Engineering Physics - II

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Module No. # 01
Lecture No. # 04

So, in this lecture we are going to conclude, the mathematical preliminaries that are required for us to start studying various electric and magnetic phenomena. If you remember in the very first lecture that I wrote the four Maxwell's equations namely, divergence E equal to rho by epsilon naught, divergence B equal to 0, then we have the Faraday law curl E plus delta B by delta t equal to 0, and then the generalization of the Ampere law which says curl B minus mu naught epsilon naught delta E by delta t is equal to mu naught J.

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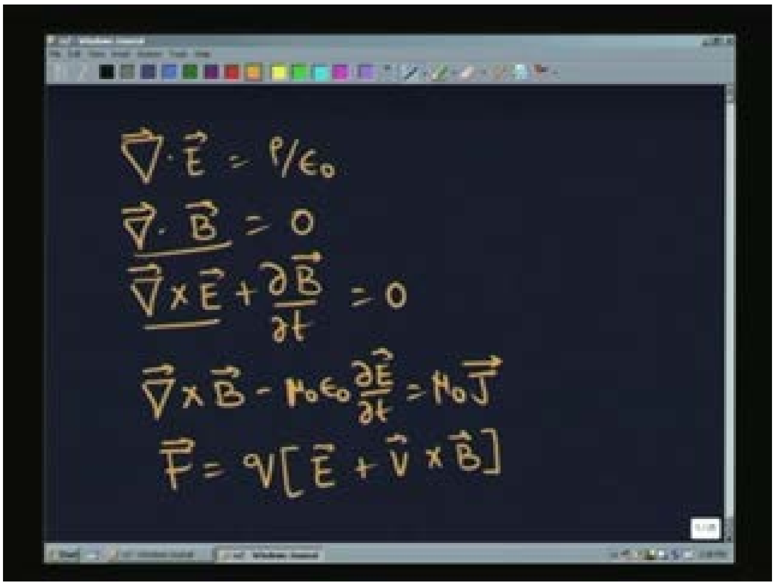

So, what have we done all this while in the last three lectures? We developed the machinery for us to appreciate the meaning of these symbols. I have the curl and I have the divergence that is what I have. Of course, all of you already familiar with the cross product which occurs in the Lorentz force equation which is given by F is equal to q E plus V cross B. However, I have not completed the mathematical preliminaries in the sense that, although in the last lecture I defined for you what the divergence means, what the curl means, I have still not given you the complete physical significance or the geometric interpretation and that is said to be done.

I motivated these definitions by saying that the divergence is a measure of the strength and the location of the source, and curl is a measure of the strength and location of some object quote-unquote which is like stirring - a stirring stick in a bucket full of water. What I want to do today is to put them on a firmer footing make them more rigorous, so that the precise meaning becomes very clear to you, whenever do whenever you do manipulations with the symbols, and let us see how to do that. Before I proceed, it is convenient for us to recapitulate the definition of the divergence and curl, and also go back to the examples that we worked out earlier.

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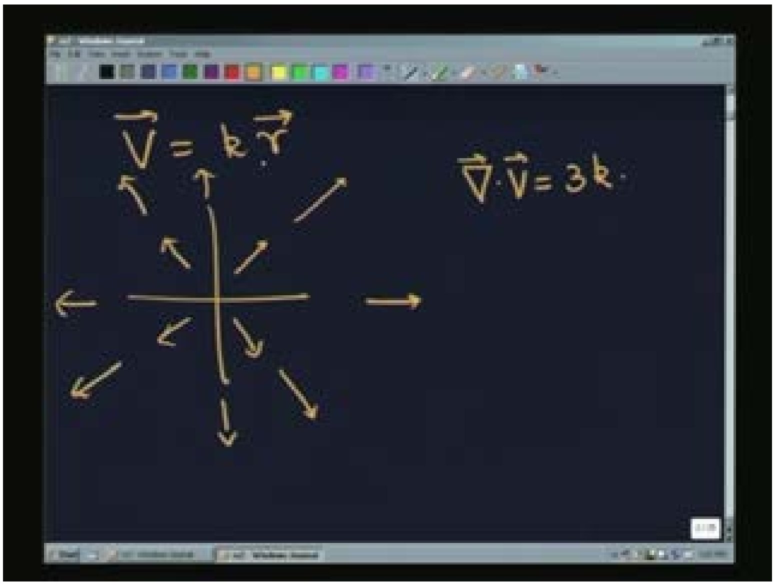

The simplest example that I worked out was that of a vector field which was simply given by some constant into the radius vector r. Well we also saw how to plot this vector field provided it is a two-dimensional example, so we set up the coordinate system and we said it is radially outward everywhere. Of course, if I were to go to a larger distance, the length correspondingly increases, and this is indeed the way we introduced our graphical representation. Now, let me formally calculate the divergence of this function, before getting into the complete geometric meaning, and let me take this to be a three-dimensional vector r by that I mean x i plus y j plus z k, then you can easily see that divergence of this vector field is simply given by 3 into k. One from each of these coordinates they add up to give you 3 and k is the total all strength. And indeed you see this is like saying that there is some kind of a constant slope, this is not exactly a slope, but still there is a derivative operating a further and further you go and accordingly this vector field is increasing linearly in magnitude. After all if I took the modulus of this vector it will simply simply turn out to be k into modulus r.

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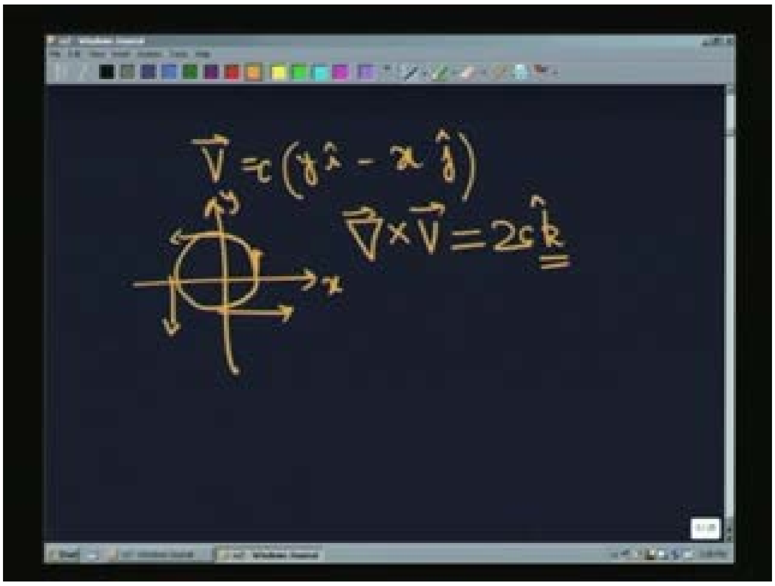

Yet another vector field that I introduced was the curly field and how did it look like. It had the form V is equal to y i minus x j. I had already plotted it in the last lecture. So, if I were to draw a circle here, here it is pointing along the y axis, here it is pointing along the minus x axis, here it is pointing along the minus y axis, here it is pointing along the x axis. Again you see that if I were to draw a larger circle then the length would correspondingly increase, and this kind of a constant rate of increase in the curly nature quote-unquote curly nature, would be summarized by calculating the curl of this vector function. I have already written the formula for you in terms of the determinant, I would not like to repeat that. If you were to do that you can convince yourself, it will be simply given by 2 into unit vector k. Of course, as in the earlier example, if I were to multiply by this by an arbitrary constant c, this would also get multiplied by a constant c where 2 c is the measure of the curliness of this vector field.

Now, if c is negative that means it is going to curl in the clockwise manner. If c is positive, for example in this diagram I sort of assume that c equal to 1 then it is going to curl in the anticlockwise manner as I have shown. Therefore, the sense is very clear from the sign of c, and k tells you that the curly nature is to be found in the x y coordinate system or equivalently in a plane perpendicular to the z direction which is given by this coordinate unit vector - basis vector k. So, we have got in intuitive picture of what a divergence is and what a curl is. Notice, divergence is a scalar field which is going to give you some kind of a rate of increase and decrease. Whereas curl is a vector field which is going to again give you a complementary information about the nature of increase or decrease of the function. Because in this case whenever I speak of curliness I have to give you a sense whether it is right handed, left handed, clockwise, anticlockwise, which is the reason why this, unit vector or the vectorial nature automatically appears, but in the case of divergence we are under no such obligation.

