Mathematics-I
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Lecture – 2
Sequences

Let us first recall what we have done in the first lecture. We have done essentially certain simple properties of the so called real line. So, what are the properties?

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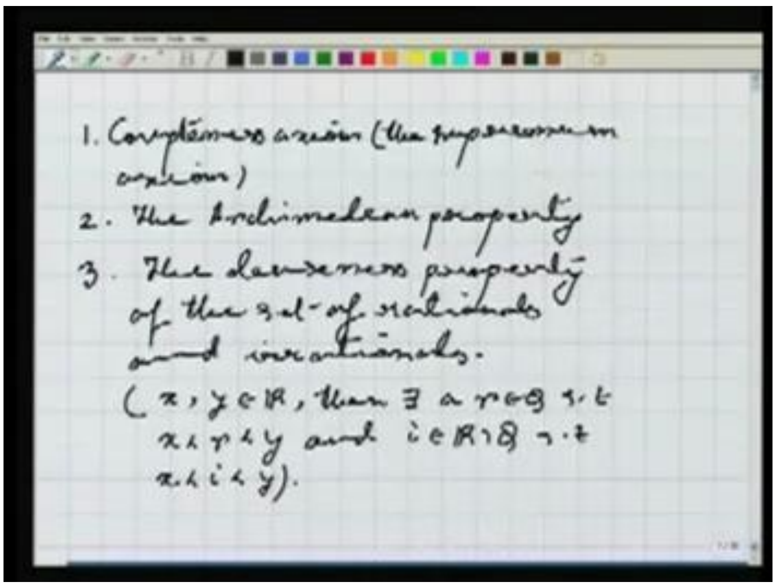

The first property was to consider the completeness axiom. We also called it the supremum axiom. Then we have studied one fundamental corollary of the supremum axiom or completeness axiom, whatever you want to say. That was the Archimedean property. Using this Archimedean property, we have proved that set of rational numbers as well as the set of irrational number is actually dense in the whole real line. So we will just write it as the denseness property of the set of rationals and irrationals and here the denseness means that if I have two real numbers x and y, then there exists a rational r such that x is strictly less than r strictly less than y and an irrational, such that x is strictly less than I strictly less than y. These are essentially the properties which we have studied in the last lecture.

Now, to continue further towards calculus we need a concept called the sequence. So, let us start with that. What is a sequence?

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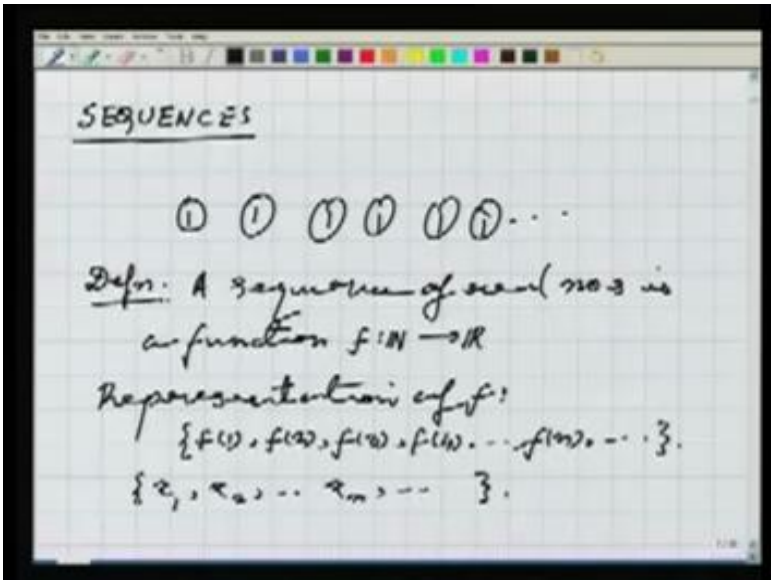

Just imagine the following thing, that suppose there is a class in which there are 10 students and then the teacher says the students should stand in a sequence, what does that really mean? It means there are essentially 10 spots, something like this and so on and one student chooses to stand here, the second one stands here, third one here, 4th one here, fifth one here, sixth one here and so on, just stand in a order so that people can count you as the first one, second one, third one, 4th one and so on.

Mathematically, sequence means exactly something like this. The only difference is, it does not stop; it goes on. So the formal definition, although it look pretty abstract but makes perfect good sense, a sequence of real numbers is a function. If it is a function, I should say tell you what is the domain and what is the co-domain. It is a function f, from the set of natural numbers to the set of reals. Then this function can be represented in the following way, I can just write it is as f 1, f 2, f 3, f 4 for an arbitrary natural number f n and so on and another symbol we use for this is, which is exactly analogous to this, we write it as x 1, x 2, x n and so on. What is the connection between this one and this one? The connection is x n is equal to f n. All these things will extremely clear once we look at examples. Let us turn towards examples.

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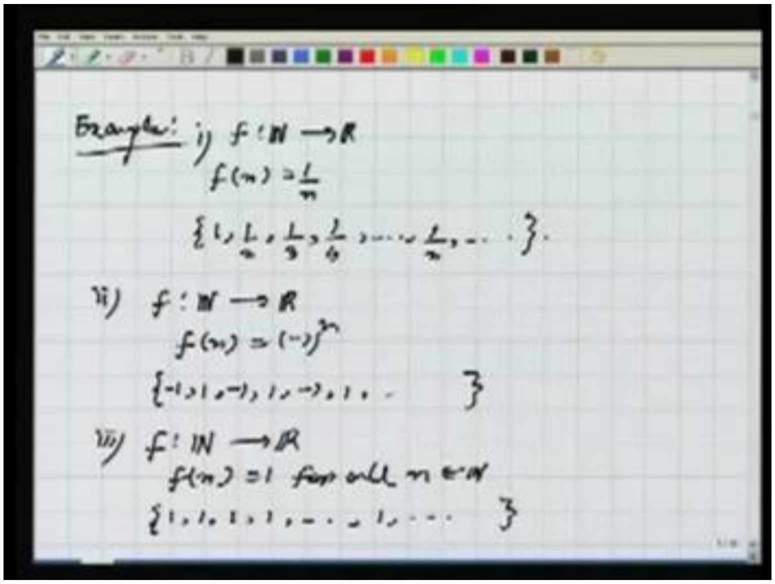

Let us define the following function f from the set of natural number to the set of reals, given by f of n, is equal to 1 by n. I am assuming that 0 is not a natural number. In that case, the another representation of this sequence should be 1, half, one third, one 4th, 1 n and so on. Let us say this is the first example. We will see some more. Let us look at the second example. I define f, again from the set of natural numbers to the set of reals, given by f of n is equal to minus 1 to the power n. Then you can that see the sequence actually means the alternative representation of the sequence is 1, minus 1. That is, at all the odd places, we have got minus 1 and at the even place, we have got 1. Similarly, I can get some more.

