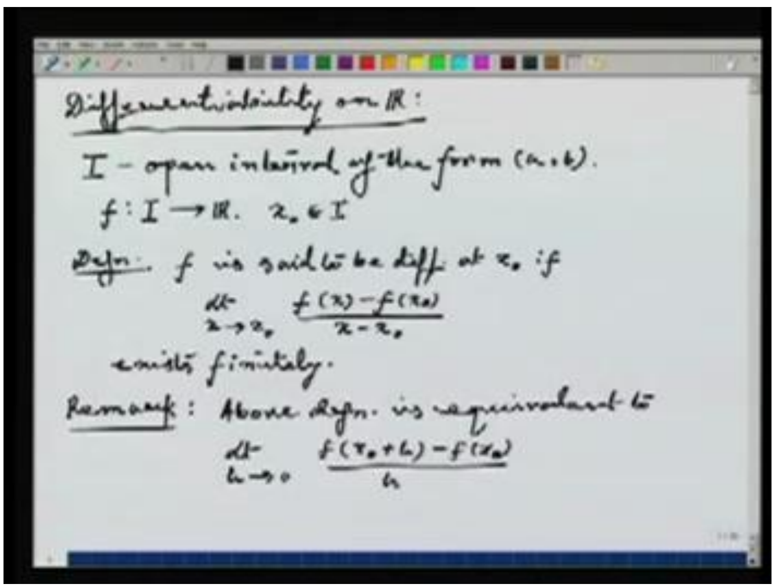
Mathematics-I

Prof. S.K. Ray
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture - 8
Differential Functions

Today we start our lecture with differentiability of functions defined on real lines.

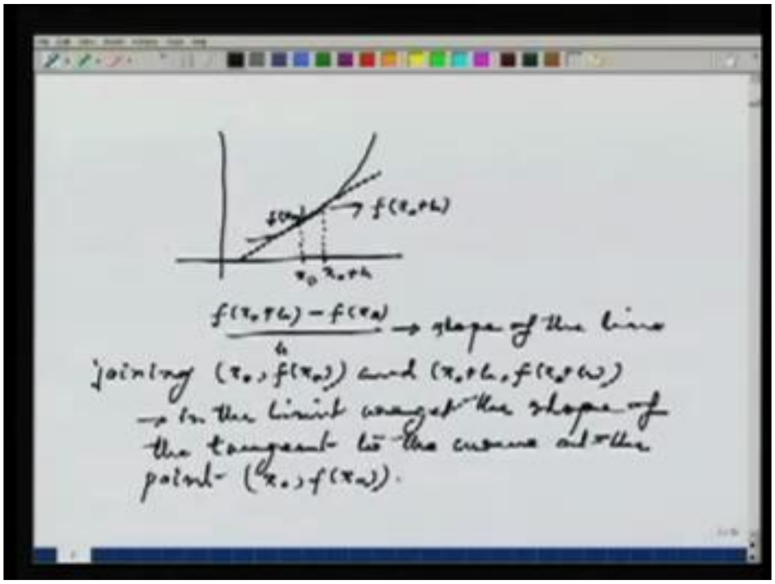
(Refer Slide Time: 00:26)



So differentiability, here the set up is as follows. Suppose I stands for an open interval of the form a, b and let us say, f is a function f defined on I taking values on the real line. Let us also suppose x naught is a point in I. We want to define, what do we mean by saying that f is differentiable at the point x naught. The definition goes as follows: f is said to be differentiable. In short, I will write it diff at x naught if the following limit exists. That is, limit x going to x naught, f of x minus f of x naught divided by x minus x naught exists finitely.

This definition can also be written in a different fashion. I will just put it as a remark, that the other definition is equivalent to limit h going to 0, f of x naught plus h minus f of x naught divided by h. Now what does this definition actually mean?

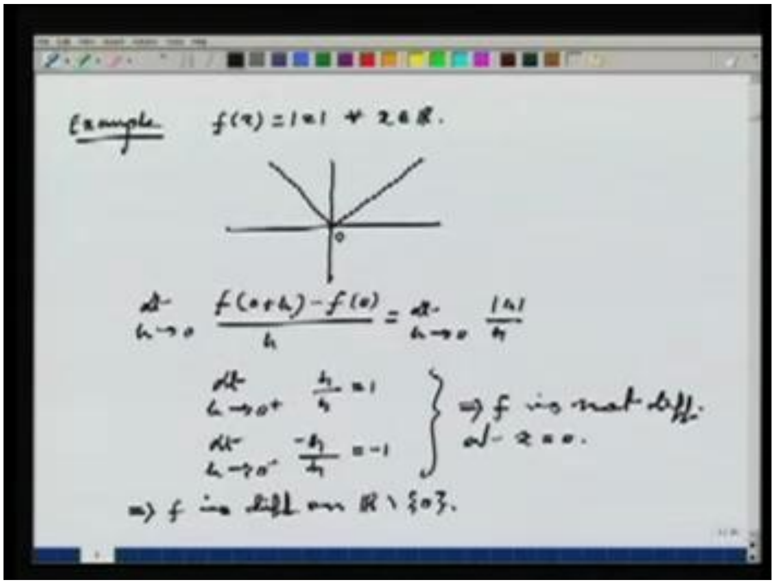
(Refer Slide Time: 02:57)



For this, we need to draw the following picture. Let us assume that the function is represented by a curve of this form. Let us say this is the point x naught. Now x naught plus h should be somewhere here. This is x naught plus h. The corresponding values we will denote, certainly, by f of x naught and then this is f of x naught plus h. Now if I look at the quotient f of x naught plus h minus f of x naught divided by h, it just means that I look at the line we join those two points. It is the slope of the line joining x naught, f x naught and x naught plus h f of x naught plus h.

