PHYSICS – I : OSCILLATIONS & WAVES
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Lecture - 02
Damped Oscillator - I

Welcome to the second lecture on this course on oscillations and waves. In today's lecture we shall be discussing damped oscillators, but before starting our discussion on damped oscillators, let us first review what we had done in the last class.

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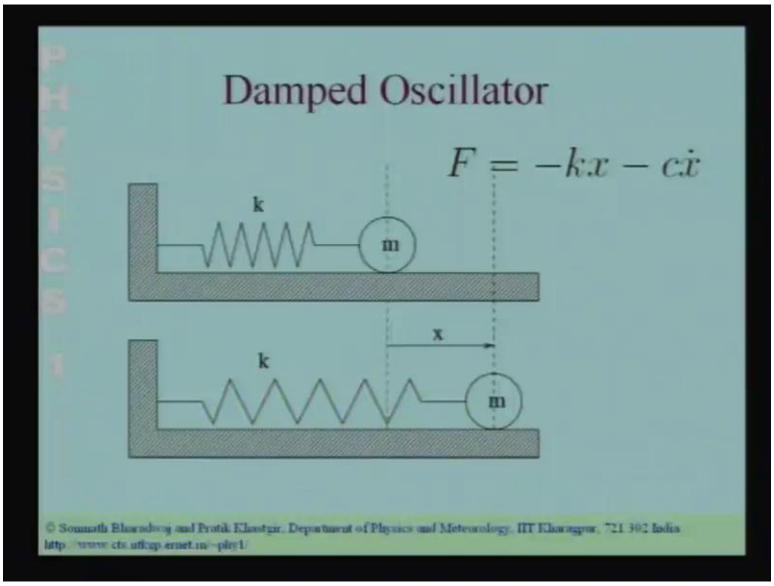

In the last class we had considered the simple harmonic oscillator we had taken the prototype simple as the prototype simple harmonic oscillator the spring mass system shown over here. We had worked out the solution to this to the motion of the particle, if it is displaced from equilibrium and the solution as we had seen in the last class is.

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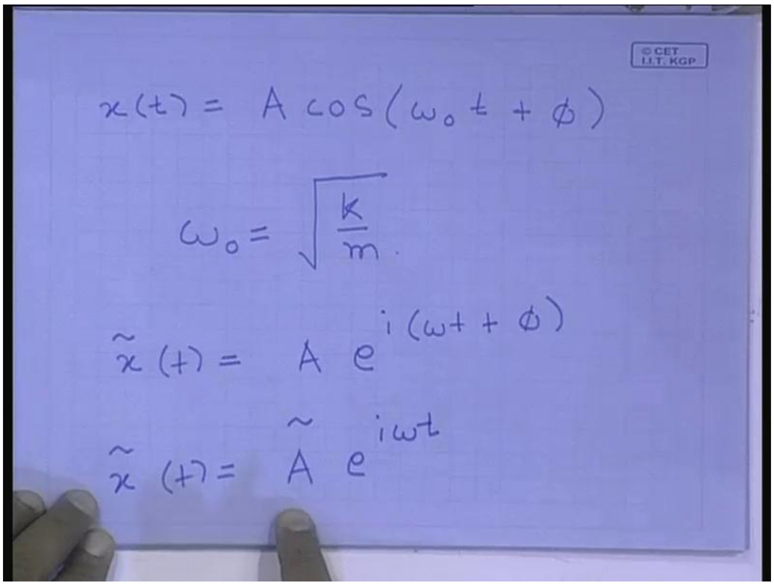

X of t is equal to amplitude A cos omega naught t plus a phase phi where omega naught is the angular frequency which is related to the spring constant and the mass as omega naught is square root of k by m. In the last class I had also introduced the complex notation we denoted complex numbers using a tilde. So, x tilde t is equal to A e to the power i omega t plus phi, x tilde t is a complex number which and it is a method for representing the same oscillation shown over here which is a solution to the motion of the spring mass system.

It is understood that, you should consider only the real part of this complex expression and the real part of this complex expression gives us the same solution x t is equal to the amplitude A into cos omega t plus phi. We can also express this expression as x tilde t is equal to A tilde e to the power I omega t. Where A tilde is now the complex amplitude which contains both the real amplitude A and the phase phi.

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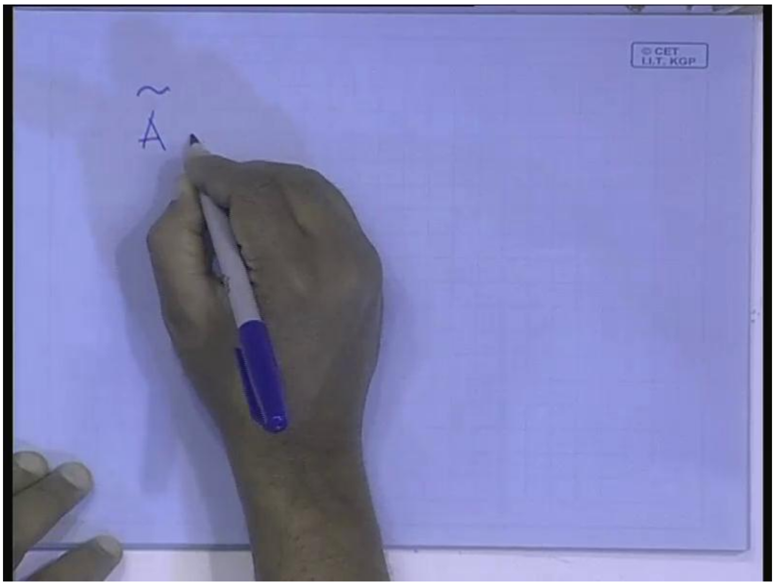

A tilde is defined as A e to the power I phi. So, in summary the motion of the oscillator can be described through a complex variable a complex expression like this where, the complex variable x of t x tilde of t which is the function of t is equal to A tilde; the complex amplitude e to the power I omega t. The real part of this represents the oscillations of the oscillator. Let us now, consider a simple example where we shall apply the complex notation.