If I look at this function, given just by f n equals to 1 for all n, then what does the sequence look like? You know all the terms are same; that is what it means. That is, the alternative representation is 1,1,1 at the nth place I have got 1 and it continues. So one important thing is, since it is a function defined on the set of natural numbers, this representation just does not stop at any stage. Whatever stage you take, there is a number sitting there, some real number. That means, it goes on but that at the same time, does not mean that the set is really an infinite set. For example, if I look at example number 3, then as I said, it is just the singleton set 1 but as a sequence, it is having an infinite representation.

Now, what is the most relevant question about the sequences which we learn or which we like to learn? Well, let me just go by the intuition. I look at the sequence 1 by n.

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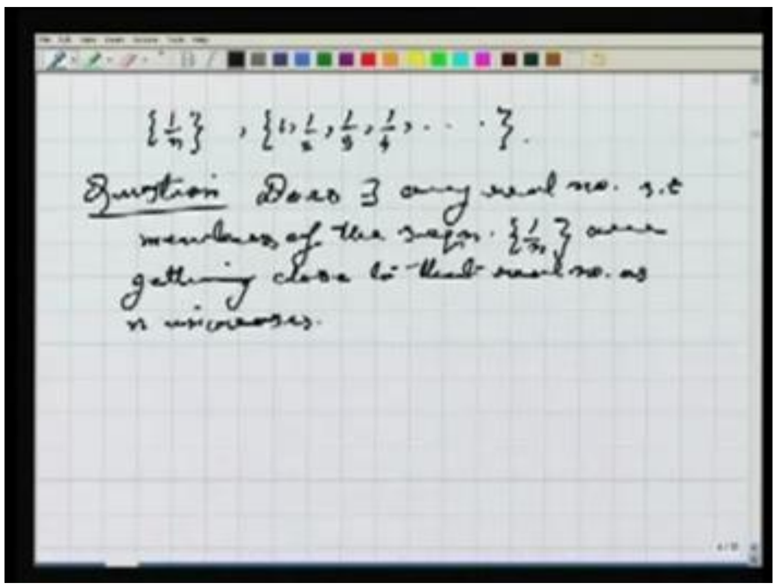

That essentially means I am looking at the sequence, one, half, one third, one 4th and so on. I just want to know, does there exist a real number such that as I go on increasing my n, the member of sequence are getting very close to the real number? Intuitively, the question we are asking is this: does there exist any real number such that members of the sequence are getting close to that real number as n increases?

If you just look at the sequence, what you see is the numerator is getting increasing as n increases. That means the whole number 1 divided by that fellow is getting smaller and smaller. So intuitively it is very clear that it is actually getting very close to 0 but every sequence only so simple to understand. This is a notion we are going to elaborate on. That is called convergence of sequences and we are bothered only about those sequences which converge.

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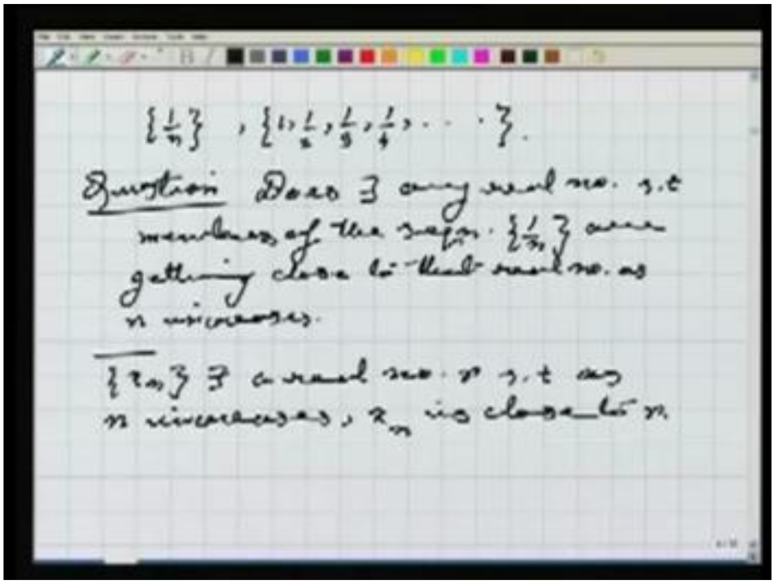

Once again intuitively convergence means what? It means for the given sequence x n there exist a real number r, such that as n increases x n is close to r. This is an intuitive definition of convergence but this is certainly, not a definition with which mathematics can be done. We have to make it quite rigorous. That is the next thing I am going to come to.

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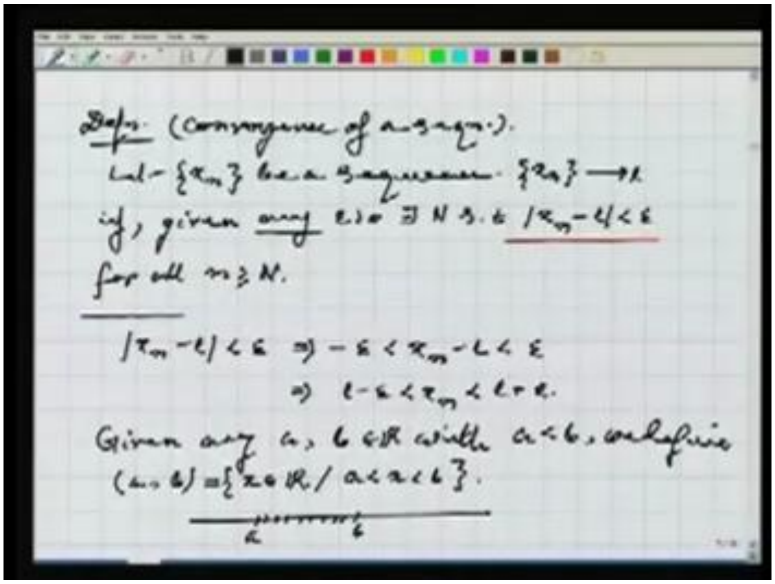

So we are going to define convergence of a sequence. So let x n be a sequence. We say that x n converges to a real number l, in symbol it means x n arrow l. If the following things happen, given any epsilon bigger than 0, there exist a natural number n such that modulus of x n minus l is less epsilon for all n bigger than or equal to N. It sounds pretty technical. We need to understand what it exactly means. Let us start thinking first about the inequality which I have here, what does this exactly mean? It means the following thing: that it implies that minus epsilon is less than x n minus l, which is less than epsilon.

This, if I rewrite, will imply that l minus epsilon is less x n less l plus epsilon. Given any real numbers a, b with a strictly less than b, we define a, b which is called the open interval a, b. This is a set. This is a set of all real numbers x in R, such that a is strictly less x strictly less b. If you think about the real line, suppose this is a, suppose this is b, then open interval means all the real numbers sitting here but they are not the end points which appear here.

