Engineering Mechanics
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Module – 03
Lecture - 01
Properties of Surfaces - I

In the previous lecture, we have dealt with forces, which act on a point. So, for example, I took a mass on which the force is acting.

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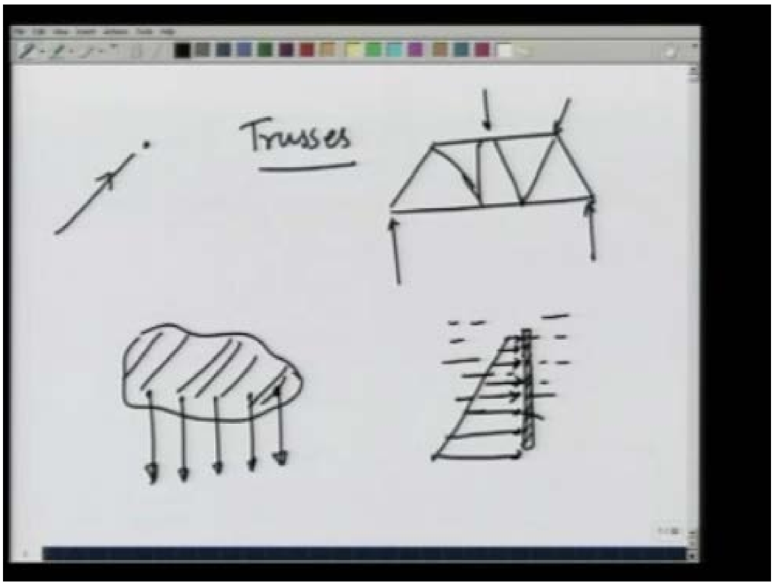

Then, we took trusses and dealt with forces acting on the joints of the trusses. So, for example, the one force could act here, one force would act this point; one force would act at this point and so on. Similarly, when we considered frictional forces we consider the forces as F, they acting at one particular point.

However, in nature the forces are distributed. For example, if I take an extended body and consider the gravitational force on it, on each point there is a force acting is net effect is felt at a certain part, which we know as the center of gravitation. Similarly, if I take a plate submerged, say in water the force acting on it due to the pressure of water is also distributed not only it is distributed its magnitude is different at different points.

So, deeper you go; larger it is near the top, the force is not that large. So, these are the arrows that are showing the force due to the pressure. This is also distributed force, which varies with the depth of water, and now we want to develop mathematical techniques to deal with such forces. In this lecture I am going to develop concepts such as Centroid of an area, which is related to the first moment of area.

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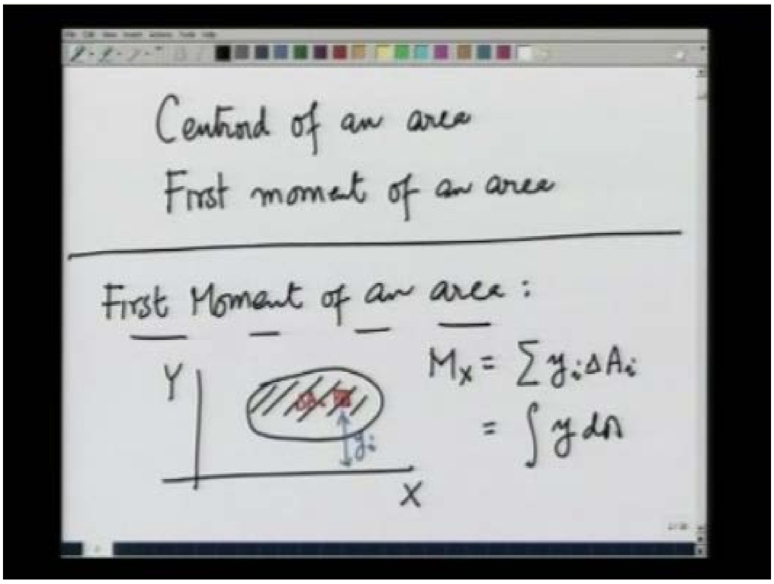

And then see how these concepts can be applied to mechanics or statics in particular. So, let us start with the concept of the first moment of area. I am first going to give you the definition, and then later we will employ in certain static situations. Suppose, I am given a plane area like this and a coordinate x axis, set x and y, I define the first moment of this area M x about the x axis as summation, y i delta A i, where delta A i is a small area. Here, delta Ai at a distance of y i from the x axis.

So, what I am doing? I am talking small element elemental area delta Ai and multiplying it by the distance from the x axis, and summing it up in a continuous distribution. You know, from your course on calculus this is nothing but, y d A. That is the first moment about the x axis. In a similar manner, we define the first moment about the y axis.

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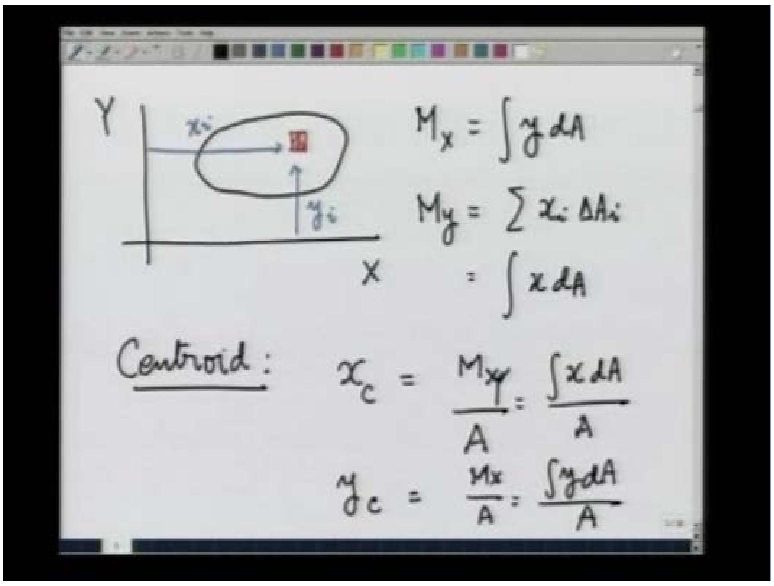

This is the area, and I took a small elemental area here at a distance y i from the x axis, and a distance x i from x y axis. This is x axis. This is y axis, define y first moment about the x axis as y d A and similarly, I can define M y. This is a first moment about the y axis as summation x i delta A i within the limit of continuous area becomes x d A. These are the definitions. So, we have first moment about the x axis and the first moment of the y axis using the concept of first moment.

I can define the centroid of this area centroid coordinate x, coordinate X c is defined as first moment about the y axis divided by the total area, which is summation x d A divided by A, And similarly, the y coordinate is defined as M x divided by A, which is summation y d A divided by A. Now, let us do certain examples using these definitions to calculate these quantities.

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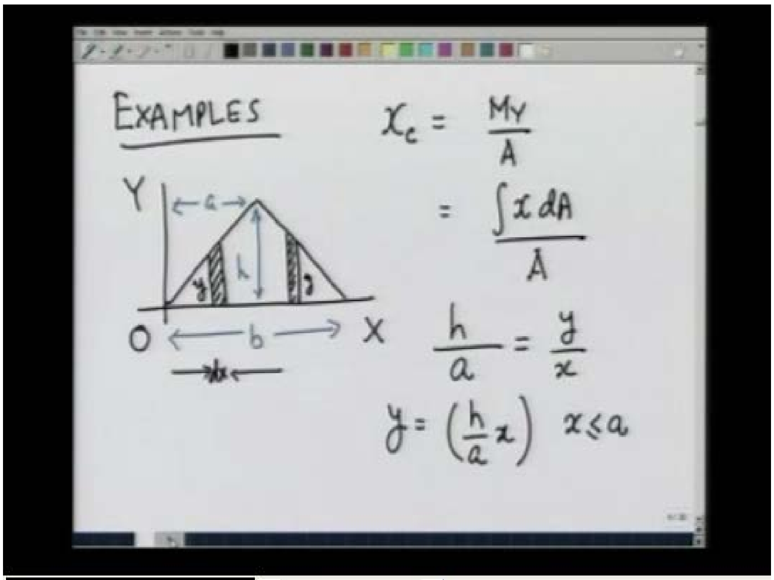

As a first example I take a triangle one corner of this triangle at a is at the origin. This is the x axis, this is the y axis. The height of the triangle is h. This distance of vertex from the y axis is a and other corner is at distance b. I wish to calculate at first moment about the x and y axis and the centroid X c and Y c. So, let us do that. Let us first calculate X c, which is related to M y divided by the total area, which is equal to integration x d A over the total area to calculate x d A or the moment about the y axis. Let me take a small area d A, like this parallel to the y axis.