Now, we should actually workout one more nontrivial example to convince ourselves that the functions are actually indeed more complicated than this kind of simple functions that we have look at looked at. So, in order to do that what I shall do is to consider not a very difficult function, it is a well known trigonometric function, we make use of that to construct a new vector field and that will have the following form.

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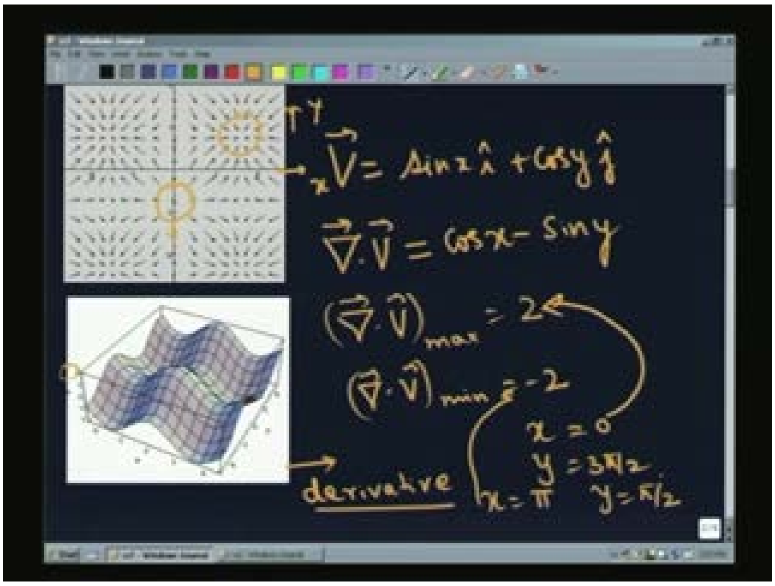

Look at this function; actually what I have done is to plot this vector field, this is a two-dimensional vector field as you can see, and this has the form V is simply given by sin x i plus cos y j. This is by no means a difficult function to imagine, because if I want to freeze the value of y and vary only the value of x, it is nothing but your trigonometry function sin x which has a sinusoidal behavior. And in a similar manner if I were to freeze the value of x and change the value of y you get a series of sin functions along the j direction; whereas, if I were to freeze the value of y I will get a series of sin functions along the x direction, what we do is to essentially super post them.

Now, what I have here is the x axis as you people can see. In this direction is the y axis and employ our earlier convention that the arrow gives you the direction of the vector field, and the length of the arrow gives you the measure of the magnitude; we are not going to say that it is up to the scale, it is scaled to some convenient value, it is multiplied by some overall constants, so that we can represented on this plane. If you did that you see there are certain interesting things happening here. If you look at this particular spot, you see all the field lines seem to be (( )) into this point. So, we would like to imagine that if this was a velocity field some kind of a flow field that there is a sink sitting here. As if, there is a hole in the garden and all the water is flowing into this particular garden.

On the other hand, if I were to look at this area, this region, all the water seems to be actually flowing out. You see all the field lines are diverging here; the line is coming here, the line is flowing out, the line is flowing out, the line is flowing out, so on and so forth. So, perhaps you would like to imagine that at this point there is a source may be there is a tap from which the water is flowing. Now, similar patterns repeat on all the four quadrants, because this is a simple trigonometry function which has nice periodicity properties. And then I would also like to see whether there is a sink here, there is a source here and so on and so forth as to where it is strong and where it is weak. Now, a good measure of that would actually to be construct the divergence of this vector field which is very easy. Divergence as you people can easily see is given by the scalar field cos x minus sin y; the derivative of cos y is minus sin y, the derivative of sin x is cos x and that is what I have got. And the figure below is indeed plotting this derivative as a surface in the two-dimensional plane.

Now, look at this scalar field. Since both cos and sin are bounded in the region minus 1 to plus 1. We know that divergence V maximum is simply given by 2 and divergence V minimum is simply given by minus 2. For example, if I look at this function cos x minus sin y, if I put x is equal to 0, y equal to 3 pi by 2, so let me write that here; x is equal to 0, y equal to 3 pi by 2 that would actually corresponds to a maximum, and if you look at this function carefully, you can go home, take a sheet of paper or sit on your computer and generate, this you will see that this values of 2 exactly correspond to that the maximum value here is actually given by 2. In a similar manner, if I were to put x is equal to pi, let me write it here x is equal to pi in which case cos x will become minus 1 and y equal to pi by 2. In that case, sin y is 1 and there is a minus sign, I will get divergence V equal to minus 2 which is the minimum value and there indeed the regions that act at this sinks.

So, this particular figure - pair of figures are actually telling me how to correlate the intuitive notion of the notion of a source and the sink with a notion of a divergence. However, this is not sufficient for us, because the minute we said that there is a certain positive divergence, we should actually show that a certain amount of fluid is flowing out. That means if I were to construct a surface which bounds a particular volume, I should show at a certain amount of fluid is flowing out. On the other hand, if the divergence is less than 0, I should be able to show for you that actually the fluid is flowing in. Unless I did that this would again only be at a naive intuitive level and it would not be rigorous to our satisfaction.

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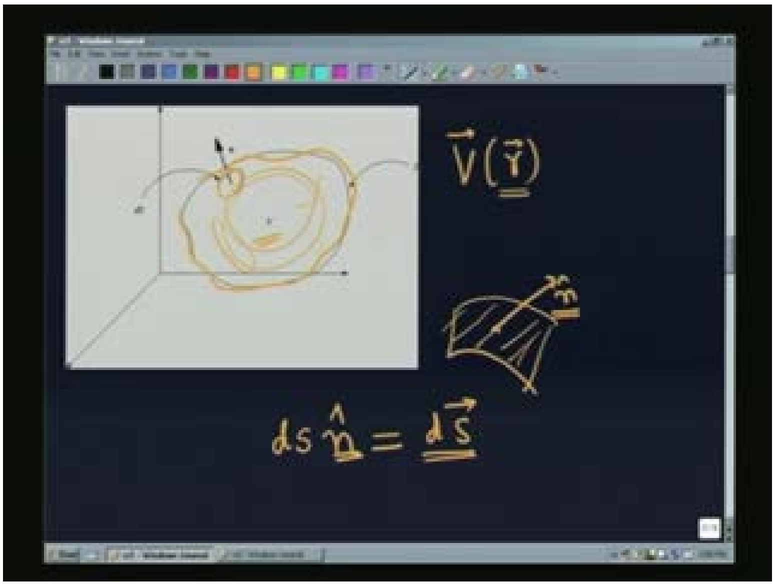

This result which we would like to capture now is actually what is contained in what is called as Gauss theorem named after the celebrated mathematician, physicist, engineer Gauss. What I am going to do is not to prove Gauss theorem for you, it is not very important for us to give a proof at this particular point. I am sure that you people would be studying it in your books and it would be done in your classroom. But we would try to get a certain amount of insight into what this theorem means.

In other words, I am not going to give you the most general proof of the Gauss theorem. It is not important for us; you can always work it out from the books. But we will see what really the meaning of that is by considering a very simple example. However, this example should not be considered to be a trivial example, because the example that I am going to give you contains the germ of its generalization, exactly like you know how to go from summation into an integration or you know how to define an integral in terms of the finite sums by looking at the Riemann sums. So, let us remember that.

So, how what does gauss theorem say? Gauss's theorem is a very, very beautiful result which simply states the following. In order to state it precisely, let me start defining, let me say that my V is a vector field defined over a certain region in space. Well for most of our purposes the region in space is all of space. So, the range of the r values is going to be from minus infinity to infinity in the x direction, in the y direction, in the z direction. It is perfectly possible that this V vanishes in many, many regions that does not concern us, but then it is define.