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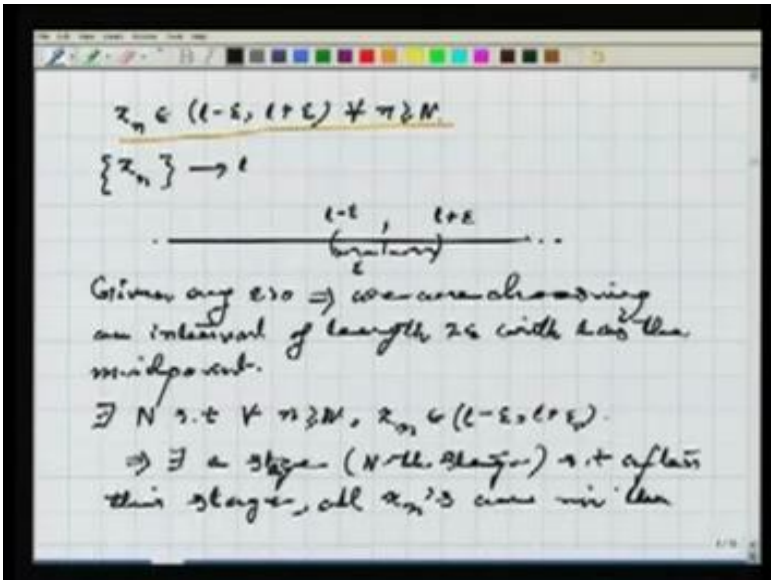

Now with this in mind, we can say that convergence implies that x n belongs to l minus epsilon l plus epsilon for all n bigger than or equals to capital N. Now let us look at the picture geometrically. Suppose x n is a sequence converging to l. I want to see the meaning of it geometrically, what exactly it means. Let us try to draw the real line. This is the real line. It does not end and suppose l is somewhere here. Then the definition says that given any epsilon, what does this phrase mean, given any epsilon bigger than 0? If you look at this expression, it essentially tells you that you are measuring how far you want to go away from l. That means, I choose the epsilon distance here and here. Then this point is certainly l minus epsilon and this point is l plus epsilon. So given any epsilon bigger than 0, it actually implies that we are choosing an interval of length 2 epsilon with l as the midpoint.

Then, let us examine the next part of the definition. What does this say? That means given epsilon bigger than 0, that is, I have chosen an interval around l, then what happens? The definition says there exists a capital N, such that for all n bigger than or equals to capital N, x n is lying in that interval. It just means, I would say in English, it means, there exists a stage, in this case it is the Nth stage, such that after this stage all x n are in the interval.

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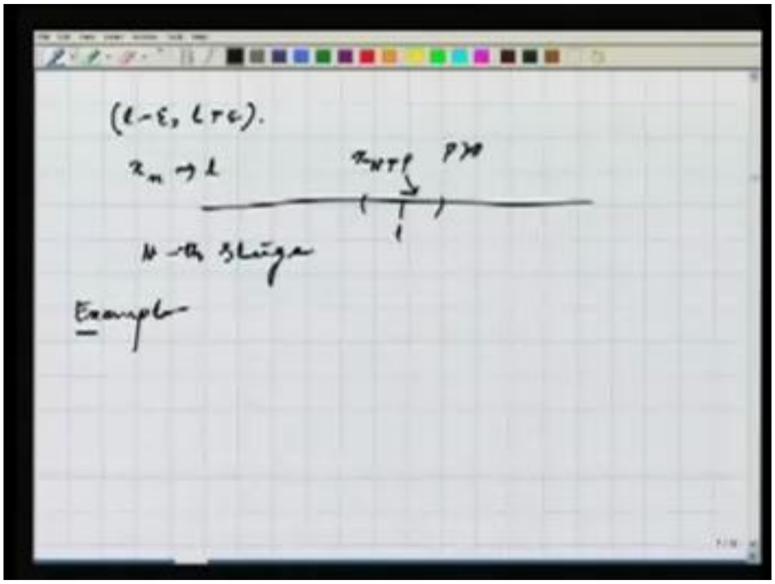

Geometrically, it means just this: that x n converges to l. Think about the real line first. This is your l. Just choose any arbitrary interval around l. You remember, the definition says for any epsilon bigger than 0 and that gives me the arbitrariness of the length of the interval. Just go on choosing any interval around l, whatever maybe its width. Then, there exists a Nth stage, such that all members of the form x n plus P, P bigger than 0, they are actually lying here. This is simply what it means by convergence of sequences.

First, let us see applying this definition, I can see the examples of the sequences which I have given, they are convergent or not. So let us come to the example first. Consider the sequence x n equals to 1 by n. What happens to this sequence? As you have noticed intuitively, it says that 1 by n is getting close to 0 and if I have translated my intuition into mathematics, it still should happen that the sequence is converging to 0.

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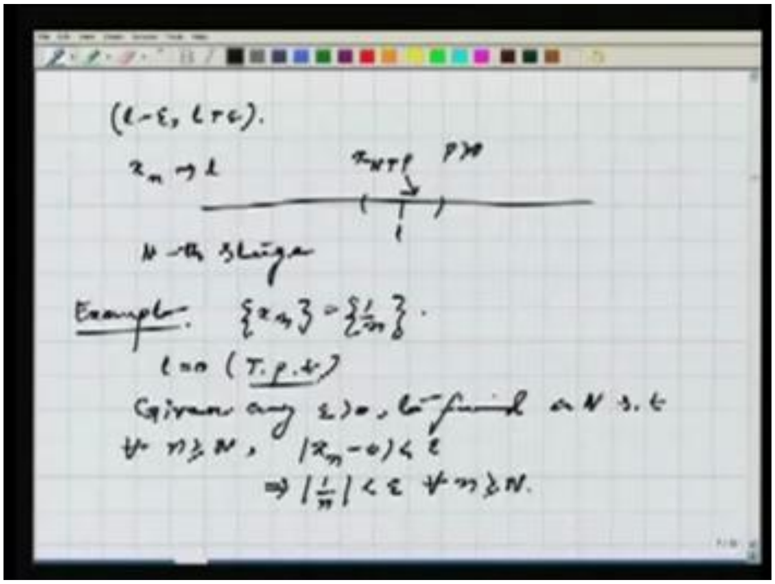

Let us check whether it is true or false. Well, l is 0. That is what I want to prove, to that l is 0. So, what do I have to do? Given any epsilon bigger than 0, I have to find a capital N such that for all n bigger than or equals to capital N modulus of x n minus 0 should be less than epsilon. This implies, if I just put the values, mod of 1 by n should be less than epsilon for all n bigger than or equals to capital N. I have to find some such n. Here we are going to use the Archimedean property.