This distance I will take to be d x, the height of this is y from similarity of triangles. I know that h divided by a is going to be equal to y divided by x and therefore, y equals h over a x. This is true as long as is x as below a for x greater than a. I take small area here, whose y is going to be different and that also we can calculate. Let me now go on to the next page. So, that the calculations become all clear. This is the y axis, x axis.

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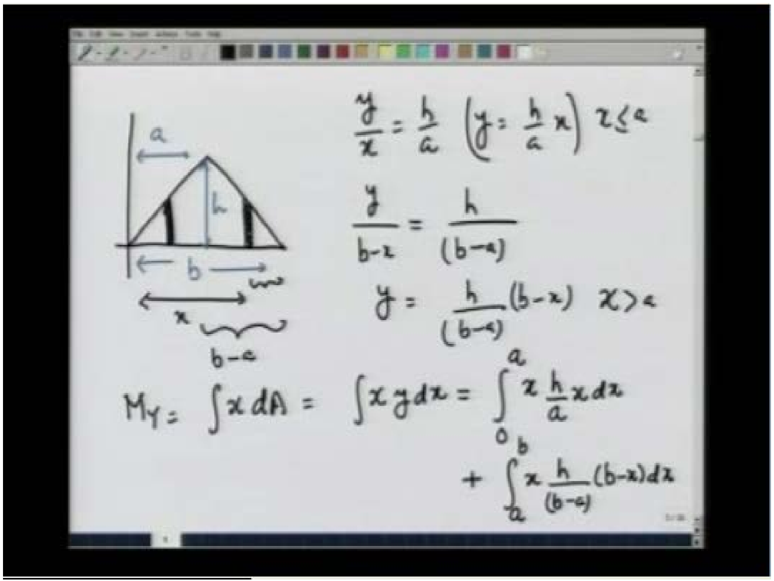

This is the triangle, this is the b, this is a, this is h 1 strip. I took here either strip, I am taking here for the first strip. I know that y divided by x, was equal to h over a or y equals h over a x less then equal to a for the second strip. I am going to have this distances x. So, I am going to have y divided by b minus x that is this distance y divided by b minus x is going to be equal to h divided by b minus a b minus a, is this distance ?

Therefore, y equals h over b minus a times b minus x. Now, I am ready to calculate the moment about the y axis moment, about the y axis, M y is equal to integration x d A, which I am going to write as integration x y d x, where y d x represents the area of this strip or this small strip. This I am going divide into 2 parts because behavior of y with respect to x is different for x up to a and for x greater than a, this by this for x greater than a. I am going to this as integration 0 to a x y up to a y is h over a x d x plus a to b x h over b minus a b minus x d x. That is M y and you calculate this integration. You can perform easily; you get an answer for M y.

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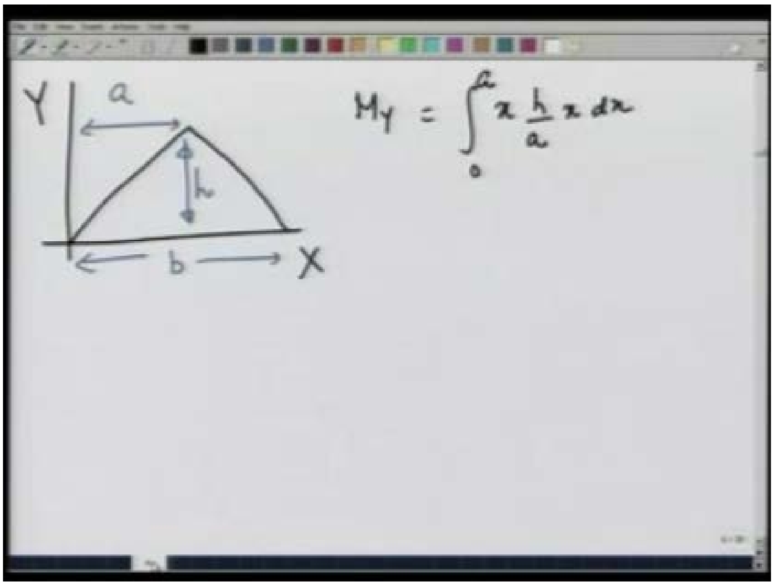

Let me make the figure again this is b, this is a, this is the height h, this is the x axis. This is a y axis and for M y, which I wrote as 0 to a x h over a times x d x. Let me see, we wrote over a.

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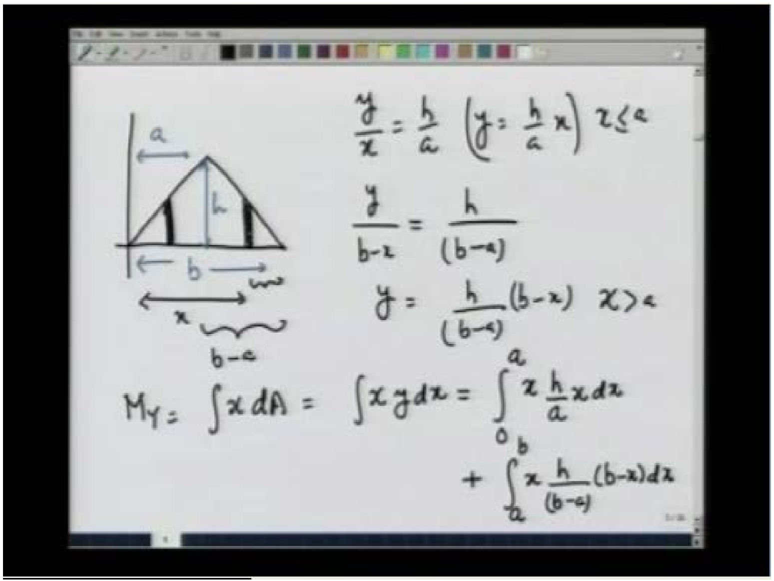

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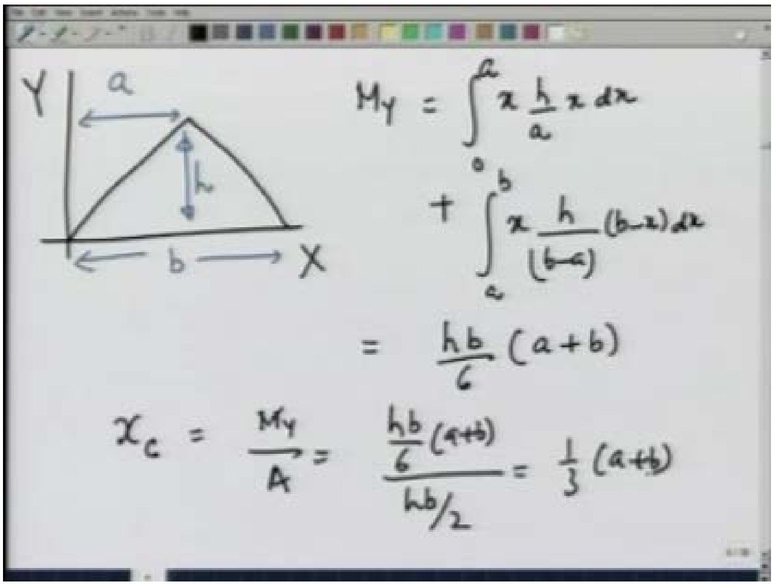

Yes, and plus a to b x h over b minus a b minus x d x, and we do this integration it comes out be h b dived by 6 times a plus b. And therefore, x centroid is equal to M y divided by the area, which is h b over 6 a plus b divided by h b divided by two. And this comes out to be one third of a plus b. That is the x coordinates of the centroid of the triangle. Let us now calculate the y coordinate.

