Engineering Mechanics
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Module - 09
Lecture - 02
Motion in Rotating Frames

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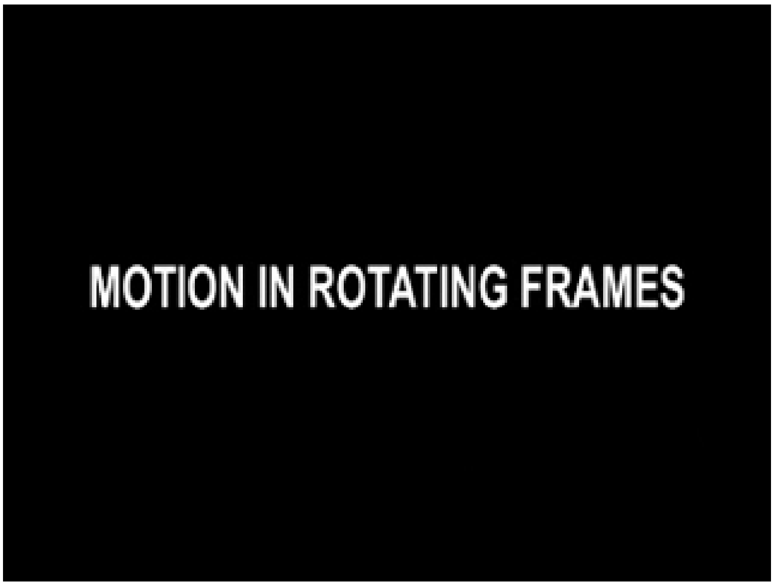

In the previous lecture, we considered description of motion in an uniformly accelerating frame of reference. We found that for certain problems it facilitated getting the solution of that problem. In this lecture, we considered the motion of a body as described in a rotating frame

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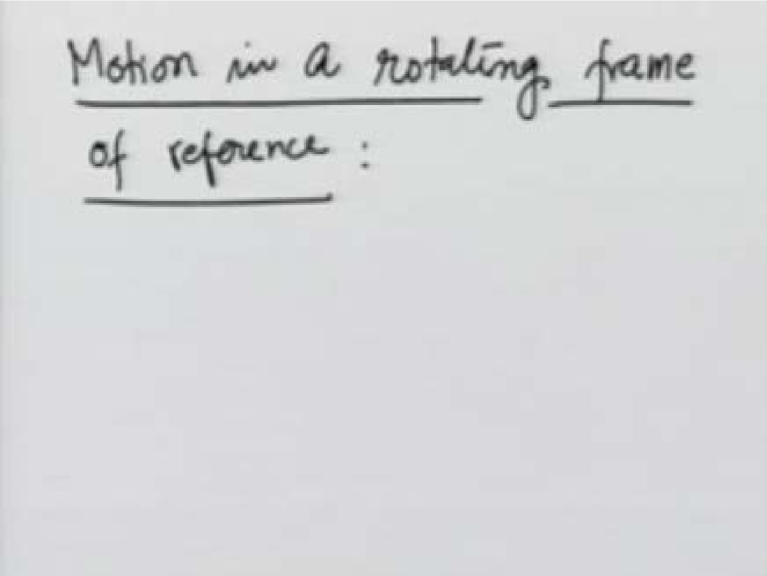

So, motion in a rotating frame of reference. A rotating frame of reference also is a non-inertial frame and it is important to deal with it because our earth itself; provides a rotating frame and therefore, its effects are significant when long distances are travelled over the surface of the earth. The first question we address is, how do pseudo forces arise in a rotating frame? We understand this by considering changes in a vector as observed in a rotating frame.