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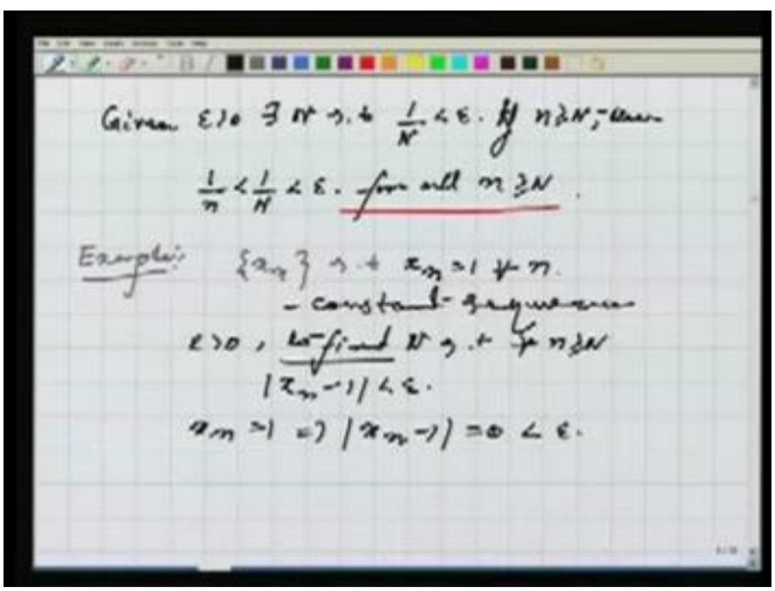

We have noticed in the last lecture that given epsilon bigger than 0, there exists N such that 1 by capital N is less than epsilon and then, if n is bigger than or equal to capital N, then I of course have that 1 by n is less than 1 by capital N which is less than epsilon. It just shows that given epsilon, I could find a capital N. That is the stage, so that after that stage, all the terms of the sequence is less than epsilon because this is true for all n bigger than or equal to N. Let us go to another example.

Let us look at this sequence x n such that x n equals to 1 for all n. This is called a constant sequence because of the obvious reason. Intuitively, we feel that this sequence converges to the number 1 because all the numbers of the sequence are 1. So, what is the real number? It should be close to it. It should be 1 and let us check from the definition of convergence whether we can verify that 1 is really the limit of the sequence. The verification is very easy. You start with any epsilon bigger than 0. To find a stage N such that for all n bigger than or equal to capital N, mod of x n minus 1 is less than epsilon. This is what I am going to show.

In this case, it is obvious because all the x ns are 1. This implies x n minus 1 is anyway 0, so obviously less than epsilon. That means, from the very first stage onwards this is happening. You do not really to struggle to find out the stage. Well, this was an easy example. Let us go to another example.

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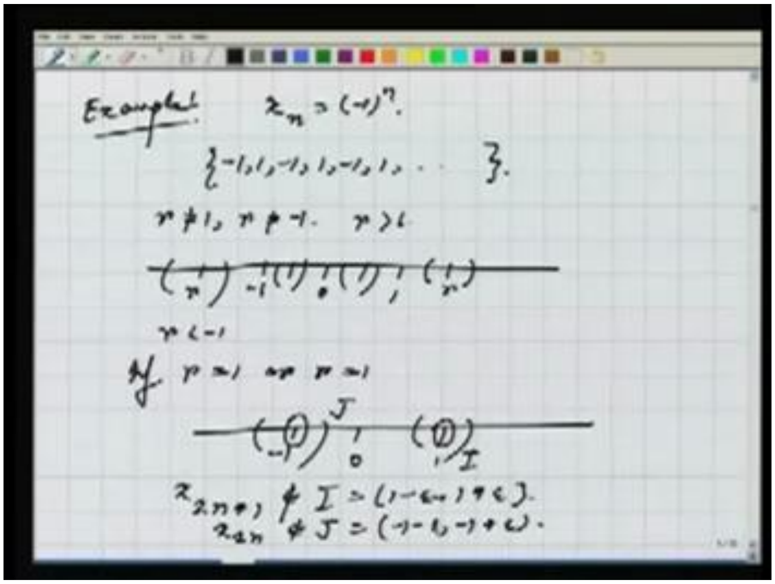

Consider this sequence. Can we see whether there is any real number so that all the terms of the sequence are getting very close to that real number? Write down the sequence. What does it look like? It looks like minus 1,1, minus 1, 1, minus 1 ,1 and so on. You feel like it is not getting close to anything. It is just oscillating between minus 1 and 1. You know nothing is happening. Now which real number it can at all getting close to? Let us say, I take a real number r which is not equal to 1 and it is not equal to minus 1.

For simplification, let us assume r is strictly bigger than 1. Just think it this way. This is the real line. I have minus 1 here. This is 0. This is 1 and let us say this is r, since it is strictly bigger than 1. For epsilon, I am allowed to choose any interval of arbitrary width, that is in my hand. I choose an interval like this. You see, this interval does not contain any term of the sequence, no stage. Well, r might be here also. That is, r is less than minus 1. I choose another interval around r and you see neither minus 1 nor 1 is there. If r is here, we choose an interval like this. If r is here, we choose an interval like this. So the problem remains. If r equals to 1 or r equals to minus 1, what happens then?

Again I draw the picture. This is r now. What I do is, I choose an interval. I see, of course 1 is there but minus 1 is not there, means what? x 2 n plus 1 does not belongs to if I call this interval I, which is 1 minus epsilon 1 plus epsilon. Hence it cannot certainly happen that after some stage, all the terms of the sequence are there because I can anyway see that odd terms are not there. The same thing gives the contradiction in this case also. That means, if I choose this and I look at interval around this, then if I call this J, x 2 n does not belongs to J. That is, minus 1 minus epsilon minus 1 plus epsilon. That means there does not exist any real number where this sequence can converge.

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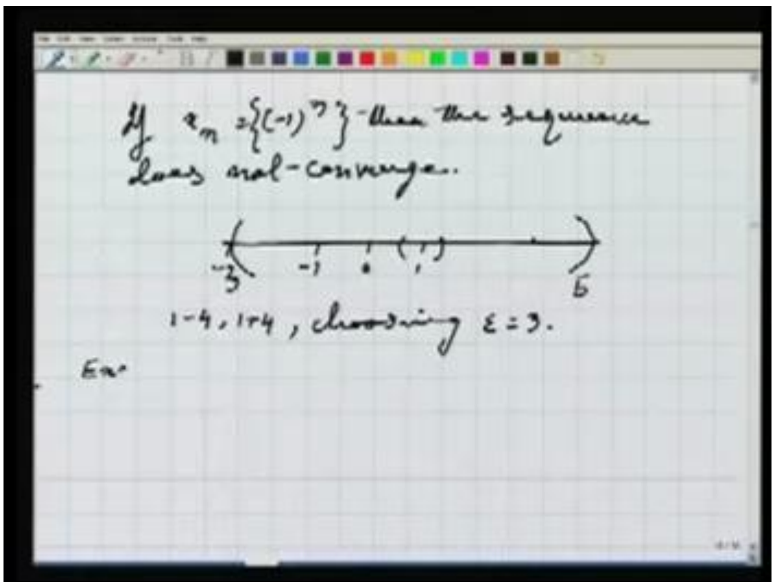

So the conclusion is that if x n is equal to minus 1 to the power n, then the sequence does not converge. That means not every sequence converges and that is the point of the definition, that there exist sequences which may not converge also and hence, somehow if we want to study only the convergent sequences, I must have some idea just looking at the sequences which converge. That means we should understand the convergent sequence in much more better fashion.

To understand the notion of convergence in a better fashion, I again want to look back at the sequence minus 1 to the power n and let me draw the following picture that this is the real line, this is 0, this is minus 1, this is 1 and I said this sequence neither converges to 1 nor converges to minus 1. What I do is, I look at 4. This is 4 and let us say this is minus

2. What I do is, I look at this interval. That is, I look at the interval 1 minus 4. Sorry, in this case, it is minus 3 and 1 plus 4 and this case, it is 5.