When you take the limit h going to 0 in the limit, then we get the slope of the tangent to the curve at the point x naught, f x naught. This is the geometrical interpretation of the derivative of the function at a point. Now it might happen then, once we know the geometric interpretation that, all such curves, they may not have well defined tangent. Analytically it would mean that there are some functions which are not differentiable at certain points. Let us look at the examples first.

(Refer Slide Time: 05:18)

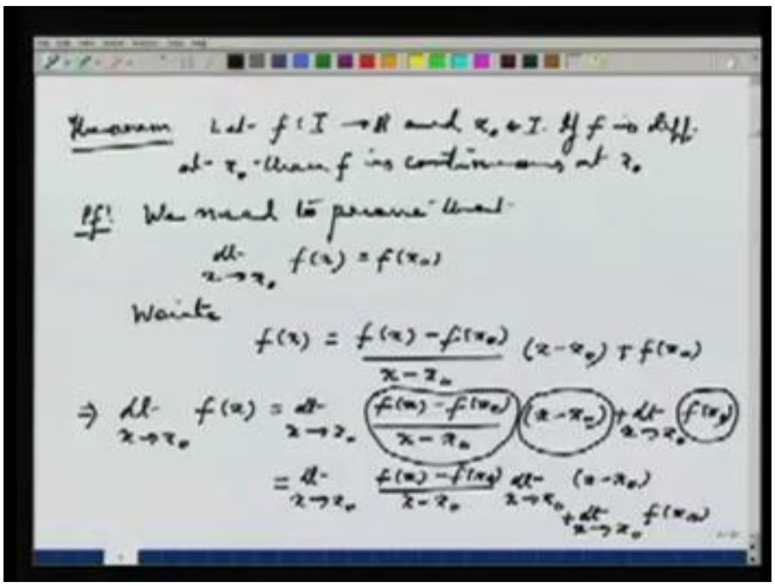


Our first example is very typical one of a function which is not differentiable at point. We look at the function f x equals to mod x for all x in R. Now you can easily draw the graph of this function. What it looks like, it is very simple function. This is the y axis, then I have x axis. This is the origin 0. Then the graph of the function is this and this and we see that at the point 0, the function has a sharp edge. As such, it would mean then that the function possibly does not have well defined tangent there. That means, our definition of differentiability should fail for the function at the point 0. Let us check why it is so.

So I look at the limit h going to 0, f of 0 plus h minus f of 0 divided by h and then I put the value of the function. Then it would mean limit h going to 0, modulus of h. Of course modulus of 0 is 0 divided by h. Now, if I look at the limit h going to 0 plus, now it means that h approaches from the right hand side. This then by the definition of modulus, it is h by h which equals to 1. But now, if I look at limit h going to 0 minus, that is, I am approaching 0 from the negative side, again by the definition of modulus, I have minus h because my h is negative but modulus of h has to be positive. So I multiply with it by minus divided by h. The answer I get is minus 1. They do not match.

This implies that f is not differentiable at x equals to 0 although it is easy exercise to check that f is differentiable all at other points. So the next conclusion is this, that f is differentiable on R minus 0. You see whenever the function has come with sharp edge, the differentiability fails there. Nevertheless, differentiability is a stronger property than something which you are already know, namely continuity of the function and that is the first result we are trying to prove. Now, that is, we will try to prove that if the function is differentiable at a point, then the function has to be continuous there. That is our next theorem.

(Refer Slide Time: 08:21)

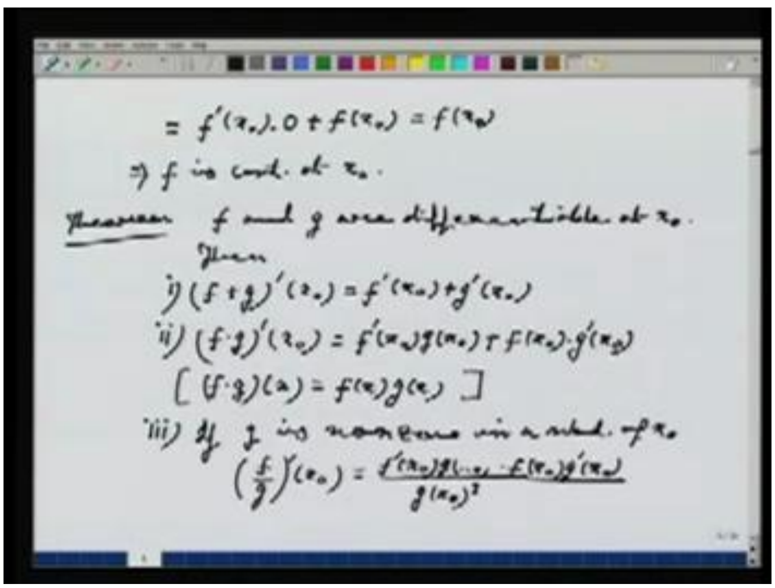


Let f be from an open interval I to R and x naught is a point in I. If f is differentiable at x naught, then f is continuous at x naught. So the proof of this is very simple. We know by definition of continuity, we need to prove that limit x going to x naught f x must be equal to f of x naught. So what we do is, we write f x equals to f x minus f x naught divided by x minus x naught and then multiply with it by x minus x naught. So the net result is f x minus f x naught. Then I cancel f x naught by adding with it. So this is the expression and now I take the limit. This implies that limit x going to x naught f of x is certainly equals to limit x going to x naught f x minus f x naught divided by x minus x naught into x minus x naught plus limit x going to x naught f x naught.

Notice that I have divided this expression of limit on the right hand side, simply because of the one thing. I know f x minus f x naught by x minus x naught. That is this quantity; it has a well defined limit as f is differentiable as x goes to x naught, this quantity also has the well defined limit. So the product limit exists. On the right side, I have a constant function whose limit as x goes to x naught is also exists. I can write it in this form.