However, when we want to state this theorem we are not going to look at the whole space, but we shall look at a certain finite volume in space. And this volume is what is indicated by the interior of this figure, you see this here, this is the interior of this figure which is denoted by V. So, I have erected my coordinate system x, y and z, and then I have the volume which is contained. How do I get a finite volume in space? Well I get a finite volume in space simply by bounding it by a surface that completely covers it. Therefore, this finite volume in space is defined by the surface that it bounds it. What this figure shows is a small element of the surface.

Now, that means if the magnitude of the surface area is to be denoted by d s, it is also characterized by the unit vector, and again as I told you in one of my earlier lectures, this unit vector is always take on to the outward normal that is a right handed sense that we always have. Therefore, I write that there is no ambiguity about that. This is what I denote symbolically as d s. In other words, the magnitude of this area element is going to give you the area contained in this infinitesimal volume element and the direction tells you the planarity - the local planarity of this particular surface, and now therefore, this is a good legitimate vectorial object. So, I have the vector field, I know how to define the surface elements and armed with these two, I am now in a position to actually state Gauss theorem for you.

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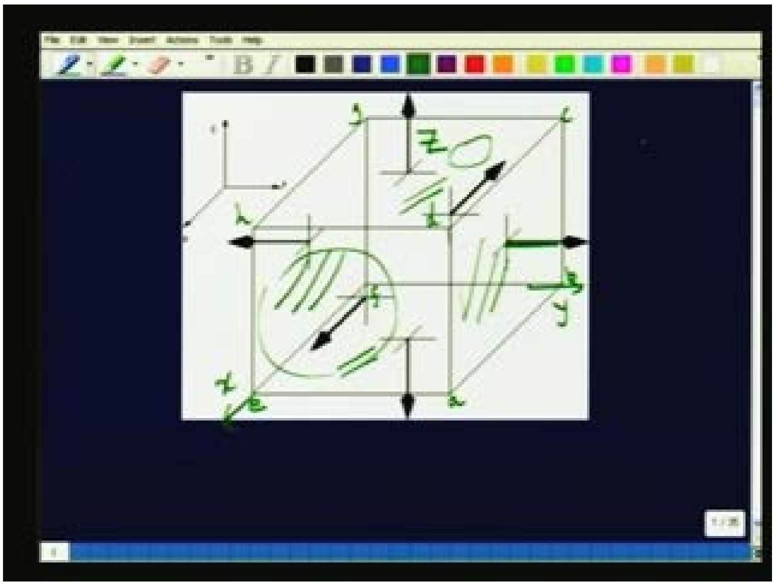

The simplest surface one can think of is a volume bounded bounded by the 6 sides of the cube as shown in the figure. What I am going to do is to demonstrate Gauss theorem for this particular volume element which has a cubical surface, and then go on to indicate how this result can be generalized to arbitrary surfaces. First let us concentrate what is given in the inset. As you can see in the inset, you see, you have the x axis along this particular direction, the y axis along this direction and the z perpendicular as we have here. Now, we have 3 planes here, 3 sets of planes to be more precise. So for example, the upper plane and the lower plane are characterized by a given value of z. This is we have the upper x y plane and we have the lower x y plane, and the z axis is perpendicular to both the planes.

In a similar manner, you have the z x plane which is indicated along this phase, we have the z x phase which is indicated along this surface and for both the surfaces the y axis is perpendicular. In a similar manner, we have the inner plane and the outer plane. So, this is the inner plane which I am indicating, and we have the plane which is entering inside the screen and for them the x axis is perpendicular. Now, given any area element like this, remember the definition that we gave earlier. We said that area element has a vector associated with it, and that vector is always outward normal, and that is what is indicated by this particular arrow. Therefore, if I want to look at this plane, let me call it a b c d, this is the plane for me. The plane a b c d which is indeed a z x plane has y axis perpendicular to that and that axis is along the positive y direction.

On the other hand, if I want to write three more coordinates - e f g h. If you look at the planes e f g h, it is again a z x plane, but the surface element has a vector which is along the negative y axis. Similarly, the upper x y plane which is given by h d c g has the axis parallel to the positive z direction whereas, the plane e a b f has an axis parallel to the negative z direction. So, what is the statement that we are making? We have three sets of parallel planes and in each of these cases the corresponding surface elements are anti-parallel to each other. If you remember this, the proof of Gauss theorem is quite a simple exercise to prove and I shall proceed how to show, how to do that.

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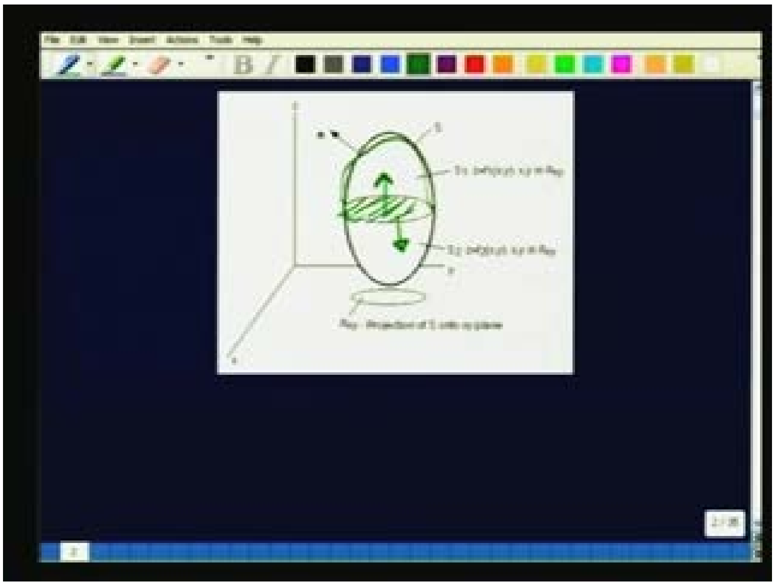

However, before I do that, it is good to know that in general a surface element is not going to be cubical like what we have shown here. It could be some complicated shape. For example, here we have indicated by an oval like surface. If you look at this oval surface and I have indicated a demarcating line here. So, there is a oval volume which has an outer surface and the equatorial plane which divides it into two parts. So, this is the upper surface and this is the lower surface. The point that I would like to make is, if you look at the volume element which is obtained from the lower hemisphere and this plane, then the surface element will be in this direction. However, if you look at the volume element obtained by looking at the upper oval and again the same equatorial plane then the corresponding surface element is in the opposite direction.

In other words, when we write down the vector associated with each surface; that is the surface vector we should remember in what sense that surface is bounding that volume. The same surface in one case is bounding the upper sphere upper hemisphere and therefore, the area element is along the negative z direction whereas, for the lower oval the area element is along the positive z direction. This is an input that you should remember, because that is what is going to help us to generalize the result that I am going to prove.

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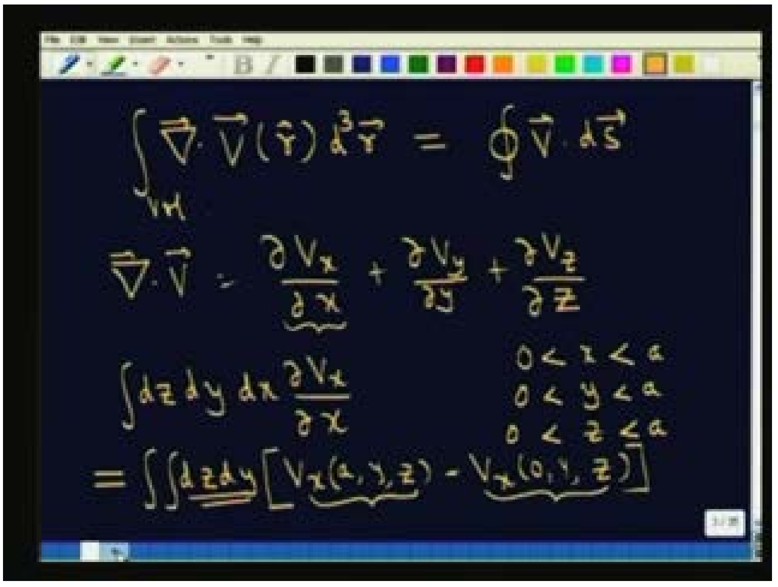

Let us see how we can demonstrate Gauss theorem for the simple cubical surface. So, in order to have everything clearly, let me rewrite Gauss theorem, you have a vector field defined over a certain volume and I first of all evaluate divergence of that vector field. And then I integrate this scalar function over that volume and I am going to write divergence V d cubed r over that volume. What gauss theorem does is to state that this is nothing but a surface integral. So, this surface is of course a close surface that bounds the volume, and we have integral V dot d s that we evaluate over the volume, and please remember this surface element d s is indeed the one that is outward normal and not inward normal that is how we have defined.