Suppose this is the situation. That means, choosing epsilon equals to 3 and then I see that all the terms of the sequence are actually in this interval. Well, is not that what the definition of convergence demands: that after some stage all the terms of the sequence should be in the interval? Well, I say no because if we say yes, then we are forgetting that any epsilon bigger than 0 in the definition. It says that whatever small interval around, in this case, it should be around the limit you choose so that all terms of the sequence should be there.

That means, if you find an interval around real number where all the terms of the sequence are there, that does not really mean the sequence is convergent. What you really need to do is, take any arbitrary small interval around the point which you think is going to be the limit of the sequence. Then after some stage all the terms of the sequence should be there and that we have seen is not happening if we choose small interval like this. That means, to understand where the sequence exactly is converging, we need to look at intervals of all possible length taking that point as the limit point which is certainly going to be a complicated business.

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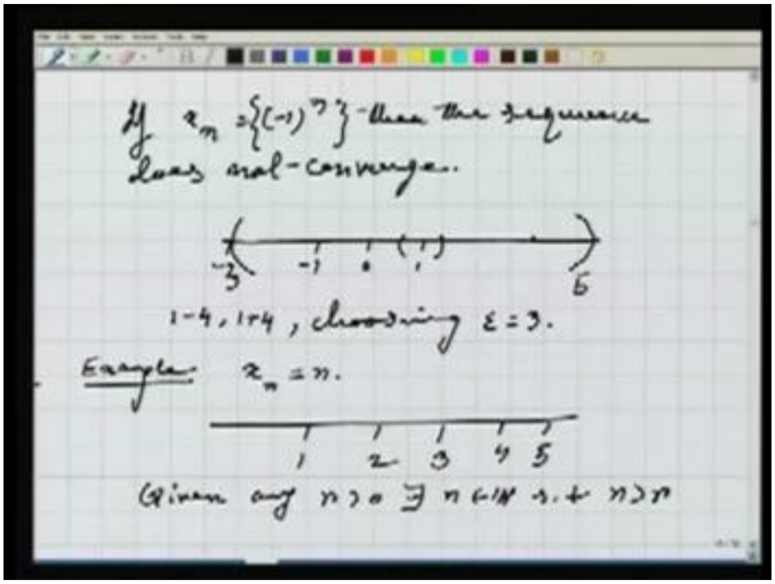

Let us look at another example. Consider the sequence x n equals to n. Does this sequence converge? Well, if it has to converge to some real number r, the whole picture of real line looks like this; goes so on. Wherever r is, given any real number r, there exists a natural number n such that n is strictly bigger than r. Well, why is this true? This actually again follows from Archimedean property but what does it mean?

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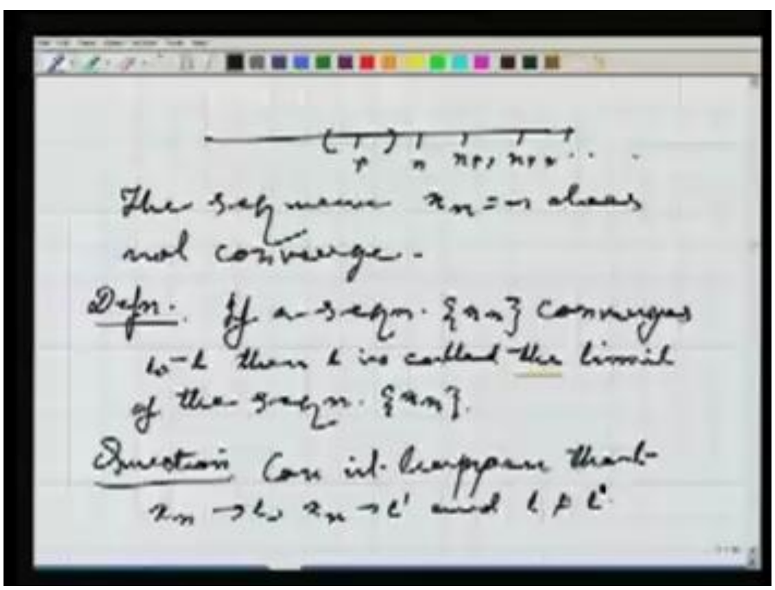

This is where my r is and this is where n is. Now if I choose an interval like this then all these terms of the sequence are outside the interval. That means r cannot be a real number where the sequence is convergent, but I have taken r to be an arbitrary real number, just any r bigger than 0, you see. That means the sequence x n is equal to n does not converge. Let us make some more definitions. If a sequence x n converges to l, then l is called the limit of the sequence.

Here what is interesting is, I would say the limit. Do I really know that there exists only 1 real number where a sequence can converge? So I will ask this as a question again. Let us ask the following question again. Can it happen that x n converges to l, x n converge to l prime and l is not equal to l prime? Because if the answer is no, then justified is that if the limit exists, it is unique.

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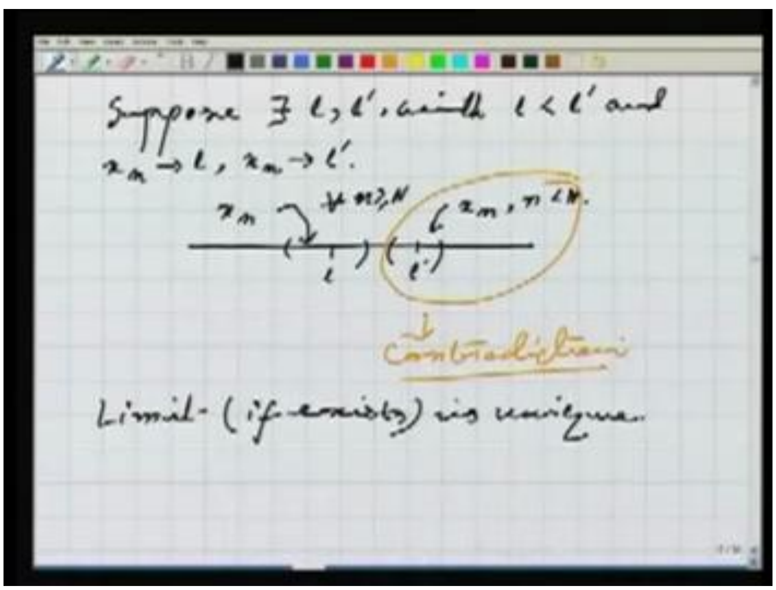

Let us see why is this true. Suppose there exists l, l prime with let us say l is less than l prime and x n converges to l, x n converges to l prime. What I will try to do is, I will try to get a contradiction out of it, you know. Again let us draw picture. This is where my l is. This is where my l prime is. What I do is, I take an interval around l. Then I know x n is here for all n bigger than or equals to N.

After some stage all the terms of the sequence are in this interval. Then, who are the members who can be here? There are only finitely many guys, that is x n such that n strictly less than N. So I got an interval around l prime where there are only finitely x ns belong but the definition of convergence says that whatever interval around the number which you think is going to be the limit you take, then after some stage all the elements of the sequence should be there but that is not happening with l prime. But I said x n converges to l prime and I say this is a contradiction. That is, if the sequence x n has a limit, then the limit is unique.