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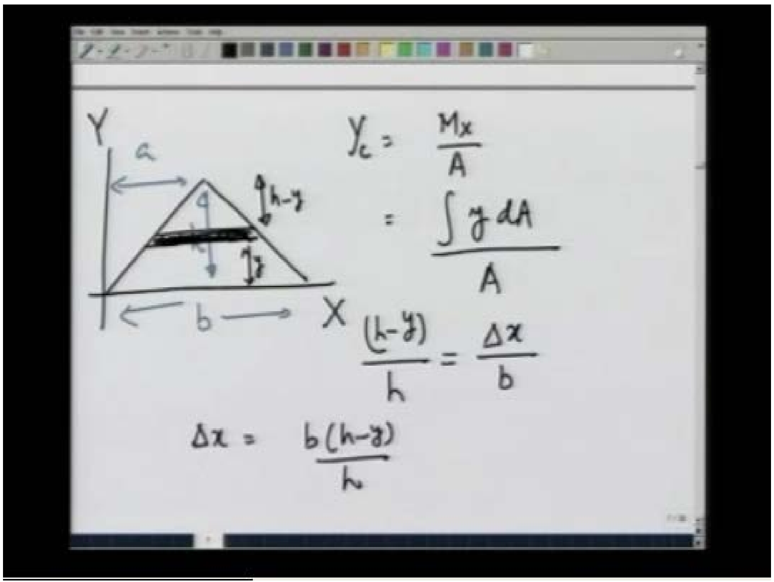

Again I make this triangle X Y b a h and to calculate the y coordinate. I take elemental area d A, like this we call that Y c is equal to M x divided by the area, which is the integration y d A divided by the area. So, add a height y. I take this is small strip multiplied by y, and then calculate the area. We can see from similarity of triangles that I am going to have Y h minus y. This is h minus y divided by h is equal to the width of strip, which let me call delta x divided by b. Therefore, delta x is equal to b h minus y divided by h. This is the length of the strip. And therefore, we can write make the picture again.

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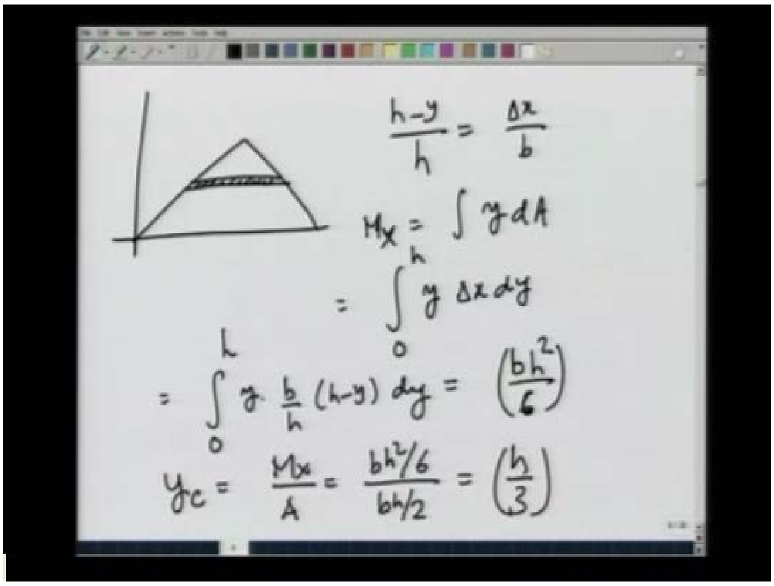

I am taking this strip. I am not writing b and a, this time we got h minus y divided by h equals delta x divided by b, and M y oh M x is going to be equal to y d A, which I can then write as y delta x d y, y varying from 0 to total height h, and this comes out to be y delta x is b over h minus y d y. And you do this comes out to be b y varies from 0 to h. This comes out to be b h square over 6. Therefore, Y c is going to be M x divided by the total area, which is b h square divided by 6 over b h divided by 2 comes out to be h over by 3. So, we have calculated this centroid of a triangle and what is it?

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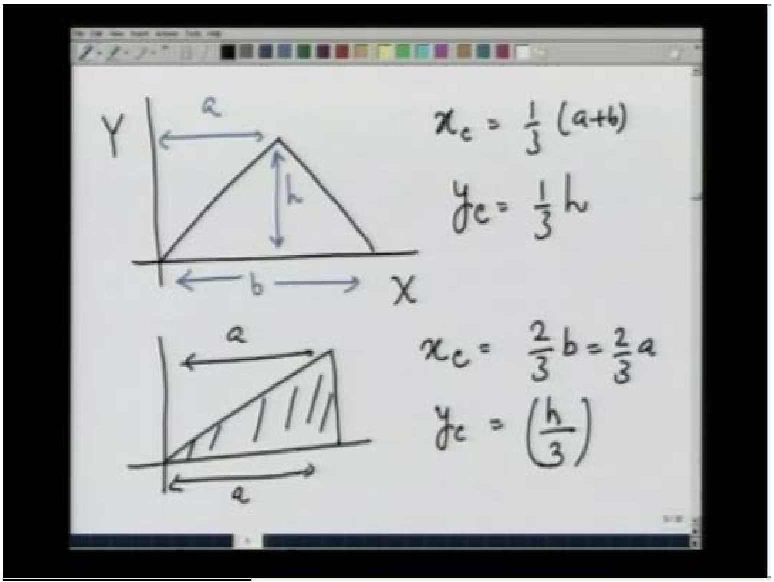

It is for a triangle, which given like this, with this distance being b, this being a, this height being h. The X Y, the x coordinate is one third of a plus b and y coordinate is equal to one third of h. We can see that y coordinate one third of the height and x coordinate is one third of b plus a. So, suppose I had a triangle like this an area, which is triangle or this shape in that case a and b are going to be the same. So, that X c in this case would become 2 thirds of b or 2 thirds of a and Y c is obviously equal to h over 3. So, if calculated the centroid for the triangle.

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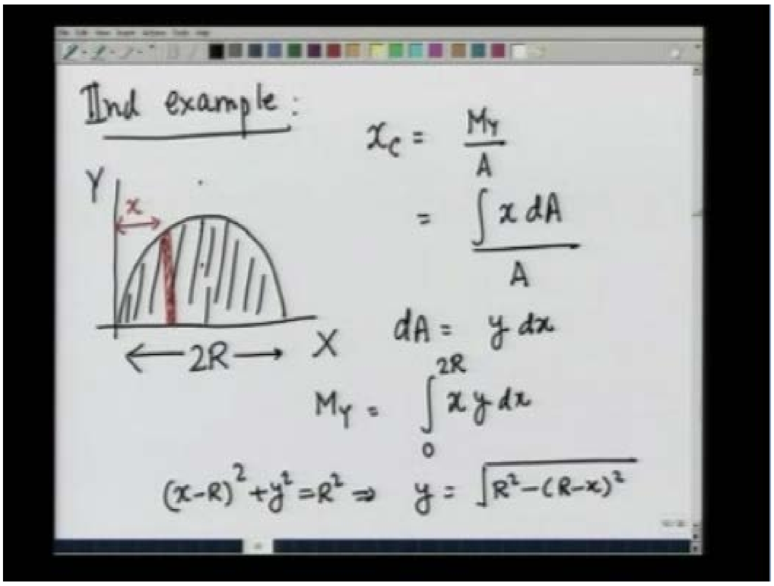

The second example let me take a semicircular area like this of radius R. So, obviously its diameter is 2 R. This is the x axis. This is the y axis, and I wish to calculate a centroid form symmetry. You can make out that the centroid must be on this axis somewhere, but, I want to explicitly calculate it to show you how to do it. So, again X c is going to be equal to M y over. The total area to calculate M y, we do integration x d A over the area. To calculate a d A, I take as strip like this, which is a vertical strip at a distance x from the y axis.

You can see that d A, in case therefore, is going to be area of the strip, which is y d x and therefore, M y is going to be integration x y d x, x varying from 0 to 2 R, what about the value of y at a given x. The equation of the circle, whose center is at R and y 0, is going to be x minus R square plus y square equals R square. And therefore, y is equal to square root of R square minus R minus x square.

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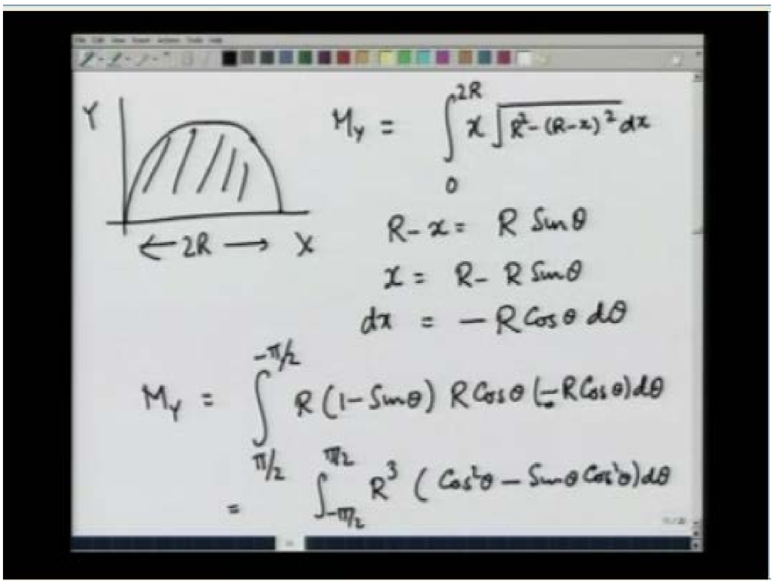

Therefore, M y for the semicircle area is going to be equal to, this is 2 2 integration x y is a square root of R square minus R minus x square d x 0 to 2 R. Let me take R minus x to be equal to R sin theta. So, that x equals R minus R sin theta. In this case d x is going to be equal to minus R cosine theta d theta. Substituting this we get M y equals, when x is 0 sin theta is going to be 1, and therefore, theta is equal to pi by 2, when x equals 2 R sin theta is going to be minus 1. And therefore, this is going to be minus pi by 2, X is R 1 minus sin theta x equals R 1 minus sin theta times R square minus R minus x square is going to be R cosine of theta.