Let us take a very simple example and see what happens in the case of a cubical surface. If I want to expand divergence of V, it has a form delta V x by x plus delta V y by y and delta V z by z. Since we are considering any arbitrary vector field, we are not going to put any restriction on that; except of course that they should be differentiable, because either in either wise I will not be able to determine the divergence, I can construct the simplest of the vector fields which has only the x component. Of course, then I can construct another field which has only the y component, a vector field which has only the z component. Since all integrals are additive, the most general vector field can be obtained by adding the three of them. Therefore, if I were to demonstrated for the first term as to how the Gauss theorem works, we have almost what they prove for the theorem.

So, let me start with the very first term. So, V x although it is V although it has a component all along the x direction is of course a function of all the three coordinates x, y and z. So, if I want to concentrate on the first term how does it look like? So, I am going to write integral d z d y d x delta V x by delta x. Now, in order to evaluate this integral I need a certain set of coordinates. So, what I shall do is to say that all my coordinates vary from 0 to a. So, I have 0 less less than a, 0 less than y less than a, 0 less than z less than a, which defines the cube for me. The origin if you feel like can be fixed at 0, 0, 0 which is one of the corners of the cube.

Now, here we have a partial derivative with respect to x and then integral with respect to x and that is something that you can evaluate by using the fundamental theorem of calculus. I cannot perform the other two integrals involving d z and d y. So obviously, we expect it is this surface integral d z d y that will be related to the surface integral on the right hand side integral V dot d s. In order to show that explicitly what I shall do is to rewrite this as two integrals d z d y, and in what is it that I have here? I have V x, I am going to evaluate the integral delta V x by d a delta x d x, and x goes from 0 to a, I have therefore, a y z minus V x 0 y z that is what I have here.

Now, I am not going to write down the figure, you people will sit down and write a cube. But you people can easily see that V x a y z is what occurs with a positive sign, V x 0 y z is what occurs with a negative sign; x equal to a corresponds to what - the plane which is to the right, whereas x equal to 0 occurs to the left, because we are obviously moving from left to right when I go from 0 to a. Therefore, for this vector element as you can for this element you can see that is d z d y the vector is along the positive direction - positive x direction whereas, for this surface integral the vector has to be along the negative x direction, and this minus sign takes care of that. So, if now we have almost proved the theorem that we wanted to show. Let me rewrite the integral that I obtained after doing the integration over x.

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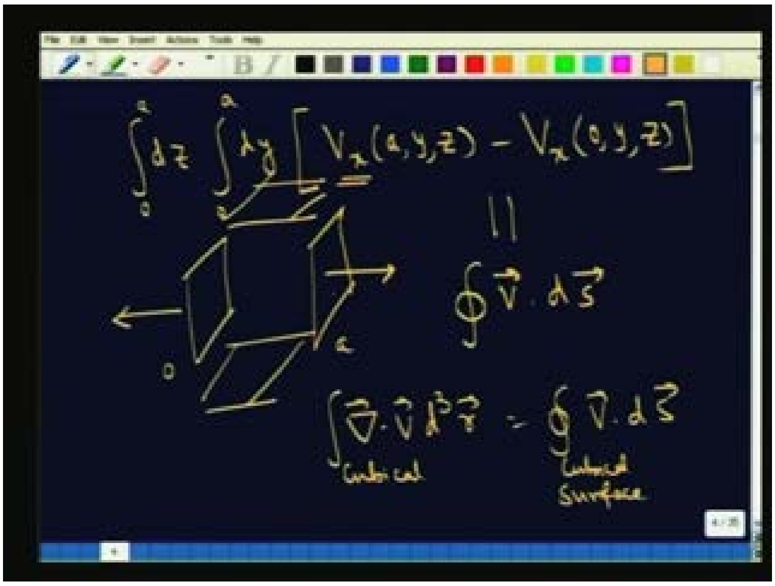

So, I have integral d z 0 to a, integral d y 0 to a, and I had V x a y z minus V x 0 y z. As I told you I am not going to write down the figure for you, but if I look at this I have a surface integral in the y z plane which is equivalent to choosing the vector along the x direction. And mind you, we have the x component that is sitting here, therefore, if I have two parallel planes, this corresponds to x is equal to a, this corresponds to x equal to 0, here we have the surface element along this direction, either you have the surface element along this direction, therefore, this object can be simply written as integral V dot d s.

I have chosen a vector field where only the x components survive; the y component was 0, the z component was 0. So, when I evaluate into divergence only V x contributed to that and since only V x is contributing to that only these two surfaces contribute to my volume integral. But then this is the closed interval, because if you feel like I can compute the cube and imagine that I have done the integral, but the vector field itself is vanishing along all the other directions. Therefore, we have shown that for a specific coordinate or for that matter for a specific component, and for a specific vector field this result holds. Having stated this result for the simplest of the cases, it is not difficult to see how it generalizes to the other cases, because if we had V y then we would have had planes corresponding to this. This corresponds to the plane corresponding to V z then if we had V y we would have had planes which come inside the screen and outside the screen, we would have got a series of a expressions like that. Each of them would have given me two terms, in all I would have had six terms.

Therefore, I have three vectors which are outward, these three other three vectors which are anti-parallel and therefore, I have a complete surface integral over the cube and what have we done. For this cubical surface we have shown indeed that divergence V d cubed r over this cubical volume element is nothing but integral V dot d s over that cubical surface. Gauss theorem for this simple example is as simple as that. There is no complication. Now, the question occurs as to what we are going to do if I had a more complicated surface. So, what I shall do is to look at a particular volume element and what of the surface.

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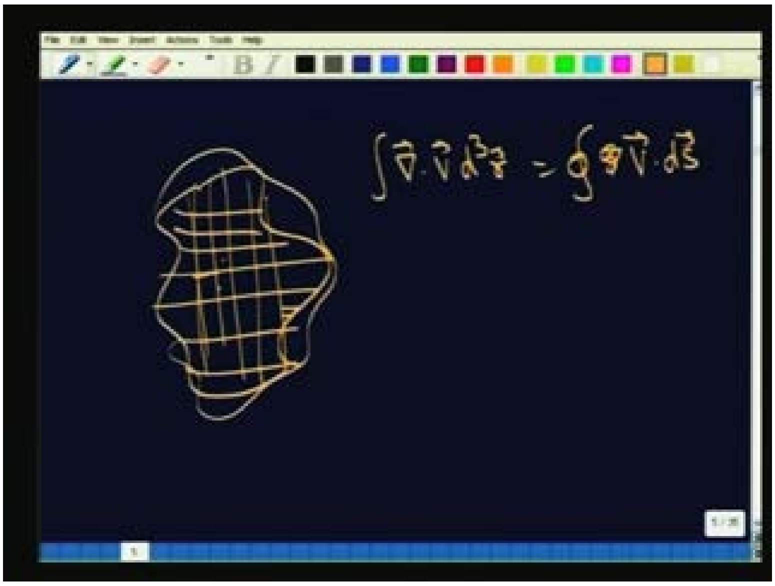

So, I have a surface like this volume like this and this is a complicated surface. Now, it might appear that the proof is very tough, but there is something that we should remember which we actually use, whenever we do numerical integration. And that is I can split this volume into a large number of small cubical volumes. Now, once I split this volume into a large number of cubical volumes, the surface is also corresponding to that. Therefore, when I am evaluating the volume integral, divergence V d cubed r over this volume, I can find out the contributions from each of these infinitesimal volumes and then add them up and then obtained them. So, there is no problem at all. Now, what about the surface integral?