All this is very simple and just followed from the definition of convergence of sequence but now I will to know more about convergent sequences. That means, can I have enough examples of sequences which converge? For that, what we try to do is try to explain arbitrary sequences, if possible, in terms of known sequences, which I know has got a limit. To do that we need something called the Sandwich theorem. Let us come to that.

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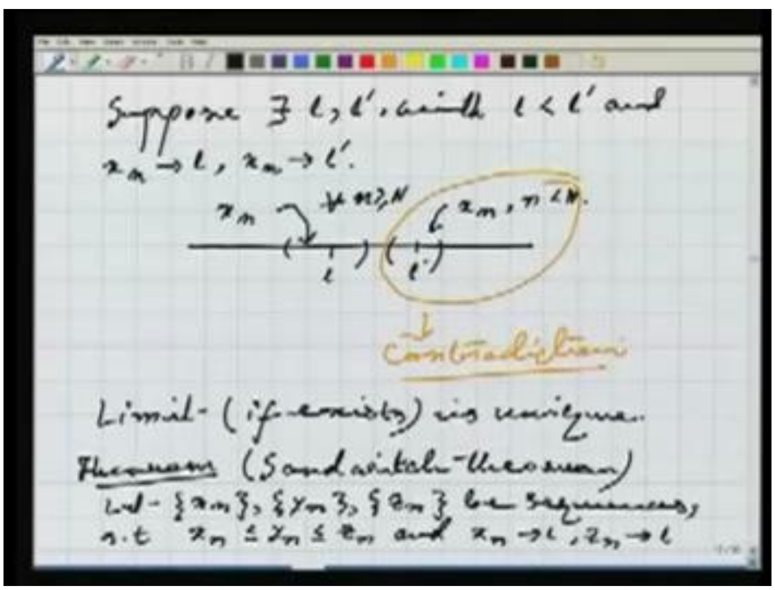

Once I tell you the theorem, using this theorem you can prove that there are many sequences which can actually converge. Let x n, y n and z n be three sequences such that x n is lesser equal to y n and y n is lesser equal to z n. You know, that means the sequence y n is actually sandwiched between the two sequences x n and z n. That is the name of the theorem. Also, suppose x n converges to a number l and z n converges to the same number l, then the theorem says y n also converges to l.

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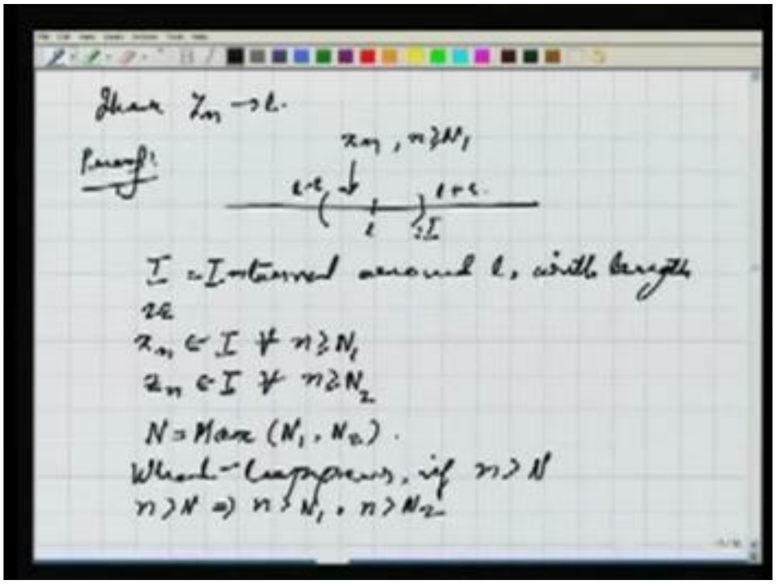

So if I have got two sequences x n and z n and y n is another sequence which is sandwiched between x n and z n and assume that the left hand sequence, that is x n and the right of the sequence, that is z n, they have got the same limit l, then intermediate sequence y n which was sandwiched, this fellow also has the same limit l. That is the statement of the theorem.

Then the proof of the theorem is very simple. It just follows from the picture. Without going into detailed proof, I just draw this picture. This is real line. This is l. I choose an interval of length 2 epsilon around l, so I is the interval. This is an interval around l with width, I would say length, 2 epsilon. That means this l minus epsilon; this is l plus epsilon. Now let us just use the fact that x n converges to l, means what? It means there is stage, let us say N 1, so that x n belongs to this interval with N bigger after the stage N 1. That is, x n belongs to I for all n bigger than or equals to N 1.

So all x n s are here; n bigger than or equals to N 1. What happens to z n? It is given that x n also converges to l. That means z n also belongs to I for all n bigger than or equal to some stage. It is not necessary these two sequences must have the same stage. I would say N 2. That is the stage for the sequence z n so that after this stage z n belongs to I. Now if I define N to be equal to maximum of N 1 and N 2, what happens if n is bigger than capital N? Well, n bigger than capital N implies n certainly bigger than capital N 1, n certainly bigger than capital N 2.

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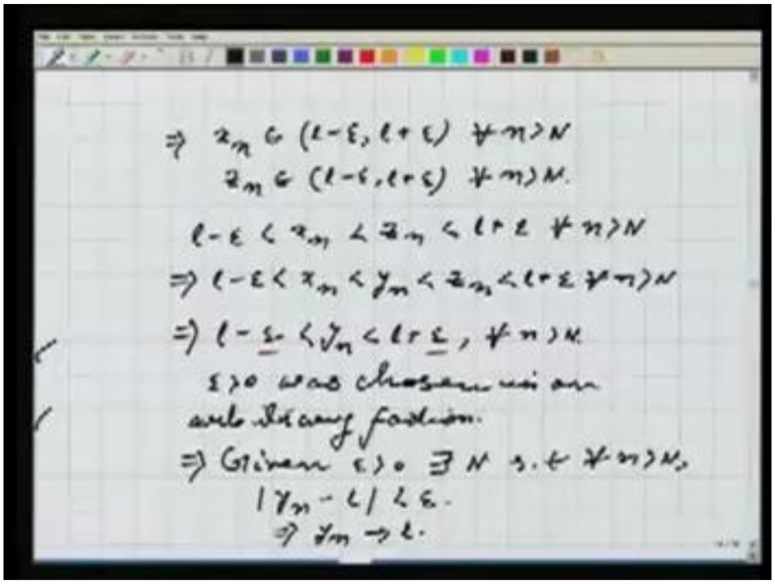

This implies x n belongs to l minus epsilon, l plus epsilon and z n, that also belongs to l minus epsilon, l plus epsilon. Now the sandwich is going to play the role. What I have is, l minus epsilon less x n but x n is any way less than z n and that is less than l plus epsilon and this is true for all n strictly bigger than capital N. Well, y n lies in between x n and z n. If I use the fact, this implies then l minus epsilon less x n less y n less z n which is less than l plus epsilon, for all n bigger than N.