Then the next line is very obvious. I just write it as limit x going to x naught f x minus f x naught divided by x minus x naught into limit x going to x naught x minus x naught plus the last limit, limit x going to x naught f x naught and then it is easy to see what is the end result.

(Refer Slide Time: 11:38)

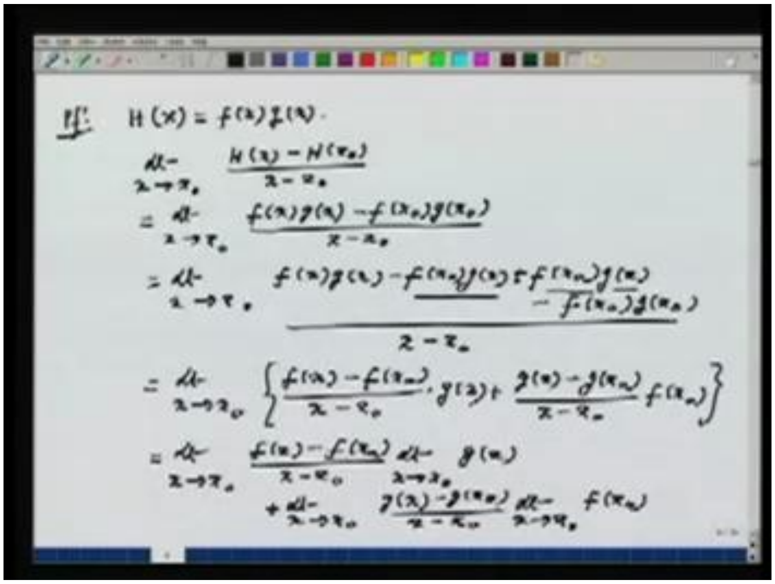


So end result is f prime at x naught because f is differentiable at x naught. So this is where I am using the fact that f is differentiable. Otherwise f may not exist. So this is the second quantity, 0 plus f of x naught. The net result is f of x naught. This is precisely what we wanted to prove. This implies f is continuous at x naught. So when are we on with this, we want to do some calculus of derivatives. That is the familiar rule because derivative after all is a operation on functions. So there are certain things which we will like to know and we write it as a theorem.

Assume that f and g are differentiable functions at x naught. Then number 1, f plus g: this is also differentiable at x naught and it is given by f prime at x naught plus g prime at x naught. Second is about the point wise product of functions. That is, f dot g prime at x naught, this is f prime x naught times g x naught plus f x naught times g prime x naught. Here, the meaning of the function f dot g is the point wise product. That is f dot g at x is defined to be f of x times g of x. The right hand side, certainly, makes sense because f x and g x are two real numbers. So I can multiply them.

The third property about the quotient, that if g is non-zero at x naught, then f by g prime at x naught is g x naught squared times f prime x naught into g x naught minus f x naught times g prime x naught. Well for this actually what I need is, it is the function g which I have written here. The precise statement should be, it is non-zero in the neighborhood of x naught. Now the first one, the number 1 is fairly simple that you just write down the definition of the derivative for the function f plus g and separate things. It follows easily. Only 2 requires bit of our attention. We will start with proving 2.

(Refer Slide Time: 15:18)

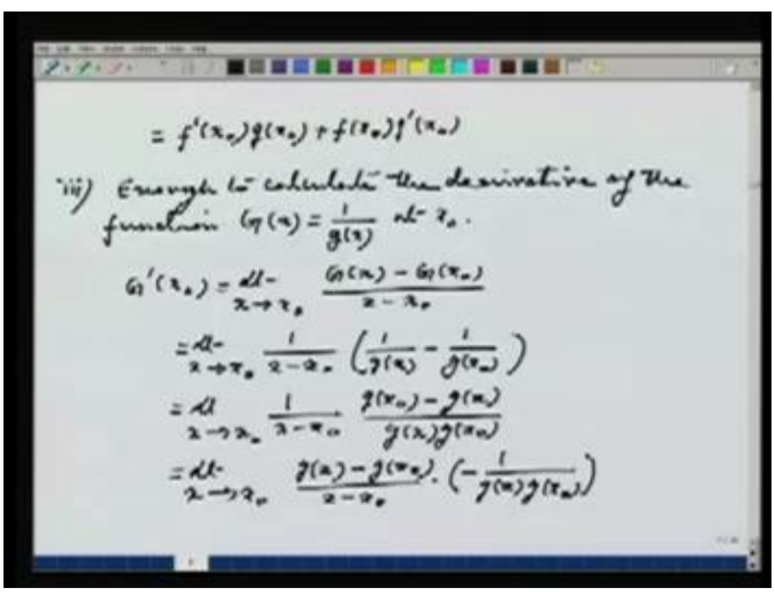


So the given function is, I call it H x, equals to f x into g x and then I want to look at limit x going to x naught H x minus H x naught divided by x minus x naught and then I write down the definition of H. That is, limit x going to x naught f x into g x minus f x naught g x naught divided by x minus x naught. Then what I do is, I write it as limit x going to x naught. I write it as f x g x minus f x naught g x plus f x naught g x minus f x naught g x naught and I divide the whole thing by x minus x naught, simple.

Notice that the middle terms cancel each other, that is, this term and this term. It has been arranged in such a fashion that they cancel each other. Now if I separate terms, what I get is this limit x going to x naught f x minus f x naught divided by x minus x naught into g x plus g x minus g x naught divided by x minus x naught into f x naught, just by separating terms.

Now notice the first factor, limit x going to x naught f x minus f x naught by x minus x naught, that is certainly f prime x naught but the added terms g x which we know, as x goes to x naught, we know that it goes to g x naught because g is continuous at x naught. So the end result is, if I separate the terms as limit x going to x naught f x minus f x naught divided by x minus x naught into limit x going to x naught g x plus limit x going to x naught g x minus g x naught divided by x minus x naught times limit x going to x naught f x naught.