Then, for d x I have minus R cosine of theta d theta Because of this minus sign here. I can write this as minus pi by 2 to pi by 2 R cubed cosine square theta minus sin theta cosine square theta d theta. This is a simple integration, which we can perform.

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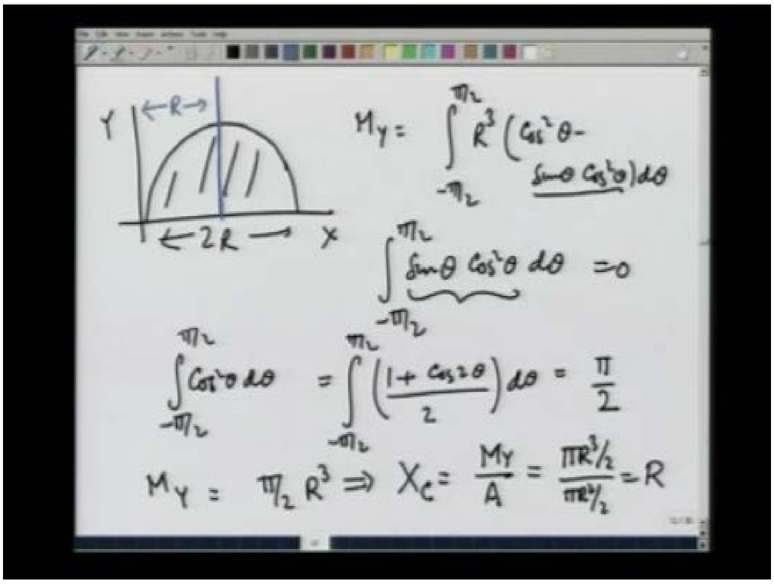

So, I have M y, for the circle semicircle of diameter 2 R, M y equals integration minus pi by 2 to pi by 2 R cubed cosine square theta minus sin theta cosine square theta d theta, cosine square theta sin theta is an odd function with respect to theta. And therefore, this integration sin theta cosine square theta d theta minus pi by 2 to pi by 2 is going to be 0, because as you go from theta to minus theta this integrant change a sign, And integration cosine square theta d theta minus pi by 2 to pi by 2 comes out be one plus cosine 2 theta over 2 d theta minus pi by 2 to pi by 2, which is nothing but pi over 2.

Therefore, we get M y is equal to pi over 2 R cubed or X c, which is M y divided by the area, is pi R square by 2. So, pi R cubed divided by 2 divided by pi R square by 2 equals R. So, as we noticed earlier the x coordinate is on this axis, this is R, how about the y coordinates of the centroid. Let us calculate that, to calculate the y coordinate.

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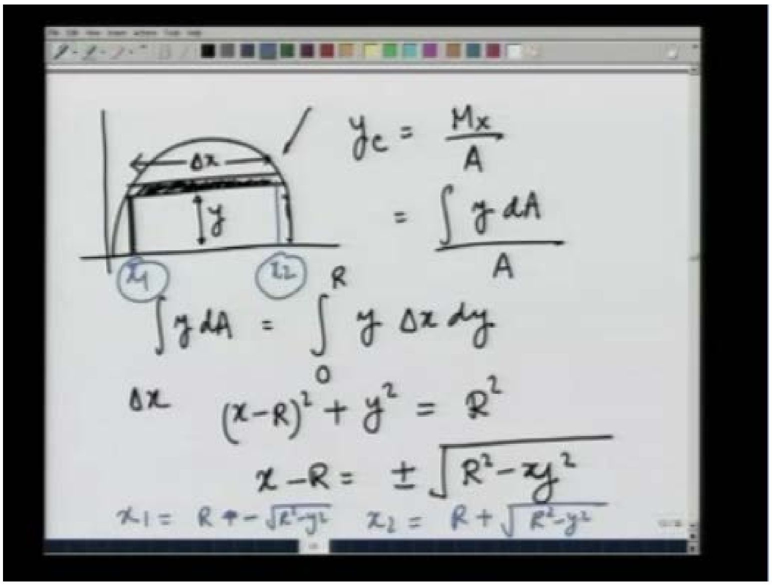

I take a strip like this at a height y. So, Y c is nothing but M x divided by the total area, which is integration y d A divided by the total area y d A. We can see, is going to be equal to the area of this strip here. This one times the y, this strip area is going to be this, if this distance is delta x delta x d y, and y varies from 0 to R. To calculate the value of delta x, we again use the equation for the circle X minus R square plus y square equals R square, and that gives me x minus R is equal to plus or minus square root of R square minus y square.

I get 2 values of x namely x 1, which I have shown here equals R plus. Sorry, minus square root of R square minus y square and x 2, which I have show here, which is equal to R plus square root of R square minus y square. And therefore, delta x again let me make this figure.

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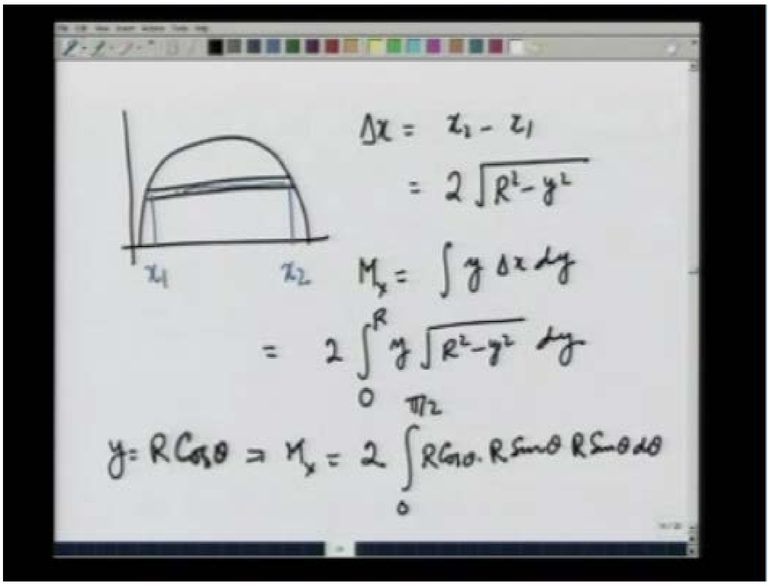

This is x 1 this x 2 delta x is going to be x 2 minus x 1, which is equal to 2 square root of R square minus y square. Remember we are after M x, which is equal to y delta x d y, which in this case is therefore, going to be 2 integration y square root of R square minus y square d y, y varying from 0 to R. Let us take y equals R cosine theta. So, that M x becomes equal to 2 integration 0 to pi by 2 R cosine theta times R sin theta times d y is R sin theta d theta. So, we have M.

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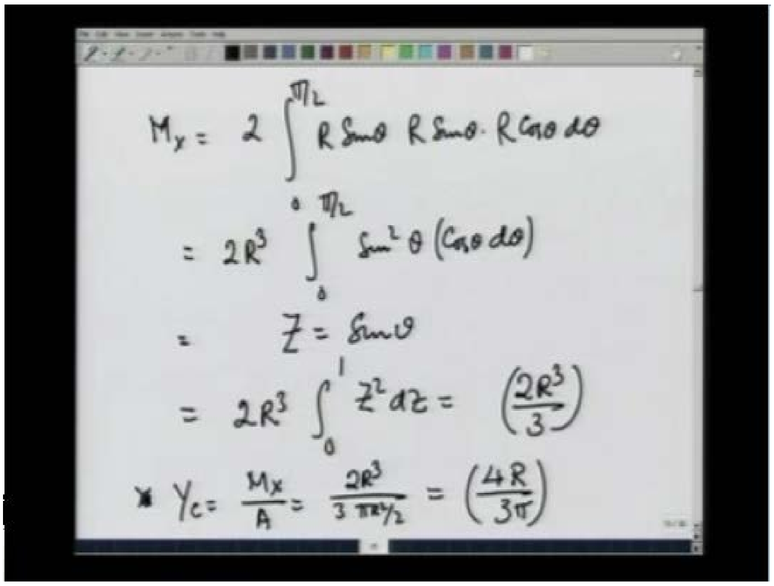

This is M x equal to integration 2 0 to pi by 2 R sin theta R sin theta R cosine theta d theta, which is 2 R cubed integration 0 to pi by 2 sin square theta cosine theta d theta, cosine theta d theta is d sin theta. So, I can write this as by substituting Z equals to sin theta. I can write this as 2 R cubed 0 to 1 Z square d z, which is 2 R cubed divided by 3. That is M x, and therefore x y centroid, which is M x divided by the total area, is going to be equal to 2 R cubed over 3 divided pi R square by 2, which is nothing but 4 R over 3 pi.

