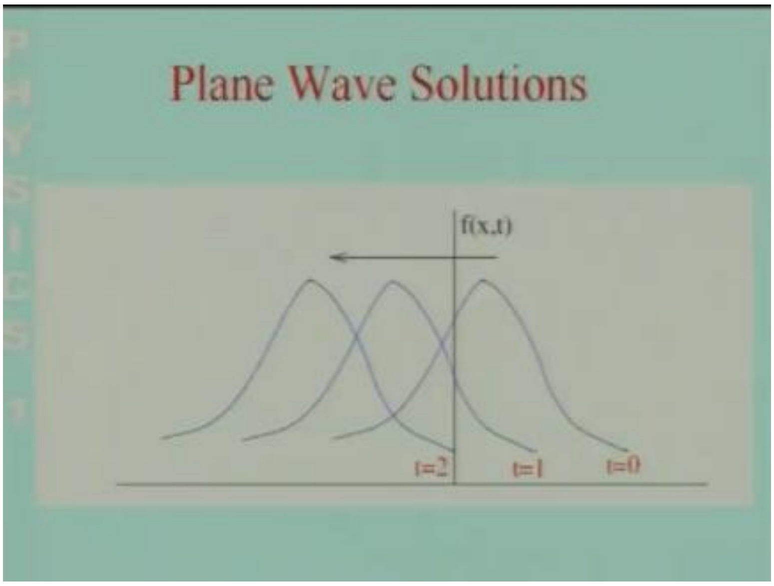

So, the disturbance was xi the variables xi represents the displacement of any element of the elastic material.

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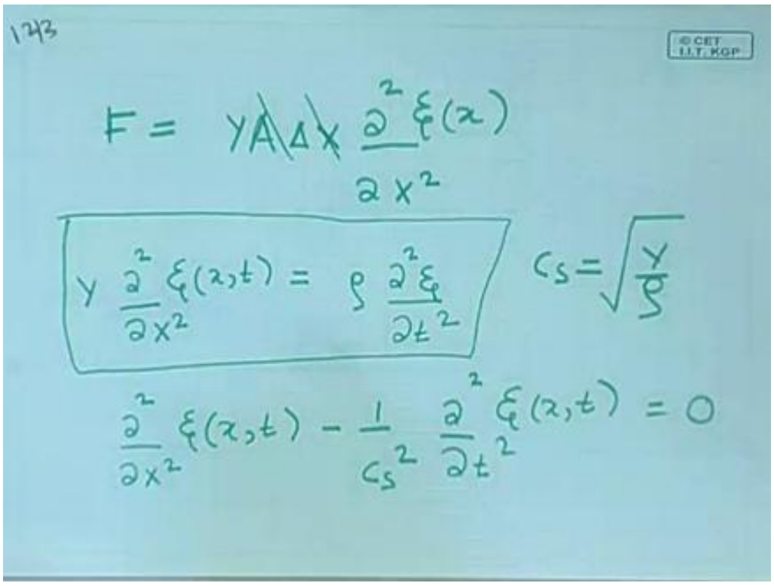

And these displacements are along the x axis and we had assumed that the displacements also vary only along the x axis. So, they xi is a function of x the only the variable only special variable that xi is a function of its x and it is a function of t also. And we had obtained this differential equation this partial differential equation which governs the evolution of xi and this partial differential equation. We had written in this fashion where we had this constant cs which was the square root of the Young's modulus divided by the density of the elastic medium. And I have told you that this equation is the wave equation. If you have a disturbance which can propagate which is which varies in all 3 directions.

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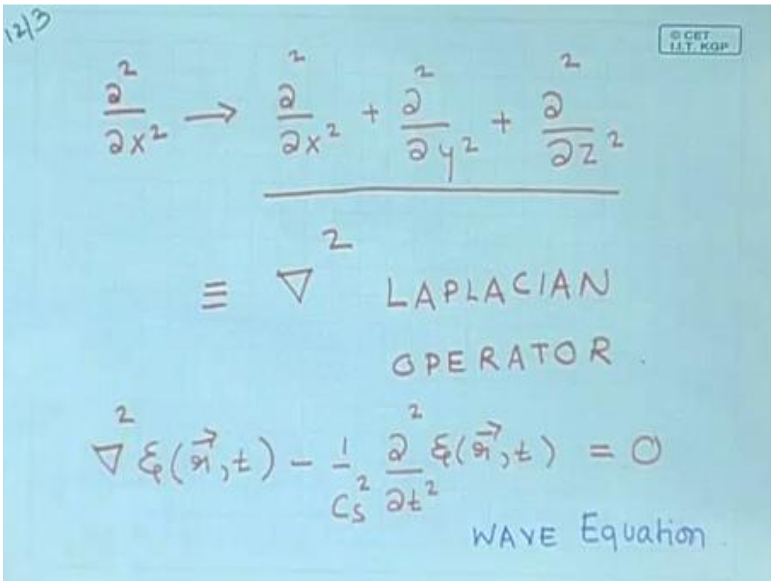

Then you have to replace which varies in all 3 directions then you have to replace the derivative with respect to y with the Laplacian operator. That is del del x square plus del del y square plus del del z square and the wave equation is now given by this. Now, I should tell you the first the first thing. I should tell you is that the waves which we had considered in the last class are what are known as longitudinal waves. So, the waves that we had considered in the last class or what are known as sorry longitudinal waves.

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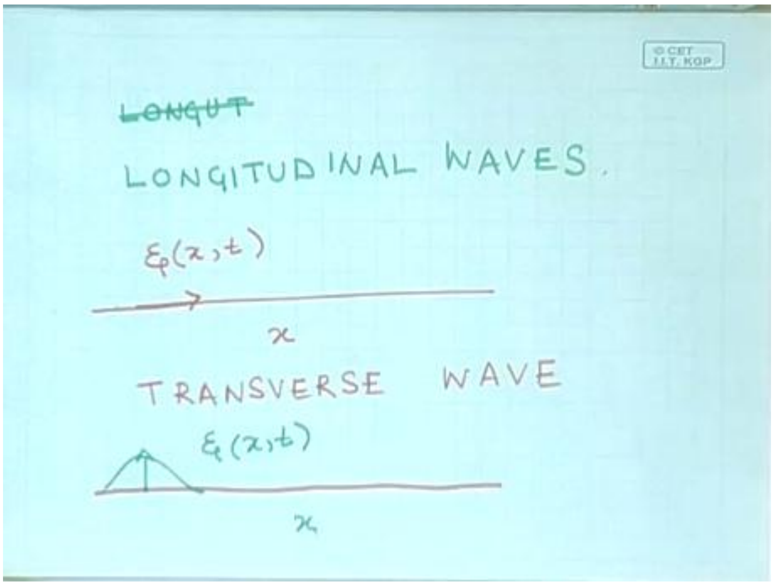

What do we mean by longitudinal waves? A wave is set to be longitudinal if the disturbance if. So, if the disturbance that in this case the disturbance is along the x axis the displacement of any point in the elastic medium xi is along the axis. And the xi itself varies along the x axis to such a wave is set to be longitudinal wave. You could also have another kind of wave called the transverse wave. Let me give you an example of a transverse wave if you have a stretched string. So, think of this line as a string which has been pulled tight it is stretched and if in this string. We introduce a disturbance which is perpendicular to the direction of the string let us say that I plug the string like this and leave it. Then the evolution of this disturbance is this disturbance is a transverse disturbance it is this disturbance is going move around along the x direction. And if I use the same symbol xi to denote the disturbance that disturbance is perpendicular to the direction in which the wave can propagate. So, this is called a transverse wav another situation where we have a transverse wave is the electromagnetic wave.

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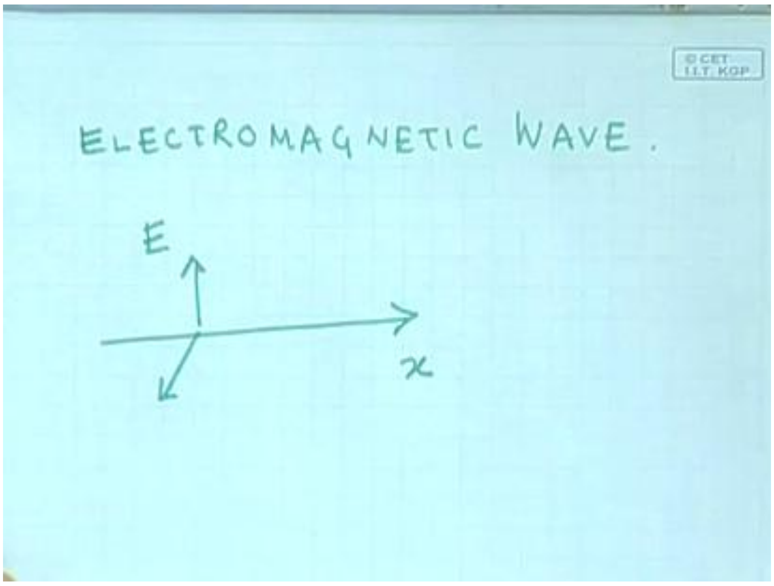

And I have already told you that for an electromagnetic wave if I have an electromagnetic wave is the disturbance in the electrical magnetic field. And if this is a disturbance in the electric field then if I have a wave which is propagating in this direction. The electric filed disturbance can only be in the directions perpendicular to the direction in which the wave is propagating. This 2 is a transverse wave the electric field and the magnetic field both have to perpendicular to the direction in which the wave is propagating. So, if the wave is going along the x axis which is the situation that we have been discussing. The electric field can be anywhere in the y z plane. So, both of these are transverse waves.

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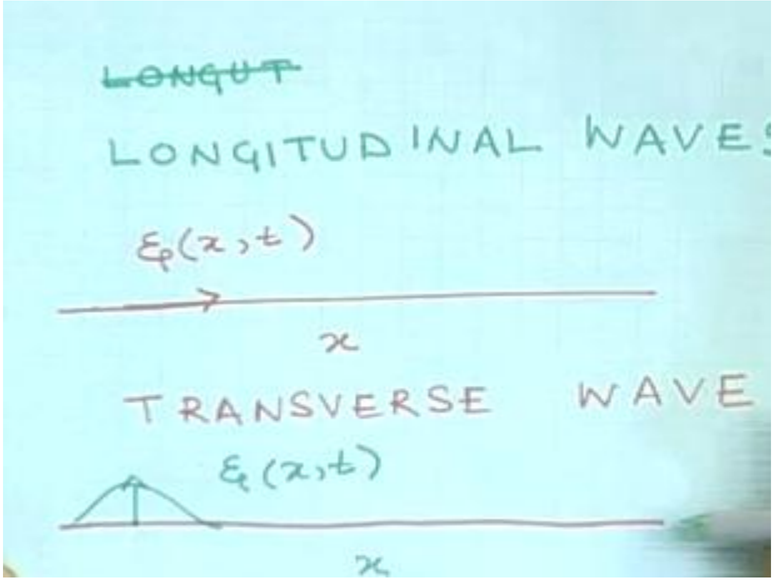

So, we have different kinds of waves possible we have longitudinal waves possible. We derived the wave equation for a longitudinal wave which is disturbances in an elastic medium, longitudinal disturbances in an elastic medium. But you could also have transverse waves like the disturbances in a string now in all such situations.

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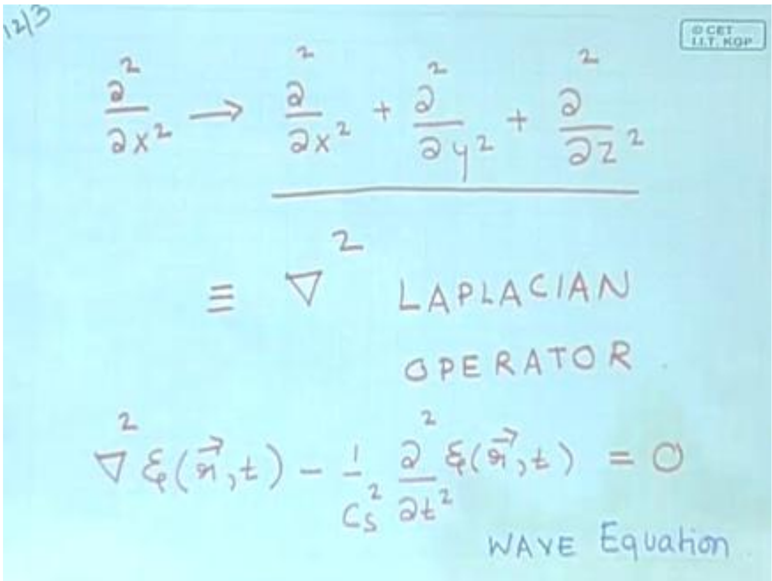

The evolution of the disturbance is typically governed by such a wave equation. So, the wave equation that we have derived is very general it does not hold just it is not that it holds just for the elastic waves. It holds in a such an equation arises in a large variety of situation. So, if you analyze the evaluation of the disturbance in the string, or if you analyze the evolution of the disturbance in the electromagnetic field in all situations you will find that the disturbance is governed. The evolution of the disturbance is governed by such a wave equation. The only difference that occurs when I go from one kind of wave to another the main difference that occur is that the speed at which the wave propagates the phase velocity of the wave that changes. So, for an elastic wave we found that the phase velocity of the wave is related to the properties of the elastic medium.