When you look at this surface integral, a little think being will show you that it is only the outer most surface that contributes. However, how does the outermost surface contributes? Since we have drawn infinitesimal cubes, so this is of course a two-dimensional section, if I consider a sufficiently small area element here, this is almost planer. Therefore, it can be looked upon as not as a curved surface, but as a plane surface and therefore, it can be looked upon a surface which is coming from a cube. That is the statement. For sufficiently, however complicated the surface may be in a sufficiently small area, it can be treated as planer. And if we did that all that we have to do is to sum over the contribution coming from all the infinitesimal contributions in the surface, they will all add up to give you integral V dot d s I am sorry integral V dot d s. That is what they are going to do and the surface is of course bounded.

What about the surfaces which are inside the volume element? Remember, I already gave you a hint when I discuss the surface element. If I consider this surface depending on whether it is the lower surface of the upper volume element or the upper surface of the lower volume element, the direction of the vector is going to change. Therefore, we can see that all the surface integrals vanish, they cancel each other, and only the outer most surface contributes. This completes the proof of Gauss theorem in as much generality as we can and as we need and as we can at this particular point.

The next theorem that we have to prove is stokes theorem and I shall proceed to do that now. What does stoke theorem state? Stokes theorem is another very beautiful result, which again does not make any assumption what so here about what so ever about the vector field except of course that we should be good, continuous and a differentiable field. What does it say? Now, we have to be careful in defining stokes theorem, because this also involves a surface integral, but this surface integral is a surface integral of a different sort. So, let me illustrate that to you again.

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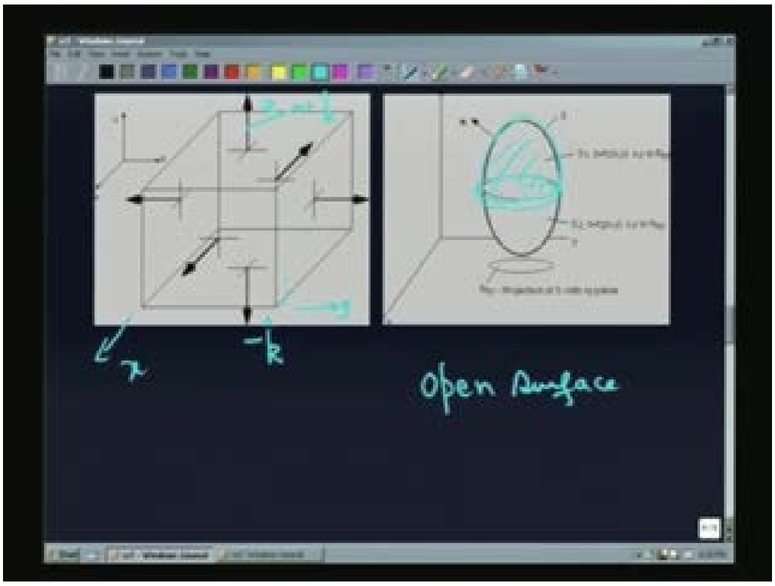

Come back to this figure. In this figure, when I want to determine the correct version, when I want to define stokes theorem correctly, what I shall do is to consider not a close surface, but an open surface. What is an open surface? Take this, I have this upper oval object and then I have this. But then when I do the surface integral, this part is not integrated over, whatever I am showing by dots is not going to be integrated over; whereas, the other covering part is something that is integrated over.

So, if you feel like the surface interval that I am going to look at is something like a bowl and not a hemisphere where the flat portion is also covered; it is not a close surface, but it is an open surface. If you want you can even imagine it to be a kind of a bag where the bag is covered of all sides except the top where you can put things and that is part is of surface integral, it is not evaluated. We are only going to evaluate this surface integral over this covered portion. There is a certain portion which is not covered which we will leave alone. That is something that we have to remember.

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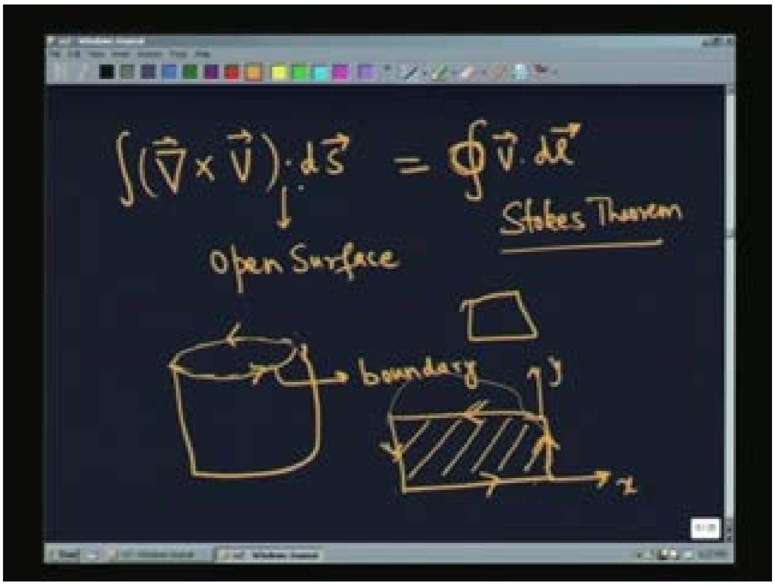

If we remember that then the statement of stokes theorem is quite simple, what does it say? It says that - give me any vector field and give me the curl of this vector field. The curl of this vector field can be dought dotted with the surface element; I have already defined the surface element for you. But this is not a close surface element, but an open surface element. This is the only catch in the statement; this is the open surface element. Now, the minute I have an open surface element obviously, there is a boundary for that surface element. For example, suppose I take a cylindrical surface, your measuring jar for instance, it is open at the top and therefore there is a boundary here, which is given by this circular ring, this is the boundary. This boundary is a one-dimensional boundary, no wonder about that.

A volume is enclosed by a two-dimensional surface. Now, we are saying that this two-dimensional surface is bounded by a one-dimensional curve; this is a one-dimensional boundary. And now I can move along this boundary that means I can actually construct a line integral along this boundary. What stokes says is that this surface integral is nothing but the line integral over the closed loop that is what I am denoting by the circle here V dot d l. So, this open surface is bounded by this one-dimensional curve, the rim of the cylinder, for example, in this particular case, what is stokes theorem states is that when you evaluate the curl of V dotted with d s over the closed part and integrate, this is nothing but the line integral, but this curve is closed. That is the statement we are making; this is stokes theorem. And this theorem is indeed a measure of the curliness of the field. We will give some examples later and see what the physical meaning of this is.

Now, again as you people might have guessed I am not going to give the most general proof, it is not required. It is convenient to consider the simplest of the surfaces, and therefore, the simplest of the line integrals. The simplest of a surface is obviously a planer surface which is squarish, so or a rectangle. So, suppose I drew a rectangular surface here, then it is bounded by these 4 lines; this line is obviously along the x direction, this is along y, this is minus x, this is minus y. So, let me denote it by my usual directions x and y. That is what I am going to evaluate. The only care that one have to take in evaluating this integral is that you see that again the line integral means something different, here if it is along the x direction, here it is along minus direction, here if it is along the positive y direction, here it is along the negative y direction, exactly we have outward normal to the surface. If we remembered it then the proof is simple.