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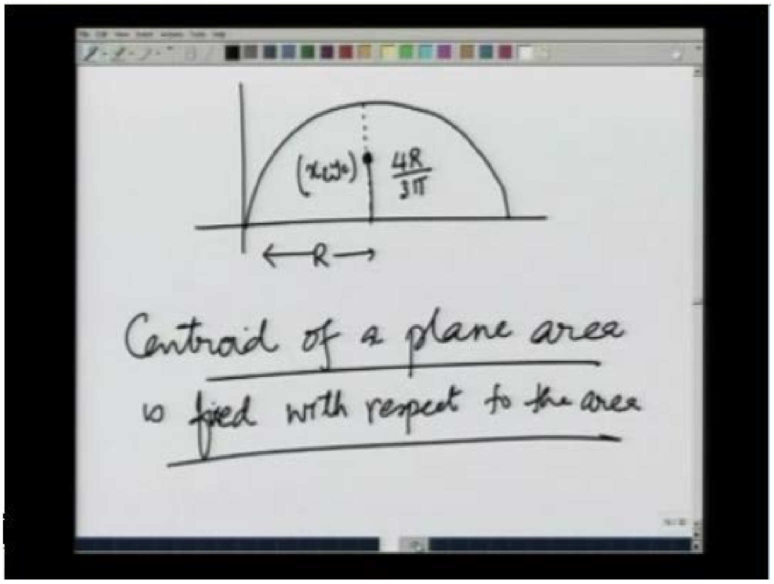

Therefore, if I look at the semicircle, the semicircular area is centroid is at a distance R and at a distance at a height of 4 R over 3 pi, slightly 4 3 pi is roughly 9. So, 4 R 3 pi slightly less than R by 2. Let us somewhere here this is X c Y c having define the centroid of a plane area, and solve to examples. Let me make a point about the centroid of a plane area, and that is that the centroid of a plane area is fixed with respect to the area. That means, no matter which coordinate system I calculate the centroid a it will always come out to be the same point in the body or in that area. So, centroid of a plane area is fixed with respect to the area. Let us see, how we can prove this?

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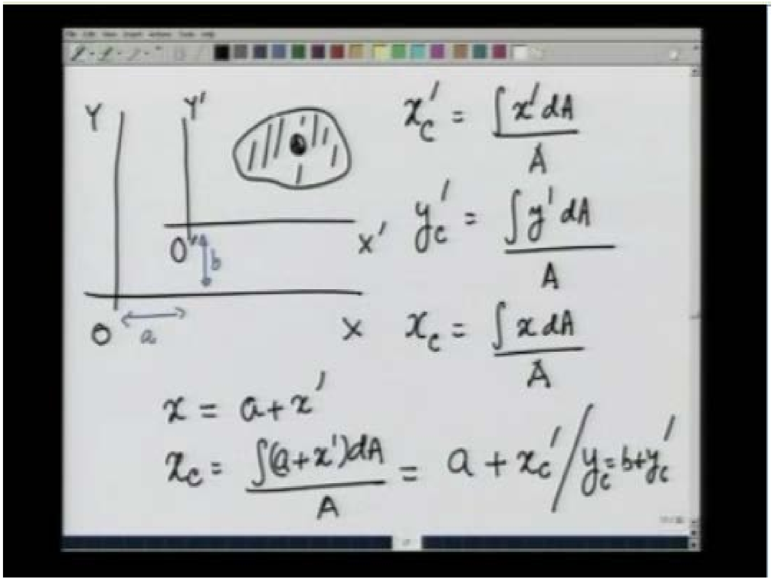

So, let us take 2 set of axis X Y with origin 0, and another set of axis X prime Y prime with origin 0 prime, 0 prime is shifted with respect to 0 by X coordinate a and Y coordinate b. Let the centroid coordinate in 0 prime b eX c prime, which is equal to x d A over the area rather x prime. Similarly, Y c prime is equal to integration Y prime d A over the area, where d A a some small area given in this plane area, which I have drawn here. Now, let us calculate X c, which is equal to integration x d A over the area, but I know that x equals a plus x prime.

Therefore, X c is going to be equal to a plus x prime d A over A d A integral a comes out integral d A is the total area. So, it will comes out be a plus X prime d A over A, is nothing but X c prime. So, you see if I calculate the centroid with respect to shifted coordinate system, the centroid shifts by appropriate amount. So, that to remain at the same point in the body and similarly, you can show that Y c will be equal to b plus Y c prime, which I leave as an exercise for you. As an important observation about centroid.

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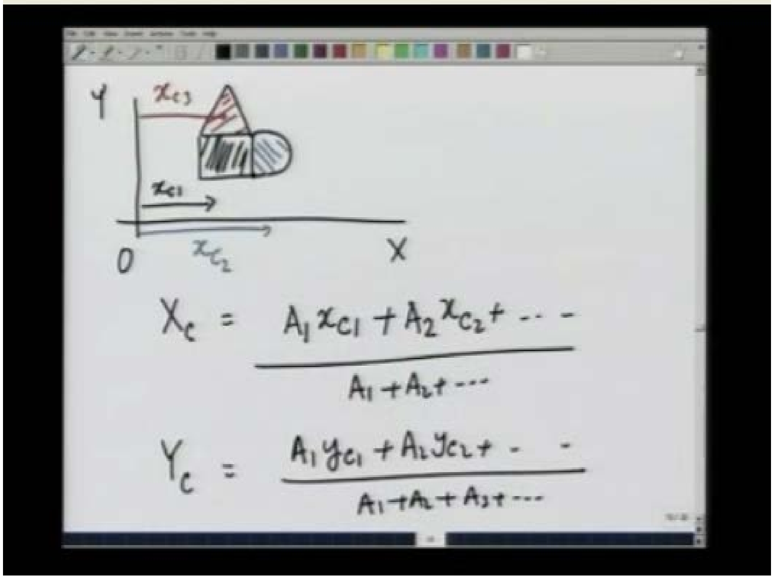

Let us now take an area, which is made up of 2 or 3 simple plane area. So, let us say a square a triangle and semicircular planar area. So, here is one kind of area here, is another kind of area and here is a third kind of area, but all of them are joined together. Then, it is very simple to prove that if I want to calculate the centroid of this whole system. Then, X c of the entire system would be equal to the area of system one times its centroid plus area of the second area X c 2 plus and so on, divided by the total area and so on.

That means, in this case if I consider the square to be area one, then X cone is known similarly, if I take semicircle to be area 2. Then, X c 2 is also known, because we already calculated it and X c 3 for the triangle is also known. I also know the formula for 3shapes and therefore, easily calculate the centroid for the entire system. Similarly, Y c for the total area is going to be A 1 times Y cone plus A 2 times Y c 2 plus so on, divided by A 1 plus A 2 plus A 3 and so on. This comes in handy, when we know several areas that are joined together and I know the formula for different areas, but to wish to calculate the centroid for the joint area. Let us do a couple of examples using this. I leave the simple proof of this for you to do.