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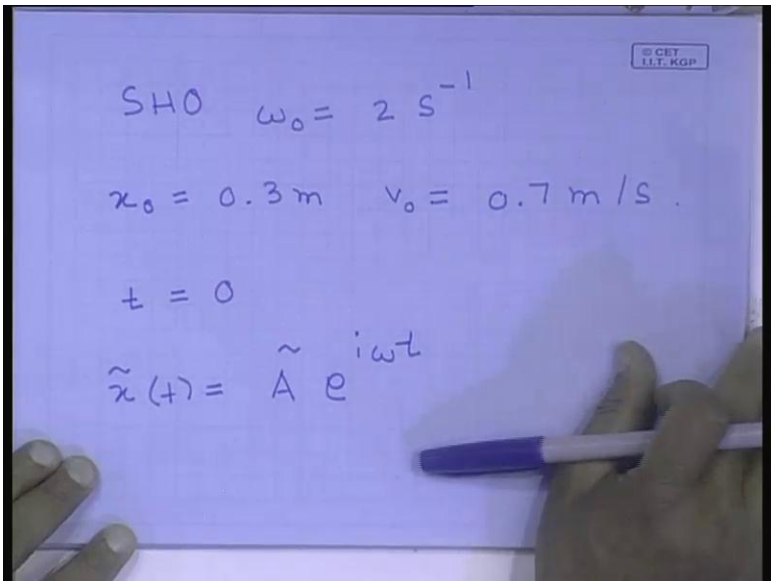

So, we consider an oscillator a simple harmonic oscillator with omega naught equal to 2 second inverse and we are given the information that the simple harmonic oscillator as an initial displacement of 0.3 meters. It has an initial velocity v naught of 0.7 meters per second. So, these are the initial conditions at the time t is equal to 0. Now, we represent the oscillation through this complex variable x tilde t A t e to the power i omega t.

The problem is that you have to determine this complex amplitude which has all the information about the initial conditions from these initial conditions which have been given.

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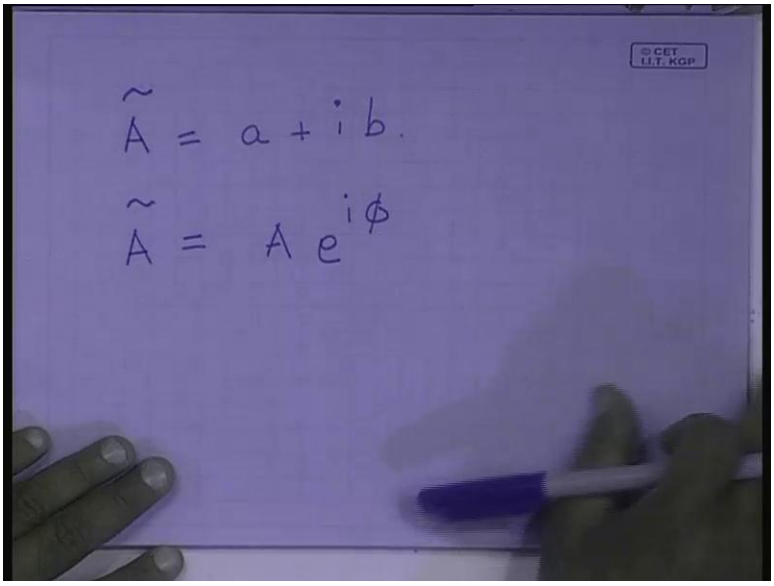

You could represent the complex amplitude A tilde as a plus i b. So, the problem is to determine these coefficients a and b in terms of the initial position and the initial velocity of the oscillator. You could also represent the complex amplitude as a real magnitude and a phase. So, the next part of the problem is to determine the real amplitude and the phase from the initial conditions.

So, let us first take up the first part of the problem where we will determine these coefficients a and b in terms of the initial conditions.

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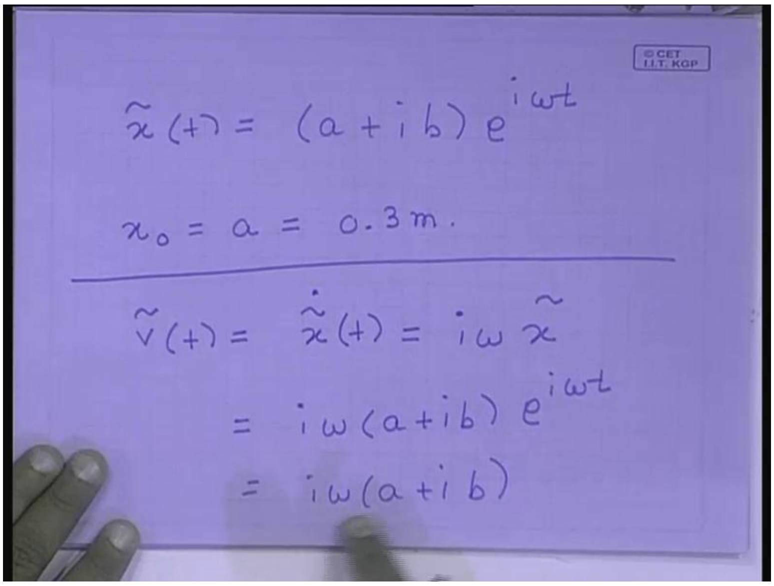

So, putting this expression for A the complex amplitude A in the expression for x of t. We have a plus i b. Now, at t equal to 0 e to the power i omega t is 1. So, x is a plus i b we are supposed to take only the real part the real part of x at t equal to 0 is a. So, we can straight away say that x naught the initial value of x is equal to a and we are given that this is equal to 0.3 meters.