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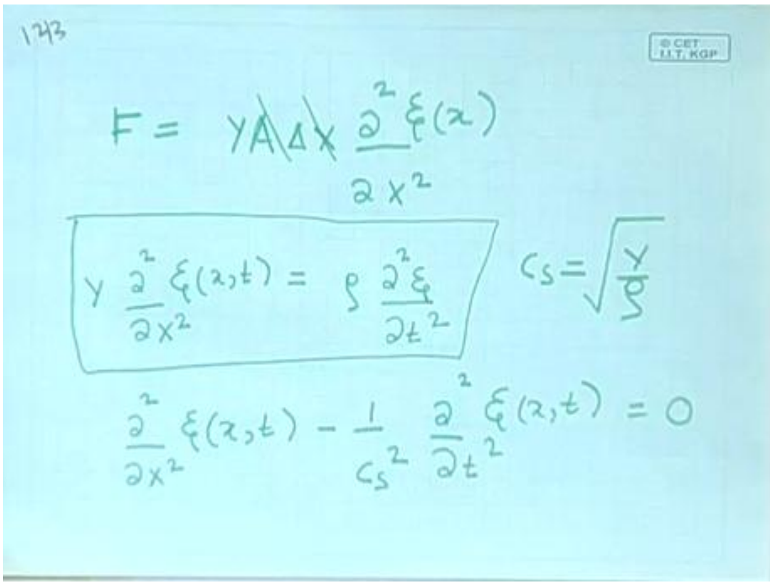

It is the Young's modulus divided by the density. If I have some other kind of wave if I have disturbances in a stretch string. Then this phase velocity is the square root of the tension per unit length of. So, this will be replaced by the tension divided by the unit

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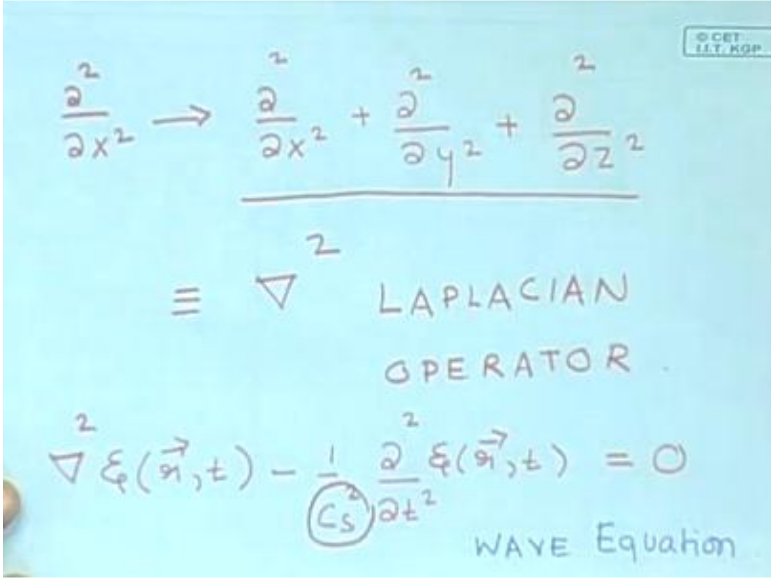

The mass per unit length of the string for an electromagnetic wave it is the speed at which speed of light in vacuum and. So, forth so, depending on the particular wave that you are considering the value of this constant the value of the phase velocity will be different. But usually the wave is you will find that the wave is governed by an equation is like this. So, this kind of an equation is a very general equation. And it arises in a large variety of situations. So, it is very important to understand the solutions of the wave equation which is what we are going to discuss in today's lecture. Now, I have already also told you that that the sinusoidal plane wave which we have discussed at great length earlier on in this course.

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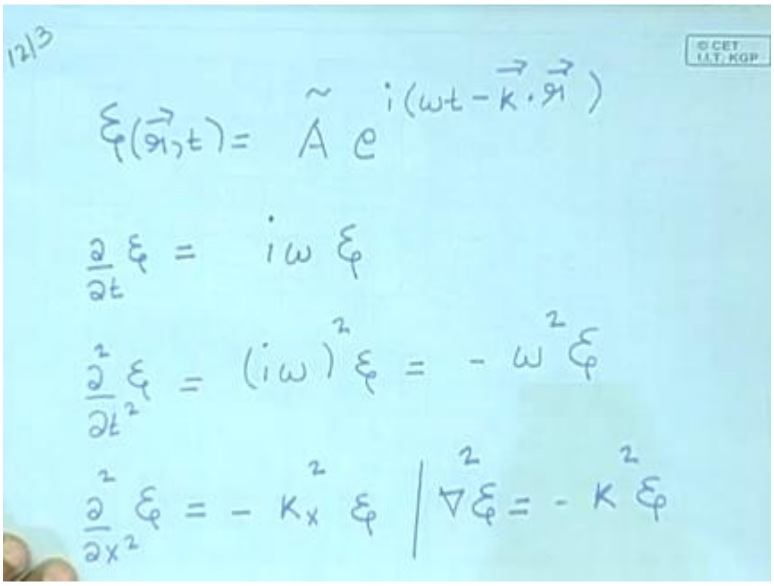

The sinusoidal plane wave which can be mathematically expressed like this I have shown you that the sinusoidal plane wave is indeed a solution to this wave equation.

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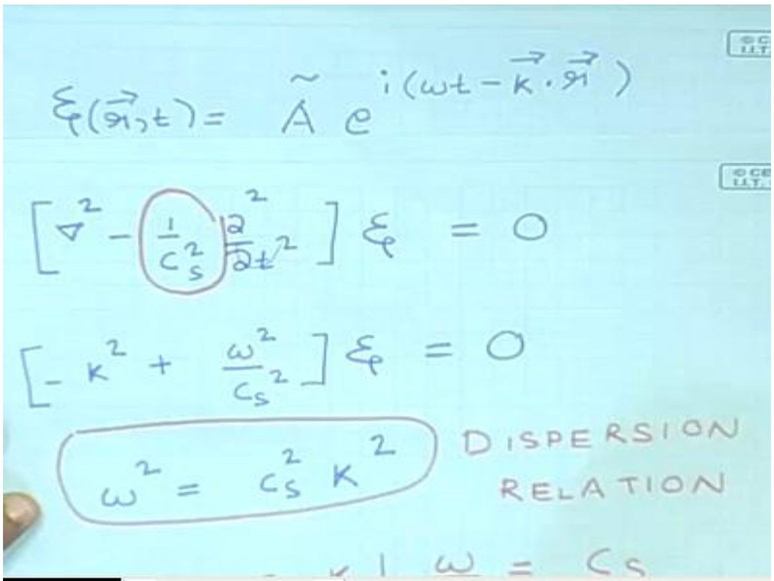

So, this is need a solution to the wave equation provided the wave number k the wave number corresponding to this wave vector and the angular frequency are related in this fashion. So, the relation between the angular velocity and the wave number is called the dispersion relation. So, provided the angular velocity and the wave number satisfy the dispersion relation corresponding to this particular wave. So, the dispersion relation for the particular wave will have the phase velocity the constant c which occurs here appearing in the dispersion relation. So, provided omega and k are related like this then this is a solution to this wave equation. So, we have already seen 1 particular solution a special solution to the wave equation. Now, let us discuss a few more general kind of a solution of the wave equation. So, the first kind of solution that we shall discuss is called a plane wave solution.

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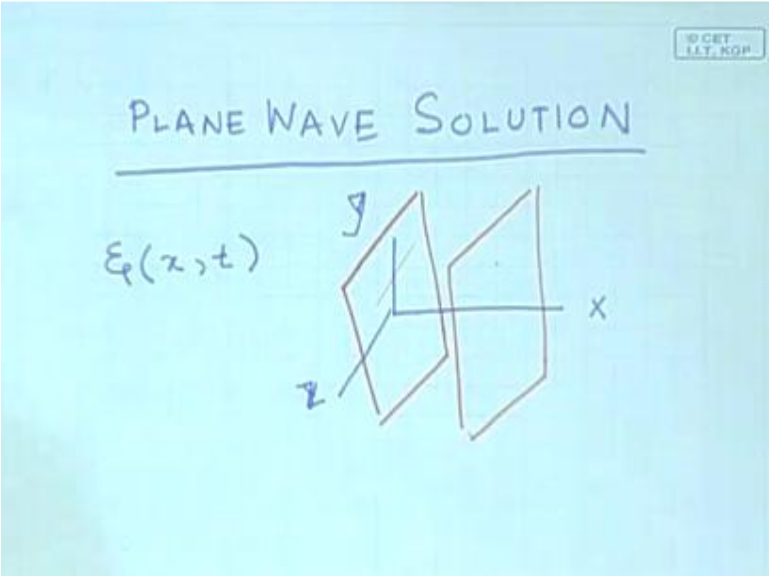

So, let us discuss a plane wave plane wave solution of the wave equation, what do we mean by a plane wave solution? We are going to assume that xi the disturbance which in general could be a function of all 3 special variables the x y z. We are going to assume that it depends on only a single special variable it. So, the xi that is the disturbance varies only in a particular direction and we will use x to denote that direction. We will assume that xi the disturbance that we are whose evolution we are studying could be the longitudinal disturbance is an in an elastic rod could be the transverse vibrations of a string. Whatever, it be we will assume that it depends only on x not on y on z. Now, if xi if the disturbance depends only on x let me draw the x y z co ordinate system this is x this is y and this z rather let me put it other way. Now, this is z and this is y to be have a right handed co ordinate system.

And if you impose the condition that xi depends only on x then you see that the value of xi is a constant on a plane which is parallel to the y z plane. So, it is constant on this plane it will be constant on this plane also it will have a different value on these 2 planes. So, xi if it is a function of x alone will be constant on this plane it will constant on this plane and the value of xi could different here and here. And it will different on every other plane in between since the value of xi is constant on planes this is referred to as a plane wave solution. That is the first point which you should note. So, when we mean when we speak of a plane wave solution what we mean is that the value of xi is constant on planes. And in this case the planes are perpendicular to the y z plane if instead of x I had chosen y then the planes would be perpendicular to the x z plane and so forth. So, in this situation when xi is a function of x alone the wave equation that we have to deal with is this.

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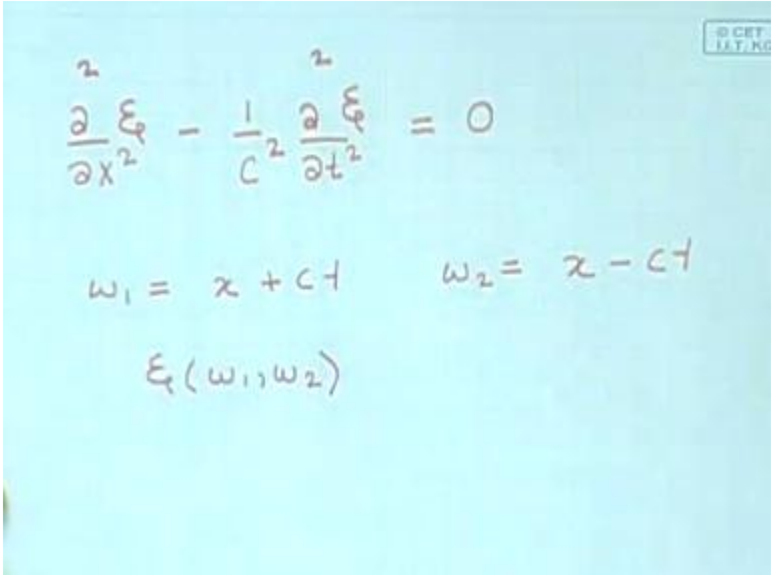

It is you have the second partial derivative with respect to x of xi minus 1 by c square. Now, please note that I am going to use c without the subscript x to denote the speed of the phase velocity of the wave.

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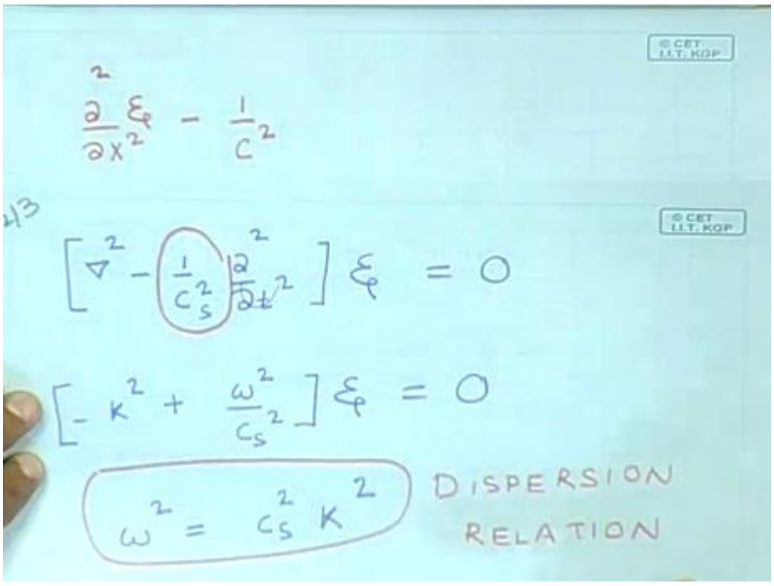

So, I am not going to put the s explicitly in all of today is lecture. So, you should you should realize that the c which is here is not necessarily the speed of light. It is the speed of light in vacuum when we refer to an electromagnetic wave in when you are dealing with an elastic wave c is the speed the phase velocity of the in the elastic medium which is the square root of y by rho etcetera. So, c is the phase velocity of that particular wave not necessarily the speed of light in vacuum. It is the same as this cs thus that I am not going to show this s explicitly.

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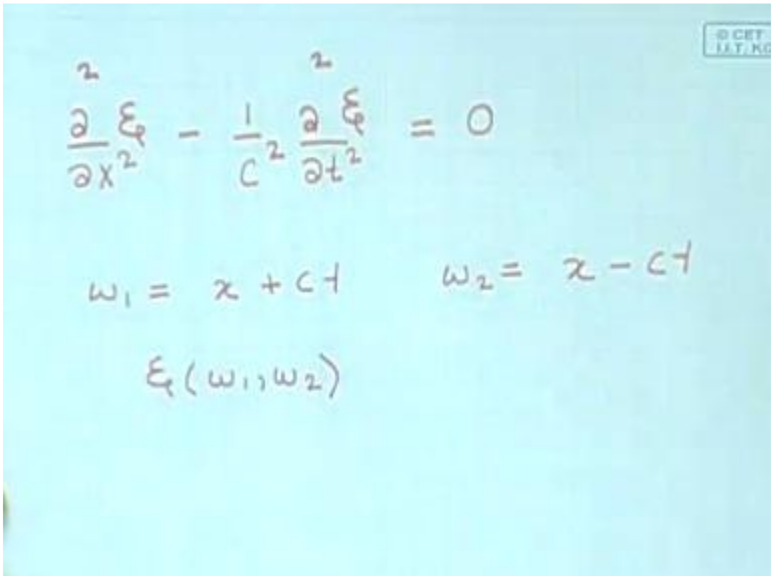

So, the wave equation is this minus 1 by c squared del by del the partial derivative of with respect to t this is equal to 0 xi is a function of x and t. And we are looking for solutions to this equation. It is convenient to introduce to new variables w 1 which is x plus ct and w 2 x minus ct. So, xi now is a function of w 1 and w 2 we can replace these 2 variables x and t position and time with the 2 new variables w 1 and w 2. So, we have to also now replace the derivatives with respect to x with derivatives in terms of w 1 and w 2. So, let me work out the derivative of with respect to x in terms of derivatives with respect to w 1 and w 2.

