Engineering Physics – II

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Module No. # 01
Lecture No. # 03

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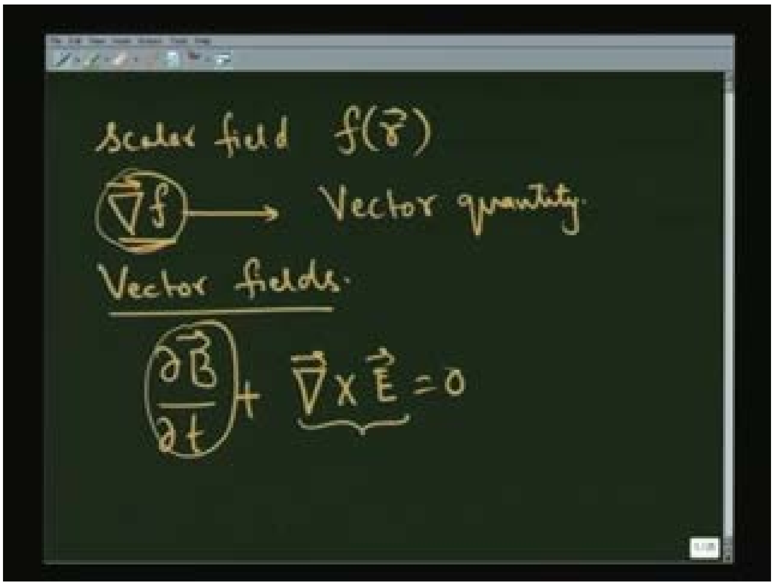

In the last lecture, we spent some time trying to understand what the meaning of a scalar field is which denoted it as f of r. And then went on to construct a very important quantity namely the gradient of the scalar function. Now, as the notation suggest as we argued in great detail, this object is a vector quantity. So, in identifying this to be a vector quantity you have actually struck upon an example of a large class of fields which transform exactly in the same manner namely, the class of the vector fields.

So, it is good to go back and actually extract the most general definition of a vector field by looking at the property of the gradient f. Because not only do we have vector fields which can be written as gradient of a vector field, like for example, the electric field is the gradient of a scalar field whenever we have a static situation. But we also have vector fields which cannot be written as the gradients of scalar fields. Like for example, if there is a change in magnetic field, so delta B by delta t is not equal to 0, then delta B by delta t plus curl E equal to 0. So, in that case this changing magnetic field is going to produce an electric field which cannot be written as the gradient of a scalar field. However, it has the transformation properties which are exactly the same as the gradient of this scalar function.

For all that matters we know that the magnetic field itself cannot be written as the gradient of a scalar field. Therefore, the most general vector field is not what is necessarily constructed out of this gradient. However, the most general vector field certainly does have the transformation properties which are exactly the same as the gradient f. So, what we shall now do is to write down the general transformation properties exactly as we argued how a scalar function would look like when it change by coordinate system from moving from one to the other. So, let us proceed to do that now.

Remember that when we spoke of a scalar, we spoke of a number associated at every point. In fact I took the example of the temperature field where at every point in the space I have one particular number namely, the temperature at that particular point. But when it comes to the vector field not only (( )) we have to speak of a number associated at every point, but also the direction associated at that particular point. One particular example is the gas or the air in your lab in your class room.

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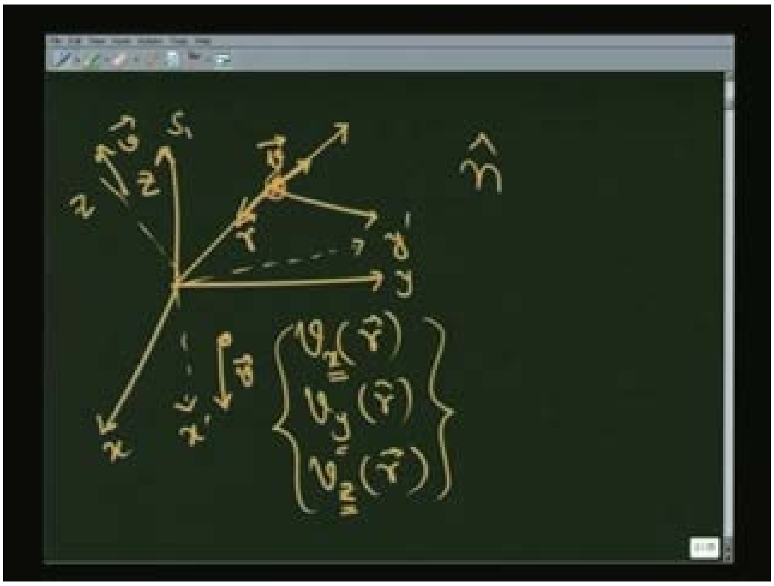

Now, you can imagine that if the windows are open, the air will be moving, therefore, at every point in your classroom which I will actually represent by points in this particular coordinate system, in this particular infinitesimal volume, the gas molecules could be moving in one particular direction. Now, of course, if they are moving in this particular direction there will also be a certain velocity associated with it. So, magnitude associated with it and that together with the direction will constitute the velocity at this particular point. Elsewhere, the velocity might be (( )) in different direction. For example, if I consider this point the velocity could be in this direction, yet another point the velocity could be in this direction, therefore when we speak of a vector field, we have in our mind that at every point we have a triplet of numbers which are actually components of a velocity.

In other words, instead of having a constant velocity which we are used to defining earlier; we have velocity which is a function of space. In general, velocity could also be a function of time. For example, at the space same point let me denote it by a position vector r, at a time t the velocity could be in this direction, at a later time t the velocity could be in this direction. In showing this arrow what I am doing is not only am I changing the direction of the arrow, I am even increasing the length of the arrow which tells you that the magnitude of the velocity has increased and so has the direction. At a later time t, it might so happen that the velocity of the air molecules at this point comes back to the original position - original direction. But then its velocity might further increase or it might start moving in this particular direction. So, what we need is to have some kind of a simple way of testing as to under what conditions I can consider a triplet of quantities to be satisfying these basic ideas that underline the concept of a velocity field. So, how shall we do that?

Well, since I am emphasizing the word velocity or vector again and again, obviously when I am going to define these functions, I am going to define a triplet of functions. So, let me denote it by v x of r, v y of r and v z of r; they are three independent functions and I am giving them the labels x, y and z. At any point r each of these are going to change independently. Now, when I say that this particular triplet is going to transform like a vector what do I mean by that. Now, in writing this as a function of r and in assigning these labels x, y and z, I obviously have in my mind a coordinate system, may be this coordinate system which I will denote by s 1. So, this as usual is my z axis, this is my x axis, this is my y axis.

Now, you naturally ask me question what would happen; if I did not consider this coordinate system, but I rotated my coordinate system in this particular manner. So, I have chosen some arbitrary axis n and I have rotated my coordinate system, because of which my y move to y prime, my x move to x prime and my z move to z prime. When I do that it does not need any great statement to say that the components are going to get mixed up, and how are they going to get mixed up? They are going to get mixed up exactly the same way the components of an ordinary vector that you define are going to get mixed up. So, let us show that now explicitly.