(Refer Slide Time: 18:51)

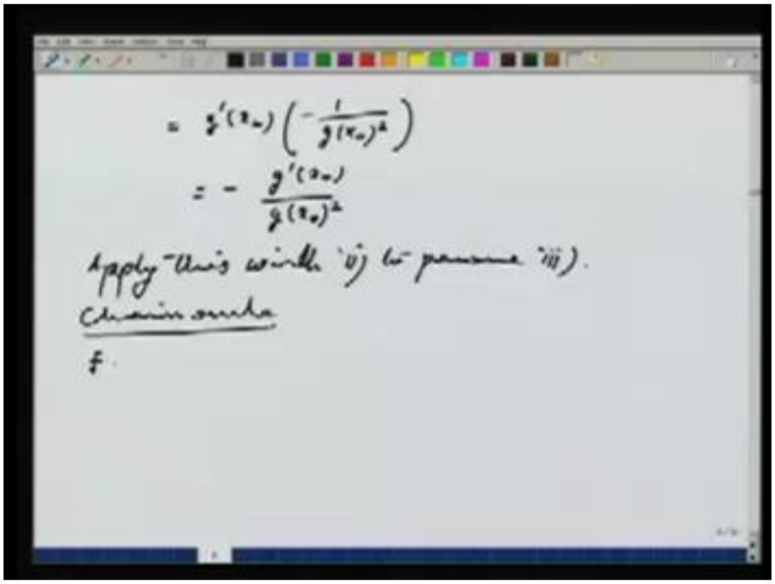


If I just put all the required values, what I will get is f prime x naught because f is differentiable at x naught times g x naught because g is continuous at x naught because it is differentiable at x naught plus f x naught times g prime x naught. So this is the formula of the product which we already know perhaps. Now for 3, what I observe is it is enough to calculate the derivative of the function capital G x equals to 1 by g x at x naught. Notice that my assumption on little g is that g is non-zero in a neighborhood of x naught. If I do not have this, the capital function G is not at all well defined.

Now once I find the derivative of capital G, then it is easy to see that by using 2, you can actually get 3. So let us first try to prove what is the derivative of G at x naught. So we again start with the definition.

G prime at x naught, by definition is limit x going to x naught G x minus G x naught divided by x minus x naught which then I write as, limit x going to x naught 1 by x minus x naught into 1 by g x minus into 1 by g of x naught, which then is limit x going to x naught 1 by x minus x naught into g x naught minus g x divided by g x into g x naught, which can now be written as, limit x going to x naught g x minus g x naught divided by x minus x naught into minus 1 g x into g x naught. Now if I take the limit, as I know since the little g is differentiable at x naught, it is continuous, so limit x going to x naught g x is actually g x naught. So limit x going to x naught 1 by g x is 1 by g x naught.

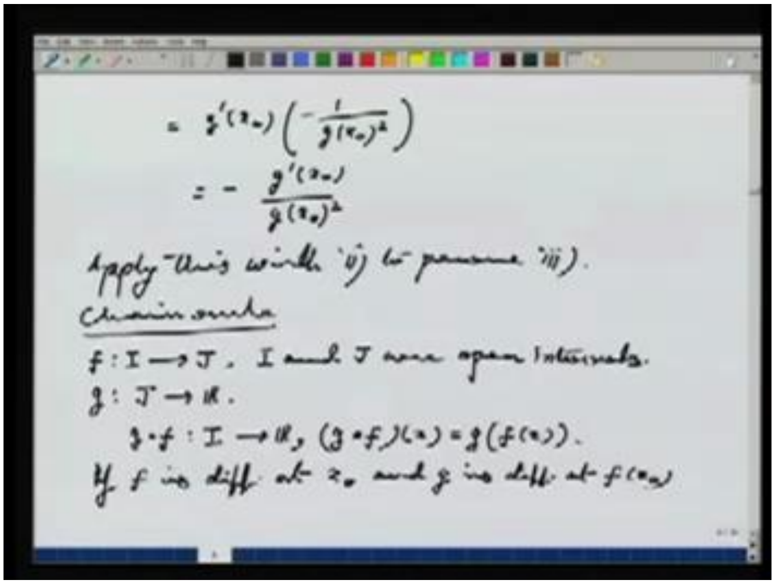
(Refer Slide Time: 22:05)



If I use this, what I get is that the limit is g prime x naught times minus 1 by g x naught squared, which we write as minus g prime x naught by g x naught squared. Apply this with 2 to prove 3. So these are the familiar rules of the derivative which we anyway know. Now I want to understand the geometry of the curve f through the derivative. See the whole point is that if I have the function f defined which is differentiable, it is not necessary that the function is given as a curve. If it was a curve, then certain of its property will be obvious which we want to derive from the differentiability of the function.

So the first thing of its kind is goes as follows: Now another fundamental property of the differentiable function, which we need for application, is the chain rule. So this is what I am going talk now. Chain rule is actually the differentiability of the compositions of functions. For example, we have a situation like this.

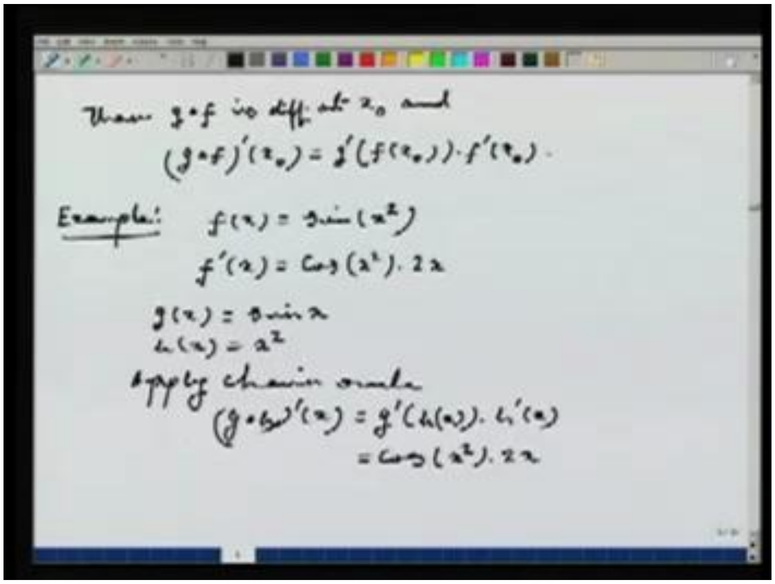
(Refer Slide Time: 23:38)