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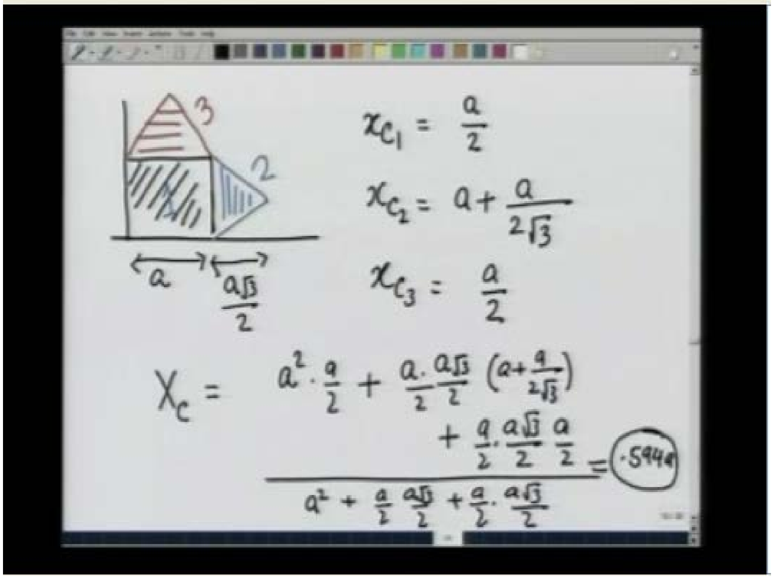

So, such as first example, I take a square and join on a 2 triangles equilateral triangle, one here and one here. I wish to calculate the centroid of this assembly. Let me call this number 1, let me call the triangle here number 2, let me call the triangle number 3. Then, I know that X cone that is the x coordinate was centroid of this square is going to be, if this side is a is going to be a divided by 2. Similarly, the height of this triangle because a side is a is going to be a root 3 by 2, and therefore X c 2 is going to be equal to a plus the distance of the centroid from the base, which is nothing but one third the height.

Therefore, a divided by 2 root 3, and X c 3 that is the x coordinate was centroid of the third triangle is going to be right on the dividing axis, and which is going to be a by 2 again. And therefore, X c of the entire assembly is going to be equal to area of the first square, which is a square times a by 2 plus area of the second one, which is going to be equal to the base, which is a times the height, which is a root 3 by 2 divided by 2.

That is the area times X c 2, which is a plus a divided by 2 root 3 plus the area of a third one, which is a by 2 a root 3 divided by 2 times a by 2 divided by the total area, which is a square plus a by 2 times a root 3 by 2 plus a by 2 times a root 3 by 2. That is the formula for X c, and you calculate this comes out to be about 0.594 times a by symmetry Y c would also come out to be the same.

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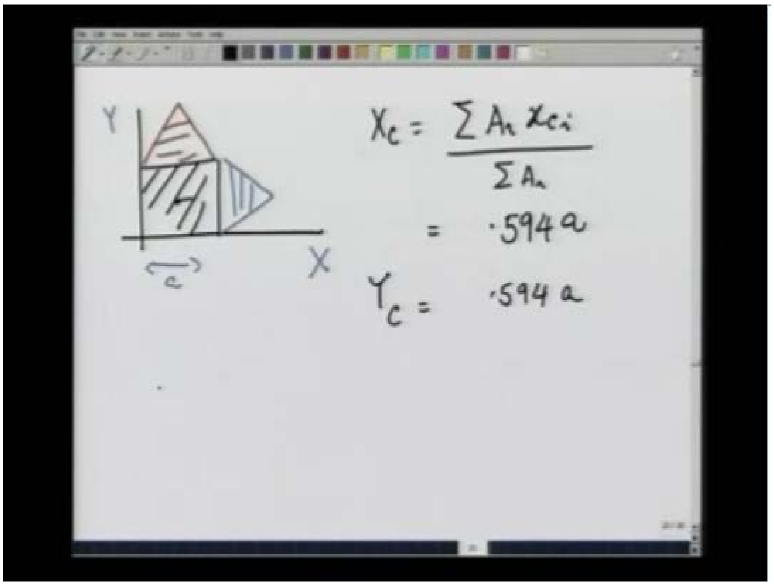

And therefore, for this assembly where I have a square of side a triangle equilateral triangle on top, and on the side this is a x y. I get X c to be which a summation A i X c i. I am writing a general formula divided by summation A I, which is comes out to 0.594 a. slightly to the right of the centroid of the square, which is expected because of the area added on to the right. It shift to the right and similarly, you can calculate Y c is going to be come out to be again 0.5944 a. So, this is how you use this observation that if I want to calculate the centroid for an assembly of plane areas for which I know individually where the centroid is going to be I can calculate the centroid for the total assembly.

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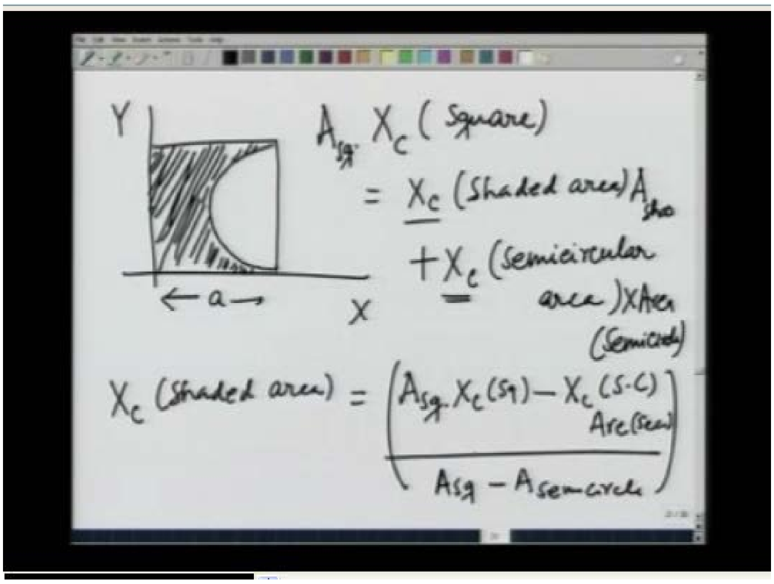

Let me take another example, where I have an area formed by taking away a semicircular area from a square of side a. So, I wish to calculate the centroid of the shaded area, where the shaded area has been formed by taking away a semicircular area on a square of side a. Again I can use the same theorem. But in a different way because now what I am going to say is that X c of the square, which I already known is going to be right in the middle is going to be equal to x centroid of the shaded area plus x centroid of the semicircle. And therefore, if I wish to calculate the x shaded of the, sorry this is the area of the square, sorry area of the square is going to be equal to X c of the shaded area times the area of the shaded area times the area of the semicircle, right? Sorry for that.

So, area of the square times the centroid of the square is going to be equal to area of the shaded area times. The centroid of the shaded area plus X c centroid of the semicircular area times area of the semicircle. And therefore, X c of the shaded area is going to be equal to A square times X c of the square minus X c of the semicircle times area of the semicircle divided by the area of the shaded portion, which is going to be area of the square minus area of the semicircle. That is what to we wish to calculate.

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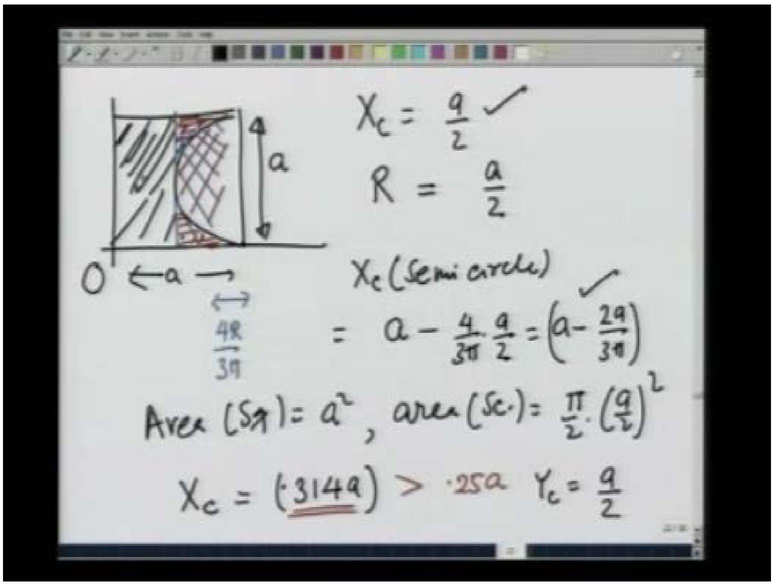

We have already calculated that for a, for a square X c is going to be a by 2, where a is a side the radius of the semicircle is equal to a divided by 2, and I know from the base from the base we already calculated this. The centroid is at a distance of 4 R divided by 3 pi, and therefore X c with respect to this origin of the semicircle is going to be equal to a minus 4 over 3 pi times a by 2, which is a minus 2 a by 3 pi.

This is X c of the semicircle. So, now we have all I know X c for the square. I know X c for the semicircle and area of both area of square is equal to a square area of semicircle, is equal to pi by 2 times a by 2 whole square. And therefore, I can calculate the X c for the shaded region from the formula that we derived earlier, and X c comes out to be to plug in the numbers 0.314 a. Notice that if I had removed an entire this half, let me make it with blue. If, I remove this half blue of the square the X c would have been here at point 0.25 a, but because of this extra area, which I am showing here to the right the centroid shifts slightly to the right, and is at 0.3148, which is greater than 0.25 a.