For your convenience, you can imagine this to be some kind of a membrane over which the field is defined. What does stokes theorem say; here is something very interesting. If this membrane is planar, you evaluate this line integral. But suppose you hold the boundary fixed, and pull up the membrane. So, you have a rectangular frame at this particular point, but this is a plastic membrane which is free to moving, suppose I pull it up. When I pull it up it is going to acquire some arbitrarily shape, and of course, the vector field is going to be defined over that surface, and you can evaluate the surface integral over either this planar surface or the distorted surface which has come, because of my pulling it up. Once I pull it up I can distort it this way or that way like I do with a balloon; never mind stokes theorem tells you that so long as this boundary is fixed, it does not matter what kind of a surface element that we are going to take. So, this is a very, very powerful result which is useful. And let us see how to prove that for this very, very simple example of a rectangular surface, the prove is not difficult at all and again we have considered the simplest of the examples. What shall we do?

Now, when I am speaking of a surface integral obviously, the simplest surface is the x y plane, but when we were peak of the x y plane the surface element is along the z direction the surface vector, therefore what shall I do?

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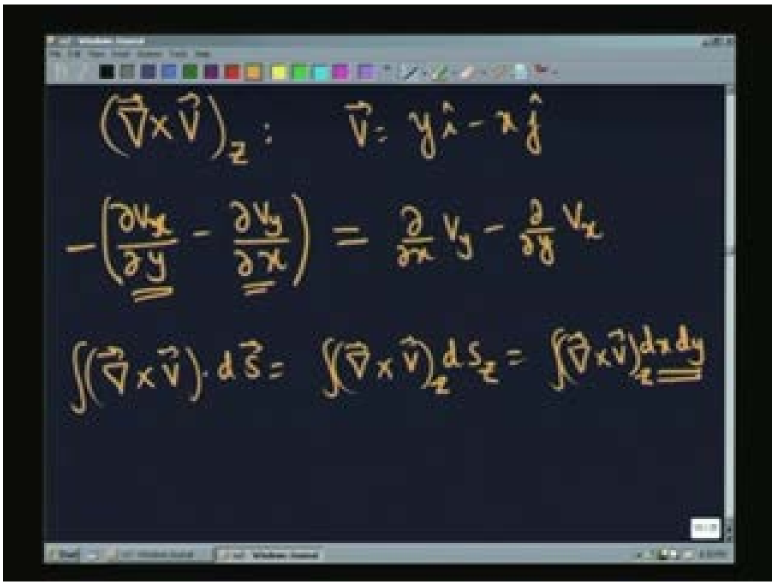

I will take a vector field for which only the z component survives only the z component survives. And I have actually constructed such an example, what was that field, if you remember we wrote V is equal to y i minus x j, this is an example where the curve survives only along the z direction. Anyway I am not going to specialize myself to this particular field, this is only to give you a feeling. Now, let me calculate the z component of this field, how does it look like? The z component is simply given by delta V x by delta y minus delta V y by delta x that is what I have here. This is the z component of the vector field.

If you go by the determinant is to to took up a minus sign and it look like minus delta V x by delta y minus delta V y by delta x which is the same as del by del x V y minus del by del y V x; I need a minus sign I should not forget that. This is the z component therefore, I am going to integrate over the x y plane, please remember that. So, what is the right hand side? The right hand side is curl of V dot d s which is nothing but curl of V z component d s z which is nothing but curl of V z d x d y. This is something which we have repeated many number of times, so that there is no confusion.

When the proof is almost staring at you, exactly as it was in the case of the divergence theorem, because again you can see, there are partial derivatives involving y and x, and there is a surface integral or a double integral involving d x and d y. Therefore, if I could manipulate them intelligently then I would have proved that the left hand side is equal to the right hand side. It is not going to take too much effort and let us prove that. How does the prove go?

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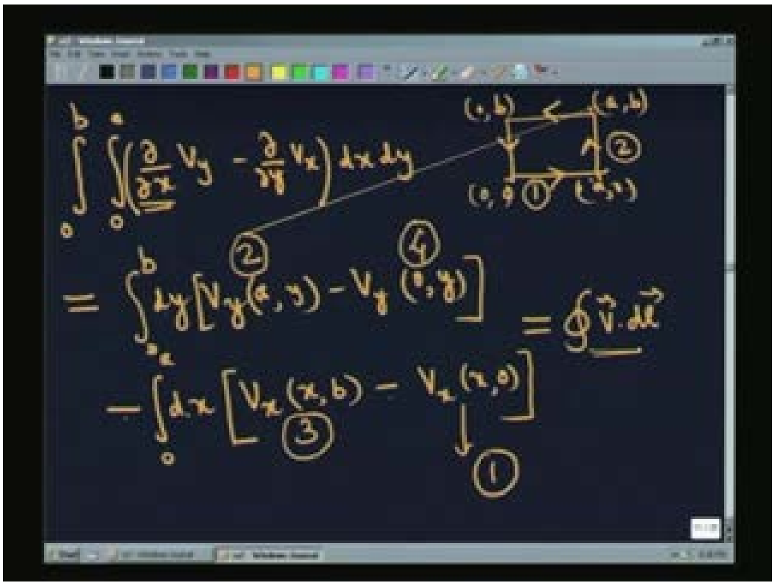

Let me write it explicitly again; I have delta by delta x V y minus delta by delta y V x d x d y. What is the range of x and y? Let me say x goes from 0 to a, y goes from 0 to b. So, this is my rectangle. So, 0, this is value (a,0), the coordinate - this is (a,b) and I have this to be (0,b). These are the coordinates. Let me do a partial integral and see how it gets. Let me look at the first term. When I look at the first term there is a derivative with respect to x sitting here therefore, I can complete the x integral by using my fundamental theorem of algebra and write is as an integral over y. So, what is the first term? The first term is simply given by 0 to b d y I have a V y. Now, I am integrating it from 0 to a therefore, it attains the values a y minus V y 0 y; that is the first term.

Now, let me look at the second term, there is a minus sign sitting here, I should not forget that. I will now compute the integration over y and retain the x integral. Since I do not know the form of V x I cannot evaluate it. So, how does it look like? This integral looks like 0 to a d x, let me open the bracket again, I am integrating over y from 0 to b. So, this integral is V x, my x is my free variable, this turns out to be b, the next term is V x my x is the free variable, the lower limit is 0 and this is what I have. Now, it is very clear that what I have is a sum of 4 line integrals; we have V y dotted with d y, we have

V x dotted with d x. So, V dot d l that form is evident for us, all that we have to convince ourselves is that it is a closed loop integral.

In order to see that it is a closed loop integral let me draw the arrows here. If I did that this is what would be a closed loop integral and see whether each term corresponds to that. Now, let me look at this first integral 0 to (a,0). This is along the x direction keeping the y value fixed. So, what am I going to integrate? I am going to look at this particular fellow. There is a minus sign here, there is a minus sign sitting here, they give you plus, I have integral d x V x (x,0) from 0 to a. In other words, if I label this by 1, this is indeed 1 this is the first line integral, minus and minus is going to give you plus, therefore that is the correct line integral.

Now, let me look at the second part of the line integral and let me label it by 2. I am moving from (a,0) to (a,b). So, where is that object? Well you see that here, this is a y equal to 0 to y equal to b and I am moving along the positive y direction, this comes with a plus sign therefore, this is indeed the second terminal line integral. What about the third term? The third term is from the point (a,b) to (0,b), this is an integral along the x direction and I am integrating from a higher value of x to a smaller value of x, therefore, indeed there is this minus sign sitting here x V 0 to a therefore, this is a third term. Sign wise, factor wise, value wise, there are agreeing term by term. The last term is of course, (0,b) to (0,0). So, let me write it as (0,0), because this is my origin, and this is this term, because there is a minus sign sitting here and it moves from b to 0 which can be written as b to 0, and it is in a negative direction therefore, this is my fourth term.

In other words, if I look at the z component of my vector field, although initially I said only the z component survives, I do not have to make that statement again anymore. Take any vector field, calculate the curl and look at it z component, do d x d l d y, d x d y integral, this is exactly equal to integral V dot d l, but what is this V dot d l. Since this is d x d y, the right hand side we will have the appropriate index namely, the integral along the x direction, y direction, minus x direction and minus y direction. Again the generalization is obvious all that we have to do is to write down the y component of the curl, the z component of the curl, you know now get integrals over d y d z, d z d x, write this coordinates planes, add them up. Since surface integrals add up on the left hand side, the line integrals add up on the right hand side, we prove the most the general version of stokes theorem for a rectangular surface or a cubical surface, open surface that is the most important thing.