Now let us just forget this x n and z n. What do I get if I concentrate just on y n? Well, I get l minus epsilon is less y n. This is less than l plus epsilon for all n strictly bigger than N. I know that epsilon bigger than 0 was chosen in an arbitrary fashion. That is, what I want to say is, whatever epsilon you choose, it does not really matter what epsilon you choose. Whatever epsilon you choose, so far I have gone through actually works, you know. That means I have proved that given epsilon bigger than 0, there exists capital N such that for all n strictly bigger than capital N, modulus of y n minus l is less epsilon. This is precisely the notion of convergence. So we could prove that y n also converges and converges to the same limit as x n and z n.

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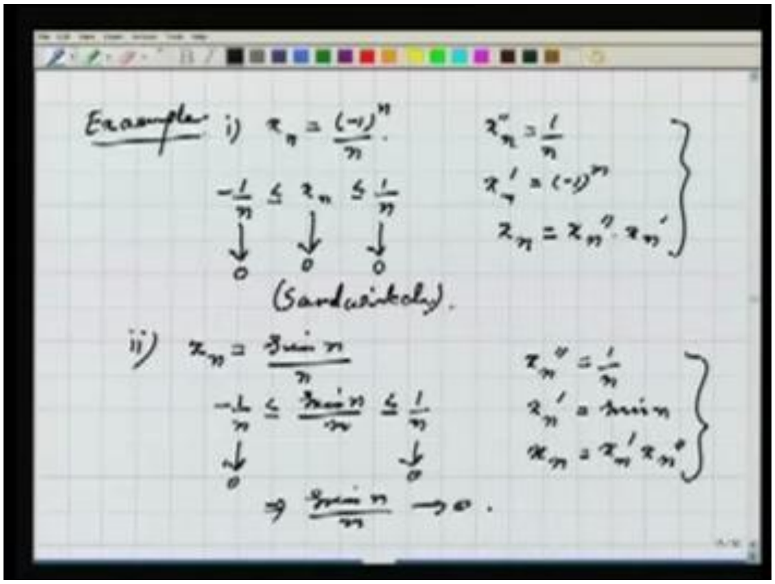

Let us look at certain applications of this result. Let us consider the sequence x n equals to minus 1 to the power n by n. Notice that minus of 1 by n is less than or equal to x n and this is less than or equals to 1 by n and I any way know that this converges to 0. That is the first thing I proved. Although I did not really prove but it is easy to prove this also converges to 0. If you apply the sandwich theorem it implies this converges to 0. This is sandwiched. Let us look at the other examples.

I take x n is equal to sine of n by n, you know. If you just look at this sequence it might seem complicated for you to guess the limit of the sequence. That is really simple and that follows from sandwich theorem. You can again prove minus of 1 by n is less than or equal to sine n by n is less than or equals to 1 by n because sin is the bounded function and it lies between minus 1 and 1. Again, the previous logic tells you this converges to 0 and this converges to 0, together gives you the fact that sine n by n converges to 0. Well, I see certain amount of similarity in this example 1 and example 2. In both the sides, I got two sequences which converges to 0 and the in between fellow converges to 0.

Notice that the first sequence was manufactured by the following way. You take the sequence x n equals to 1 by n. Take another sequence x n prime equals to minus 1 to the power n and then call this double prime. Then the sequence x n was defined as x n double prime into x n prime. In this example, if I take again x n double prime equals to 1 by n and x n prime equals to sine n, then x n was nothing but x n prime into x n double prime. So certain similar kind of constructions has taken place. Now what I would like to do is, try to generalize this. Well, which way? Let us see. First, I define something.

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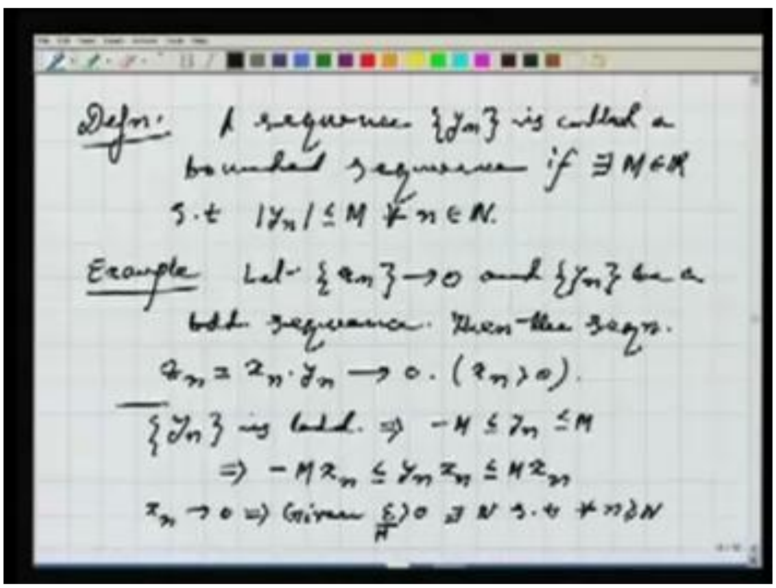

A sequence y n is called a bounded sequence if there exists a real number capital M such that mod of y n is lesser equals to M for all n. It essentially means that the function f which defines the sequence is actually a bounded function on the set of natural numbers and then I want to prove, you will see, that this example actually is the generalization of the previous two examples which I have given. The example is this.

Let x n be a sequence which converges to 0 and y n be a bounded sequence. Then the sequence z n which is defined as product of x n and y n; this converges to 0. This is a very simple application of sandwich theorem. I would like to do exactly what I have done in the previous two examples, you know. I notice that the y n is bounded, implies minus M less or equal to y n less or equals to M. Well, for the sake of simplicity let me also assume that x n is bigger than 0.

Now once I have this, this implies minus M into x n is less than or equal to y n times x n. It is less than or equal to M into x n. Now I know that x n converges to 0. Now x n converges to 0 implies, given epsilon bigger than 0, well, instead of epsilon I take epsilon by M, there exist n such that for all n bigger than or equals to capital N mod of x n is less than (E by n).

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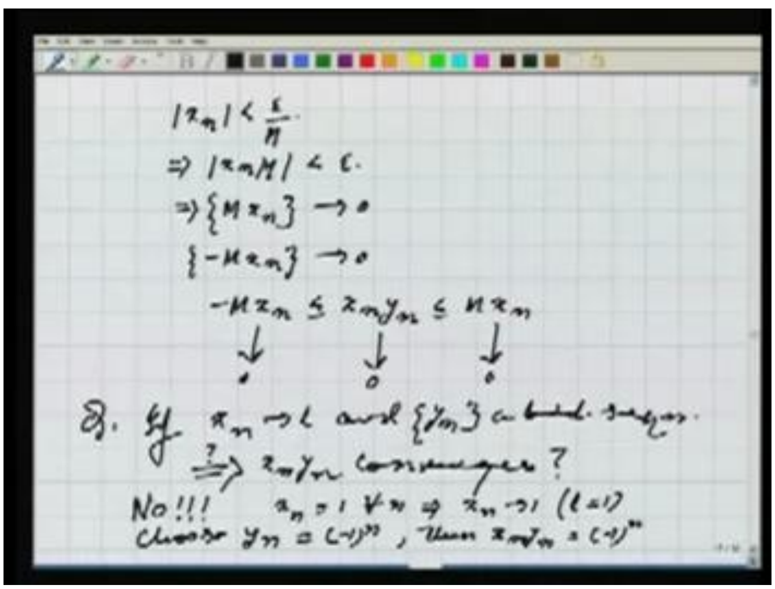

This implies mod of x n times M; this is less than epsilon. This implies the sequence M x n also converges to 0. By the similar reasoning, we can show that the sequence minus M x n also converges to 0. Let us come back to the situation, minus M x n less than or equal to x n y n less than or equal to M x n. I know that this converges to 0. I know that this converges to 0. By sandwich theorem then this converges to 0. Now using this result, what you can do is, you can construct many more sequences out of just 1 sequence, so that it converges to 0. You just choose y n to be a bounded sequence and go on multiplying it with x n. You get a sequence which converges to 0.