That would have been the case if I had removed this entire half, this entire half of the circle Y coordinate. Obviously, because of symmetry remain sat a by 2. So, I given you 2 examples of how to use this, the observation that the centroid of a given area actually awaited combination of centroid of the area is making. This joint area to calculate centroid of complicated shapes, having given you the concepts of first moment of an area.

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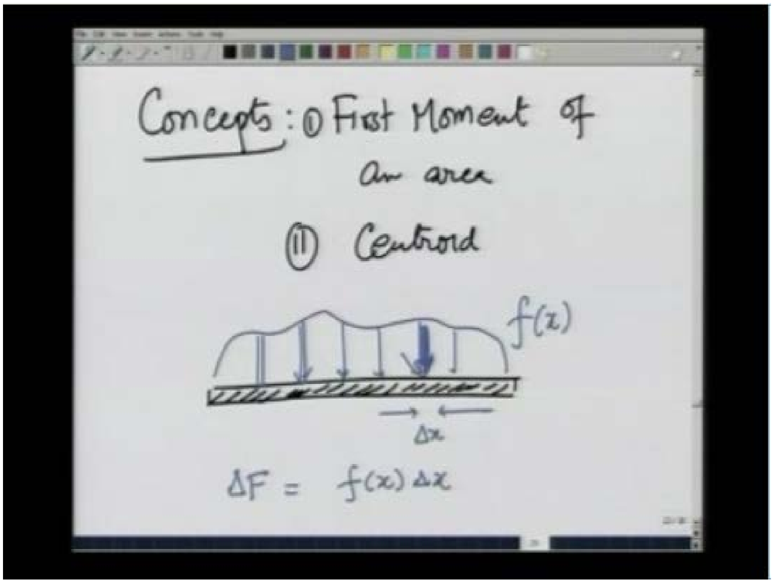

And the concept of the centroid, we are now ready to apply the concepts to mechanics. So, the question we ask is, suppose I have a beam on which I do not apply force at a certain given point, but the force on it is distributed. So, there is some force per unit area. Let me show this by a graph, sorry per unit length.

So, there is force acting at each point. Let me call this function f x, so that if I take a distance a small section of length delta x here, the net force acting on the point delta F is going to be given as f at x delta x, what we wish to, now find is what is an net force acting on this beam and where does it effectively act? That means, if I were like to suppose, apply an opposing force to balance it, where should I apply it. So, that this force is balanced.

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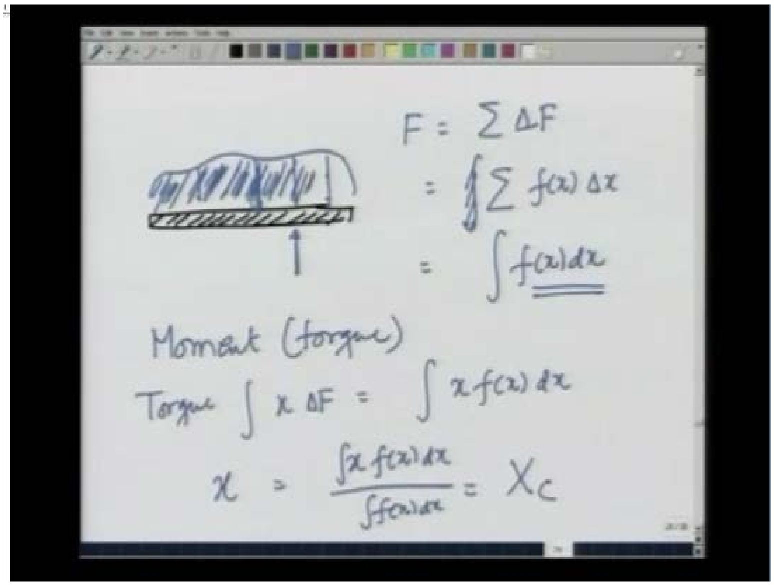

The answer comes in terms of the quantities just defined. So, if there is this force fx distributed force acting on its beam the net force F is going to be summation delta F, which is nothing but integration. Let me write it one more strip with summation delta F is nothing but f x n delta x, and this nothing but integration f x d x, which is the area under this curve. That gives you the total force for a distributed force per unit length given, how about the moment or the torque is created by this force that is; obviously, going to be x delta F integrated, which is nothing but x f x d x.

This is the torque, if I want to calculate apply an opposing torque with a point force that force has to be equal to the net force, which is being applied, which is the area of this curve. And it has to be at a point x, which is nothing but integration x f x d x divided by integration f x d x, which is nothing but the centroid of this curve. That describes force per unit length. So, we can say that effectively the force acts at the centroid of this area, this given area, which describes the force distribution. So, you see immediately the connection between the first moment or the centroid that we define with a problem in mechanics. As an example of things to come, let me end you see this lecture by solving a problem of calculation.

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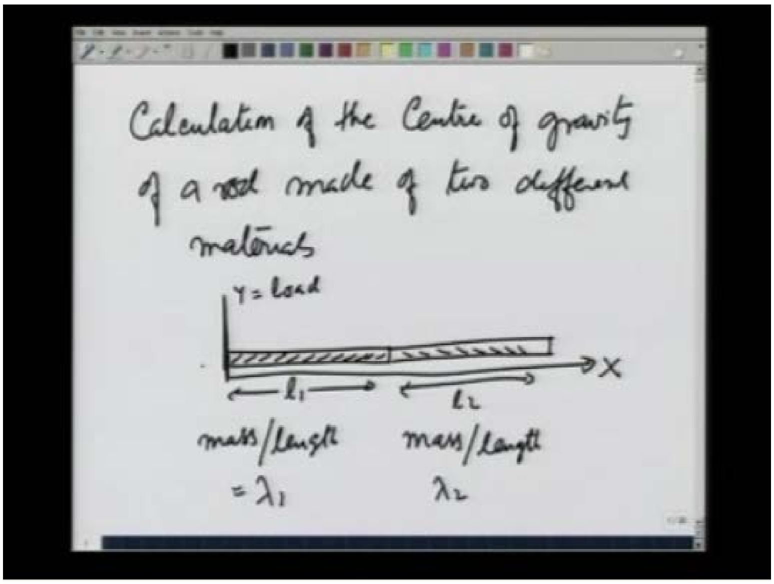

Of the center of gravity of a rod made of 2 different materials. So, we take a rod, which is made of 2 different materials, one material here is of length of l 1 and its mass per unit length is lambda 1, the other rod here is of length l 2 and its mass per unit length his lambda 2. We wish to calculate a center of gravity starting from this point. Let us say this is our axis, so that this is our x axis, this is our y axis or on the y axis I will be showing the load and this is x equal to 0.

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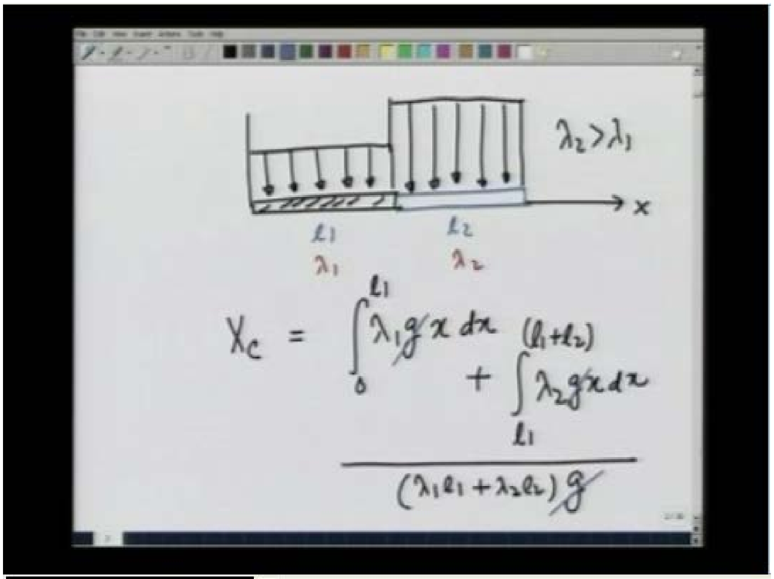

Let me go the next page and show this rod black here of length l 1 and blue part for the length l 2, l 1 mass per unit length is lambda 1 here, lambda 2 here axis is starting from here going this way, if I plot the load curve it will be constant lambda 1 g per unit length here. And if you assume, lambda 2 to be greater than lambda 1, lambda 2 greater than lambda 1, qnd I wish to calculate the center of gravity by definition. The center of gravity is the centroid of this load curve.