I have function f defined from I to, let us say, to another open interval J, where I and J are open intervals and let us say, our another function g defined from J to R. Then it makes sense to talk about the composite function g compose f. We write in this form, which is defined from I to R, the following way. The definition is g compose f at x is g of f x. See, it makes sense because f x lies in J where g is defined. So f x actually belongs to the domain of g. So I can apply g on that.

Now suppose the condition is this, that if f is differentiable at x naught and g is differentiable at f x naught, then the new function which we have formed g compose f, this is differentiable at x naught.

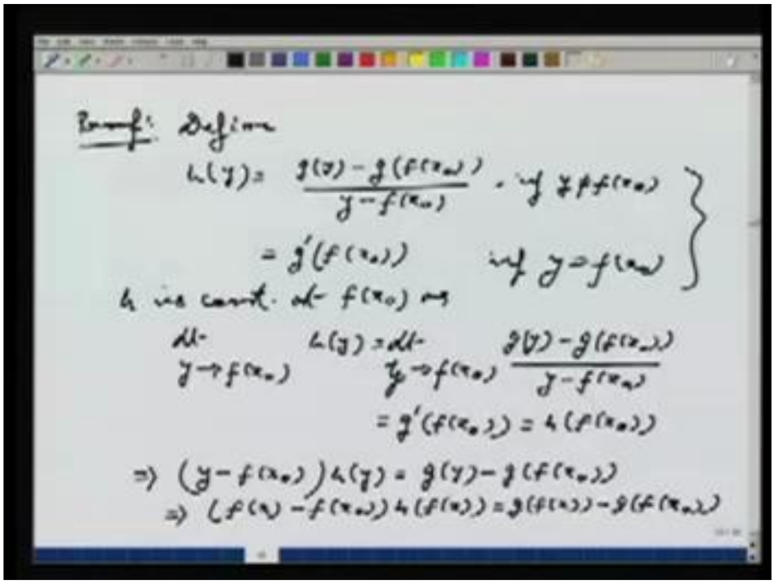
(Refer Slide Time: 25:16)



Not only that, we can calculate the derivative also and the derivative g compose f prime at x naught is g prime at f x naught times f prime at x naught. This is the familiar chain rule. Now let us look at an example first before proving the result. Suppose I look at this function f x equals to sine of x squared. Then many of you know it is true that f prime at x is actually cosine of x squared times twice x. How does it follow? It follows actually from the chain rule. In the following way, just take g x to be equal to sine x and h x to be equal to x squared.

Apply chain rule. How, because g compose h prime at x, this is g prime at f x, sorry at h x times h prime x. Now I know what is g prime. g x equals to sine x g prime is cosine x, which we will derive after some time. So it is cosine of h x is anyway x squared times h prime x is twice x. We get the formula. This is technical rule which we will be needing but it requires proof which we are going to supply right now. The proof goes as follows.

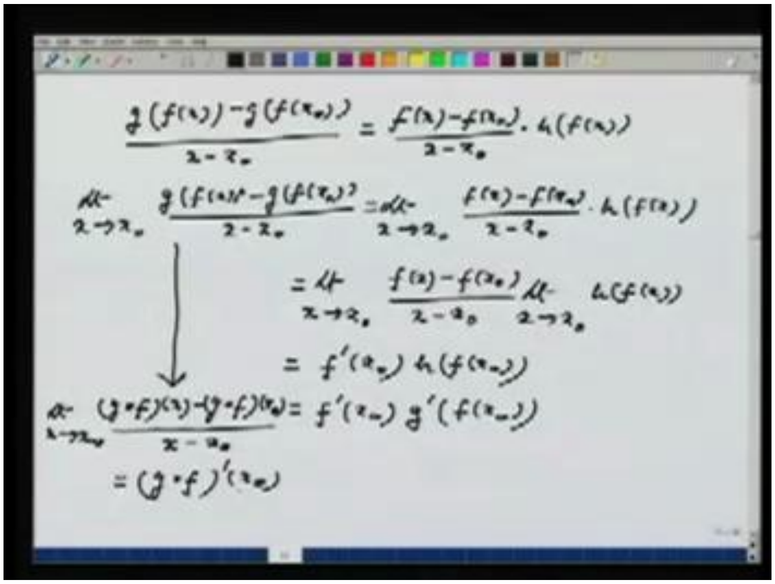
(Refer Slide Time: 27:44)



I first define a function define h of y to be equal to g of y minus g of f x naught by y minus f x naught, if y is not equal to f x naught. See if y equals to f x naught, we will have the problem of having 0 in the denominator, which we want to avoid. Then how do you define the function h, for all y? Suppose, y equals to f x naught, we define it as g prime at f x naught which we know, exists. So this defines the function h on the whole real lines. Then I say h is continuous at f x naught, as limit y going to f x naught h of y is same as limit y going to f x naught g of y minus g of f x naught divided by y minus f x naught.

This, by definition of, derivative of g at f x naught is g prime at f x naught which by our definition is h of f x naught. Now this then implies that y minus f x naught times h of y equals to g y minus g of f x naught. This implies then if I put y equals to f x for an arbitrary x, I get f x minus f x naught times h of f x equals to g of f x minus g of f x naught.