So, the initial position determines the real part of the complex amplitude namely a and we have a value 0.3 meters, it is exactly equal to the real part of the complex amplitude. Let us now, look at the velocity in the complex notation the velocity v as a function of time is the time derivative of the complex position variable. Now, if you differentiate this you pick up a factor of i omega.

So, the velocity is especially i omega into x tilde which is given over here x tilde of as a function of time is given over here. So, the velocity is i omega a plus i b e to the power i omega t and t equal to 0 e to the power 1 omega t is 1. So, the velocity is i omega a plus i b and we have to take out only the real part of this. The real part is the physical quantity corresponding to the velocity the real part is minus omega into b. So, the velocity at t equal to 0 is minus omega into b.

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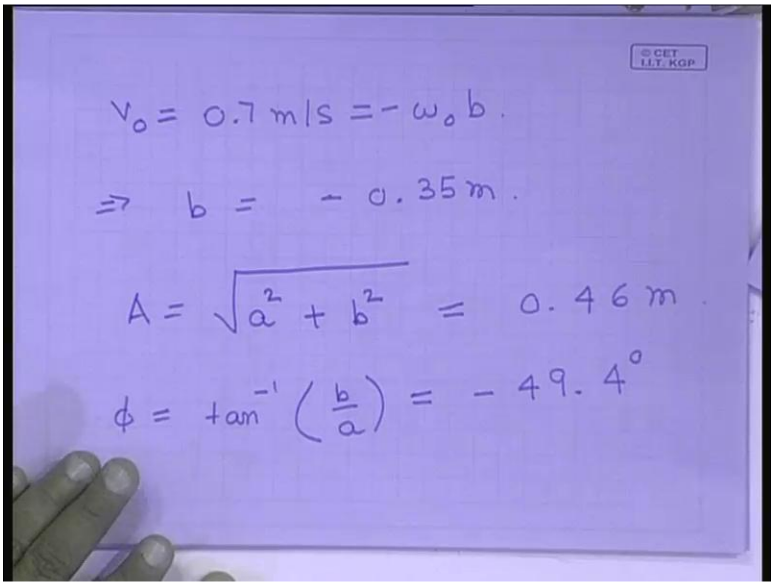

We are told that this has a value 0.7 meters per second this is equal to omega naught into b and omega naught has a value 2. So, v has a value minus this is a minus sign here minus 0.35 meters. So, we have worked out the 2 coefficients which completely determine the complex amplitude. And these coefficients have a value a which is equal to x naught it has a value 0.3 meters and b has a value minus 3.5 meters.

Now, next we shall work out the complex amplitude in terms of the real amplitude and the phase. This again is quite straight forward the real amplitude is the square root of a square plus b square. So, a is point three b is minus 0.35 and this gives us a value 0.46 meters. The phase phi is related to a and b as follows: phi is tan inverse of b by a and putting in the value minus 0.35 here and 0.3 here, we find that this is minus 49.4 degrees.

So, in this example we have worked out how you can use the initial conditions to determine the complex amplitude. And this complex amplitude encodes all the information about the initial conditions and we can now put this back into the solution. And you have here you have a the full information about the future revolution of the simple harmonic oscillator. Having taken up this example, let us now go back to the topic of today’s lecture which is damped oscillators.

So, the system which we had considered in the last class we had a spring and we had a mass.

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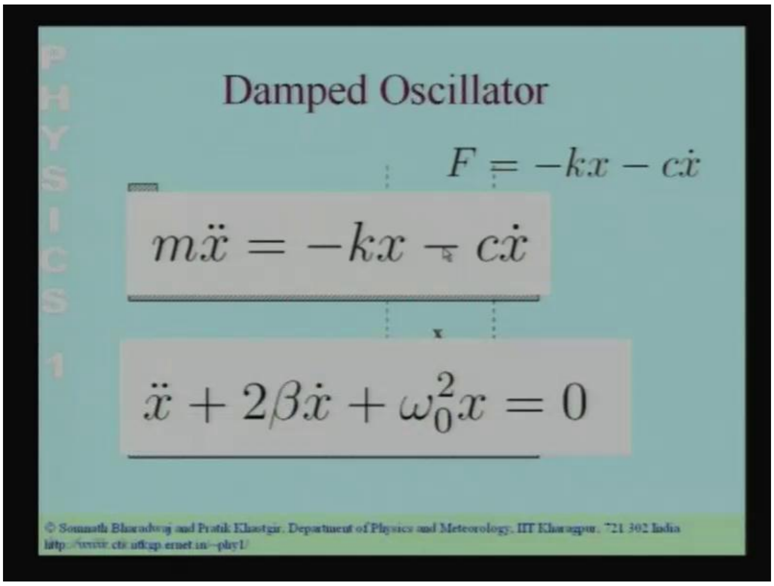

If you pull the mass the spring exerted a force F equal to minus k x, but in reality you will also have damping. Whenever, there is any motion there is usually some damping some force which opposes the motion and to bring the moving particle to rest. Here, we shall consider the simplest possible situation where the damping force is proportional to the velocity. So, when you displace this particle from the equilibrium position and leave it the force now has 2 parts: 1 part of the force arises due to the spring and that is minus k x.

You also have the damping force which is proportional to the speed of the particle; the damping force tries to bring the particle to rest. So, it opposes the motion it is the direction opposite to the velocity. So, it is minus constant of proportionality into the velocity. The force has these 2 components, putting these 2 components of the force into the equation of motion. This is what we get mass into the acceleration is equal to the force, the force now has these 2 components: 1 from the spring and 1 from the damping.