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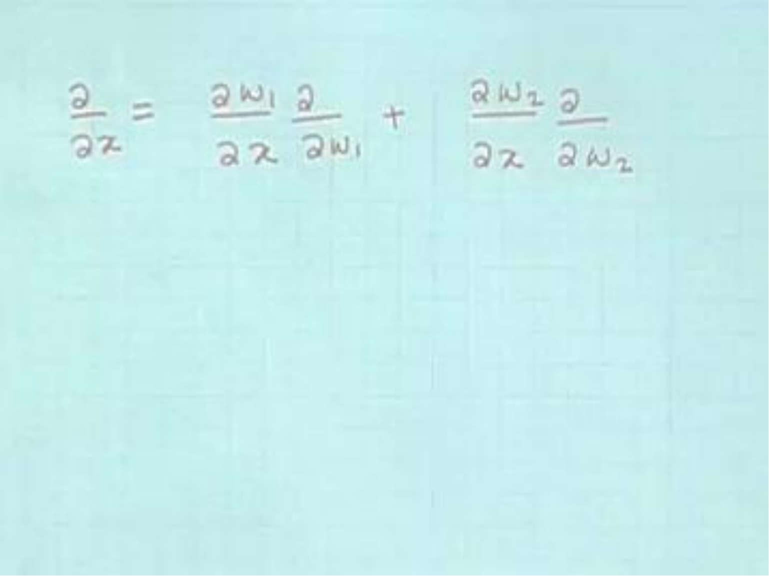
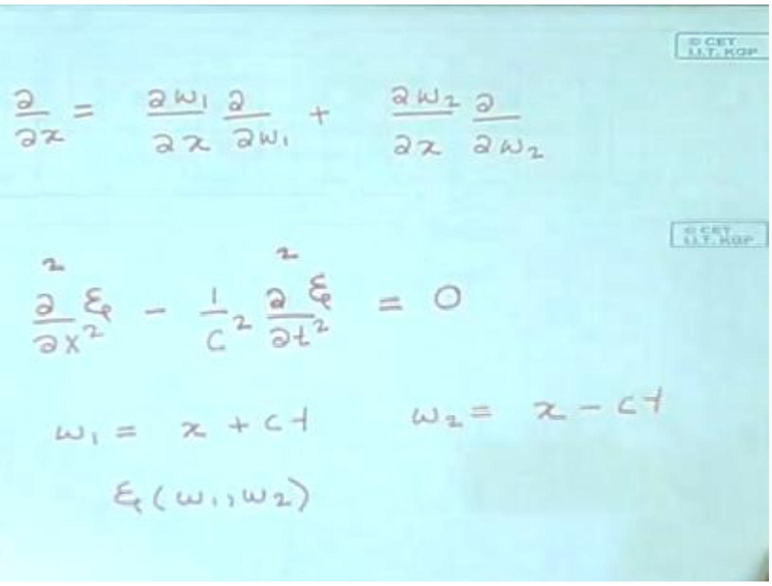


So, del del x is equal to del w1 so, the derivative with respect to x the partial derivative with respect to x. You can express in terms of partial derivatives with respect to w 1 w 2 using the chain rule of differentiation and. So, you have to now calculate these derivatives the derivative of w 1 with respect to x and derivate of w 2 with respect to x. (Refer Slide Time: 16:17)



So, let us go back to our expression for w 1 if you differentiate this with respect to x you get 1. Similarly if you take the partial derivative of this with respect to x you get 1. So, you find that this equal to this.

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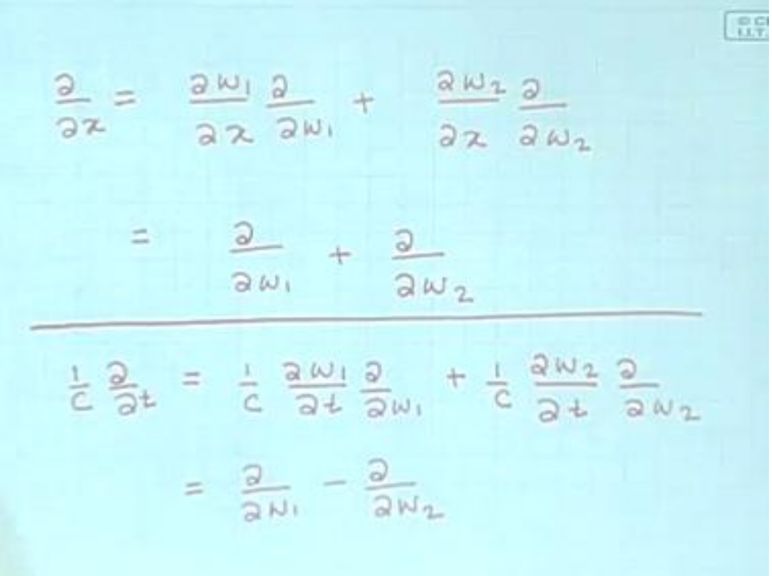

So, the derivative with respect to x can be written in terms of derivatives with respect to w 1 w 2 like this. Now, let us also work out the derivatives with respect to t. So, the quantity which is convenient to deal with is 1 by c partial derivative with respect to t again we will do the same thing. So, I will put a 1 by c here del w 1 del t del del w 1 plus 1 by c w 2 del t. So, we have now what we are doing is we are expressing the derivative with respect to t the partial derivative with respect to the in terms of partial derivatives with respect to the new variables w 1 and w 2. So, we have to calculate this partial derivative first.

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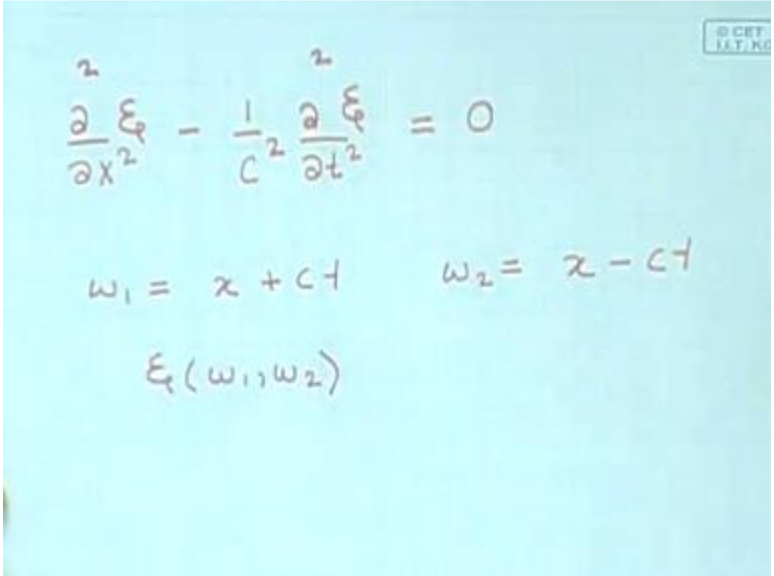

So, what happen when I differentiate this with respect to t and divided by c? I get plus 1 if I differentiate this with respect to the and divide c I will get minus 1.

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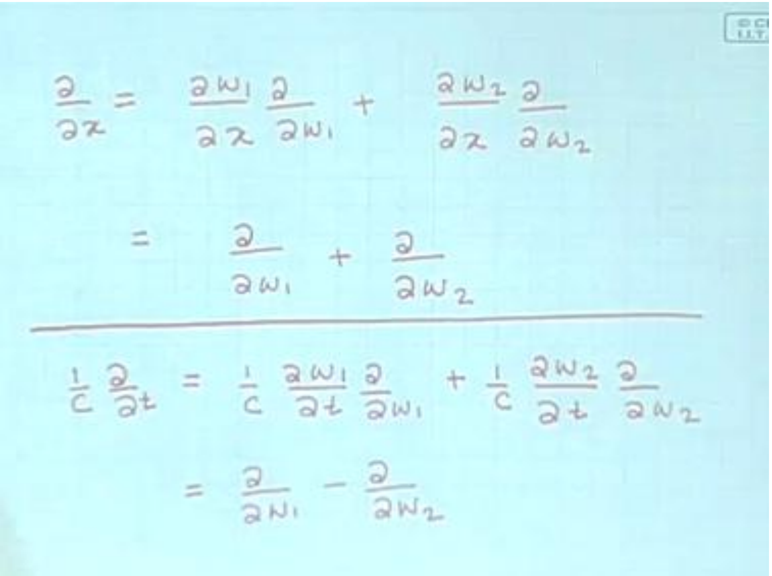

So, we see that this is equal to so, we have worked out the derivative with respect to x and t in terms of derivatives with respect to the new variables w 1 w 2. Let us now go back to our wave equation.

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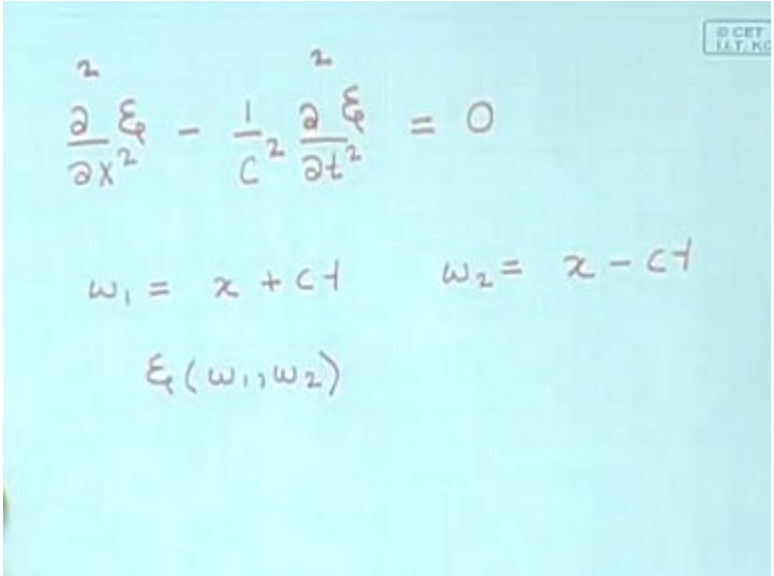

So, this is the wave equation, we are we originally started with in this wave equation. We have to now replace derivative with respect to x In terms of the derivative with respect to w 1 and w 2.

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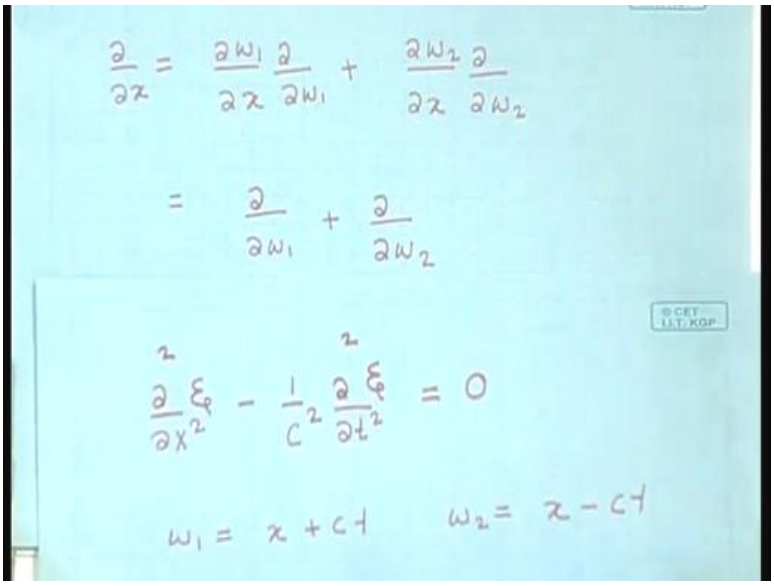

We also have to replace derivative with respect to t in terms of derivative with respect to w 1 w 2. So, we have to square this so, when you replace that we do it for you explicitly instead of its quite straight forward. So, the derivative with respect to x the equation that we get is the derivative with respect to x squared.

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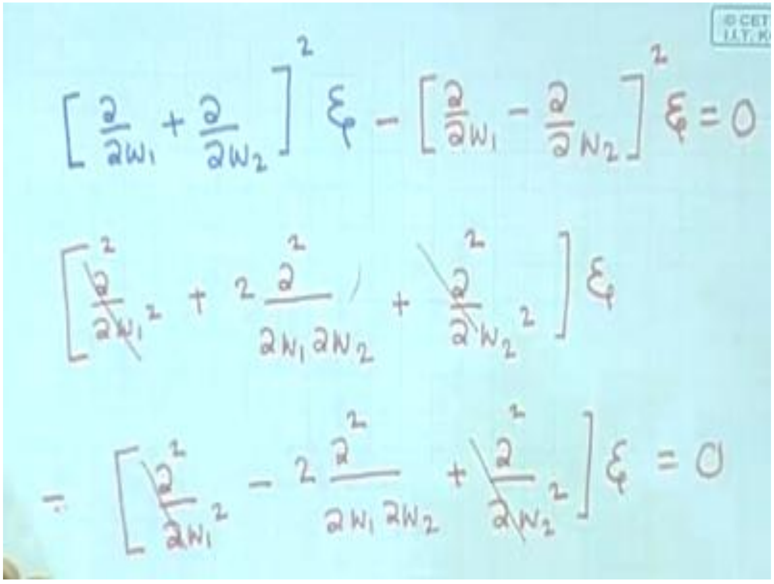

So, we have partial derivative with respect to w 1 plus partial derivative with respect to w 2 that is the derivative with respect to x the square of this acting on xi.

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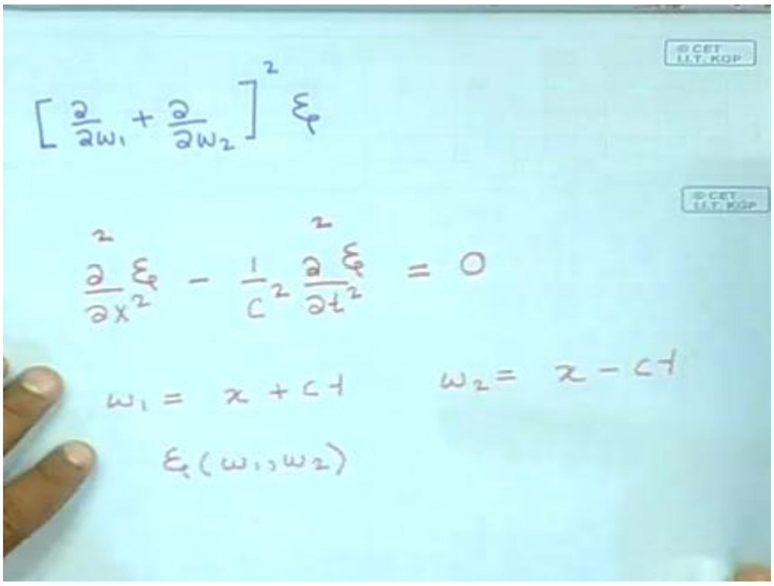

So, I have written down the first term over here the square of the operator that is the partial derivative with respect to x is acting on xi. And then I have to subtract out this.