(Refer Slide Time: 31:02)

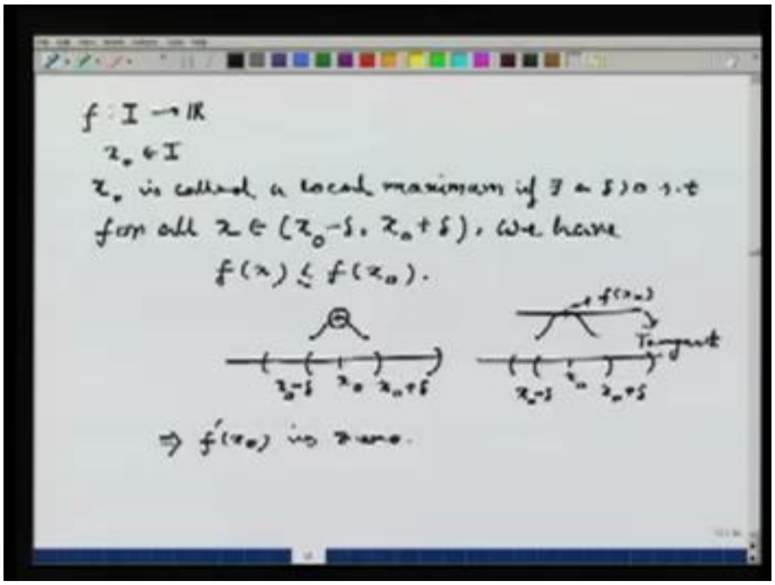


If I divide now both sides by x minus x naught, what I get is g of f x minus g of f x naught divided by x minus x naught equals to f x minus f x naught divided by x minus x naught times h of f x. Now I am in a perfect condition to take the limit as x goes to x naught. Let us see what happens. So limit x going to x naught g of f x minus g of f x naught divided by x minus x naught equals to limit x going to x naught f x minus f x naught divided by x minus x naught into h of f x, which again I can write as, limit x going to x naught f x minus f x naught divided by x minus x naught into limit x going to x naught h of f x.

Now the first quantity is nothing but f prime at x naught, by definition, as f is differentiable at x naught. What about the second quantity? Well, I have proved that h is a continuous function. So x goes to x naught since f is also a continuous function at x naught, I have that the second limit is actually h of f x naught and now I go back to the definition of h. What is h of f x naught? By my definition, it is g prime at f x naught. So what I get is f prime at x naught times g prime at f x naught.

Notice the left hand side. What is the left hand side? Well, the left hand side here, if you just apply the definition of composition, it is limit x going to x naught g compose f at x minus g compose f at x naught divided by x minus x naught, which is same as g compose f whole prime at x naught. This is precisely what we wanted to prove. This is called the chain rule of the derivative. Next, we turn towards certain properties which are related to the geometry of curves.

(Refer Slide Time: 34:27)



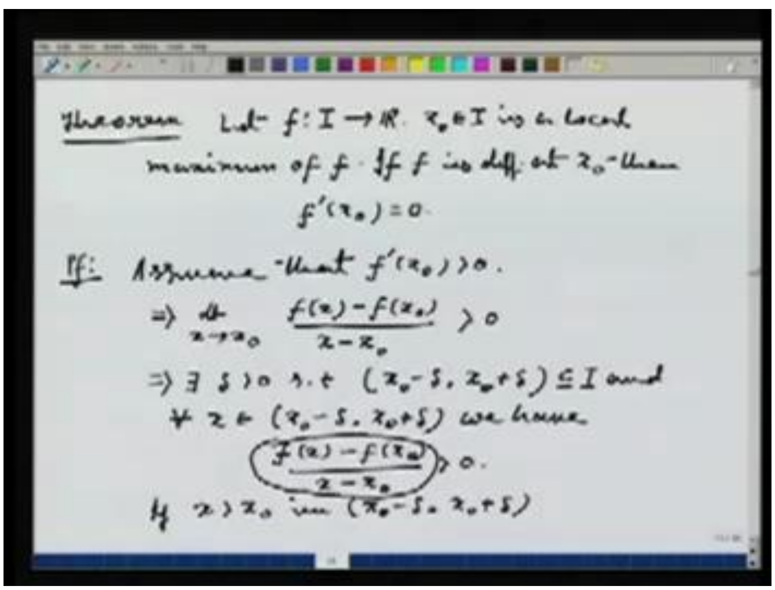
So first I define, if a function f from I to R is given, let us say, x naught is a point in I, then x naught is called a local maximum. If there exist a delta bigger than 0 such that for all x in x naught minus delta x naught plus delta, we have f x is less than or equal to f of x naught. That is, x naught is called a local maximum if there exist a region around x naught, such that, for all x in that region f x naught is the largest one. All the other values are less than f of x naught.

If I draw the graph of the function, if it is possible, let us say this is my I. Suppose, x naught is here then I have delta, such that, these two points are x naught minus delta and x naught plus delta. There, the graph of the function looks something like this. That is, x naught is the highest point. This is what local maximum means. Now suppose the picture is like this only. So again I will draw a replica of the picture. This is x naught and this x naught minus delta and x naught plus delta. The graph of the function, as I said looks like this.

If this is the case, when you see the point f of x naught, this is the point f of x naught. If I draw the tangent at this point, it will look exactly like this. That is, this is the tangent. We have that the tangent there is actually parallel to the x- axis. What does this mean in terms of derivative because the derivative f prime at x naught, if it exists, it tells you the slope of the tangent at the point f x naught? If it is parallel to the x-axis, that means the slope is 0. That is, this implies f of is 0.

Now to conclude this, do I really need to know how the graph of the function there looks like because the functions there might be so complicated that I cannot draw the graph of the function? Nevertheless, the conclusion, that makes the sense. We can examine whether it is true. So that is the next result. So the theorem is this.