Now, it is convenient to recast this equation and write it as follows as shown over here. Omega naught square is k by the mass which we had defined earlier for the simple harmonic oscillator the new thing is this beta. Beta is essentially this coefficient c which appeared over here which determine how much the damping force was. Beta is related to c beta is c divided by the mass. Beta is c divided by the mass and there is the factor of 2 which has also been put in which makes things convenient as we shall see as we go along.

So, we have written the same equation over here with just the variables the constants constant coefficients been redefined. Now, next we shall discuss how to solve this equation.

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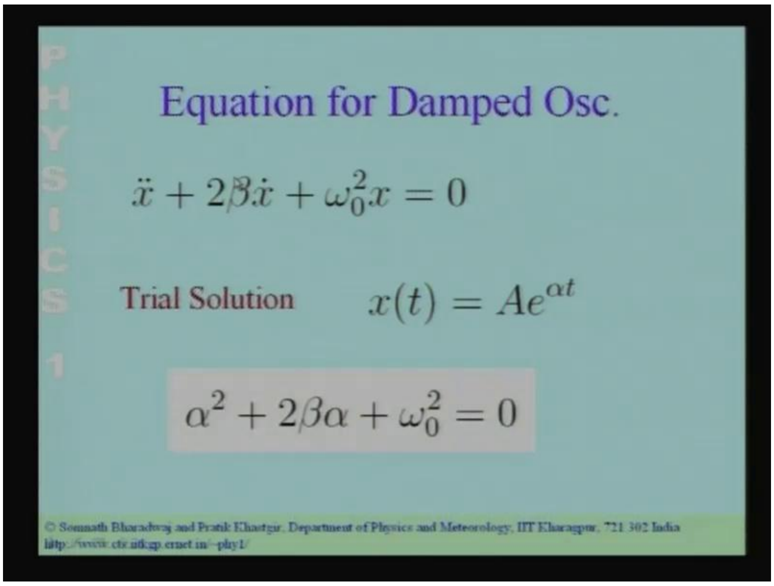

So, this is the equation which governs the motion of a damped oscillator. Notice that, this equation is a second order differential equation and it is homogeneous in the variable x. So, we proceed to solve this equation by considering a trial solution x t x is the function of time and the trial solution is a constant e, e to the power alpha t. So, the method to solve such second order homogeneous differential equations is to consider such a trial solution which is an exponential of time.

The exponential has an unknown coefficient alpha which will be determined from the differential equation. So, you have to now put this trial solution into this differential equation.

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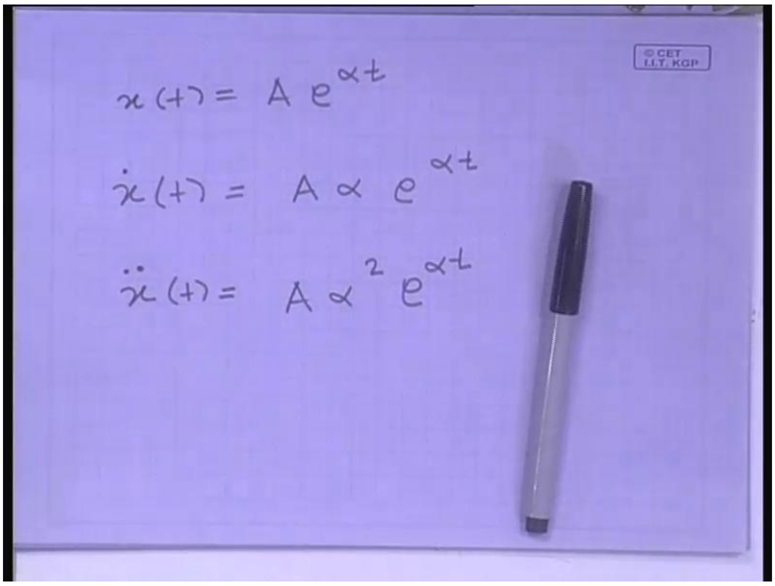

Let me do the steps involved on the paper over here. So, we first consider so this is the trial solution which, we shall be considering and we have to evaluate the first derivative. So, the first derivative if you differentiate this function once with respect to time you have A alpha e to the power alpha t. So, differentiating with time essentially pulls down a factor of alpha. So, if you differentiate this twice you will get 1 more factor of alpha and what you have is A alpha square e to the power alpha t.

So, now we have to plug in these expressions for x the first derivative of x and the second derivative of x into the differential equation which is given over here. This is the differential equation which governs the motion of the damped oscillator. So, if you plug in this trial solution with the expressions for the derivatives of x into this equation. You will notice that, the constant A and e to the power alpha t will cancel out through out.

And you all left with this second order differential equation for second order equation. It is a quadratic equation essentially left with this quadratic equation for alpha and the equation is alpha square plus 2 beta alpha plus omega naught square equal to 0. So, essentially every derivative of x is replaced by an alpha. You have to now the solve this quadratic equation which is very easy to solve.