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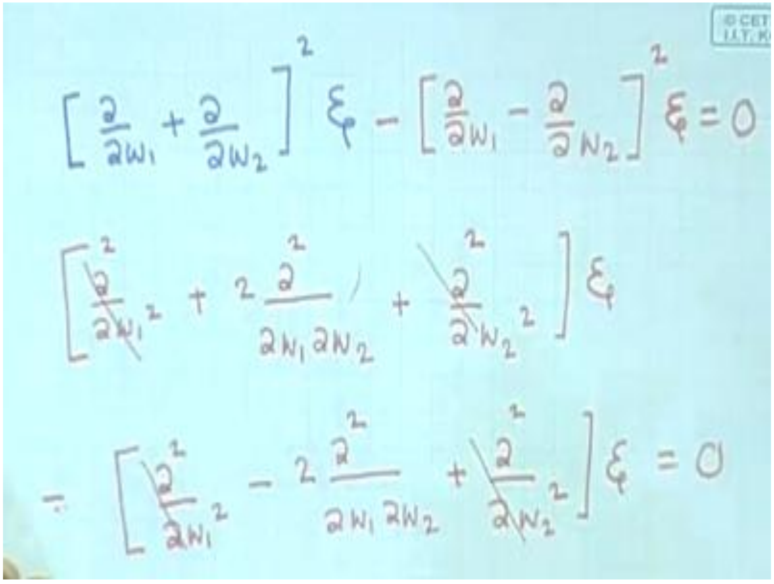

So, I have to subtract out let me have to. So, I have to subtract out del w 1 minus the partial derivative with respect to w 2 square into xi this is equal to 0 the this is equation. This is the same wave equation.

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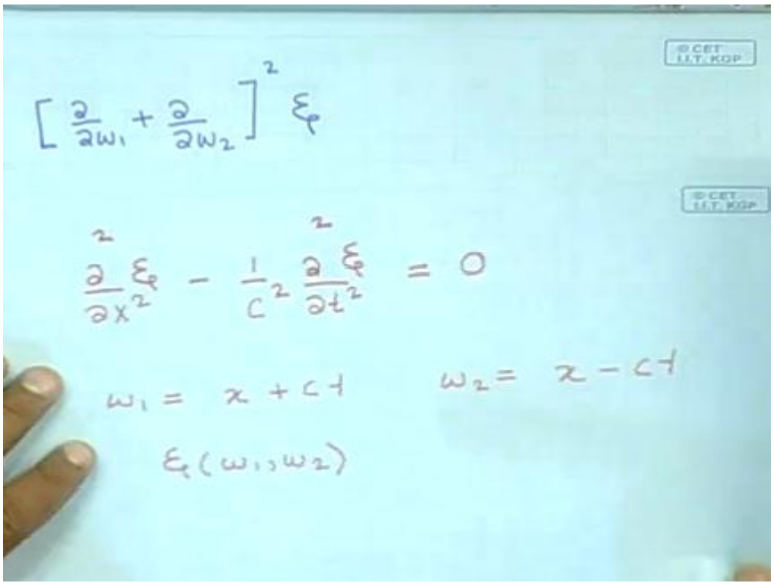

So, what we have done is we have written this wave equation in terms of derivatives with respect to w 1 and w 2. This is the derivative with respect to x which has been written like this is the derivative to the respect to t divided by c square, the second derivative which I have written in this fashion. So, the same wave equation is now this equation over here in terms of w 1 and w 2.

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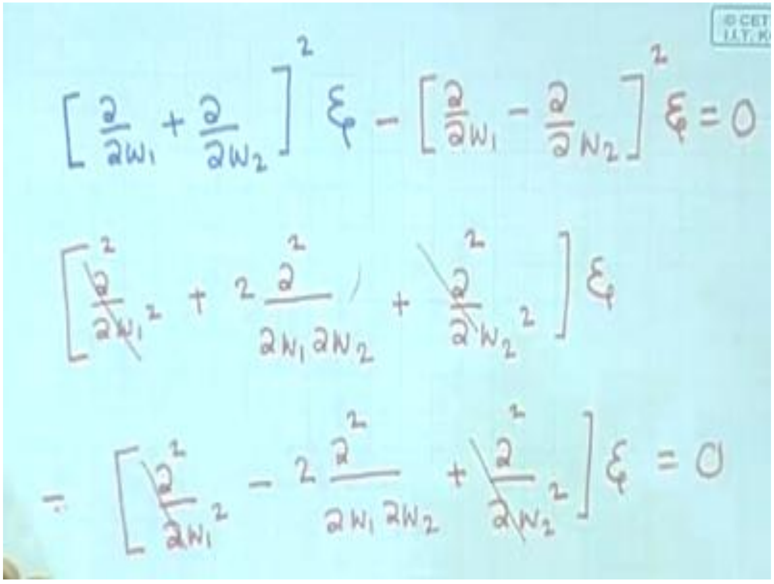

Now, when I expand out the first term let me expand out the first term. So, the when I expand out the first term what I get is partial derivative with respect to w 1 the square of this plus 2 times partial derivative with respect to w 1 w 2 which is the cross term between the, this . And this plus the square of the second term this whole thing acting on xi minus now when squared this I will have I will get the same thing as this, but with an extra minus sign because there is a minus sign over here. So, I will have minus 2 del square plus this acting on xi is equal to 0. Now, when you add these 2 terms there is a relative minus sign here. So, this will cancel out with this this will cancel out this and you are left with the equation. The equation that you are left with let me.

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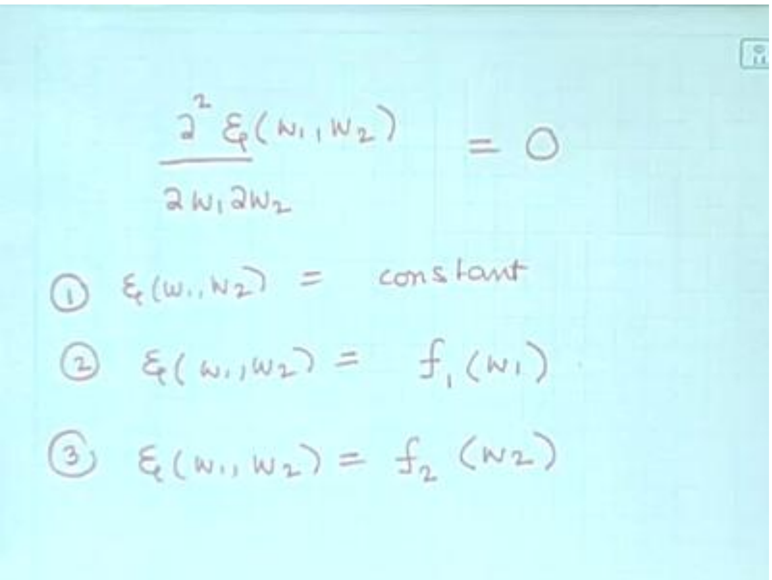

So, the equation that you are left with is that the second derivative of xi which is a function of w 1 and w 2 with respect to w 1 this is equal to 0. So, what we have done is we have a changed variables we started off with xi is a function of x and t. We have changed variables gone over to new variables w 1 and w 2. And we have a written the wave equation in terms of these 2 new 2 2 new variables and this is the equation that we get we have to now look at the solutions to this equation. Now, you see that this equation the finding the solutions to this equation is what simpler you can guess the solutions to this equation much simpler than the original equation that we had. And there are 3 possibilities you can straight away think off the first the possibility is if xi w 1 w 2 is a constant. But this possibility does not give us a propagating the disturbance it is not a wave. So, it is not a not a very interesting solution though it is a a mathematically permissible solution.

The second possibility which is interesting now is the situation where xi is a function some arbitrary function of w 1 alone. So, if xi is a function w 1 alone the function should be differentiable. And we would like the function to the vanish at infinity plus minus infinity otherwise there is no other restriction. So, if I if you choose xi to be a function of w 1 alone then when you put this here the derivative with respect to w 2 is 0. So, this equation is satisfied so, this is you see this gives you a particular a possible solution to this equation. And you have an another possible solution which is a situation where xi is some other function f 2 of w 2 alone. So, if you have some arbitrary function of w 1 alone or some arbitrary function of w 2 alone. Then both of these are solutions to the wave equation. So, let us go back to the variables which we started out with. So, what we see is that the solution to the wave equation can be written in the following way.

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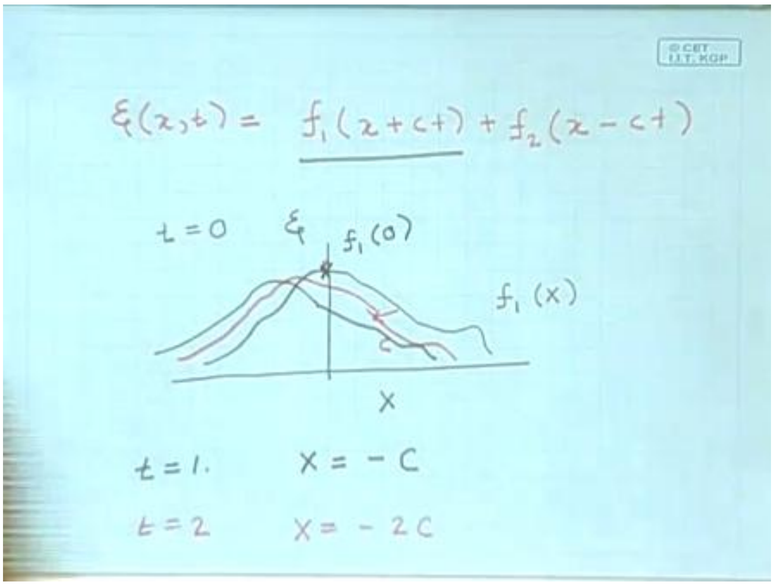

It could be some arbitrary function f 1 of w 1 and w 1 is x plus ct plus some other arbitrary function f 2 of w 2 which is x minus ct. And this combination linear superposition of 2 such functions is solution to the to this wave equation which for which you are trying to find a solution.

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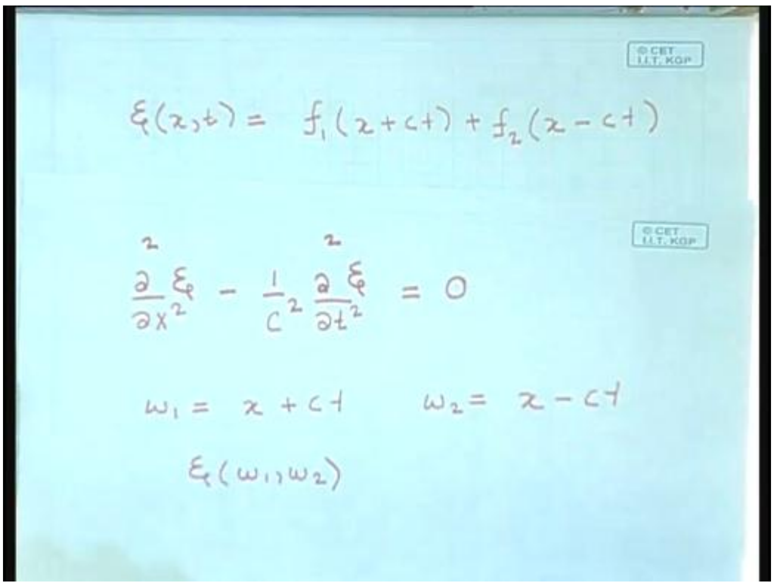

So, we have obtained the most general plane wave solution to the wave equation plane planar solution to the wave equation.

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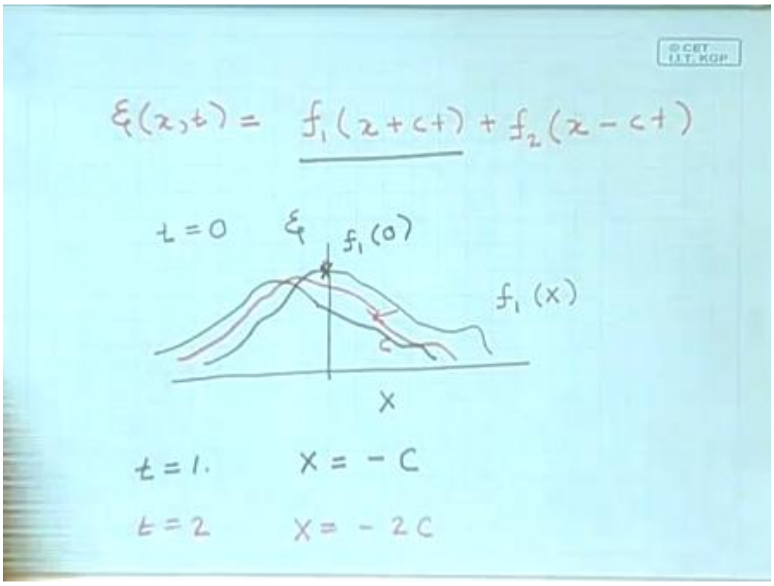

Now, let us interpret this Solution let us first focus on a situation where we have only the first function. So, we have only let us focus on this particular solution let us said this to be 0. And let us focus on this particular solution. So, at t equal to 0 let us plot this function f 1 could be some arbitrary function which is differentiable and we would like it to vanish far away. So, let we plot some such function you could choose any function that you wish. So, I will ask you to plot a function your choice I will plot a function of my choice. So, let we plot a function so, the function that I will plot looks that looks something like this. So, at t equal to 0 the function looks like this. So, this f 1 x which is xi at t equal to 0 we have only this we do not have this part we are not considering this part we are only considering this part. So, xi at t equal to 0 xi is f 1 x which looks like this some arbitrary could chose some of the function which for your of your choice and draw it.

So, now we have drawn xi at t equal to 0 now let us ask the question what will xi look like at t equal to 1. Let us ask the question that xi has a particular value f 1 0 at this point at t equal to 0 where will xi have the same value at the time t equal to 1. So, at t equal to 1 we have to make the argument of this function the same as x as the as 0 you have to make the argument of this function 0. So, what you see is this that the argument of this function become 0 at the point x is equal to minus c at the time t equal to 1. The argument of this function become 0 when x is equal to minus c. So, this kind of an argument applies to all the points. So, what you can say is that the whole function at the time t equal to 1. The whole function has shifted by the amount c the whole function is shifted to the left side. So, the whole function has shifted by an amount x equal to minus c.

That is what happens as time involves. So, at the time t equal to 1 the whole function has shifted. So, you have exactly the same thing repeated. But it is now all shifted by an amount which is equal to c at t equal to 1 the whole thing has shifted by an amount c to the left at t equal to 2. So, at t equal to 2 if you ask the question, where has where is the value of xi the same as the value at as it was at this point at t equal to 0. So, you want to make the argument of this function 0 at the instant t equal to 2. So, this is going to happen at x is equal to minus 2 c. So, what you see is that the at t equal to 2 the whole curve has shifted by minus 2 c. So, it has shifted to the left by 2 c. So, at t equal 2 the same curve is going to be the the value it is going to be the same curve.

The value of xi is going be described by the same curve all that has happened is that the thing the curve has shifted by an amount 2 c to the left. So, this so, what we see is that this part of the solution represents a wave propagating to the left at the speed c let me repeat again. We have determined general solution to the wave equation and the general solution we found is a sum of 2 parts. The first part is some arbitrary function of x plus ct. The second part is some arbitrary function of x minus ct we have been interpreting the the significant of the first part which is some arbitrary function of x plus ct. And what we saw was that this corresponds to a wave propagating to the left with the speed c. So, the functional form of xi does not change all that happens is that it shifts keeps on shifting to the left.