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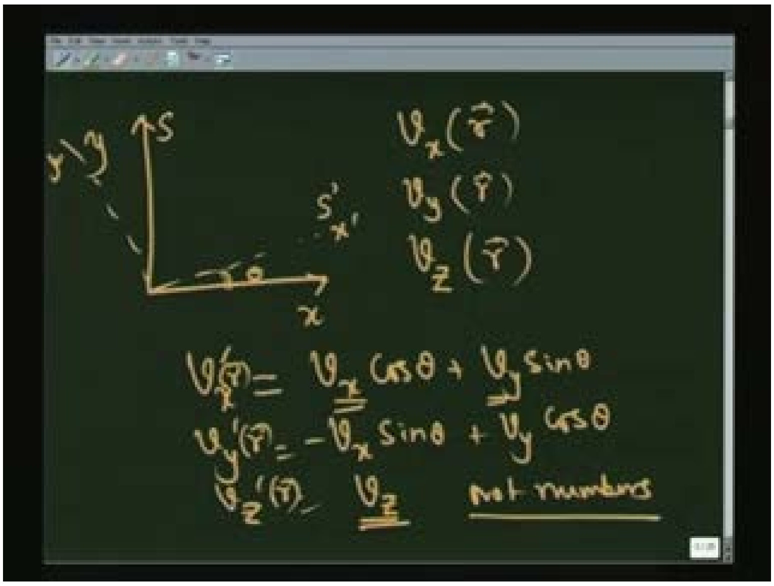

As usual we are not going to clutter up our rotation by writing down the most general rotation matrix; in fact we have never done that so far. So, what we shall do is to consider the simplest example of the rotation in the (x,y) plane rotate the z axis. So, I do not have to write this z axis at all, I have my x, I have my y and I am going to rotate my coordinate system about the z axis by an angle theta; so, this becomes x prime and this becomes y prim. Now, if I look at my components v x, v y, v z of these triplets of functions; how would I say that these three components are going to mix among each other under this rotation? Well, we would say that they would behave exactly the same as the position vector would do. Therefore, we would write v x prime is nothing but v x cos theta plus v y sin theta.

Similarly, v y prime is given by minus v x sin theta plus v y cos theta and then v z prime is given by v z that remains unchanged, because I am rotating about the z axis. So, it appears that this is free of all kinds of some hidden non trivialities, because all that I did was to reproduce the formula for the vector that we have known earlier. But that is not entirely the situation; we have to exercise a little caution. Because these objects v x, v y and v z they are not numbers, they are functions. They are going to change from point to point. So, remember that this is a function of r, this is a function of r and this is a function of r. So, given the fact that these three are the functions of r we have to be a little bit more careful in writing down this transformation formula and we should not just stop here. Let me explain that.

Now, in the right hand side where v x, v y and v z are written in the original coordinate system s; the rotated coordination system shall be denoted by s prime; the v a component v x, v y, v z are also functions of r which are x, y and z. But now when I made a transformation to the new coordinate system s prime, well there is a correct functional dependence in terms of the rotational angle theta, you have v x prime, v y prime and v z prime. But in order to be consistence we should ensure that v x prime is written as a function of x prime, y prime, z prime; v y prime is a written as a function of x prime, y prime, z prime and so is v z prime written as a function of x prime, y prime, z prime.

If I did not do that then I would not be doing a complete job, because I have rotated my coordinate system, I claim to have gone to a new coordinate system. But then I am still employing the (( )) coordinates and that is as unacceptable as staying to use the currency of one country in another country; once you move on another country we should know how to go on to how to make an exchange of the currency from one country to the another country, but then we should use the currency that is used in that particular country. So, what are we going to do? In this transformation formula, this is still a function of r, this is still a function of r, this is still a function of r; now, I have to eliminate r in favor of r prime that is the most important thing to be done. If I did that then we have done our job.

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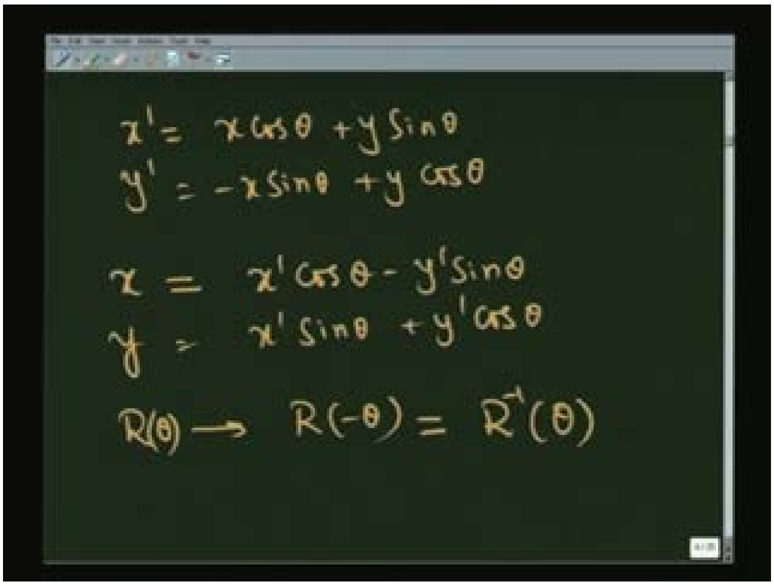

Now, how are we going to eliminate r in from in terms of r prime? That is not difficult at all, because as we saw in very first lecture. If I write x prime is equal to x cos theta plus y sin theta, and write y prime is equal to minus x sin theta plus y cos theta, I know how to write down the inverse transformation and that is simply obtained as x prime x is equal to x prime cos theta minus y prime sin theta, and y is given by x x prime sin theta plus y prime cos theta. In fact, if you wrote this transformation as a rotation matrix we said that you have R theta which will go to r of minus theta. I am sorry I employed a wrong notation here; this theta should come as an argument and not as a label. So, R theta will go to r of minus theta which is exactly the same as r inverse theta. It is good to remember this notation. So now, I am going to make use of this and rewrite my transformation formula such that I am able to completely display what happens to the components when I go from one coordinate system to another coordinate system.

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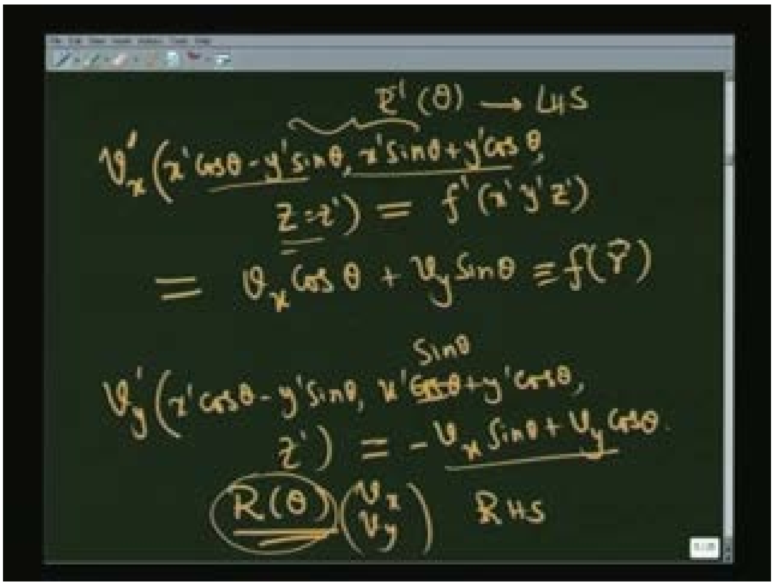

So, let us look at the x component of the field again, I have my v x, v x when I wrote in terms of cos thetas and sin thetas became v x prime; this is in the new coordinate system. But it is still display as a function of x, y, z which should become a function of x prime, y prime and z prime. But just now we wrote down the transformation formula for x, y, z in terms of x prime, y prime and z prime, therefore, what is x? x is nothing but x prime cos theta minus y prime sin theta comma y prime was given by x prime sin theta plus y prime cos theta and then of course, z which is equal to z prime.