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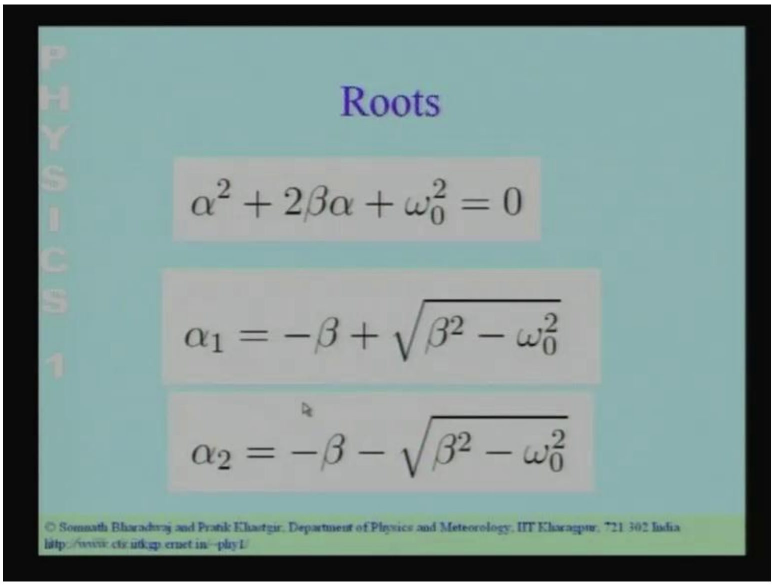

So, this is the quadratic equation which has to be solved and we all know that it has 2 roots. The 2 roots are minus beta plus square root of beta square minus omega naught square and we have another root which is minus beta minus square root of beta square minus omega square. So, there are 2 roots to this quadratic equation and the coefficient alpha is the exponent appears in the exponent of the trial solutions. So, the trial solution is a constant e to the power alpha t and alpha we saw as 2 roots: alpha 1 and alpha 2 which are defined as follows.

Now, if you look at these 2 roots you will see that the behavior the value of these roots depends crucially on the values of beta and omega naught. Let us first just touch upon the situation where beta is equal to omega naught. If beta is equal to omega naught then, this factor over here exactly cancels out for both the roots and both the roots are exactly equal. This situation is referred to as critical damping. We shall come back to this situation later, but the point to note is that the situation where beta is equal to omega naught marks a dividing line between 2 kinds of solutions.

You could have beta less than omega naught if beta is less than omega naught then, this turn beta square minus omega naught square is negative and the square root over here will give you an imaginary number. Whereas, a beta is more than omega naught then you would get a positive number and the square root would be a real number. So, if beta is less than omega naught you have an imaginary number over here and you have 2 complex root. Whereas, a beta is equal to omega naught or beta is more than omega naught you have real roots.

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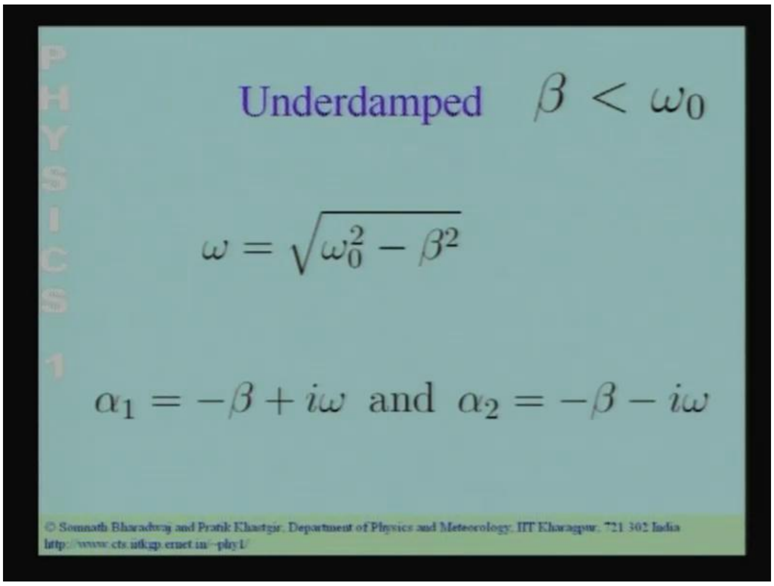

So, we shall first take up the situation where beta is less than omega naught this situation is referred to as under damped. So, if beta is less than omega naught you have under damped oscillations. If beta is less than omega naught this term beta square minus omega naught square is negative and the square root of this negative number is imaginary. It is hence convenient to take the minus sign out and define a positive real number omega square, omega naught square, minus beta square. The square root of this is defined to be omega.

In terms of this variable omega the 2 roots: alpha 1 and alpha 2 can be written as minus beta plus i omega, that is alpha 1 and alpha 2 is minus beta minus i omega. Where omega is again I will repeat omega is square root of omega naught square minus beta square. Omega naught square minus beta square is positive. So, the square root is a real number. So, omega is a real number and alpha 1 and alpha 2 are the 2 complex roots of that quadratic equation.

Now, we have to put in alpha 1 and alpha 2 back into our trial solution x as a function of time is equal to a constant amplitude e to the power alpha t. So, we now have 2 values of alpha; alpha 1 and alpha 2. So, x of t will be a super position of those 2 solutions.

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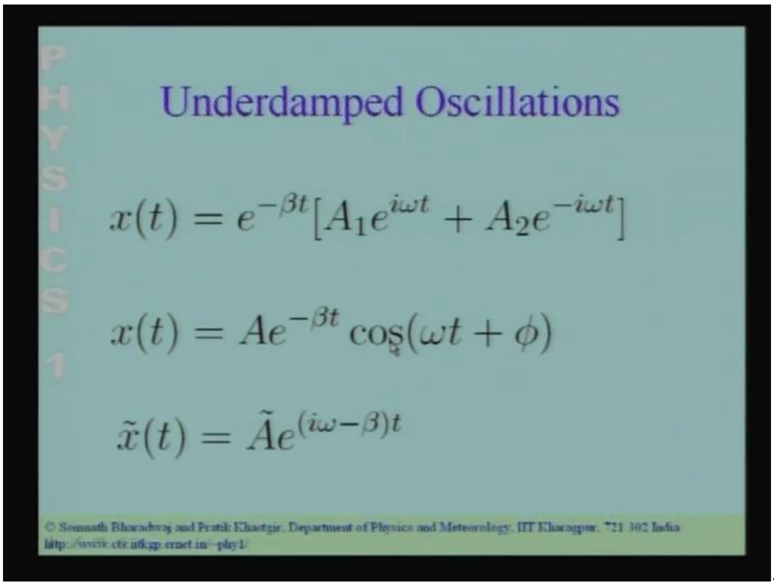