(Refer Slide Time: 38:20)



Let f is from I to R, x naught belonging to I is a local maximum of f. That is given to me. That was the previous situation also. If f is differentiable at x naught, then f prime at x naught equals to 0. Once I prove this theorem, it will show at least one thing, that to draw the conclusion, that at local maximum the tangent is parallel to the x- axis, for that I need to draw the curve of the function. Without drawing the graph, I can conclude the same. That is the power of having the formal definition of derivative. Let us start with the proof of this. How do go about it?

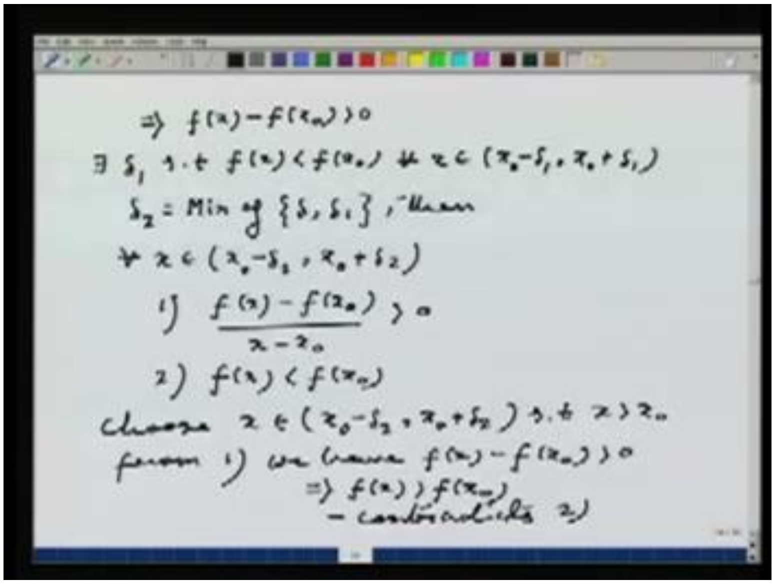
Well, I know that f prime at x naught exists. That is in the supposition. Then there are two things which can happen. Either f prime at x naught is positive or it is negative. If none of these things are true, the only possibility to remain is that f prime at x naught is 0. So what we try to do is, let us assume that f prime at x naught is positive and try to understand what it means.

Now assume that f prime at x naught strictly bigger than 0. This then implies by our definition that limit x going to x naught, f of x minus f of x naught divided by x minus x naught. This limit exists and is strictly bigger than 0. Now while dealing with continuity, we have noticed one thing which we are going to use now. That if I have the function whose limit at a point x naught is strictly bigger than 0, then there exist a neighborhood of that point x naught where the function is strictly bigger than 0. That is the observation I am going to use now.

This would imply then that there exist a delta bigger than 0, such that, this whole interval x naught minus delta, x naught plus delta is contained in I and for all x in x naught minus delta, x naught plus delta, we have f x minus f x naught divided by x minus x naught is strictly bigger than 0. Now suppose I choose x strictly bigger than x naught in x naught minus delta and x naught plus delta. That is, in this interval I choose a point x which is right hand side of x naught.

Then look at the quantity here. I say the denominator x minus x naught is positive and the whole quantity is positive. That means the numerator also has to be positive.

(Refer Slide Time: 43:11)

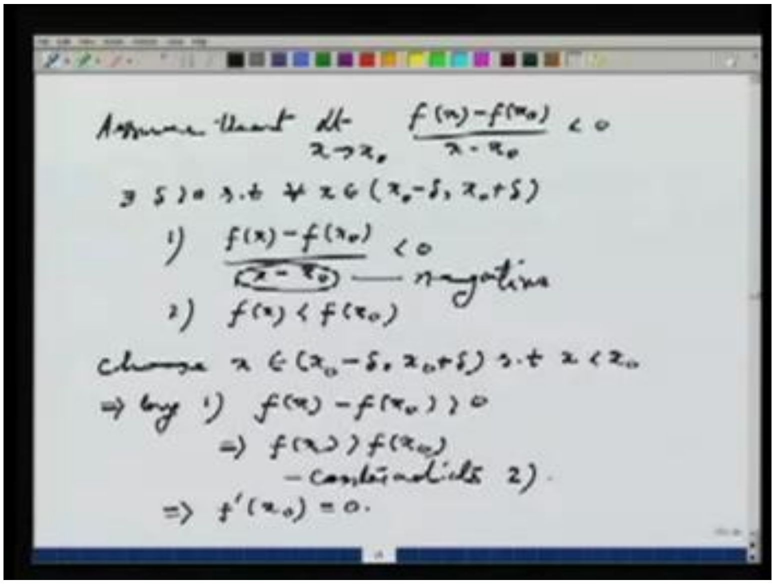


That is, it then implies f x minus f x naught is strictly bigger than 0. Now I could have chosen the delta to be the delta 1, which satisfies the local maximum property. To be precise, there exist delta 1, such that, f x is less than f x naught for all x in x naught minus delta 1 and x naught plus delta 1.

Let us choose delta 2 to be equal to minimum of delta and delta 1. Then I have that for all x in x naught minus delta 2, x naught plus delta 2, I have two conditions. Number 1 is f x minus f x naught divided by x minus x naught strictly bigger than 0. Number 2, f x is less than f x naught. Now choose x, as in the previous case, in x naught minus delta 2, x naught plus delta 2, such that, x is bigger than x naught. Then 1 tells me, from 1, we have f x minus f x naught is strictly bigger than 0, which implies f x is bigger than f x naught which certainly contradicts 2 because I know f x has to be less than f of x naught.

Why did this contradiction occur, because I have assumed that f x minus f x naught by x minus x naught, the limit is strictly bigger than 0. This cannot happen. Then the other possibility is f x minus f x naught by x minus x naught, the limit is less than 0. Let us examine that case.