So, this is going to be lambda 1 g is the load per unit length in the first part times x d x integration 0 to l 1 plus integration l 1 to l 1 plus l 2 lambda 2 g x d x divided by the total area, which is going to be lambda 1 l 1 plus lambda 2 l 2 times g. You see, g cancels from the denominator and the numerator.

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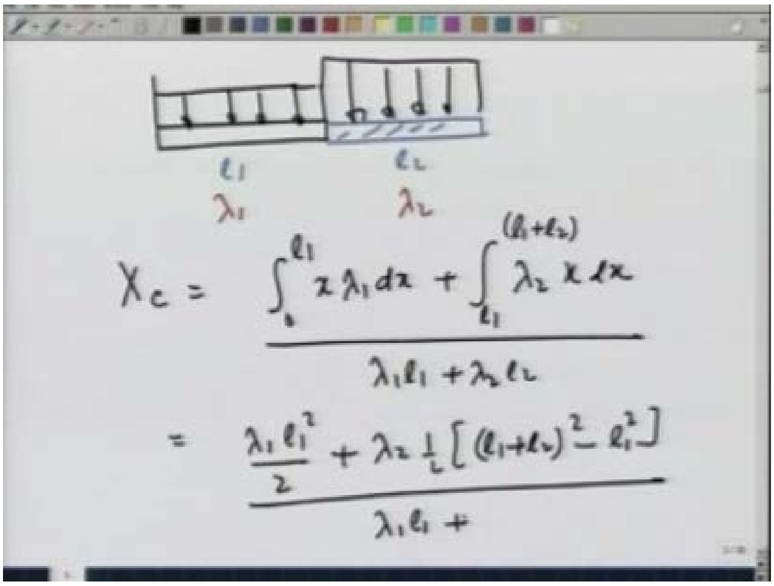

And I am left with the center of gravity of this rod l 2 l 1 lambda 1 lambda 2. Let me again show you the load curve is something like this a center of gravity is 0 to l 1 x lambda 1 d x plus l 1 to l 1 plus l 2 lambda 2 x d x g. We have already cancelled over lambda 1 l 1 plus lambda 2 l 2, and this comes out to be lambda 1 l 1 square divided by 2 plus lambda 2 1 over 2 l 1 plus l 2 whole square minus l 1 square divided by lambda 1 l 1 plus lambda 2 l 2. You simplify this and you get X c, which we just calculate to be.

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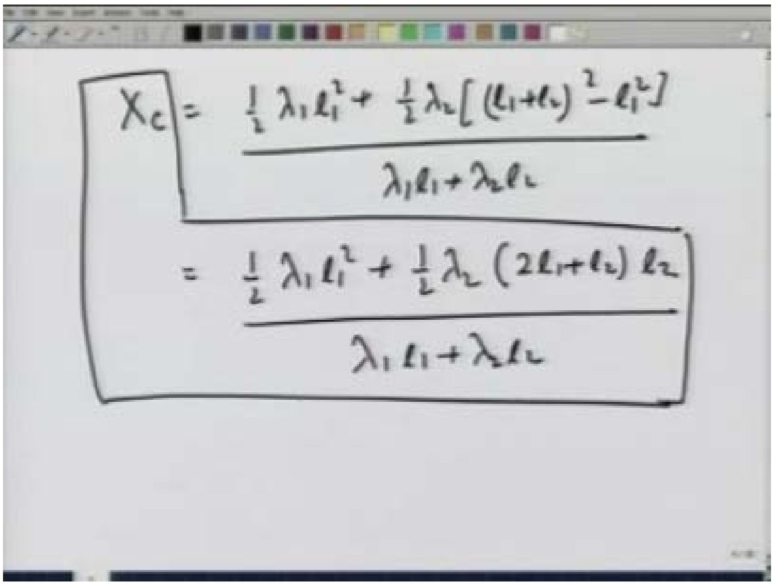

One half lambda 1 l 1 square plus one half lambda 2 l 1 plus l 2 square minus l 1 square over lambda 1 l 1 plus lambda 2 l 2 to be one half lambda 1 l one square plus one half lambda 2. I can simplify this will come out to be 2 l 1 plus l 2 times l 2 divided by lambda 1 l 1 plus lambda 2 l 2. So, from the left hand side end of the rod this is where the center of gravity lies. We can also calculate the center of gravity by using our observation earlier that if there are 2 different areas given the center of gravity can be calculated from those 2, if I know the center of gravity of each one of them.

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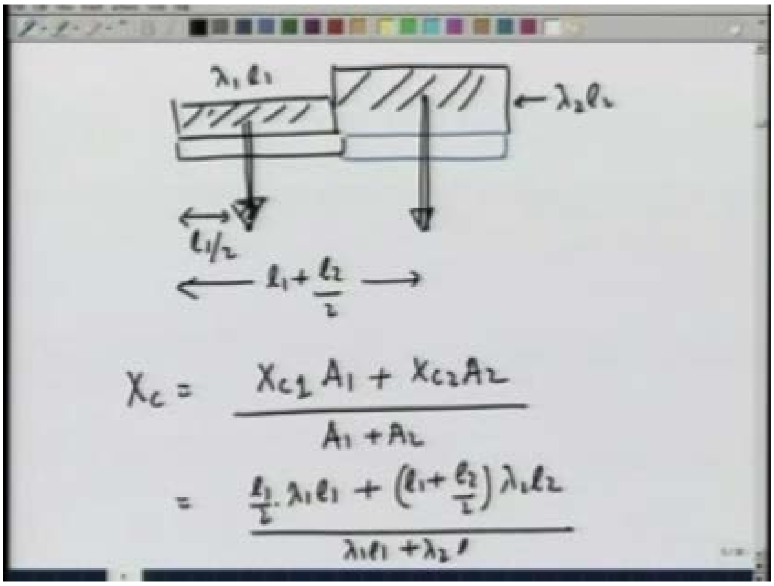

So, if I were to use that you see the load curve on the rod is something like this. I know the center of gravity of this load, which is like here at a distance of l o1 by 2 and center of gravity of the other load is here, which is at a distance of l 1 plus l 2 divided by 2. The total area here is lambda 1 l 1, total area here is l lambda 2 l 2. So, X c is going to be X c 1. That is a part times area 1 plus X c 2 area 2 divided by area 1 plus area two, and that is going to be l 1 over 2 times lambda 1 l 1, g would cancel from the top and the bottom plus X c 2, which is l 1 plus l 2 divided 2 lambda 2 l 2 over lambda 1 l 1 plus lambda 2 l 2. So, we have calculated X c to be lambda 1.

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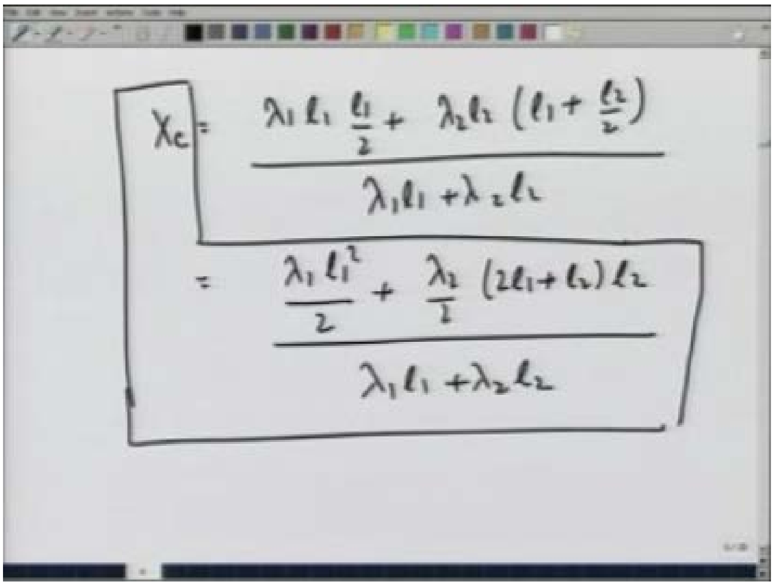

L 1, l 1 over 2 plus lambda 2 l 2 l 1 plus l 2 over 2 divided lambda 1 l 1 plus lambda 2 l 2, which is same as lambda 1 l 1 square divided by 2 plus lambda 2 over 2 l 1 plus l 2 times l 2 divided by lambda 1 l 1 plus lambda 2 l 2, which is a same answer as we obtained earlier when we calculated the center of gravity by integrating over the entire area. So, this is a simple example of how center of gravity is related to the centroid of the load curve. In the next lecture, we will be doing more such examples.