In writing this they have replaced the quantity x everywhere by x prime cos theta minus y prime sin theta; you have replaced the quantity y everywhere by x prime sin theta plus y prime cos theta; we have replaced z by z prime of course, the value is unchanged, because we are rotating about the z axis and this object will be given by v x cos theta plus v y sin theta. So, please notice that. Whereas, the right hand side is completely a function of x, y, z, so this if I call as a function, it is completely a function of x, y, z; whereas, the left hand side is completely a function of x prime, y prime and z prime. So which I will write as equal to sum other f prime of x prime, y prime and z prime.

Whereas in the case of a scalar, f prime of x prime, y prime, z prime was equal to f of r, here it is more complicated, because this f of r is neither v x nor v y, but a specific linear combination of v x and v y. In a similar manner, you can write down the transformation formula for v y and v z. Well if you started with a v y, you know what to do, again the same arguments will get repeated, x prime cos theta minus y prime sin theta, x prime cos theta plus y prime x prime sin theta plus y prime cos theta and z prime, and but on the right hand side you will get minus v x sin theta plus v y plus cos theta. That is what we do. So, now we know how to transform a vector from one coordinate system to the another coordinate system.

Now, there is a small point that we might as well take note of at this particular juncture and that is the following thing. Look at the transformation property on the right hand side and look at what left hand side and look at what appears on the right hand side. Now, if you were to look at what is in the right hand side, what operates is simply r of theta; r of theta is actually acting on v x and v y in order to generate the right hand side. Whereas, if you look at the left hand side, look at the argument, I wrote x prime cos theta minus y prime sin theta, x prime sin theta plus y prime cos theta, which is actually if you remember r inverse of theta.

So in other words, when the components are going to transform according to the rotation matrix say r, the arguments shall transform according to the inverse. This is required, because when the components transform according to this rotation matrix r, I still have my function in terms of the old coordinates and those old coordinates have to be written as in terms of the new coordinates. The new components are written in terms of the old components on the right hand side, whereas the old coordinates are written in terms of the new coordinates on the left hand side, therefore, if r of theta occurs in RHS are inverse of theta occurs in LHS. Now, this feature of course is not specific to a rotation in the (x,y) plane; it is characteristic of rotation about any axis by any angle theta, therefore this may be taken to be the characteristic, the defining property of a vector field. Having defined this vector field it is now natural whereas to go on to work out a few examples and study what is it yet that we can learn about a vector field.

Remember when we looked at the scalar field, we asked where does this slope increase, where does this slope decrease, where does it acquire a maxima, where does it acquire a minima; you also ask what could be a saddle point and we evolved a criterion in terms of the gradient of the scalar field to understand that. Now, we need to study in a similar fashion, the properties of this vector field to the extent possible. And two defining properties and two very, very important properties for us are what are known as the divergence of the vector field and the curl of the vector field. But before we go on to that what we should do is to work out a few examples which not only make us familiar with what a vector field is, but there will also be examples which illustrate the basic ideas behind the basic geometry concepts behind a where the divergence of a vector field and the curl of a vector field. So, let us start with a few examples.

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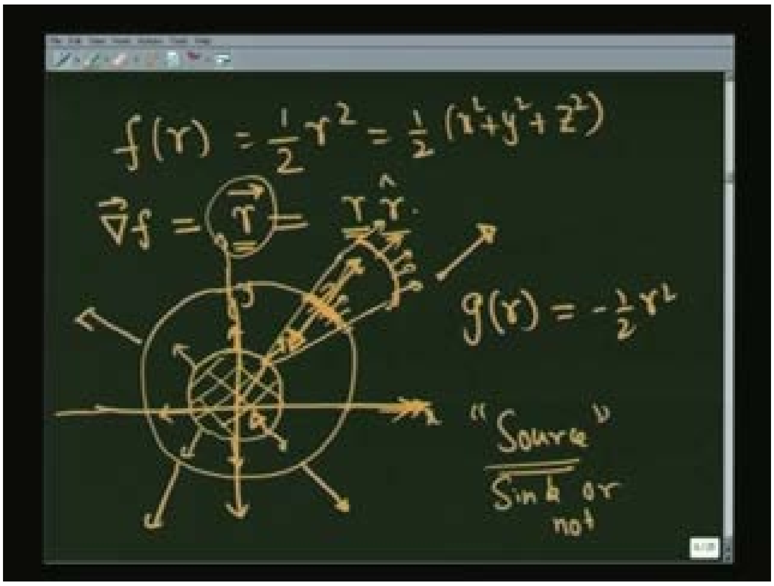

Now, a natural way of generating although not in exhaustive way of generating a vector field is to start with a scalar field and construct its gradient. Now, let me start with the simplest of the scalar functions, is that ok, no square roots, no complicated derivatives. So, let me start with the scalar field f of r which is given by r square; to be somewhat now to make our life somewhat simpler what I shall do is to put a vector of half, you will see why, now let me calculate the gradient of this function. Well this object is nothing but half x square plus y square plus z square by definition. Therefore, what will my gradient will be given by? My gradient f will simply be given by when I do delta f by delta x I get a 2 x, 2 x divided by 2 is x, 2 y divided by 2 is y, 2 z divided by 2 is z, which is the reason I smuggled in this factor of half. So, what shall I do? This object will be nothing but your position vector r; no surprise about that.

Now, what is the meaning of say that this is a vector field? Let us graphically depicted which you are already familiar with and then go on to explore the meaning of it. This of course represents the position of a vector that what does it mean. So, position of a point I mean, so let me draw the (x,y) plane, again my example will be drawn in the (x,y) plane, although it can be generalized to two three three dimensions; so I have my x axis here, I have my y axis here. If I locate the magnitude of gradient f that is nothing but r and the direction is of course radially outward. Therefore, I will write it as a r, r hat, r hat or r cap is nothing but the unit vector in that particular direction which you are already familiar in the ordinary polar coordinate system in two dimensions. Now, what do I do?

Let me draw a circle of radius r, so this unit vector r at this particular point if residing here, it is radially outward. Now, let me solve that assignment given length which is actually proportional to this, in principle it should be equal to the length of this object. Similarly, the radial vector is here, this vector is here, this vector is here, sphere it is here, at this point it is here, at this point it is here so on and so forth, all that we have done is to draw perpendicular to the circle all over the place. Now, let me consider a circle of a larger radius. So, let me elongate this, let me elongate this, let me elongate this. So, now I draw a circle of a larger radius, please imagine this to be a circle.