So, if you put in those 2 values of alpha that is alpha 1 and alpha 2 in the trial solution what you get this x of t is equal to e to the power minus beta t. Notice that, minus beta is common to both the roots. So, you can pull a factor of e to the power minus beta t out you can pull this factor out from e to the power alpha t. And what you are left with are the 2 different roots e to the power i omega 1 t omega t and e to the power minus i omega t. And these 2 solutions could have 2 different coefficients A 1 and A 2 amplitudes A 1 A 2.

So, the combined combination of these 2 gives you the total solution. Let us now, spend a few minutes and understand the behavior of this solution. How do we expect this solution to behave with time? Let us first look at the terms in the square brackets over here. The terms in this square brackets namely A 1 e to the power i omega one t plus A 2 e to the power minus i omega 2 t, if you look at these 2 terms you will recollect that this represents the simple harmonic oscillators.

A 1 and A 2 are the unknown amplitudes which have to be determined from the initial conditions. And the combination A 1 e to the power i omega t plus A 2 e to the power minus omega t is essentially an amplitude into cosine of omega t plus phi. The major difference is that you have this extra factor e to the power minus beta t outside. The role of this e to the power minus beta t is that it causes the amplitude of the oscillations to go down exponentially in time.

So, you could take this solution and write it in this form where this term in the square bracket is essentially A cos omega t plus phi and you have this e to the power minus beta t outside. So, you have a sinusoidal oscillation with the amplitude A. So, A cos omega t plus phi represents a sinusoidal oscillation the simple harmonic oscillation, which we had discussed earlier but now, you have an extra feature the amplitude of this oscillation e to the keeps on falling exponentially as the oscillation proceeds.

And this is the big change which occurs due to the damping. Damping causes the amplitude of the oscillation to decay exponentially in time. Now, we can combine this real amplitude A and the phase phi you can combine these 2 terms and also write this solution as x using the complex notation. So, if you write this solution using the complex notation it becomes x tilde t is equal to A tilde e to the power i omega minus beta t. Remember that A tilde has got both the real amplitude and it also has the phase.

So, A tilde is essentially A t to the power i phi. So, it has both the amplitude as well as the phrase of the oscillation.

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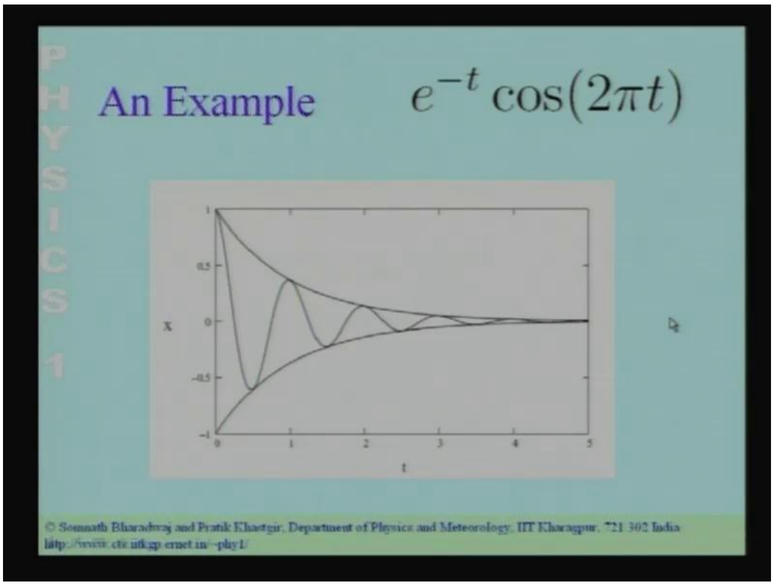

So, let us now have a look at what this oscillation looks like. So, this shows you an example of the under damped oscillations. The particle continues to oscillate just like in the situation where you had no damping here the particle till does oscillations. So, you have these oscillations, but you have this feature that the amplitude of the oscillations falls exponentially. So, in this particular case the amplitude falls as e to the power minus t and this is what is shown in the graph over here.

The amplitude falls as e to the power minus t and the motion is sinusoidal oscillation just like the simple harmonic oscillator just that the amplitude falls exponentially as time evolves. And it slowly gets smaller and smaller and finally, it will turn to 0 as time goes to infinitive. So, let me now summarize briefly 2 of the main important consequences of this damping the first consequence of damping is as follows.

The first consequence of damping can be seen in the expression over here. If we have no damping the undamped oscillator would oscillate at an angular frequency omega naught. You would not have this term in the differential equation governing the oscillator and the oscillator would oscillate at a angular frequency omega naught. Now, the consequence of introducing a damping is that it changes the frequency at which the oscillator oscillates.

If you introduce, a damping notice that the angular frequency for the damped oscillator is smaller than the angular frequency for the undamped oscillator. Further the more the damping the larger the difference between the damped oscillator and the undamped oscillator. So, damping causes the oscillator to damps slower to oscillate slower and if you increase the damping more and more the oscillations become slower and slower. This is the first consequence of introducing of having damping in the oscillation.

The oscillator no longer oscillates at omega naught it oscillates at a slower angular frequency, which is related to omega naught through this expression. The second consequence is that the amplitude of the oscillations decay exponentially with time through this factor e to the power minus beta t. So, it is of great interest sometimes to quantify this damping directly from the oscillations.