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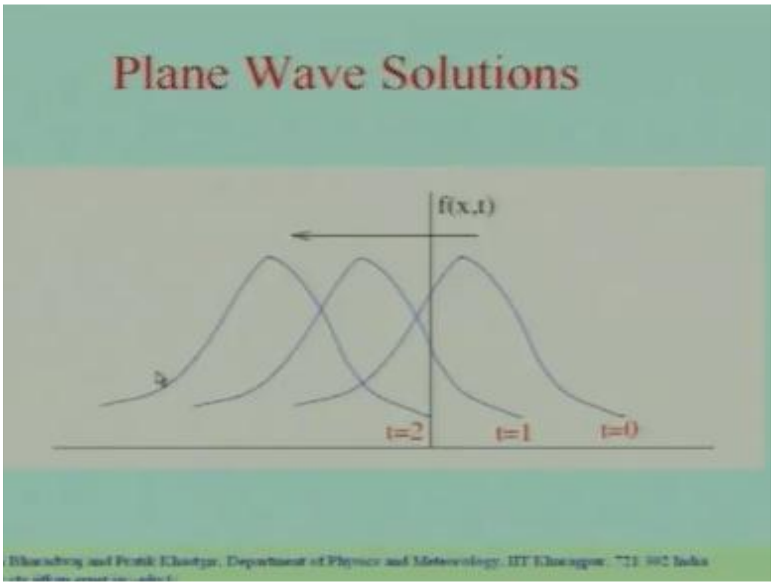

So, this picture over here shows you the evolution of the left propagating plane wave solutions. So, this is the solution at t equal to 0 at t equal to 1 the whole solution has shifted to the left and at t equal to 2 it has shifted further to the left the shift being. So, from here to here shift is going to be c from here to here shift is going to be 2 c and at t equal to 3 it could have shifted to 3 c.

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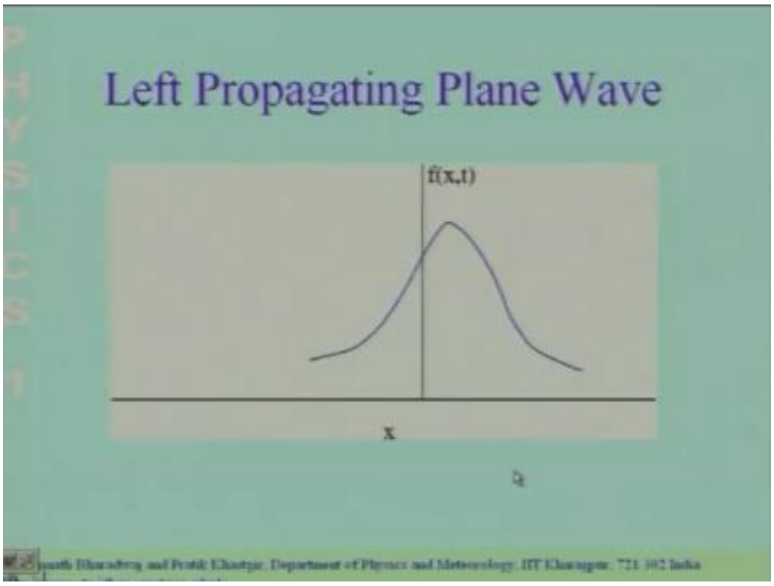

This shows you animation of the left propagating plane wave. So, this and it moves keeps on moving as time evolves. So, let me show you this again. So, as time evolves this is xi at t equal to 0 at t equal to 0 xi is the function fx f 1 x in this case and then at as time involves the whole pattern moves to the left with speed c which you what you see here. So, this part of the solution represents a wave propagating to the left it is a left propagating wave.

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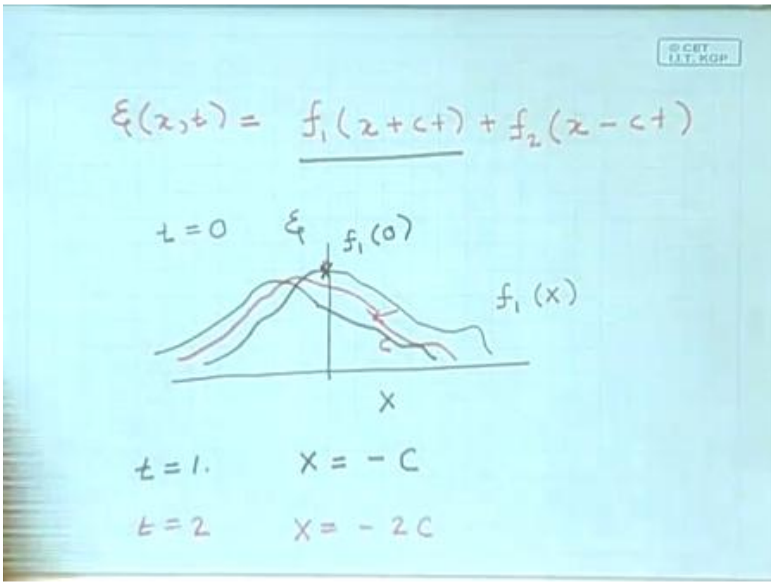

Now, you could ask the same question what does this part of the solution represent it is quite clear. What this part of the solution represents f 2 again could be some arbitrary function. And as time evolves at t equal to 0 the xi x let us forget about this and focus only on this at t equal to 0 xi x is f 2 x as time evolves the whole pattern. Now, propagates to the right with a speed c which is what you can this picture shows you?

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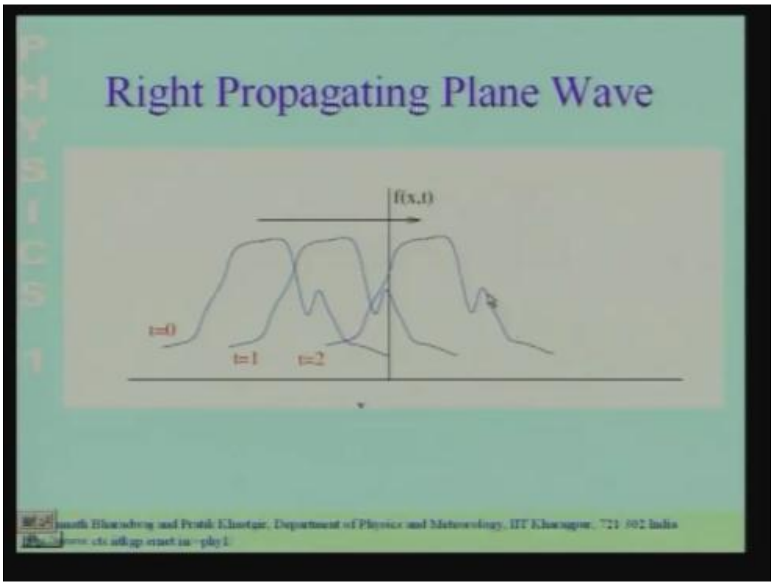

So, we have f 2 which some other function some arbitrary function of x this is the functional format t equal to 0 as time evolves. So, this is the whole thing shifts to the right. So, this is the functional form of xi at t equal to 1 and this is the function form xi at the equal to 2.

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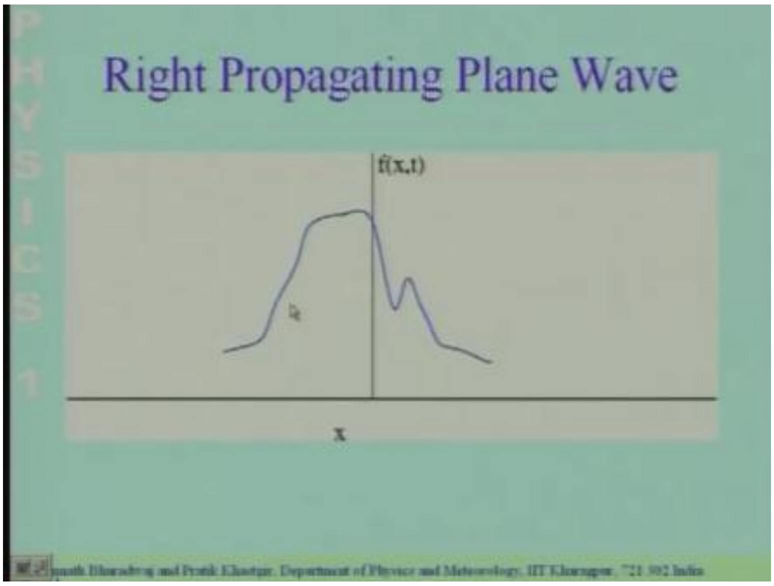

So, this shows you an animation of how right propagating plane wave involves. So, this shows you xi as a function of x at t equal to 0 it is described by a function f 2 x at t equal to 1 2 3 the whole thing keeps on shifting as time involve and shifts forward in x with the same speed c. So, let me just recapitulate what we have done. We wanted to look for solutions of the wave equation the disturbance the wave disturbance xi could in principle be a function of all three special coordinate x y z. We restricted our attention to a particular situation when xi depends on only 1 special coordinate and we chose it to be x. So, xi is constant on planes perpendicular to the x axis this called plane wave.

And we found solutions to the plane wave equation to the wave equation and we found that there are 2 kinds of solutions possible. The first kind of a solution is could it could be some arbitrary function of x plus ct this represents a wave propagating to the left or it could be some other arbitrary function of x minus ct which represents the wave propagating to the right. And in general the any arbitrary solution could be a linear superposition of to such functions. Now, the plane wave the sinusoidal plane wave which we have discussed extensively earlier the sinusoidal plane wave.

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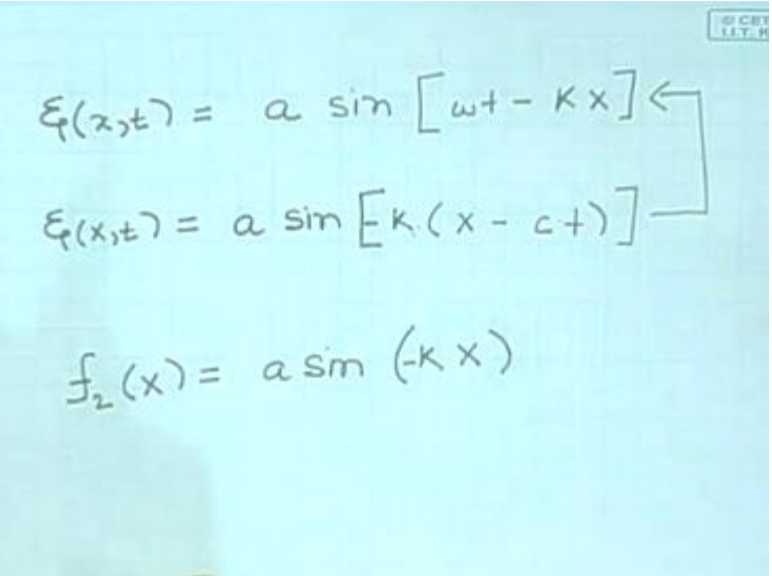

In purely real notation can be represented like this a sin omega t minus kx. See the sinusoidal plane wave which we are already quite familiar with now let us check that the sinusoidal plane wave is special case of this solution.

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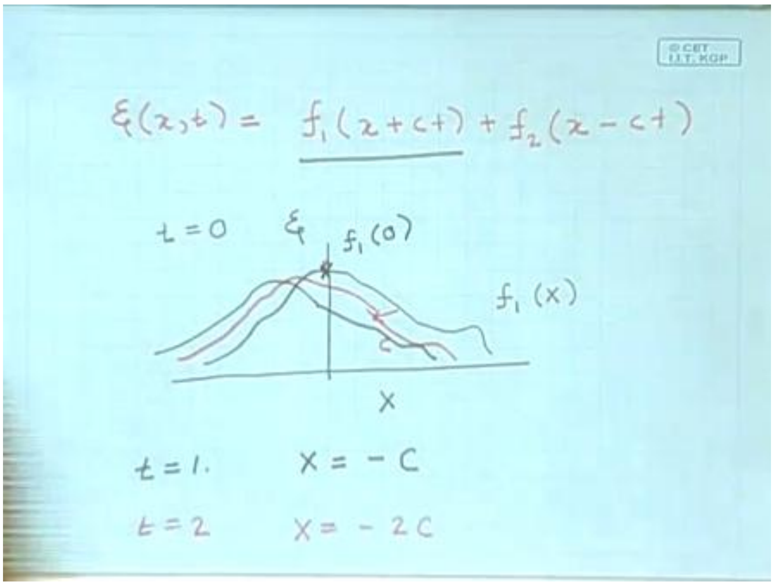

So, to show you that all that we have to do is we have choose f the function f 2 to be A sin x.

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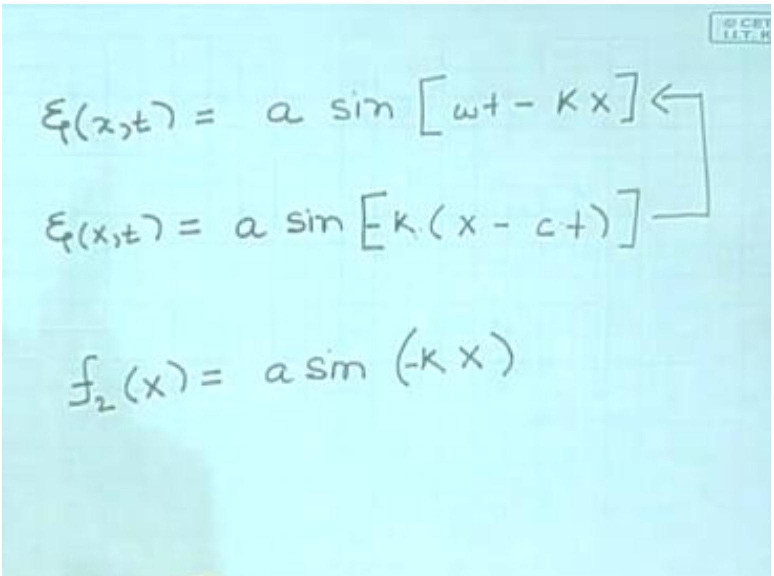

So, with that choice so, I have set f 2 to be a sin x so, with this choice sin x into kx so, let me write it here f 2 I have chosen f 2 to be a sin kx. Then the solutions xi with this choice of f 2 the solution xi is going to be f 2 of the function of x minus ct. So, it will be k x minus ct. Which we see is exactly the same as this provided we identify I can put a minus sign here. So, this will be minus and i i have minus here. So, this exactly same as this represents the left propagating wave and all that you have to do is you have to identify omega the angular frequency with k into the speed c. So, what we see is that the sinusoidal plane wave which we have been discussing is particular example of a plane wave solution where the function is sin. But you could have a much more general solution and any arbitrary function provided it vanishes far away. Vanishes at infinity and it well behavior that infinity need not vanish sin x is not vanish provided the function is well be keep that infinity and it is differentiable then we could it could be a solution, right.