The next thing we want to ask is, what is so special about the 0. Isn't it true, if the sequence x n converges to l and y n, a bounded sequence, does this imply x n y converges? The answer is no. 0 is very special and we have actually seen the answer. Let us take x n to be equal to 1 for all n. There is the constant sequence 1. This implies x n converges to 1. So it is the case, l equals to 1 and I choose y n equals to minus 1 to the power n. Then x n into y n is minus 1 to the power n and this does not converge.

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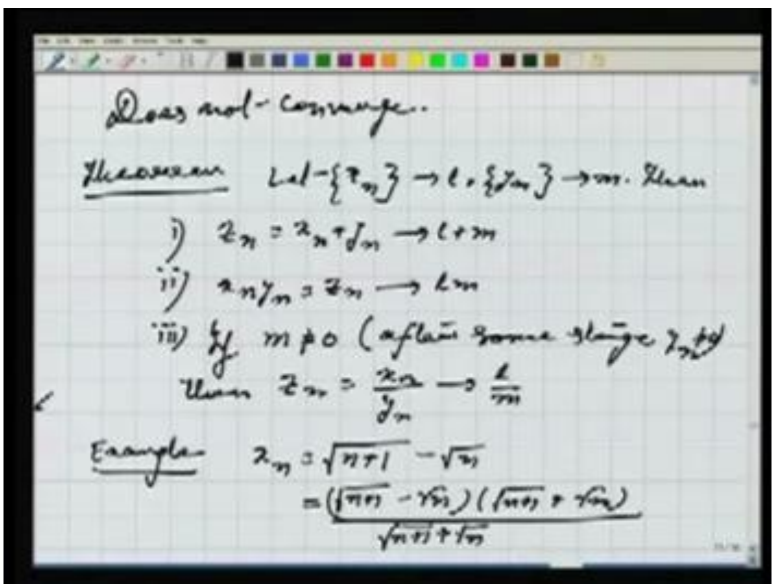

Now since we have got lots of examples of convergent sequences, we like to know certain natural questions about sequences, that suppose I have two convergent sequences and each term, suppose I add with the other one, does the resulting sequence converge? Or if I have two sequences which converge x n and y n and I go on multiplying terms, that will produce another sequence, does that fellow converge and questions like that. We have the following result. The proofs of all these results are very easy. You just apply the definition of convergence, the results will follow.

So let x n be a sequence which converges to l and y n be a sequence convergence to m. Then if I define the sequence z n to be equal to x n plus y n, it converges to l plus m. If I look at the sequence x n into y n to be equal to z n, this converges to lm and if I assume, if m is not equal to 0 which means after some stage y n not equal to 0, then the sequence z n equal to x n by y n converges to l by m. Only the little bit problem is I said after some stage y n is not equal to 0 but the stages before that y n might be equal to 0. In that case what I mean x n by y n is, from the term onwards where y n is not equal to 0. The proofs are very simple; follows from the definition.

Now we are going to use all these properties to manufacture some more sequences which converge. Now using this last theorem, let us try to manufacture some more examples of convergent sequences. So let us look at this example. Let us define x n equals to square root of n plus 1 minus square root of n. We will just look at the sequence. It look likes as n goes on increasing, the terms of the sequences are getting bigger and bigger. You tend to forget that there is minus root n sitting there to decrease those things. We will actually see that this sequence converges. What we do is a very simple thing, I write it as root n plus 1 minus root n into root n plus 1 plus root n and then I divide it by root n plus 1 plus root n.

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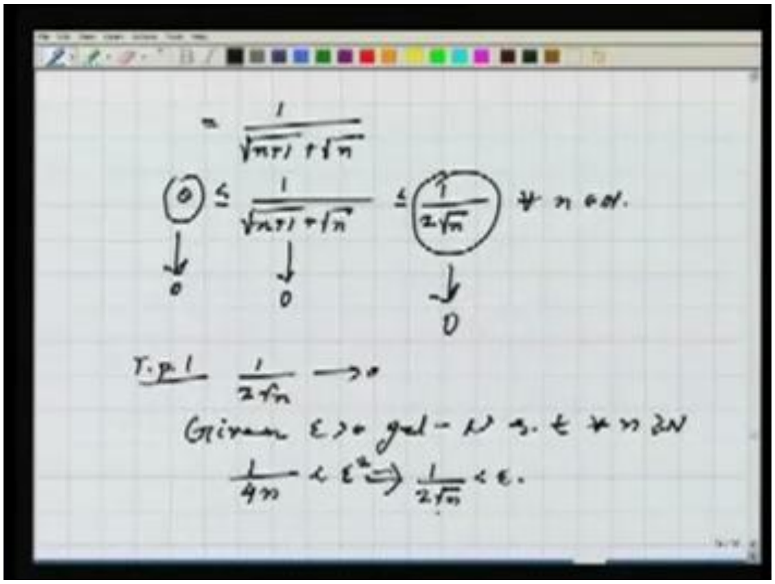

Now the numerator is simply 1. So what I get is, this is equal to 1 by root n plus 1 plus root n and I want to show this converges. Well, I then know certainly 0 is less or equal to 1 by square root n plus 1 plus square root of n and which is less or equals to 1 by 2 times square root of n for all n. Now this 0 here stands for the constant sequence 0. So it converges to 0. If I can somehow prove that this sequence converges to 0, I will get by sandwich theorem that this converges to 0. The question is does this converge to 0. If I am to prove, then what should I do? What I will do is, given epsilon bigger than 0 get capital N such that for all n bigger than or equals to N, 1 by 4 n is less than epsilon. That is simply the Archimedean property. I can do that. This implies 1 by 2 root n is less than epsilon but that is what meant by that the sequence 1 by 2 root n converges to 0.

So, to some up what I have done in this lecture is, we have defined the notion of limit in rigorous fashion. Then we have shown that if a sequence has a limit, the limit has to be unique. That means, there cannot exist two limits of a sequence which converges. Then we have proved sandwich theorem and have seen some of it is applications. The most prominent one is to calculate limits of certain sequences by applying Sandwich theorem. In the next lecture, we will go further about sequences and its limits.