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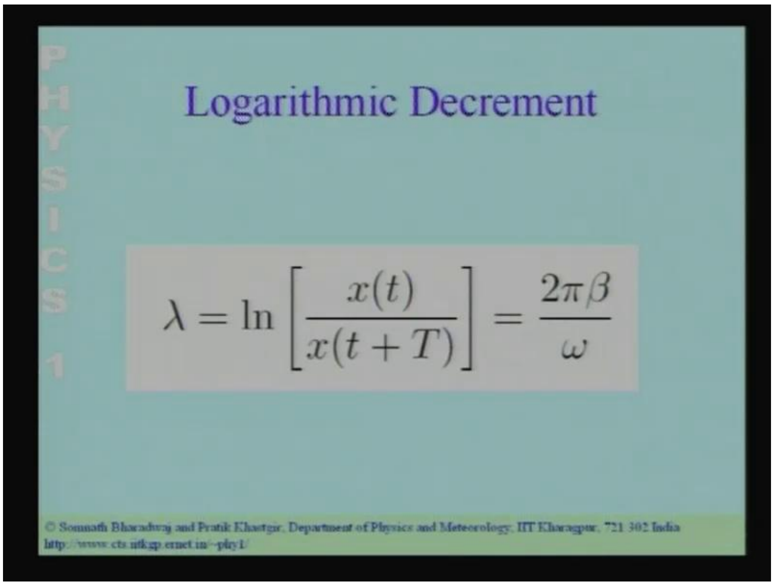

And this is done through a factor called the logarithmic decrement. So, let me now explain to you what the logarithmic decrement is the logarithmic decrement is a measure of how much the amplitude falls in 1 oscillation that. So, the logarithmic decrement is a measure of how much the amplitude falls in 1 period of the oscillation. So, you should take the amplitude at a time t this is x as a function of t at a time t is is the amplitude at a time t.

Now, also look at the amplitude after one time period of the oscillation if the time period is T. So, look at the amplitude of the oscillation after 1 time period. This ratio the natural logarithm of this ratio is going to give us what is called the logarithmic decrement. The logarithmic decrement is very useful and you can estimate this directly from the oscillations. So, let us now consider the particular oscillation shown over here this oscillation is e to the power minus t cos 2 pi t.

So, we have to look at the amplitude of the oscillation after 1 whole period of the oscillation. So, let us take the time instant t equal to 0 first. So, the amplitude of this oscillation at time t equal to 0 is 1. Now, let us ask the question when what is the time period of this oscillation? This oscillation has omega equal to 2 pi the time period is 2 pi by omega. So, the time period is 1. So, we have to also look at the amplitude of the oscillation at t equal to 1.

So, you have to take the ratio of the amplitude here divided by the amplitude after one time period which is the amplitude here. So, this is very useful if you are observing an oscillation you have to record the amplitude and then record it again after 1 time period find the ratio of these 2. In this case the ratio of this divided by this as you can see from here the amplitude falls as e to the power minus t. So, the amplitude here is going to be e to the power minus 1 that is 1 by e the amplitude here is 1.

So, if you divide 1 by 1 by e you get e and you are supposed to take the natural logarithm of that. So, in our situation you have the natural logarithm of 1 the natural logarithm of e the natural logarithm of e is 1. So, the logarithmic decrement for this oscillation is 1 in general the time period is 2 pi by omega. So, the amplitude if you ask the question how much does the amplitude go down in the time period.

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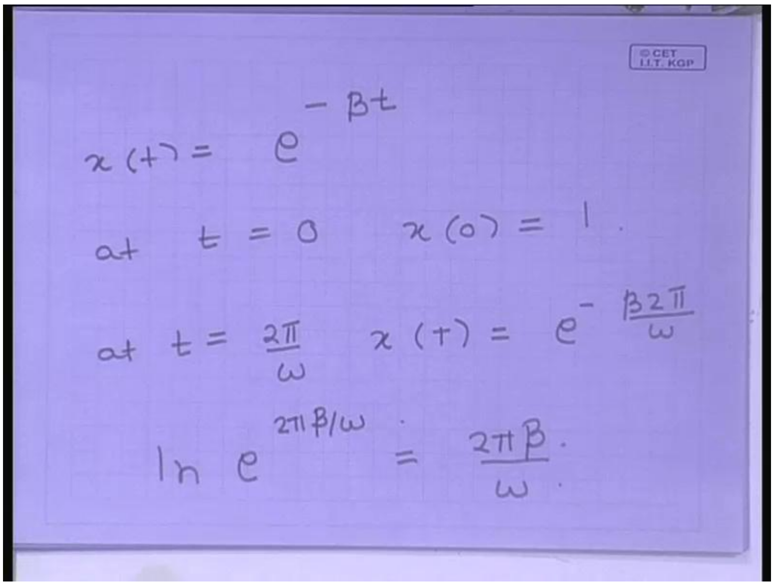

The oscillation the amplitude of the oscillation falls like this at t equal to 0. We have x 0 which is equal to 1 and at after a time period time period is t equal to 2 pi by omega. So, x after a time period will have a value e to the power minus beta 2 pi by omega. You are supposed to take the ratio of this divided by this. So, if you divide this by this you will get e to the power 2 pi beta by omega and then if you take the natural logarithm of this you will get 2 pi beta by omega.

So, this is the natural logarithm it essentially quantifies how fast the oscillations are dying down. And it tells you the decay of the oscillation in 1 time period of the oscillation. So, this brings to an end our discussion of the underdamped oscillation. To just remind you once more, what the underdamped oscillation is this corresponds to a situation where beta the damping coefficient is less than omega naught. Beta equal to omega naught is the critical damping situation and beta greater than omega naught is the overdamped situation.