(Refer Slide Time: 46:13)

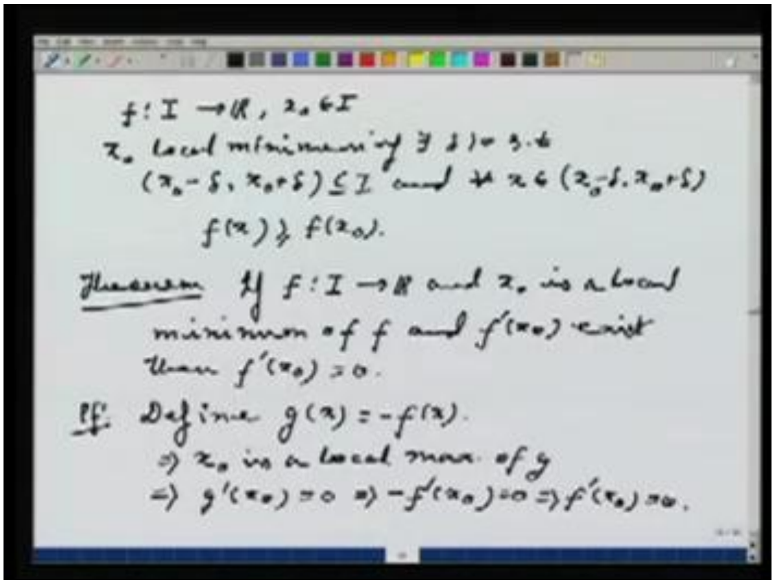


Now assume that limit x going to x naught f x minus f x naught divided by x minus x naught is less than 0. As in the previous case, I can say that there exist delta bigger than 0, such that, for all x in x naught minus delta x naught plus delta, I have two conditions. Number 1, f x minus f x naught divided by x minus x naught is less than 0 and f x is less than f x naught.

Now choose x in the set x naught minus delta x naught plus delta, such that, x is less than x naught. Then the denominator here is negative but the whole quantity is positive that means the numerator is positive. So this implies by 1 that f x minus f x naught is strictly bigger than 0. That is, f x is bigger than f x naught, which certainly again contradicts 2. That is, if I assume the derivative of f at x naught is bigger than 0, then there is a problem. If I assume that the derivative of the function f at x naught is less than 0, then also there is a problem. So the only possible way out is the conclusion which I want. That is, f prime at x naught is actually equal to 0.

Now the question is, now in this whole argument what is so holy about the local maximum. I could have defined the local minimum also analogously.

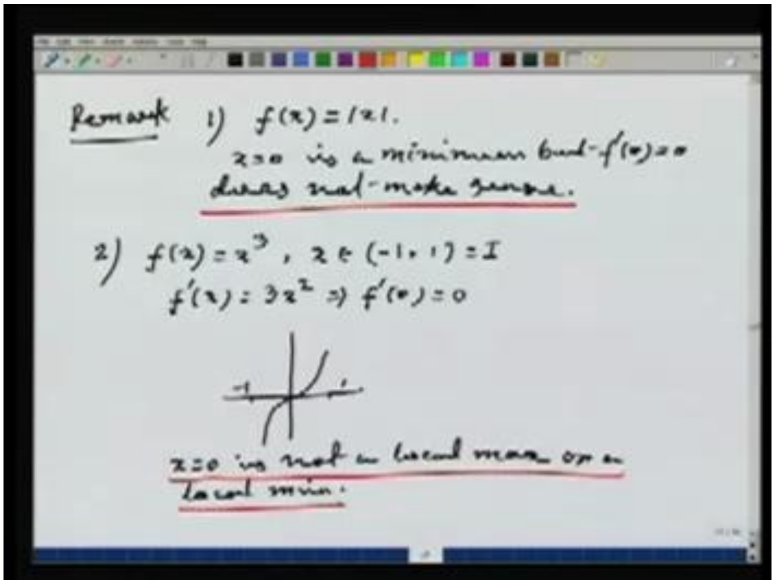
(Refer Slide Time: 49:05)



So I define, if f is from I to R, I is an open interval as always. x naught belongs to I, then x naught is called the local minimum if there exist a delta bigger than 0, such that, x naught minus delta x naught plus delta is contained in I and for all x in x naught minus delta x naught plus delta, we have f x to be bigger than or equal to f of x naught. That is, there exist a neighborhood of the point x naught, such that, the value f x naught is the smallest all the f x s. Then I would like to show this is also true.

If f is from I to R and x naught is a local minimum of f and f prime at x naught exists, then f prime at x naught is 0. Well, this proof follows very easily. Proof is just one line Define g x equals to minus f x. Notice that this implies that x naught is a local maximum of g. By the previous theorem, it would imply g prime at x naught equals to 0, which would imply that minus f prime at x naught is 0, which certainly implies that f prime at x naught is 0. We will like notice certain things here. The first is as follows.

(Refer Slide Time: 51:56)



I will put it as a remark. Look at the function, f x equals to mod x. Then 0 is the minimum. Then x equals to 0 is a minimum and hence a local minimum but f prime equals to 0 does not make sense. This is simply because the function is not differentiable there. That is, the condition that the f exists is very fundamental. The second one is the converse of the result is certainly not true. That is, if I have a function f whose derivative vanishes at a point, let us say, x naught, then that point has to be either a local maximum or a local minimum. That is not true.

I look at the function f x equals to x cube, where x belongs to the open interval minus 1 and 1 which I call I. Then notice f prime at x is 3 x squared, which implies, f prime at 0 is certainly 0. That is, the derivative of the function at the point 0 vanishes. If I look at the graph of the function, it looks like this. The point 0, x equals to 0 is not a local max or local min. That is, the converse of the result that, if f is differentiable at x naught and the derivative vanishes at the local maximum or the local minimum, the converse of this result is not true. From the zeros of the derivative, you cannot conclude that the point is the local extreme of the function but the observation that if f is differentiable at the local maximum or the local minimum, then the derivative is 0. This has profound applications.