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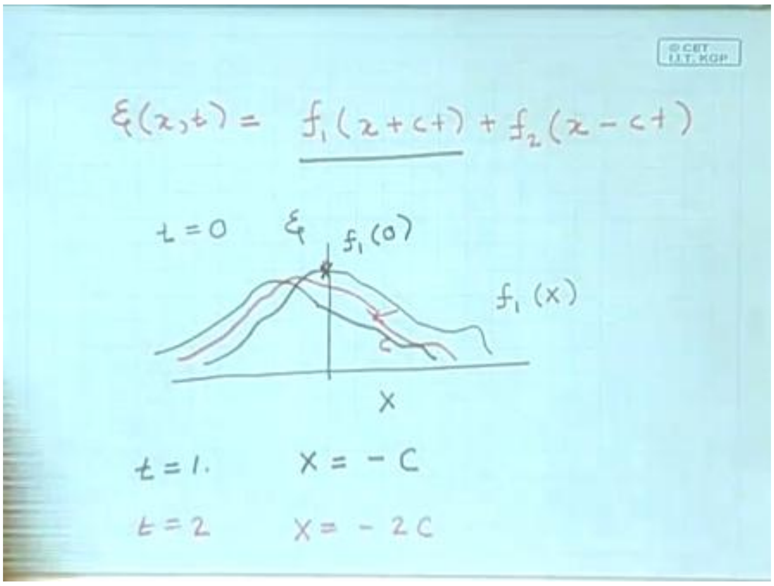

So, this very special solution where this is a sin function, but in principle it could be any arbitrary function provided it is well behaved at infinity well behaved everywhere and it is differentiable everywhere. So, these were the plane wave solution to be wave equation now let us we will discuss another kind of solution today. So, the other kind of solution that we are going to discuss is what is called as spherical wave solution.

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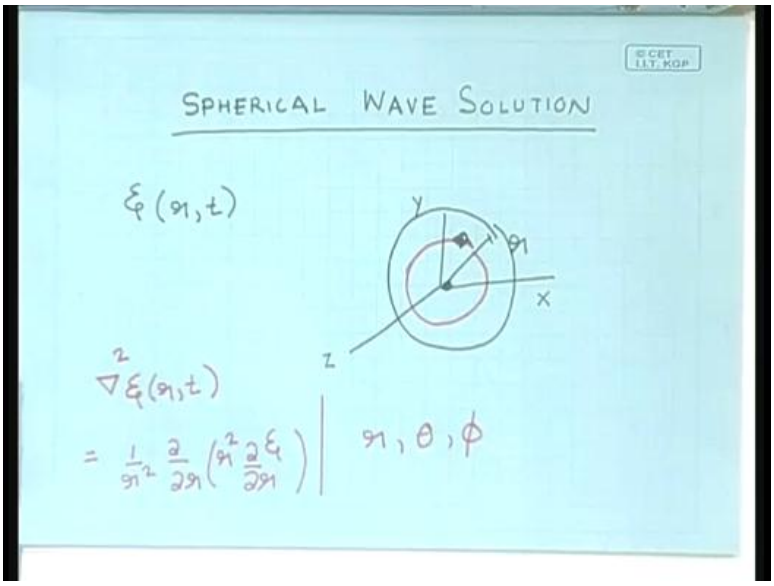

So, the next kind of solution that we have to going to discuss spherical wave. So, we are looking for solutions to the wave equation.

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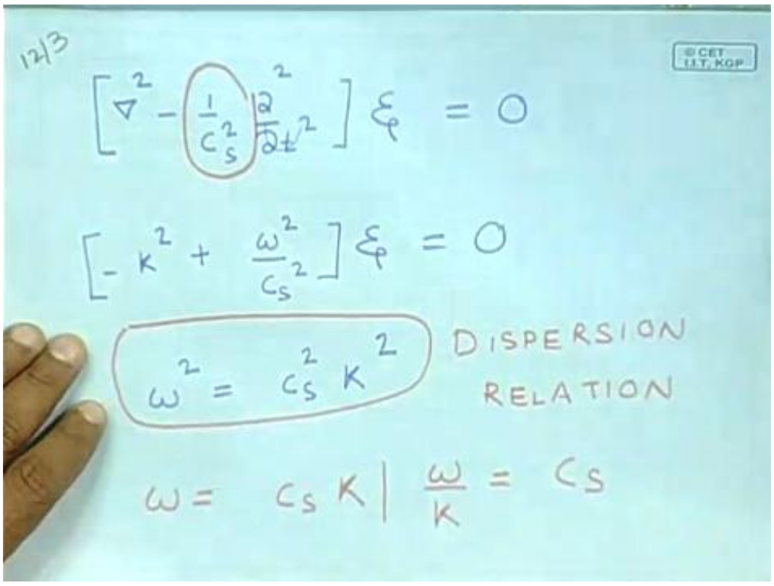

So, let me show you the wave equation again we are looking for solutions to the wave equation. this is the wave equation and xi could be a function of all 3 special variables x y z and time. Now, we again going to impose us an certain symmetry.

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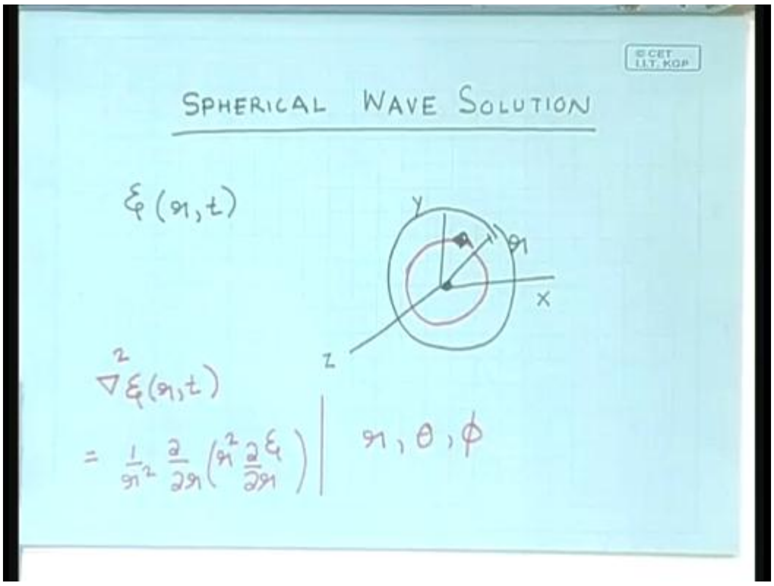

The plane wave solution comes when you impose a planar symmetry. We again going to impose us a symmetry we are going to assume that xi depends only on the distance from the origin. So, we have chosen a particular coordinate system x y z. And this is the origin of the coordinate system we will assume that the wave that that we have a situation where the value of the wave disturbance depends only on the distance r from the origin. So, this is the distance r from the origin we will assume that xi depends only on this. So, xi as constant value on spheres centre on the origin. So, the value of xi is the same on this xi will have a different value on this sphere, but it will be same all over this sphere.

So, depending on the radius of the sphere xi will have a different value. We are assuming that xi depends only on the distance from some fixed point which is the origin. So, this is the value of xi is constant on sphere. So, that is why it is called as spherical wave solution spherical wave. Now, I am sure you are familiar with the polar spherical polar coordinate system. So, the spherical polar coordinate system has 3 variables r theta and phi instead of using the Cartesian coordinate x y and z. It is equally possible to equally well described this whole space all point on this space using spherical polar coordinate system r theta and phi. So, you could suppose we are using spherical polar coordinate system. So, we do not have x y we are not using x y z we are using r theta and phi.

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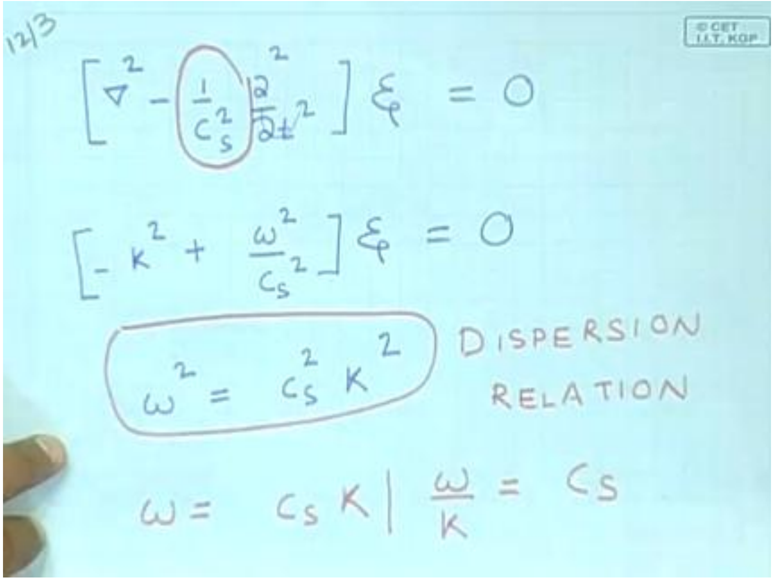

The question now is that we have to represent the Laplacian operator. The Laplacian operator you know is the some of the partial derivatives.

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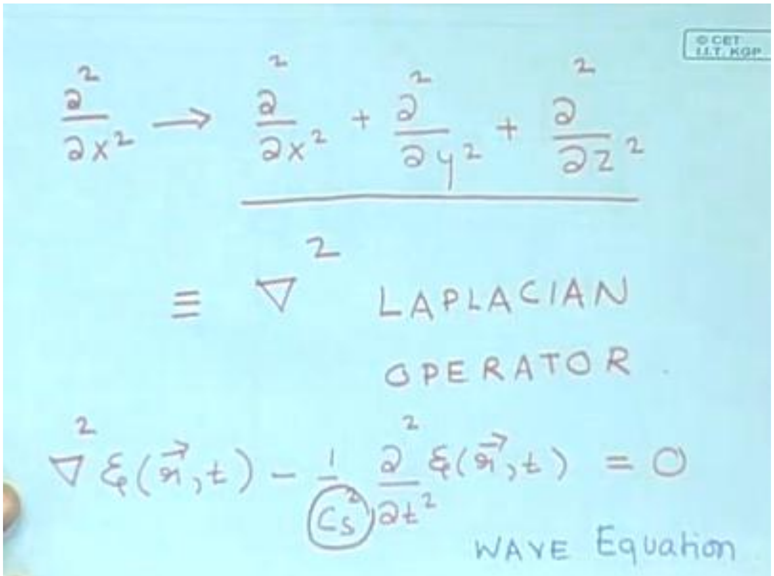

So, the Laplacian operator is the second partial derivative with respect to x plus the second derivative with respect to y plus the second derivative with respect to z. So, the Laplacian operator is this some it has derivatives with respect to x y z when you go over to this spherical polar coordinate system r theta and phi.

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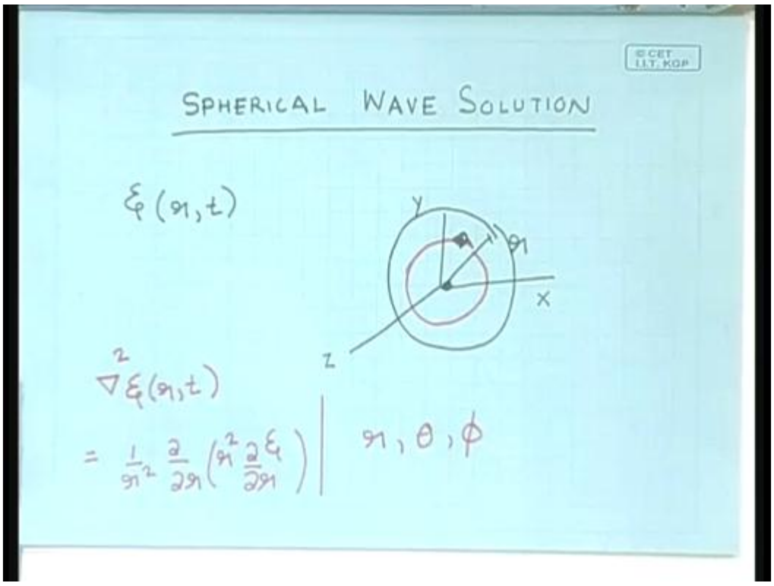

You have to represent this Laplacian operator in terms of derivatives not in terms of x y of x y and z, but derivatives in term of with respect to r theta and phi.

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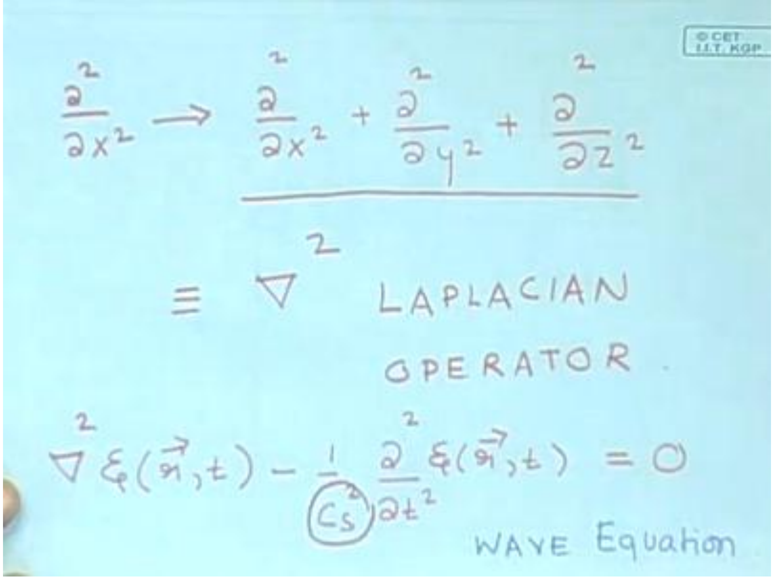

So, briefly you have to now write down x y and z in terms of r theta phi. And then transform these expression just like we wrote it in terms of w 1 and w 2 you have to use the chain rule of differentiation and write this in terms of derivatives with respect to this. Now, I shall not go through the algebra it is a little t d s and lengthy. So, I shall not go through through the algebra. But the point is that it is possible to write down the Laplacian in terms of the the polar spherical polar coordinate system.