Now of course, again all that I have to do is to repeat for you that a regenerate surface can be written as a sum of small planer surfaces; prove this for this particular each of them, add them up and the result goes on to that arbitrary surface at hand. In a sense, what we have done is to indicate how stokes theorem and gauss divergence theorem are going to work. And now you can easily see that when curl of V was not equal to 0 and when you did curl of V d x d y that is a measure of the curliness. This curliness comes with a sense which is given by this integral, whatever V dot d l is, and that gives a measure of what we normally call as the verticity. When it comes to divergence we will to we speak to we like to speak the language of source and sink, but when it is comes to curl we like to speak the language of verticity. Because in the very first lecture I gave you an example of stirring of a liquid in which case actually a vortex is formed at the center, if the stick is only at the center. Now, this essentially answers the question as to how many vertices are there how they are distributed and what their strengths are over the space. All that I have to do is to look at appropriate surface element and calculate the appropriate plane elements.

In short divergence theorem tells you about the nature and distribution of the sources. By source I also mean the sink, whereas, stokes theorem tells you information about the distribution and nature of the vertices. The vertex could be located at one point, the vertex could be located at many points; there could be an anti vertex, in a sense that the sign could change, there could be a sink which is anti source in the sense that that is not going to send the vector field out that it is going to observe the vector field, this intuitive feelings have been put on a rigorous basis by these theorems. But in order to get a new and better feeling, what we shall do is to work out one or two examples, and convince ourselves that is indeed the geometric import of these theorems.

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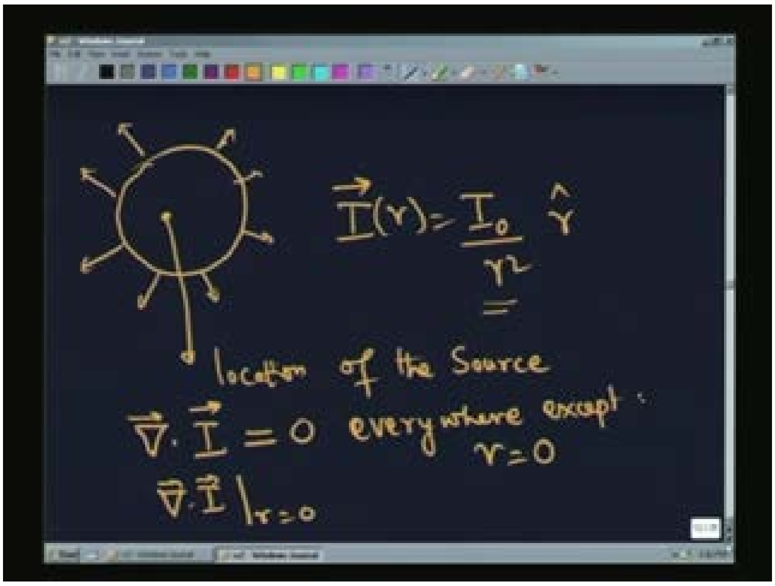

Let us consider the simplest of the examples that we already know. What is that example? Let me imagine that there is a point source which is emitting radiations in all directions. The radiation is emitted at a uniform rate. The source is sitting only at the center; this is the location of the source of the source. The field is radial everywhere, but however, if my notion of divergence is correct, the divergence should vanish everywhere except at the origin, because that is where the source is located. In order to verify that what we shall do is the following. We shall ask how would the intensity vary if there is the point source sitting at the center. Well the intensity as a function of r would vary as whatever that I naught is divided by r square r hat; that is something that we know. Because it is flowing outward therefore, the intensity falls off like I naught r square by r hat.

Now, I invite you people to calculate the divergence of this vector field. Now, I am not being very consistent in rating this, because intensity is scalar, but I am interested in the intensity flux. Therefore, I should actually put this vector sign. Therefore, I naught r squared over r hat is what I have. This vector field is going radially outward at everywhere. However, if I want to calculate divergence of I, this will be 0 everywhere except r equal to 0. How do I know that? It simply by explicit verification. Now, there is a small problem here, which actually tells you how careful you should be when you are evaluating derivatives, and simultaneously also gives you some information about some indication about the power of gauss divergence theorem.

Now, you were like to come and tell me very well divergence is equal to 0 everywhere, but there is a source. Therefore, I want to calculate divergence I at r equal to 0. Now, when I am into trouble you cannot calculate divergence I at r equal to 0, because this vector field is actually blowing apart r equal to 0, it is becoming infinite. In other words, since I do not know how to assign a value at r equal to 0, I do not know how to calculate the divergence, therefore I need a criteria, I need a handle in order to substantiate my statement that indeed there is a source sitting here. And this is where there gauss divergence theorem is going to help us. In fact that is the reason why coulomb law is also called as a gauss's law. How do we do that?

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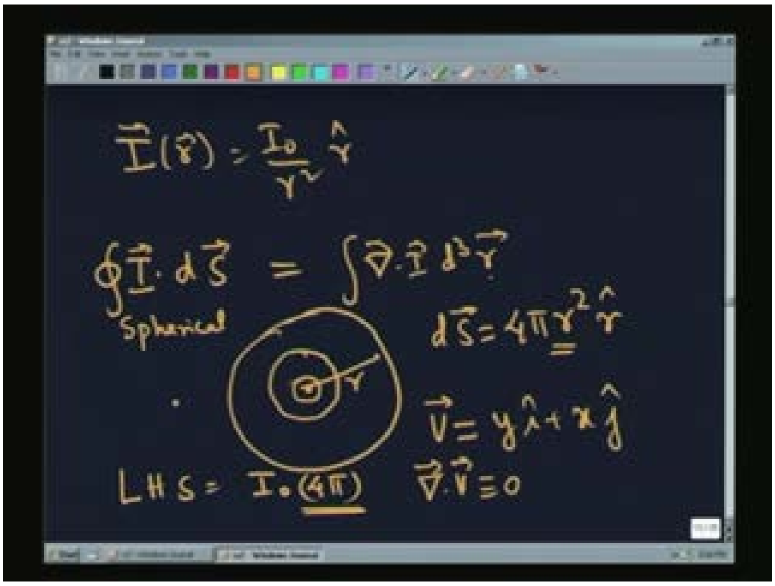

Take the same field again I of r is equal to I naught divided by r squared r hat; I now make use of gauss divergence theorem. Now, I assumed that I have proved the divergence theorem for arbitrary volumes and arbitrary surfaces. Since it is a spherically symmetric example, what we should do is to use a spherical surface. So, what shall I do? I integrate this div I dot d s over the spherical surface; this is a spherical surface. And by gauss divergence theorem, this must be given by divergence I d cube r. Notice the right hand side is 0, so long as r is not equal to 0.

In other words, suppose the source is located here and I constructed a spherical surface here, and I did an integration over the right hand side this would be identically equal to 0, because r equal to 0 is not included and this would also be identically equal to 0. But now I want to make sense of the fact of the notion that the source is actually located at r equal to 0 therefore, I put the source here and I want to evaluate this integral. That means my surface is going to enclose this integral. If I want to enclose this integral, this integral I dot d s is very easy to evaluate, because my vector d s is simply given by 4 pi r squared into r hat at a given radius r; r squared and r squared cancel each other, so what does this integral give r dot r is equal to 1 therefore, you find that my left hand side is simply given by I naught multiplied by 4 pi. It is a purely geometric factor. So, if you wanted that to be I naught, I could have defined my original field with a factor of 1 over 4 pi, so we are not going to be bothered by that. But what we got is that left hand side is given by I naught into 4 pi, which indeed tells you that there is a source.