Now, at this point I will again represent them by radial points. So, this goes like this, at this point it goes like this, at this point it goes like this, at this point it goes like this. In written this what I have done. I am again showing the radial components all right, but I have actually increased the length of the vector. When I have increased the length of this vector, I am saying that at this point the magnitude of the vector is not only radially outward, it is proportional to; in fact it is equivalent to r. In writing this picture, actually we have given a general prescription, although in general it is not easy to write these diagrams of representating representing vector fields graphically on a surface. So, what do you do? Choose any point that you want, the direction of the vector field pick that out, find out the magnitude of the vector field; so, in the direction of the vector field, draw a line or an arrow to be more precise which is having exactly that length.

So, in space at every level point there will be arrows moving in all possible directions. So for example, instead of considering f of r equal to half r square, if I had considered a function which is it is negative, g of r is equal to minus half r square, what would I have done? I would have I would have drawn the vector inwards. Similarly, at this point I would have drawn the vector inverse. So, in this convention an outward vector tells you that the field is spreading out, whereas, the inward arrow will tell you that the field is actually going in. This is one particular example. In fact this is the most familiar example that we are have so far. Because this as I told you in a two-dimensional example is nothing but the unit radius vector.

Although we started with the unit radial vector, it is not necessary for us, but you would always represent the position vector of a given point. It can actually be some kind of a fluid field, the velocity of a fluid. Or if you have a source, for example, if you imagine this to be a lamp, I can imagine it emitting radiation in all possible directions and therefore, in this particular direction it can be the intensity that is going out in this particular direction at this solid angle. Now, obviously I am specifying the direction and I am saying the intensity which is a certain magnitude and the (( )) represented by a vector. All these quantities - the momentum density, the intensity flux, the velocity field all of them can be by represented in this particular fashion.

Now for a moment, suppose I imagine that this is a velocity field; now, what would I conclude from this particular velocity field? Again look at this function r of r, the velocity is increasing as I keep on growing radically farther and farther away, in fact it is increasing linearly with the distance. That means as the velocity fluid moves outwards as it is expanding spherically, if its velocity is increasing that means there is a source – quote-unquote source for the velocity. What is this source doing? The source is increasing the magnitude of the velocity. Now, I need a certain prescription, I need a way of handling, I need a way of finding out what the nature of this particular sources. I can take yet another example for instance, imagine there is a lamp which is of this particular size and it is emitting radiation. Now, what would be the property of this radiation?

Let me consider this solid angle, so at this radius this is a solid angle, at this radius this is the solid angle, whatever is the energy in this angle is also the energy in this particular angle. So, there is some energy that is flowing out, there is some energy that is flowing out. In this case, although the flow is radial, I would not say that there is a source at this particular point. In fact there is neither a source nor a c at this particular point, and how do I know that is not there, because I know that the intensity will fall off like 1 over r squared. Therefore, whether or not there is a source depends not only on the direction of the vector field, it also depends on the magnitude of the vector field at various points as it keep moving inwards and outwards. If I look at the function g of r minus half r square, it is the other way round. The fluid can be imagine to be moving towards this particular centre and as it moves it is accelerating.

It is accelerating its acceleration become right, because it is in this particular direction, and it is accelerating not in the sense of absolute magnitude, but it is accelerating inwardly although its magnitude is decreasing, therefore, I would say in that case what I have is not a source, but I have is a sink. So, the lesson that you have learnt here is that this simple particular example of the position vector r can be used to represent many, many quantities and depending on how the function behaves, it can be we can ask whether there is a source or a sink or not. Even before we give a precise definition of whether there is a source or sink, let me give a general prescription of constructing this functions and that is not difficult at all.

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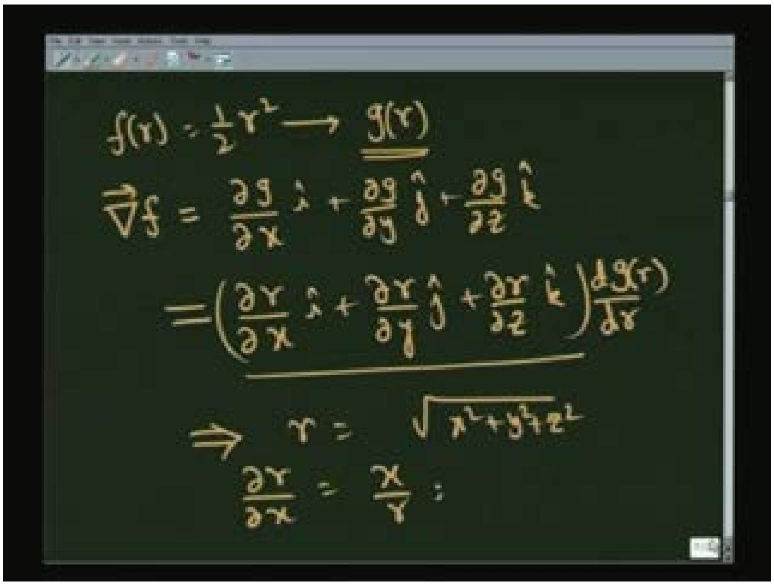

After all when I wrote my f of r is equal to half of r square, there is no special reason to restrict myself to that. If I wanted radially out flowing or an inflowing function, because I can replace this by any general function which I shall say is in g of r - any generic function. What do I do? I calculate the gradient of this generic function in order to find out the vector field. So, what is the gradient of this function? I will leave it as a simple exercise for you people to work it out. Because there is nothing much to do except differentiation, what do we do? This object will be simply given by delta g by delta x i plus delta g by delta y j plus delta g by delta z k, and this g is only a function of r. Therefore, if I know what my delta r by delta x is, if I know what my delta r by delta y is, if I know what my delta r by delta z is, I can calculate these. So, what am I trying to say?

This object is nothing but delta r by delta x i cap plus delta r by delta y j cap plus delta r by delta z k cap multiplied by simply d g by d r. The ordinary derivative gets replaced by the sorry the partial derivative gets replaced by the ordinary derivative and the vector nature is sitting here. So, please simplify this. This object will be simply given by simply… Let me work it out, r is simply given by root x square plus y square plus z square therefore, delta r by delta x is simply given by x by r so on and so forth. So, if I did that my gradient of the function has a very simple form. What is the simple form of this gradient function?

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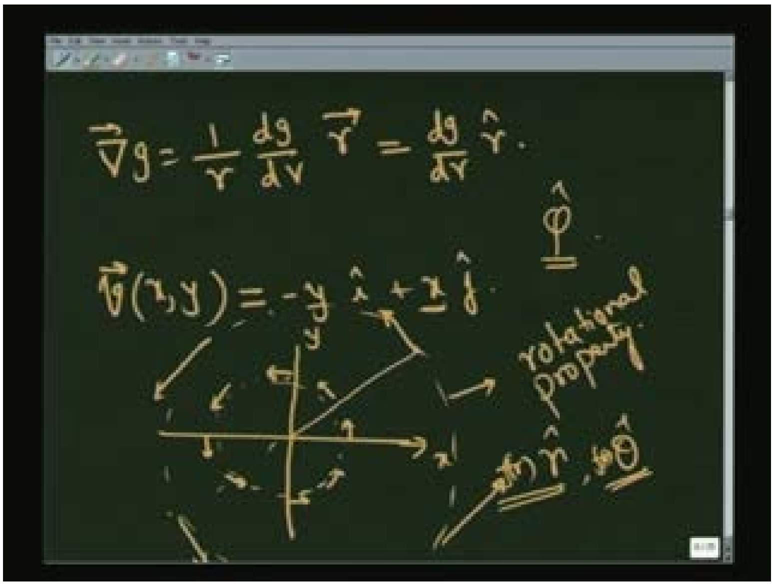