Here, we have consider the underdamped situation in the in the underdamped situation the oscillator continues to oscillate like the simple harmonic oscillator where there is no damping. The effect of damping is twofold it causes the oscillations to be slower than if that damping were absent. It also causes the amplitude of the oscillation to decay exponentially the time. Let me now, give you a problem on underdamped oscillations and I shall close the class at the end of the problem.

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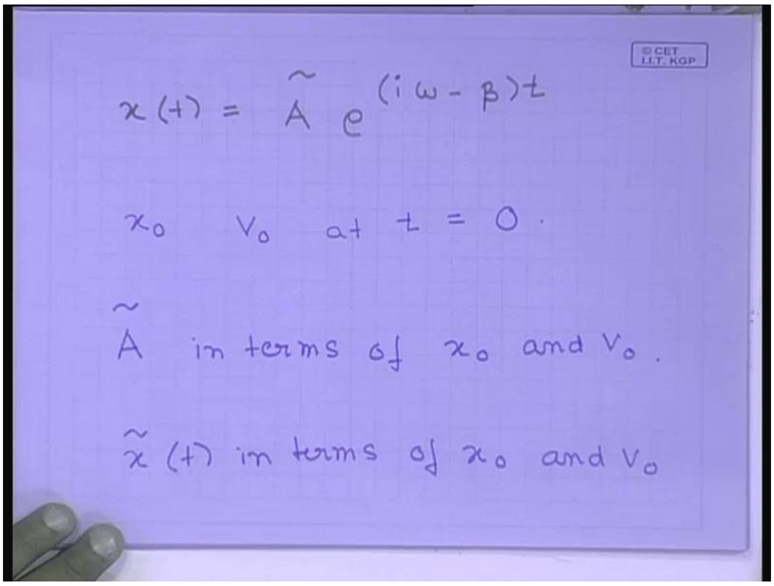

The problem is as follows: you are given an underdamped oscillation x t A tilde e to the power i omega minus beta t. This represents an underdamped oscillation. You are also given the information that the particle that the oscillator has an initial position x naught and a initial velocity v naught at t equal to 0. So, you have given the initial conditions at t equal to 0 now, the problem is that you have to find the coefficient, the amplitude, the complex amplitude A tilde in terms of x 0 and v 0.

So, you have to find an expression for the complex amplitude A tilde in terms of the initial position and the initial velocity. Once you have found this you have to find an expression for x tilde the motion of the particle in terms of x naught and v naught. So, you have to determine the motion of the of the damped oscillator underdamped oscillator in terms of the initial conditions x naught and v naught given at t equal to 0.

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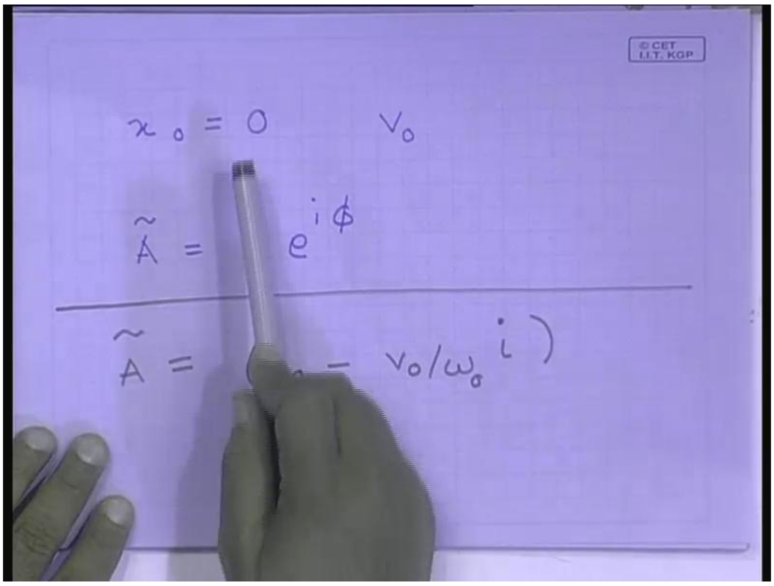

The next part of the problem is as follows: you are told that for the same oscillator the particle is initially at rest. And the particle has a velocity v naught initial velocity v naught. So, the particle is the oscillator is initially at rest and it has just a velocity v naught. So, you have to now find for this for these initial conditions you have to find the phase of the oscillator the phase of the oscillator come can comes in the complex amplitude which can be written as A e to the power i phi.

So, the problem is you have to find the phase of the oscillator for this particular situation where the oscillator is initially at rest and it has an initial velocity v naught. To give you some motivation for this problem let me just go back to the problem which we had taken up sometime earlier. Where, we had considered a simple harmonic oscillator and for without damping. For this simple harmonic oscillator without damping we had worked out exactly the same thing. We had a oscillator for which x naught was given and v naught was given.

We had worked out the amplitude; the complex amplitude for the oscillator in that situation. And if you go back to your notes the solution for A tilde in the absence of damping. In the absence of damping the solution if you go back to your notes A tilde is determined in terms of the initial position and velocity as per as shown here; x naught minus v naught by omega naught into i. This is for the undamped oscillator. So, if the oscillator is at rest to start with is not displaced has a initial displacement 0 as a velocity only to start with. Then this term is 0 it is the amplitude is totally imaginary, so the phase is pi by 2.

Now, the question is does this think that the real part is uniquely determined by the position initial position and the imaginary part is uniquely determined by the initial velocity, does the same thing still hold when you put in damping. That is the question which is interesting and which you will you will arrive at the solution if you work out the problem which, I have mentioned just now. We shall discuss this in the next class.