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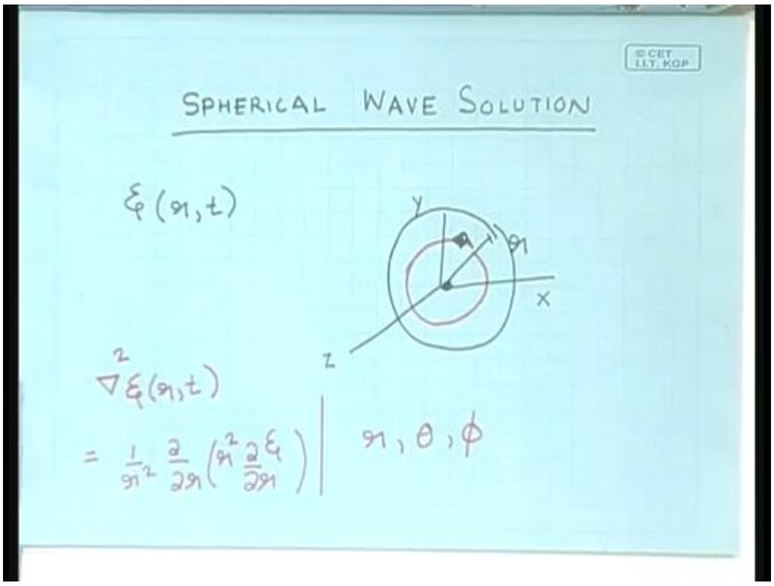

And you will terms involving derivative of r derivatives of theta and derivates of phi.

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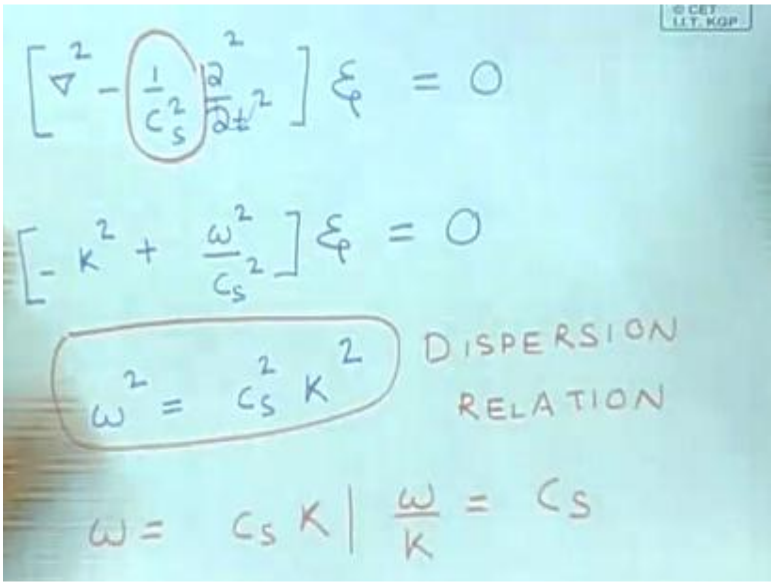

Now the equation which we are trying to solve is this the Laplacian of xi minus 1 by c squared time derivative of xi. So, when you have the Laplacian acting on xi remember that xi is been assumed to be the function of r alone.

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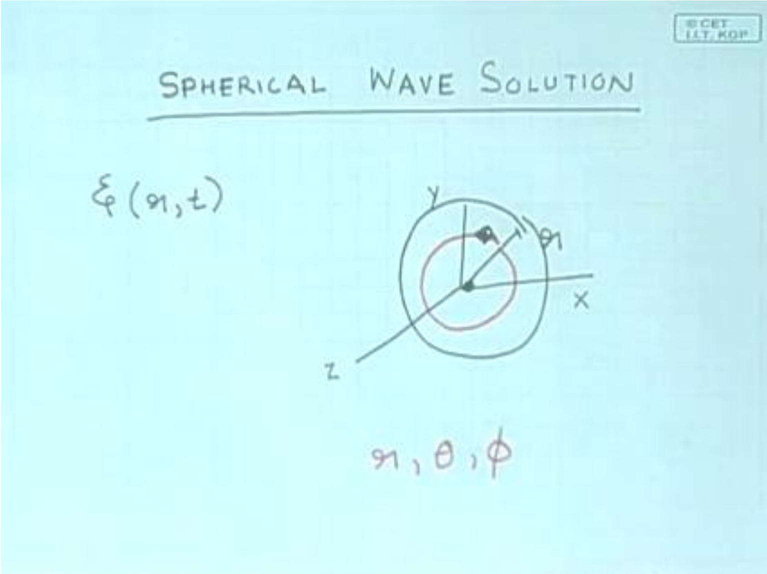

So, when the Laplacian acts on xi the derivatives with respect to theta and derivatives with respect to phi are not going to matter. It is only the terms involving derivatives with respect to r which are going to be important. So, I can write down the Laplacian the point is that I can write down this in the spherical polar co ordinate system. And if in general the Laplacian in the spherical polar coordinate system will have derivatives with respect to r, derivatives with respect to theta and derivative with respect to phi. You can get the expression in any book on mathematical physics or any book. Let us see on electro dynamics, but the point here is that the derivatives with respect to theta and phi are not going to be important.

Because we are assuming that xi does not depend on theta and phi depends on r alone. Theta and phi referred to different points on the sphere we have assumed at xi has a constant value on the sphere. So, it is only the r derivatives at we have to retain we can ignore the theta and phi derivatives in the Laplacian. And if you retain only the r derivative the Laplacian is 1 by r square del del r partial derivative with respect to r, r square partial derivative with respect to r of xi. So, this is the Laplacian written in the spherical polar coordinate system. So, you have derivative not x y z, but with respect to r theta and phi and we have dropped the terms involving derivatives with respect to theta and phi. So, this is what the Laplacian operator acting on xi becomes.

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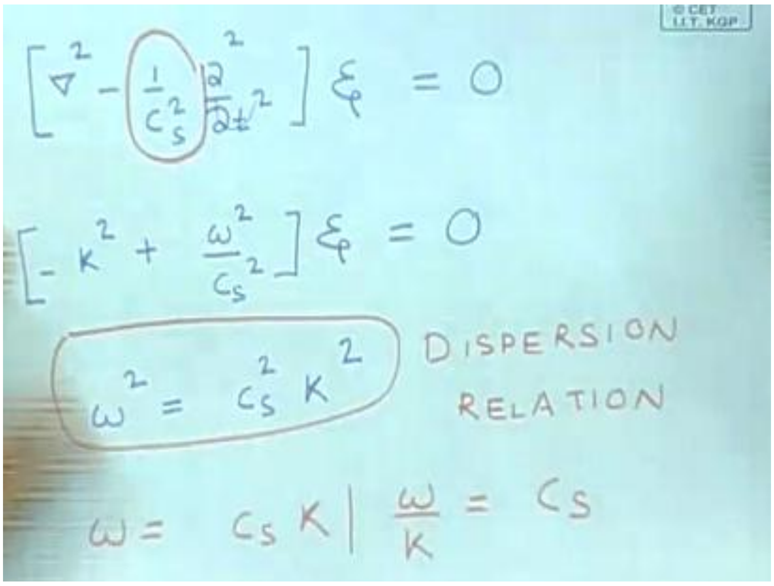

Now, we replace this in the wave equation. So, the wave equation now reads as follows. So, for a spherical wave the wave equation now becomes 1 by r square partial derivative with respect to r.

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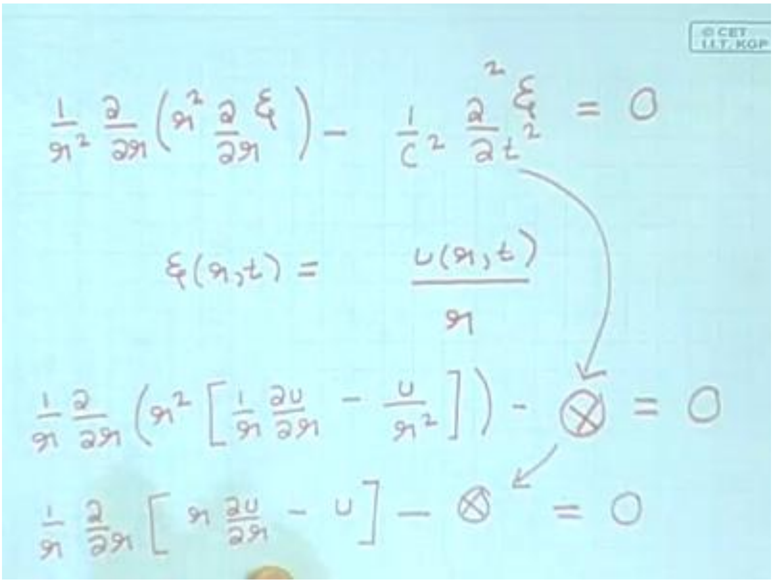

And then I have r square partial derivative with respect to r xi minus 1 by c squared partial derivative with respect to t square of xi. This is equal to 0 we are looking for solution to this equation xi is a function of r and t. Now, it is convenient to introduce a new variable here so xi is a function of r and t it is convenient to introduce a new variable u which is also a function of r and t such that xi is equal to u divided r. So, with this new function introducing new function we have differentiate this once with respect to r. So, let we do it here 1 by r partial derivative with respect to r and then I have r square and let me write down the derivative of xi with respect to r. So, I will have 1 term which is the derivative of u divided r and I will have 1 term which is the derivative of r.

So, I am going to get u divided by r square this is the first derivative of xi this minus the term over here which I am not going to write down explicitly again I am just indicating it. So, this is the same time derivative term which here which is here. Just that you have to place xi in terms of u now you can simplify this terms. So, what will happen is that this factor of 1 by r. So, let me write it here 1 by r. And I have r into del u del r minus u that is what we have minus again the time derivative I am just carrying it unchanged is equal to 0. So, this is the same thing as this now we have differentiate with respect to r again notice that if when I act on this I will get two terms 1 term is the derivative of r into. So, which is 1 so, 1 term is going to be del u del r which is exactly going to cancel out with that term that I get when differentiate this. So, there is only finally 1 term which remain which is r into the second derivative of u. So, we can now write down the same equation here.

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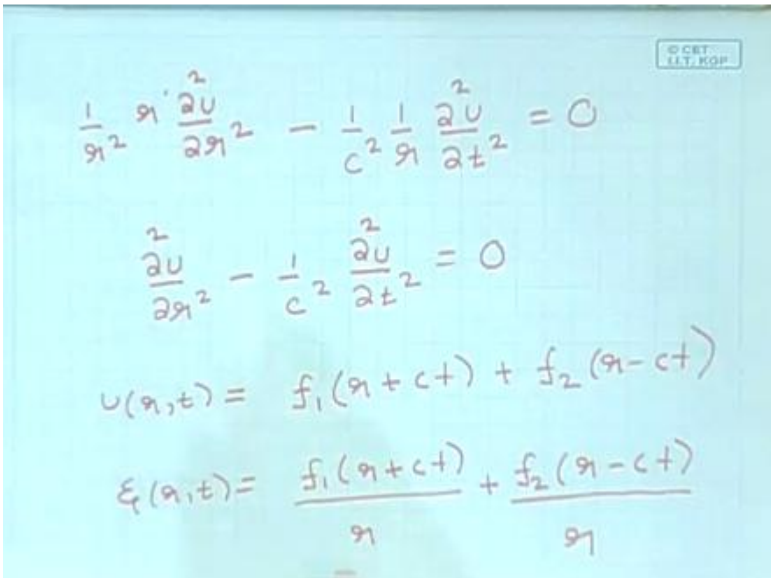

So, what we have is 1 by r square and r into the second derivative of u.

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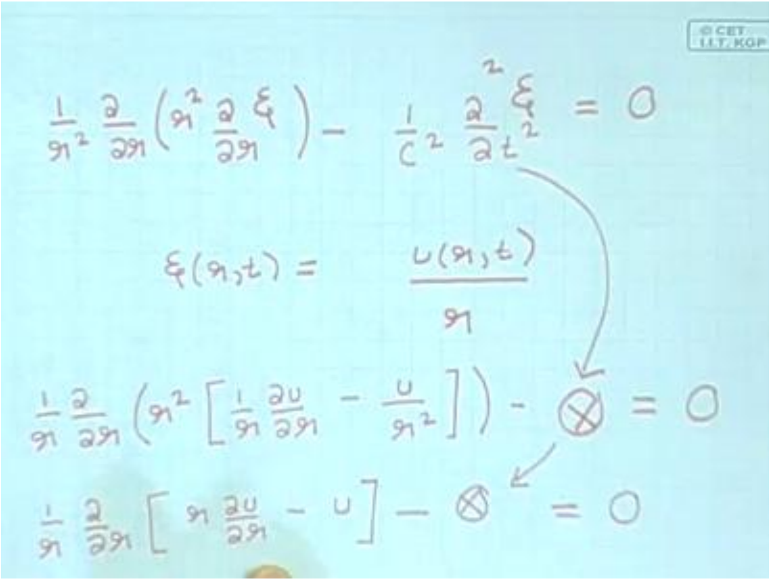

So, this is the, what we get from the whole term over here. And I have to also write down this term over here replacing xi with u by r. So, r 1 by r comes out.

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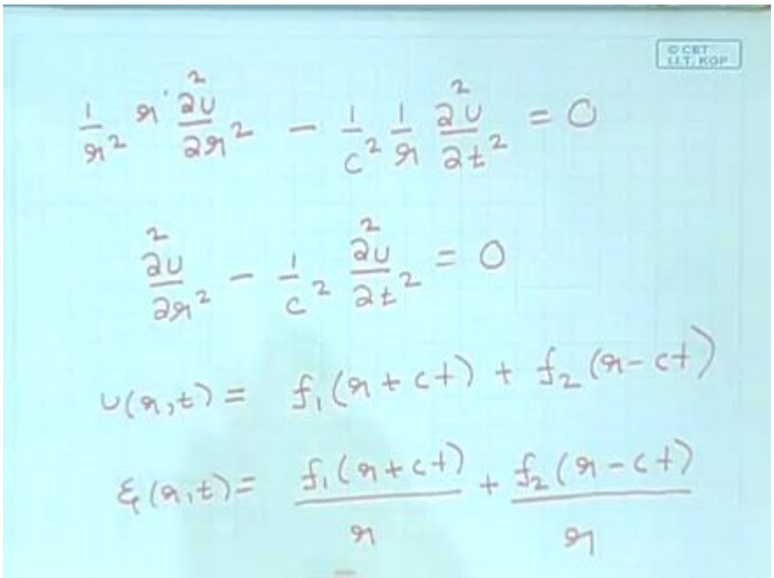

And what I have is minus 1 by c square 1 by r del u del t square is equal to 0. And the factor of 1 by r gets cancel out and finally, what we have is del u del r square minus 1 by c square del square u del t square is equal to 0. So, what you see is that u the wave new variable u which you have introduce satisfies exactly the same wave equation the same 1 dimensional wave equation which we have just solved earlier. And we have seen that there are going to 2 solutions I can straight away write down the solutions for u, u as function of r and t is going to be a some 2 solutions. The first part is f 1 r plus ct plus f 2 r minus ct we have already studied the solution to this wave equation which is what I have written down here. And I can straight away write down xi as a function of r and t this is going to be f 1 r plus ct divided by r plus f 2 r minus ct divided by r. So, we have worked out the spherical wave solutions and again we see that the spherical wave solutions are some of 2 parts. That it could be some arbitrary function f 1 well behaved arbitrary function whose derivatives of also define of r plus ct divided by r plus some other arbitrary function of r minus ct divided by r. So, we have 2 possible solutions now remember that r is if this is the origin r is the radial coordinates.