Now, the interesting thing about this theorem is the following thing. I did not tell you what the radius r is. W ell suppose I imagine that there is a point source approximate this bulb by a point source, I consider a sphere of radius 1 meter, evaluate this integral, you get I naught into 4 pi. Take a sphere of radius half a meter, you get I naught into 4 pi; take a sphere of radius 1 centimeter assuming that the source size is less than 1 centimeter you get 4 pi. In other words, the value of the left hand side is independent of the volume that surrounds that point. Irrespective of what the volume is, however small it is, however large it is, it is always going to give me 1 and the same value I naught into 4 pi, and since I say I can consider sufficiently small volume elements, it can be made as small as I please. This tells you that the source is entirely lying at a given point, which is the reason, why, the divergence is going to vanish everywhere except at the origin.

There is actually a mathematical way of representing this by using what are called as the delta functions. But that is something that we need not get into at this particular stage, but gauss theorem actually allows you to make sense of divergence even at points where it is not defined d by alternative rules of differentiation. That is something that we have to remember. But on the other hand, if you had considered a vector field like y i minus x j, I exhibited that field through a very nice figure, you can see that the divergence of this object could be identically equal to 0, because that field was something which was completely curly.

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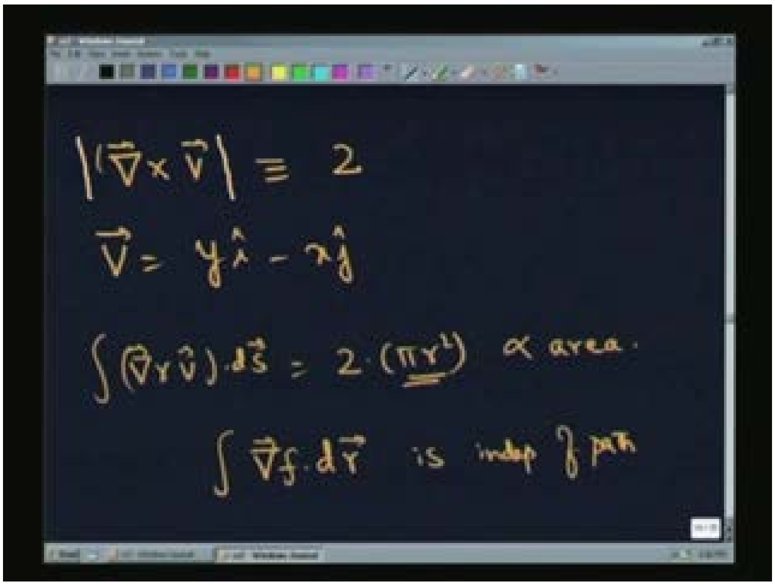

But on the other hand again, if you consider this vector field which is completely curly what is that vector field. Curl of V is what I want to calculate where V is given by y i minus x j, you find that curl V is identically given by 2. Therefore, if I want to integrate over a circular surface for example, integral curl V dot d s will be given by 2 into pi r squared where r is the radius of the circle. And of course, you can go home, write down this as a line integral and verify that it is going to be the same, you find that integral V dot d s is increasing with area, it is proportional to area, which tells you that as you consider larger and larger areas the values increasing which means there are vertices which are sitting everywhere. It is not difficult to construct the analog of a point source even in here this particular case where curl V dot d s is independent of the size of the surface that contains, but we shall not get into that. Later when we all going to discuss magnetic fields I will come back to that.

Now, it is a good time to summarize whatever we have learnt. We started with the simplest notions of the coordinate system; we saw how to construct basis vectors in curly linear coordinate systems in spherical polar and cylindrical polar in particular. And then went on to define scalar fields and vector fields, and determine their property in terms of the gradient, in terms of the divergence and in terms of the curl. At this point in order to be complete for the shape of the course I should also mention that there is a celebrated gradient theorem, which tells you that gradient of a vector function dot d r is independent of the path. I did not spend any time discussing this, because this is a most familiar result from your mechanics. Because you know that if you have a conservative force, it can be written as the derivative of a potential, and the work done is independent of the path, you might as well remember that. But apart from that we worked out the gradient and we worked out the curl, we worked out the divergence and we also got what its geometry or physical interpretations.

Now, a good question asked at this point is what are the mutual constraints that the divergence, that the curl and the gradient put on each other. For example, suppose I know that a vector field is the gradient of a scalar field. What can I say about its curl? Suppose I know that a vector field is a curl of another vector field, what can I say about its divergence or some other property? If I know that a vector field is written as a gradient of a scalar field or a vector field is given by the curl of another vector field, the natural question for us to ask is what can I say about other properties. Well these are very easy to evaluate and I am not going to work them out, but I am only going to state the results, please work them out for yourselves. Because they are straight forward and simple and that is the following.

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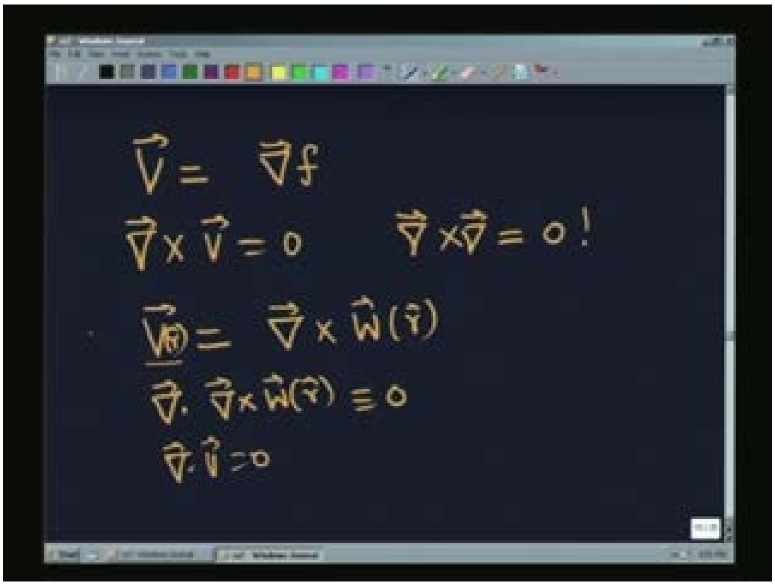

If V turns out to be the gradient of a scalar field then we know for sure that curl of V is equal to 0. So, it is as if we can write curl of gradient cross gradient is equal to 0 like a cross V equal to 0. This is something that you should verify explicitly. This is indeed the reason why we are able to introduce potentials, because for electrostatic fields we know that curl of V is identically equal to 0. There is the complementary result involving curl and that is the following. If I can write a vector field as curl of another vector field w of r, so there is r sitting here. Then we know that divergence of this vector field is identically equal to 0. So, this should remind you of a dot a cross b is equal to 0. So, it appears that this formula can almost be taken over. This is the next thing that we can verify.

Again this is something that is going to be useful for us. Because we know that divergence B is always equal to 0 and we would like to write it as curl of another field. So, what am I saying? What I am saying is that I know that whenever a vector field is a gradient of a scalar field curl of V equal to 0. That is something that you can verify. The converse is also true. Whenever curl of V is equal to 0 it can be written as the gradient of a scalar function. Similarly, whenever the divergence of a vector field is 0, divergence V is equal to 0, it can be written as a curl of another vector field which you do when you introduce the vector potential.

The last question that remains is that suppose you give me a vector field, you give me the divergence, you give me the curl, what do I know about the vector field. Well, the answer to that is remarkably simple, although the proof is not completely straight forward; it is not tough, but it is a little bit tedious. It says that if you give me the curl and if you give me the divergence of a vector field, you essentially know all about the vector field provided you put some reasonable conditions. What is a reasonable condition for us engineering wise, physics wise? That the fields should be well behaved, the fields should be continuous, the field should be differentiable, and if I go far away the fields better die of to infinity. I do not want a velocity field which survives, as I go to r equal r tending to infinity, if I did that it indeed turns out that the divergence and the curl completely characterize a vector field.

So, what we have done is to build up the basic preliminary machinery that is required for us to embark on the study of electromagnetic phenomena and that we shall start from the next lecture onwards.