Well my gradient g will be given by 1 over r d g by d r into the gradient of r, which is simply the same as d g by d r r hat. So, whether there is possibly a sink or not depends on the sign of d g by d r. It is completely radial. And whether there is absolutely a sink or not, obviously also depends on the variation of the magnitude of d g by d r. But at this point let me not get into that, because that is what will be given by the divergence of my vector function. But I shall go on to give one more example which is sort of complimentary to the function that I have written here. So, what would the function be? In order to write down that function I shall not consider of three dimensions, but I shall consider only a two-dimensional example and that is the following. So, let me take (x,y) only I am not working in three dimensions, so what happens. Consider this function v(x,y) given as follows, which is nothing but minus y i hat plus x j hat; do not be surprised by this function you are all familiar with this function, in fact this is nothing but representative of the variable theta or the vector that represents the variation in the theta, the theta vector in your polar coordinate system, after that matter the vector phi in this spherical polar coordinate system, please remember that. But now I have a different view point altogether in writing this particular function and what is that.

So, again I want to represent this function; so, I draw my x coordinate and my y coordinate, so how am I going to do it. Let me look at the x axis, along the x axis this vector is along the j direction, because this is x, y is equal to 0 and its length depends on the value of x. As I keep on increasing the value of x its magnitude keeps on increasing. Therefore, if I look at this particular point the vector is like this. Let me come to the y axis then what happens, if I come to the y axis, it is the x component x and y and notice there is a minus sign here, therefore along the y axis positive y axis my vector is along this particular direction. Well come to the negative x axis, I come back here, but it did not except a minus sign, it comes here, come to the negative y axis then the vector is here, somewhere in between the vector is here, the vector is here, the vector is here, the vector is here. In other words, this is a vector which moves along a circle.

So, if I could draw a circle which I shall denote by a curl E by a dotted line, all that we have done is to draw tangents to this circle. This is what this vector is going to represent. Now, what is going to happen if I am going to increase the radius of my vector? Well draw a circle here if you feel like, so I am drawing a section of a circle here. Here at the same angle my vector will have a larger magnitude, because it is proportional to x. Whereas, if I come to this point it will be along this direction, if I reach this point it will be along this direction, at this point it will be along this direction, so on and so forth. They continue to remain tangential to the surface, but their magnitude is increase. What is the difference between the earlier radial vectors, which is generically represented by d g by d r into r hat compare to this; here there is a certain flow.

Here again you can imagine this to be some kind of a velocity field or a flow field, but here the flow field is completely circular, whereas in the earlier example the flow field was completely radial. So, in one case, you can actually imagine may be there was a tap in your garden, which actually spreads water radially outside, whereas here this can be imagined to be a bucket of water which you spread the water is flowing outside. In one case, the there is a out flux of water from the inner region to the outer region, and of course, if there is a sink there is an out flux of water towards the sink from the outer region to the inner region, whereas here there is no out flux at all, there is no water that is going flowing out side of the surface, but it is going round and round. In other words, this has a rotational property.

So, we have actually reconstructed two simple vectors namely, r hat, theta hat; given them from arbitrary magnitude of a function of r, and then we are saying that their distinct properties, one of them has a radially out flowing or in flowing property, another has a rotational property. I also gave you a physical example of taps and sinks and rotations, however, the question that we have to ask is how do I find out by looking at various properties of these vector, whether there is a tap at a particular point namely, whether is a force at a particular point, whether there is a sink at a particular point. Or if I imagine this particular rotational flow to be coming, because of this stirring of a stick; suppose there is a bucket full of water, I take a stick deep in and stir it only close to the center of the bucket, there of course it will cause a flow all over the place, but the flow at any given point is simply because of the stirring close to the center of the bucket.

Or it might so happen I take two sticks, three sticks, four sticks or I may take a broad stick like in a grinder which keeps on stirring I need a property, and these are what are given by the grade and divergence and the curl. So, we shall go on to discuss that. So, we are now ready to discuss two more important properties of the vector fields namely, the divergence and the curl. However, again before I proceed I jump on to discuss them, there is yet another point that we have to find out and that is what is the meaning of the components of this vector field, and want do they tell you even we before we start differentiating them.

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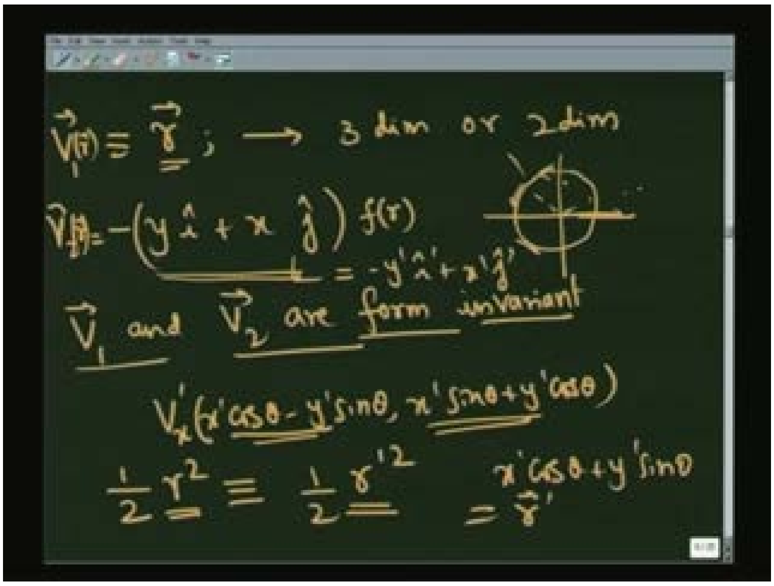

It is no accident that I took this three examples two examples; one is proportional to r, this is in any dimension including three dimensions - three dimensions or two dimensions. And the other example was what I wrote as the theta. So that I will write it explicitly, I will write minus y i hat plus x j hat. What is important for me is not this minus y and x, I can put any f of r here, because it can keep on increasing or decreasing. What is the property of these vectors? These vectors have a remarkable property and that is these vectors. So, let me call it as v 1 of r and let me call this as v 2 of r. These have the remarkable property that v 1 and v 2 are form invariant. This is a very, very important thing. What do I mean by form invariant?

Remember, when I define the transformation properties of the vector, I said you have the quantity v of x prime and I ask you to explicitly replace x by x prime cos theta minus y prime sin theta, replace y explicitly by x prime sin theta plus y prime cos theta, when we do that in general the functional form changes. No surprise about it, because even in the case of a scalar function, the functional form changes. For example, when we looked at the temperature dependence in our previous lecture, if we gave a dependence on z and the minute we rotate it our coordinate system, it acquired dependence not only on the z coordinate, but also on the y coordinate.