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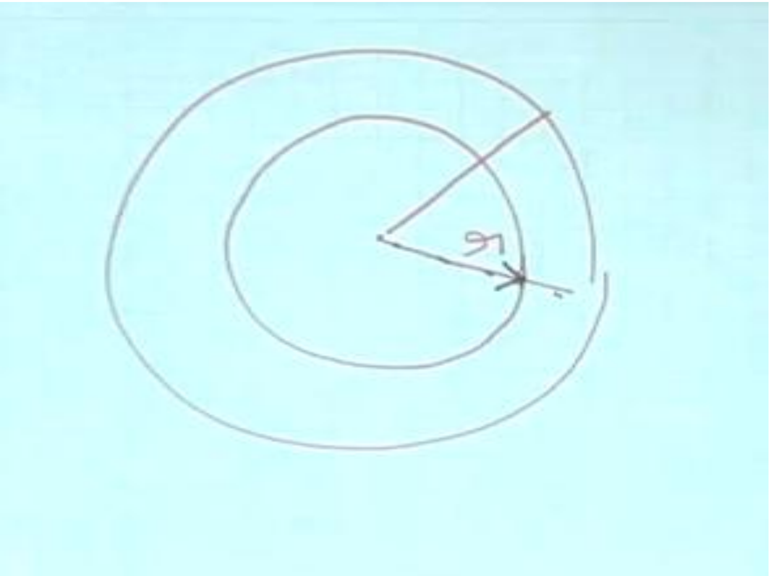

So, r changes in this directions as you go from this sphere to this sphere the value of r increases.

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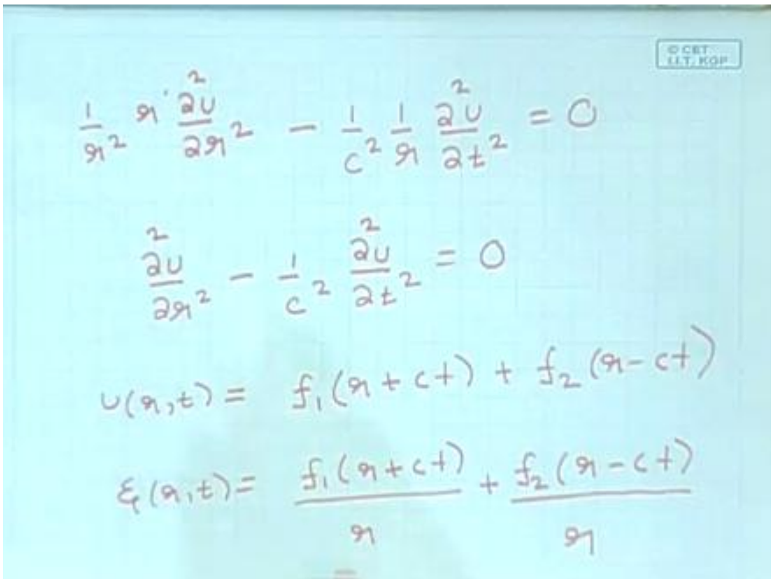

So, for a plane wave these 2 solutions corresponded to left and right propagating waves in this particular case just see as t increases the value of r has to go down. If you are following the point where f has a certain value at this is as a certain value the value of r has to go down. So, this represents the wave which is propagating inwards. So, the first part represents the wave which is propagating inwards that is decreasing r, the second part represent the wave which propagating outwards that is increasing r. Now, a point to note here is that s the wave propagates inwards or out words its amplitude changes as 1 by r. So, if you have an out word propagating wave outward means as timing increases r increases. If you have an output propagating wave this represent an out word propagating wave.

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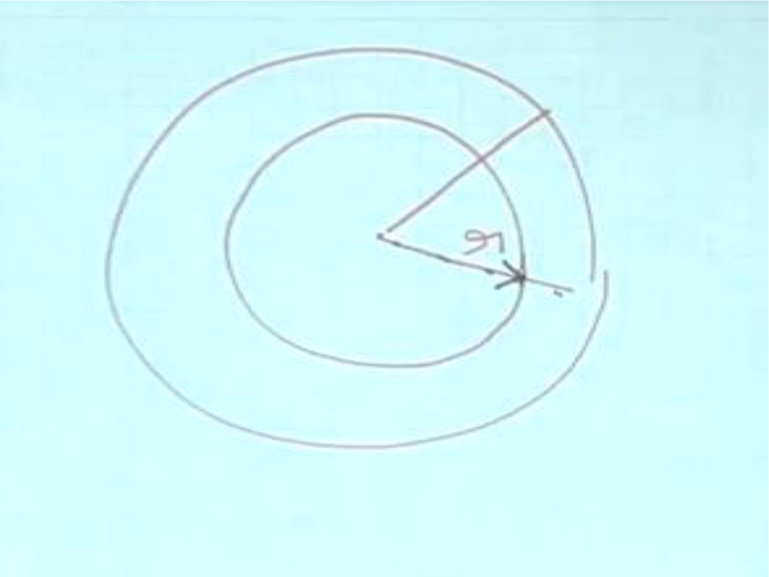

As time increases the wave goes in this way or value of r increases.

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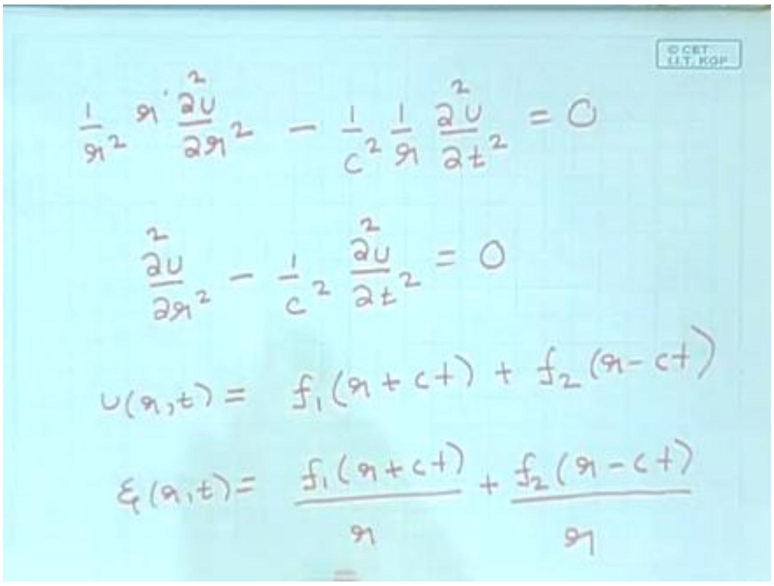

And as the wave propagates outward which amplitude falls as 1 by r. So, let me show you an out word propagating wave over here.

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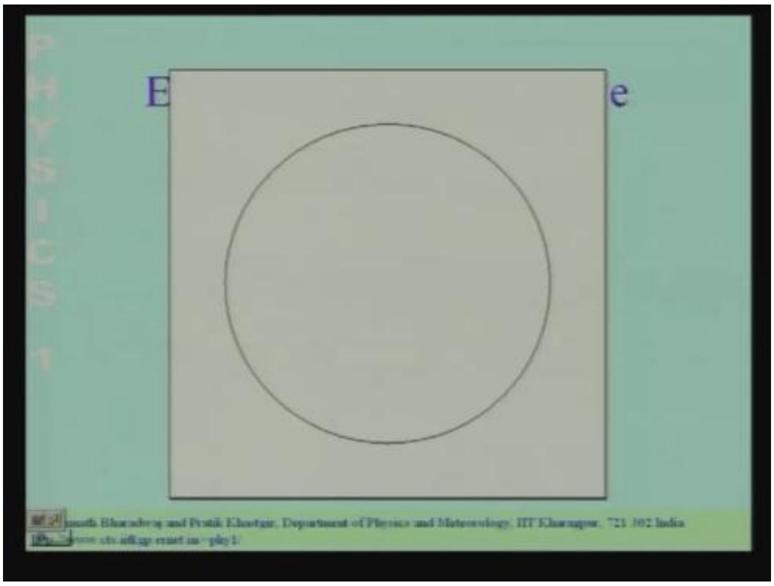

So, the value of the wave has a fixed value on this sphere some time at a later time that same value has shifted out. But the amplitude is fallen by 1 by r and then at the later time the whole thing is again shifted out. The amplitude falls again as 1 by r and the amplitude keeps on falling as 1 by r. So, this is an outward propagating wave and if you want this c in invert propagating wave this shows you an inward propagating wave. Now, we could ask the question what context do you have out word and inward propagating waves.

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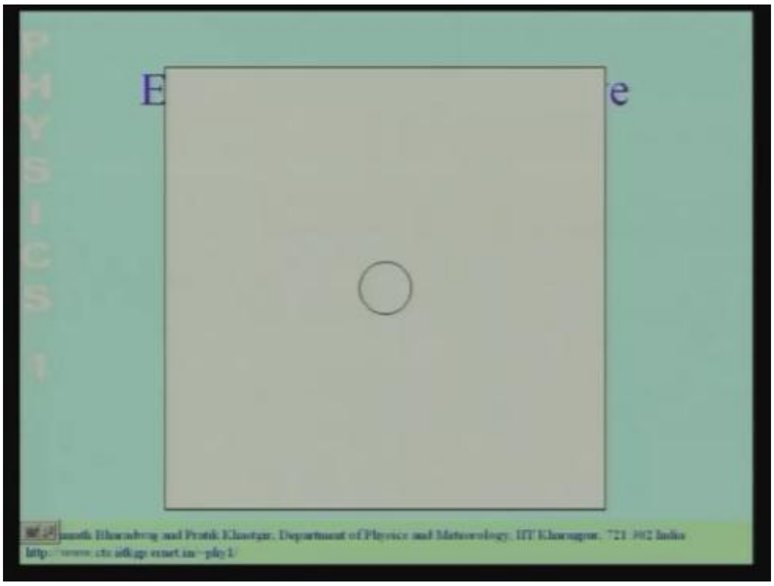

Now, if I had a point source from which there is some wave coming out. So, there is a source which is localized point and there is some wave coming out from that. Well, this could be represented using the outward going spiracle solutions.

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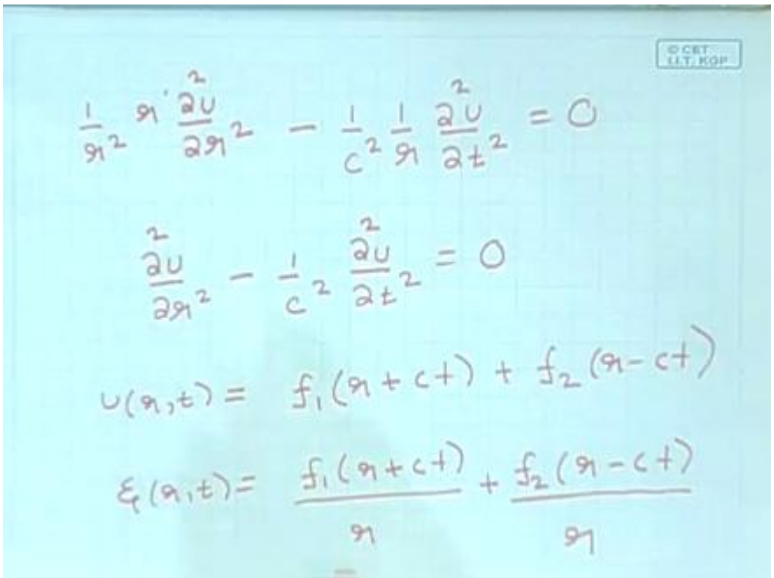

So, this will represents such a solution if I have source located at the centre over here.

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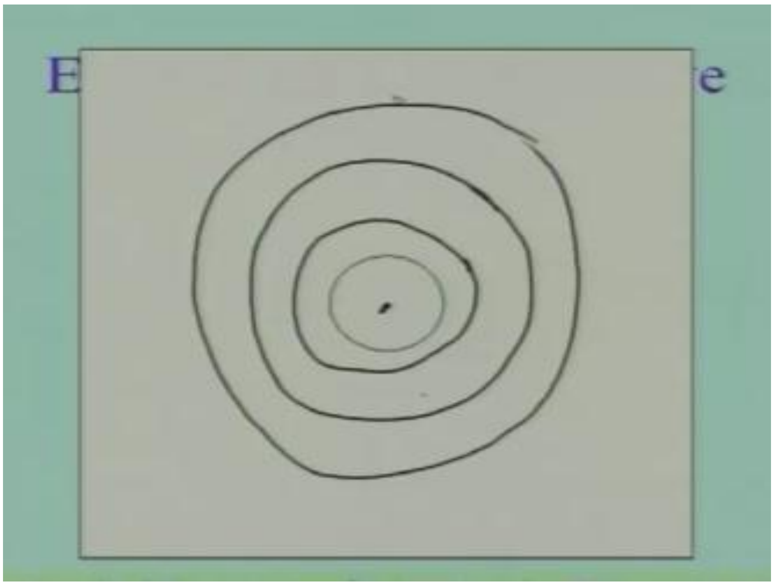

So, let me show you this. So, if I had a source located at the centre over here and there was wave coming out from this the evolution of the wave would be described by an outward going waves. So, as time involves the wave would get evolved into the larger and larger spheres like this. And the amplitude would fall as 1 by r or if you ask the question how does the intensity fall? The intensity would fall as the in the intensity goes as the square of the amplitude. So, the intensity would fall as 1 by r square. So, this is the outward propagating wave the inward propagating wave there is it also is has applications. Suppose I set a light through a lens which focuses a light to a point. Then once the light comes out from the lens I can represented by a spherical wave and that spherical wave is going too slowly. That sphere is going to slowly contract it is an inward going wave intensity of the light is going to go up. And at the focus it is going to infinity and then light is going through the focus. So, the light going towards the focus and then it is going to go through the focus and it is going to come out.

So, the inward going wave is going to collapse and then it is going to come out as an outward going wave. So, in today is lecture we have a discussed solutions of the wave equation. The wave equation itself is very general it appears in a large variety of situations. And today's lecture, we have discuss 2 kinds of solution to particular kind of solutions. The first solution applies in a situation when we have planer symmetry if the wave depends only on 1 special direction. Plane wave; we have plane waves and the second solution that we consider we have spherical symmetry. So, it you can applied to a situation where you have waves coming out from a source. If you are sufficiently far away from the source this is the point which point you should remember that if you are sufficiently far away from the source sufficiently far away from the source. So, when the wave becomes quite large you can you can represent this quite well using a plane. So, spherical wave so, sufficiently far away can be well approximated by a plane wave. And we have studied only 2 kinds of solutions there are verity of other solutions possible at different situation. We could have a cylindrical solution and whatever could impose different kind of symmetries and get the different kind of solutions. So, we shall end our lecture here today and move on to something new in the next.