But if you look closely at this function v 1 of r proportional to r, it was obtain from the scalar function half r squared, and half r squared is the same with respect to all rotations, because this is identically equal to half r prime squared; whatever may be the angle of rotation, whatever may be the axis of rotation. So, this property that the scalar function has the same form in all coordinate systems is simply tells you that under rotation, the equation of a sphere remains the same, it is not going to change is inherited by this vector property also. Because the minute I write x prime cos theta plus y prime sin theta you already identify that to be r prime, you do not have to again try to rewrite r in terms of r prime. In other words, the transformation for more on the right hand side accomplished for you, this transfer this exercise that you have to perform automatically, you do not have to repeat it.

Now, if this is going to be fairly obvious to you, I would invite you people to verify that it is also true for this function which is not all that obvious. Geometrically of course it is obvious, because as I told you if I am going to write a circle and if I am going to write tangents it does not matter whether I am going to look at this coordinate system of this rotated coordinate system. However, it would be good for you to take your paper and pencil, and explicitly verify that this is also form invariant. What do I mean by that? That this object can be written as minus y prime i prime plus x prime j prime; it automatically has a form invariance therefore, you do not have to worry about writing these things.

In other words, whenever you have a function which is proportional to r, like this, a vector field which is proportional to r then you can know that there are no preferred directions for that particular function; there is only a preferred point of origin. But on other hand, whenever you have a function which depends only on i and j of course, it has to be independent of z, so that is why going to the cylindrical polar coordinate system. Then you know it is something like the theta vector and again there is no preferred coordinate axis. That is something that you can remember. In contrast let me consider in even simpler vector field apparently, but transformation wise which can be more complicated in this and that is simply a constant vector field. What is that constant vector field?

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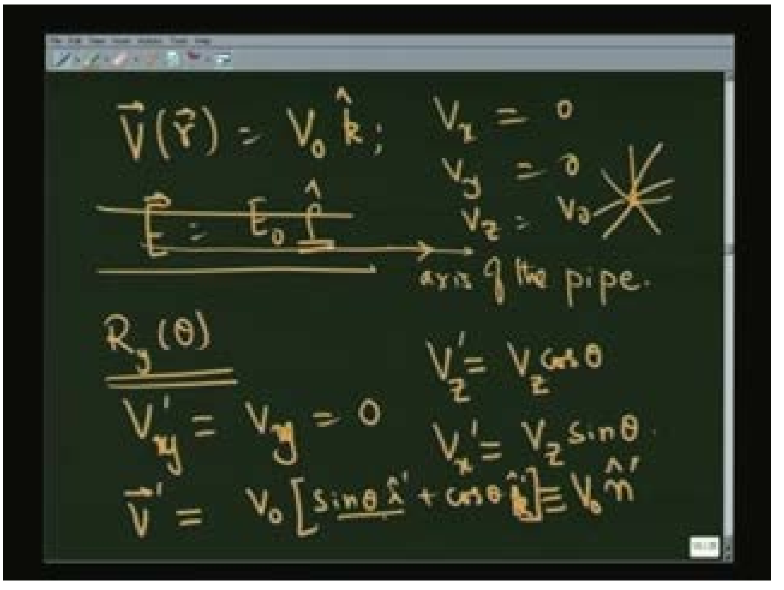

I will now write v of r is nothing but some constant which I shall denote the v naught and let us say it is along the k. Look at this function, this for example can represent the velocity field of a fluid flowing in a straight pipe; incompressible, nice, smooth flow. So, at every point the velocity is given by v naught k where k is actually the axis of the field - axis of the pipe. Or for example, it can represent the electric field produced by an infinite sheet of charge in the (x,y) plane to the right of the fluid let us say; the right of the plane let us say.

Now, suppose I were to rotate my coordinate system, what happens, when I look at it component wise; well I need a axis and I need an angle, since it is already around the z direction, let me rotate about the y axis by an angle theta. So, when I am doing that my x component remains the same. If I were to open it up and write it completely component wise, I am going to get v x is equal to 0, v y is equal to 0 and v z is equal to v naught. That is the meaning of this statement; this is the compact notation. Now, when I rotate it what am I going to do? I am going to express v x prime, v y prime, v z prime in terms of v x, v y, v z according to the transformation formula. Let us write that explicitly. Now, v x prime is simply equal to v x is equal to 0, because the x component read does not change, I am rotating about the x axis, I am sorry I am rotating about the y axis, I made a mistake there.

So, what is happening is that, my v y prime is equal to v y is equal to 0; I am rotating about the y axis. The x and the z are going to get mixed up with each other; now, let me write down the z component here, what is my v z prime? My v z prime will be simply given by v z cos theta, well it is going to be plus v y cos sin theta but v y is identically equal to 0 therefore, I get v z prime equal to v z cos theta. On the other hand, my v x prime will be simply given by v z sin theta. These are the new components. If I were to now write down my vector in the new coordinate system, and v z is identically equal to v naught, my v prime is simply given by v naught sin theta i hat plus cos theta j cap; that is what we did. This is a very, very simple example again where I do not have to worry about expressing my argument, because there was no dependence on the argument to start with, this was a constant vector field, because the magnitude was constant everywhere. We will come to slightly more complicated examples at a later time.

Now, what is sin theta J i prime plus cos theta j prime? Again I made a mistake. This is k prime; this must be k prime I am sorry. What is the meaning of this? This is nothing but the component of my vector k in the new coordinate system. And notice if I am going to look at this particular form in the original frame I had v of r equal to v naught of k, whereas here my v is not v naught of k prime, that would have been the form invariance, it is given by sin theta i prime plus cos theta k prime which I will write as v naught unit vector n prime. And this n prime is not the same as k prime. If m prime were the same as k prime then it would have been a form invariant vector, whereas n prime is not the same as k prime, but m prime is actually the same as the original k, we see that there is a complicated mix up. Now, why is it that in the original example that we had a form invariance and in this example we do not have a form invariance. The answer is fairly simple, then the original example what we had is what we call as isotropy; all directions were equally preferred and it was radially flowing outwards.

So, when you rotated there is no reason for you to chose x, y, z either in this particular fashion or in this particular fashion or in this particular fashion, because there is a spherical symmetry in the problem. But whereas, in this particular problem, if you look at the velocity flow, there is a preferred direction namely, the axis of the pipe - axis of the straight pipe and that is reflected in the transformation formula sin theta i prime plus cos theta k prime. In other words, the long and short of the moral that we have learnt from this is that what we should do is to look at the components of each of them and study whether they are form invariant or not, and more often than not the vector is not going to be form invariant. That means there are implicit directions which are there already in the definition of the vector field, which tells you something about the physics of the problem. For if this what to represent as I told you the electric field, then you immediately know that somewhere far to the left, there is an infinite sheet of charge which is producing this field.

For example, if I write my electric field for example to be given by E naught into rho cap, rho cap is the unit vector radially in the (x,y) plane, then you immediately know that at the origin there is an infinite line charge distribution, there is a preferred direction. Whereas, when you write your electric field from a point charge or a spherical distribution of charge it will be radically outwards, because there is no preferred direction. Many times it is convenient, it is advantageous, in fact it is advisable for us study the geometry of these kinds of configurations before we solve the problem, because it will tell you how the currents, how the charges and how the sources must be distributed; so let us remember that.

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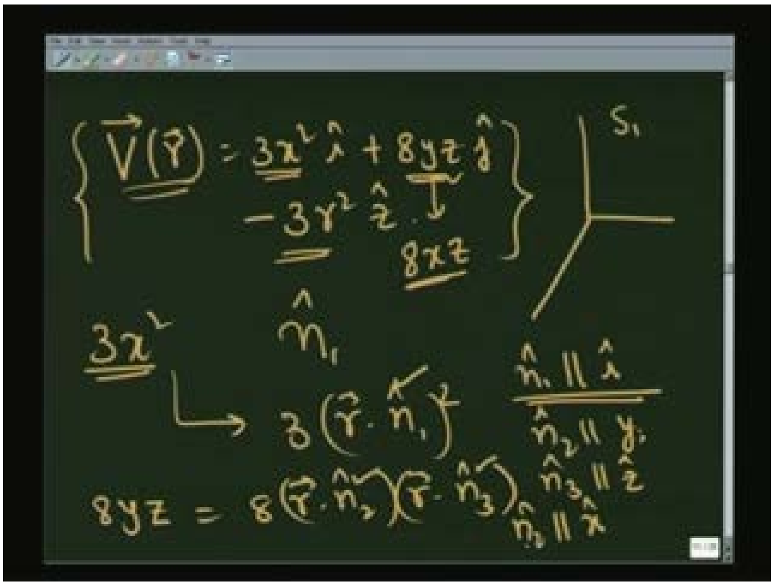

Now, you can consider a complicated example of this kind. For example, if you look up a book like Griffiths of Purcell, they will give you all kinds of vector fields and they will ask you to calculate divergence, curl or line integral, so on and so forth. So, let me write one vector field at random. So, let me write a vector field v of r to be 3 x square i cap plus 8 y z j cap minus 3 r square z cap. Now, well I have given you three components 3 x square is the x component of the vector field, 8 y z is the y component of the vector field, 3 r square is z component of the vector field. Now, you might ask me what is the meaning of this; of course, you know how to transform, the minute I rotate my coordinate system v x, v y, v z mixed up with each other, you will write 3 x square cos theta plus 8 y z sin theta, then you replace x by x prime and y prime and z prime; y z by y prime, z prime, x prime, so on and so forth. That is not what I am interested in.

The question that we can ask is what is the underline geometry if I were to give this kind of a vector field. The clue to that is to remember that whenever I expand a vector in terms of the basis vectors, these objects have to be scalars, in other words they have to be scalar fields. Because we know from our vector algebra that if I take a vector and if I take another vector, and if I construct a linear combination that is also a vector. Except that the co-efficiency of linear combination are changing from point to point in space and perhaps with time too.

If I remember that then it is not going to be difficult at all. Let me consider 3 x square, how can I generate this from 3 x square. Well I would say I got this because there is one preferred direction n 1 inherent in this problem; what the physical significance of this n 1 is that is something that will revel to us, only if I tell you the nature of the vector field, whether it is a momentum density or whether it is a velocity or a magnetic field at this point I do not know what it is, but all that I know is this n 1. And in that case 3 x square can be simply written as 3 r dot n 1 whole square. Now, what is the nature of this n 1? In this coordinate system that I wrote, so let me call it as the coordinate system s 1, in this coordinate system n 1 turned out to be parallel to the i axis. Is that right? That is what we have done.

Now, let let me look at 8 y z we immediately know what to do that; 8 y z will be simply written as 8 of course is a number, you do not have to worry about that, I will say r dot n 2 into r dot n 3. Now, I know what the directions of n 1, n 2 r, n 2 is simply parallel to the y axis and n 3 is simply parallel to the z axis; r square is invariant and it remains in that particular form. Therefore, you know if I gave you the vector field and I ask you to calculate curl, divergence, line integral and what not, you know that whatever may be the source, whatever be the nature of this vector field, the physics of this field, the geometry of the field is exhausted by giving these vectors n 1, n 2 and n 3.

Now, we should remember that there can be multiplicity of vectors, there is no reason where there should only be 2, 3 or 4 or 5 there might be any number of vectors that define the physical properties of these vector fields, but those can be easily obtain by looking at this. So, in this particular coordinate system n 1 is parallel to y, n 2 is parallel to y, n 3 is parallel to z. For example, if I had replaced this 8 y z by 8 x z, what would have happen? In that case, n 2 would not have been parallel to y, but n 2 would have been parallel to x. There are two axis n 2, both of them are parallel to n 2 or to put it even simpler fashion n 2 is parallel to n 1, there are only two independent vectors and not three independent vectors. So, there will be a large number of vectors which define what the nature of these coefficients will be and they will actually reveal to us something about the physical and geometric origins of the vector field.

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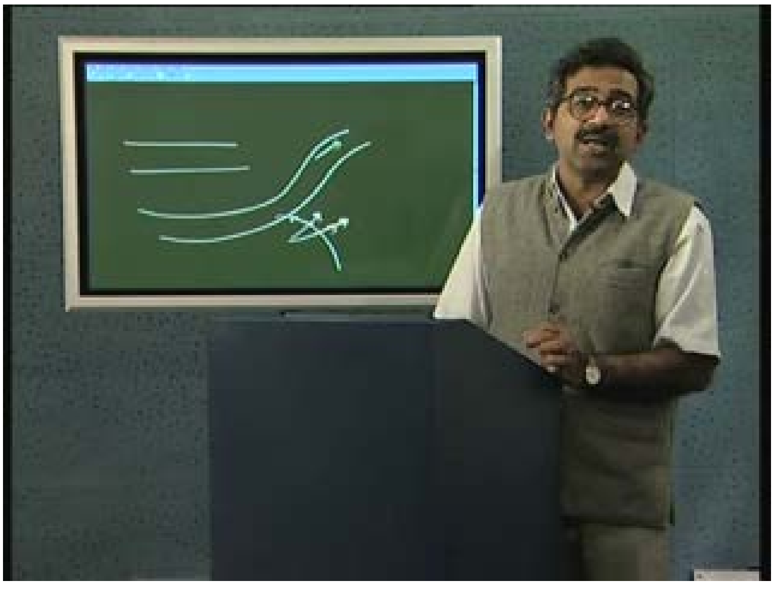

Now in general we know that vector fields are not going to be form invariant, because there is more than one axis that is going to define. May be my my pipe is not straight, but it moves in this particular direction let me not mention this figure. May be my pipe is not straight, but it curves in this particular direction and therefore, there will be many, many axis. If it is still only in the (x,y) plane then there will be a preferred direction namely, n which will tell you something about the property of the vector field. Whereas, within the plane both x and y will be playing an important role. Therefore, this is something that we should always analyze before solving the problem.

And physics wise, jargon wise, this is what we say when we say that is what we mean when we say that a problem has axial symmetry or cylindrical symmetry, the problem as spherical symmetry, a crystal is also is has no preferred direction, in a material has no preferred direction, a crystal is uni-axial, a crystal is bi-axial, so on and so forth. All that is reflected in the nature of the vector field that we are going to write. Of course, if it turns out that this vector field can be written as the gradient of a scalar field, it will also be reflected in the nature of the scalar field that we have in mind.

So, now we have… Therefore, seen that in general the vector fields are complicated in terms of their functional form; the vector fields are also complicated in yet another sense and that is they neither like to be radial or rotational. At any given point if I look at a vector field and if I were to draw a surface, it is perfectly possible that it has a tangential component and that it has a radial component. The tangential component is what is going to give you the curl of that vector field, the rotational property of the vector field, whereas, the radial component is what is going to give you the divergences property or the radial properties of the vector field, we call it as the longitudinal property of the vector field.

Now, we are not simply interested in the rotational or the radial flow of the vector field, we also want to know whether there is a source or a sink. That is what I said at the very beginning of this lecture. In order to study that we know formally introduced what is a divergence, what is a curl and then proceed to study their properties. What I will do in this lecture is to merely give you the definition of the divergence and curl. In the next lecture, I will study its properties in great details by looking at various integral theorems namely, gauss divergence theorem and stokes theorem and that will completely give us the geometry properties. What is the divergence of a vector field give us; let us ask this question divergence of a vector field.

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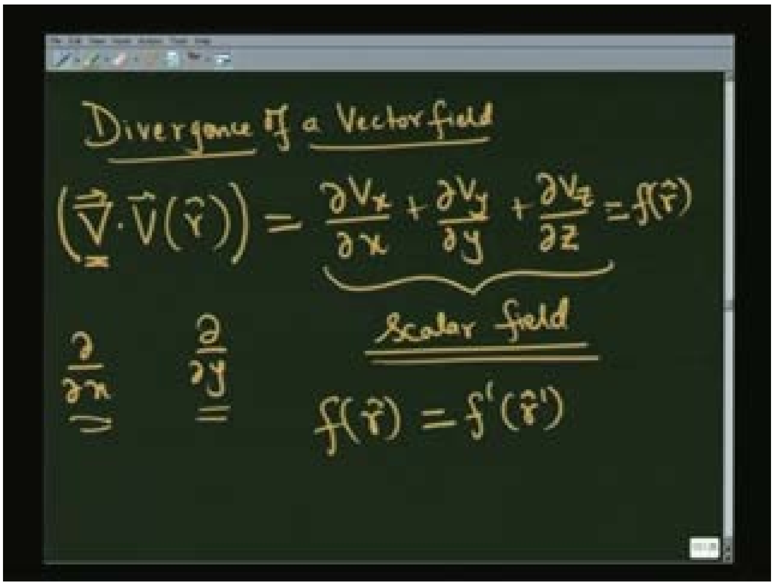

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Let me make a vector filed V of r; and now I am not going to construct an ordinary derivative of this vector field, but I am going to construct, what is called as a divergence, and I shall denote it by this notation. So obviously I am constructing a scalar quantity. In the last lecture, when we discuss the gradient, I told you that you should be very careful as to where we replace this del, whether we replace, whether we write it to the right of a function or the left of a function; here I am writing it to the left of a function, and in fact to be doubly careful, I am going to enclose it in a bracket.

What is going to happen? This gradient operator, which has vector properties is going to act on this vector field, which is a triplet of functions, to produce for me a scalar field; and how do I define that; that is simply given by delta V x by delta x plus delta V y by delta y plus delta V z by delta z; if he did this, then we have produced the scalar field. Now a field does not become a scalar, simply because I employed a notation; we have to explicitly demonstrate that.

Remember when we looked at the components of this gradient, delta by delta x delta by delta y, I actually employed the properties of partial derivatives, when from one coordinate system to another coordinate system, and show delta by delta x and delta by delta y transform exactly like x and y. Same thing works out with V of r; in a similar manner, what you should now do that is something very, very important, before we go on to study the curl and the divergence in the integral theorems, is that you should construct this object divergence of vector field and convince yourself that the right hand side is indeed a scalar field.

So this definition satisfies all the properties of a scalar field, meaning if I denote it as f of r under a rotation, f of r will be numerically equal at every point, except that it might have a different functional form, which I will represent it by f prime of r prime; this is something which is essential; this object is what is going to tell us, as to whether there is a sink which is located at a point or a source which is located at a point or there is neither a sink or a source located at the particular point. Let me also formally give the definition of the curl, before we go onto discuss other things, and that is the analog of the cross product of a vector.

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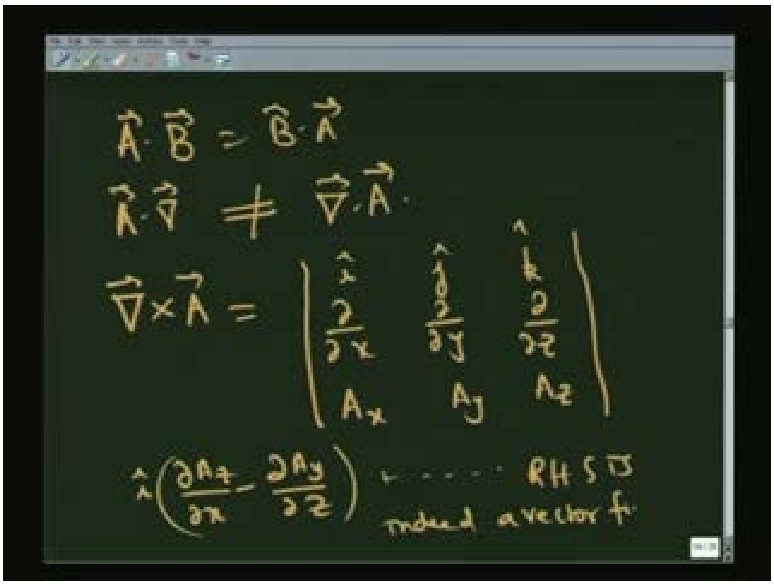

The divergence of V is the analog of the dot product of two vectors A dot B, except that We should be sensitive to the ordering of A and B; for two ordinary vectors A dot B is equal to B dot A; A dot B is equal to B dot A, but if one of them is an operator like a derivative, then we know that A dot del is not equal to del dot A, we know that we already discuss the meaning of this. So in a similar manner let me define what is called as the curl of a vector field; what is that object; curl of a vector field is exactly like a cross product except that if I define something like B cross A; I would have written B x

B y and B z, but in this place, I am going to put the partial derivatives del x, del y and del z.

So the simplest way of representing this is obviously in terms of the determinant i j k; I will put a delta by delta x here; I will put a delta by delta y here; I put a delta by delta z here; and now I have my A x; I have my A y; I have my A z; this is what I have. Well we know how to expand the determinant; this will be unit vector i del A z by del x minus del A y by del z plus so on, so forth; this object constitutes the vector field. Again I urge you that please go home, take a paper and pencil, and verify for you that RHS is indeed a vector field; in fact something more, for those of you are enterprising, you should verify not only is this a vector filed, in fact it is an axial vector field; that is under reflection, it does not change its sign. If you do not know what are axial vector field is at this particular point, you need not worry too much about it; but if you are already familiar with it, it is good to verify.

So in summary, what we have done today is to define carefully what a vector field is, made a distinction between what is a form invariant vector field and what is not a form invariant vector field. A form invariant vector field tells you that there are no preferred directions in the physics of the problem or the geometry of the problem. If a vector field, which does not a form invariant tells you that there are inherent physics directions, inherent direction to the problem; then without describing the meaning, without discussing the physical or geometric significance, we have defined two objects namely the divergence of the vector field and the curl of the vector field, as to what their physical significance is is something that will take up in the next lecture. And the next lecture is going to conclude the mathematical preliminaries required for our course, and then we launch on to study various phenomena in electricity and magnetism.