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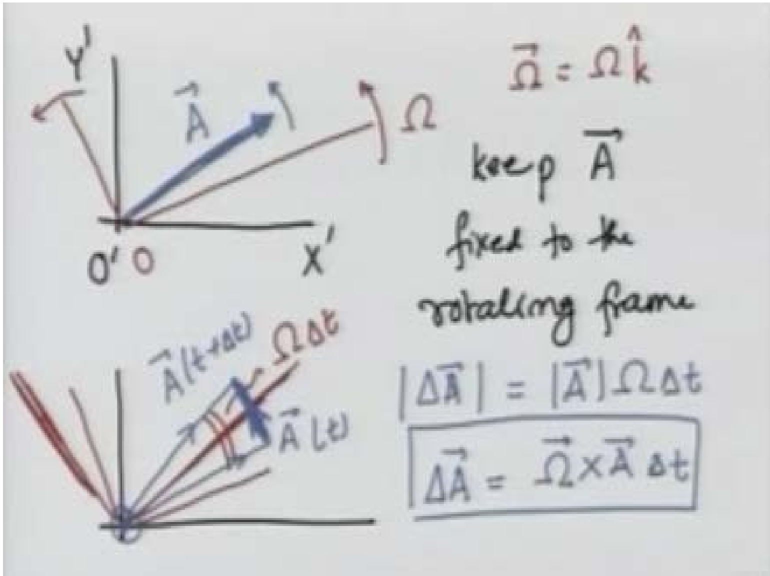

I first explain this in two dimensions I first explain this in 2 dimensions. So, i consider an inertial frame with X prime origin at O prime Y prime axis and consider a vector A in this and look at this vector from a frame which is rotating in this direction with angular velocity omega obviously; in the vector form omega vector is omega k because direction is going to be in the Z direction. We have considered the frames as that their origins coincide.

Now, let us keep A fixed to the rotating frame. If A is fixed to the rotating frame A would also be moving with velocity omega. So, let us consider a 2 times different times what is the position of A? At 1 time t the position of A would be here. Let us write A t this is vector A by there and at slightly later time the position of the vector if, it does not change in the rotating frame would be this the rotating frame initially was like this and after time t it has become like this.

Let us make it slightly thicker, where this angle here is omega delta t the blue 1 is showing you vector A at time t plus delta t. So, the vector has changed roughly by this amount here this is delta A. In a forward to write delta A magnitude you would see that for small times this is nothing but, A magnitude times omega delta t the angle. If I do write the vectorial form then you will see that delta A can be written as omega cross A delta t because omega is coming out of the plane. So, omega cross A gives you; the right direction.

So, omega cross A delta t gives me the change in vector in the inertial frame, when it is rotating with the fixed with the frame which is rotating. Let me summaries this.

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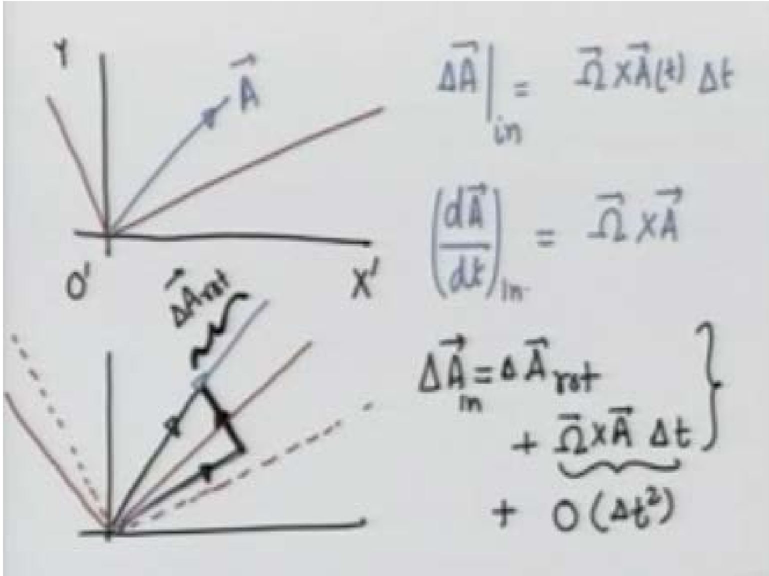

So, we considered an inertial frame X prime with the origin o prime Y prime and considered a frame which is rotating and considered a vector which is fixed in the rotating frame then we found that delta A in the inertial frame is equal to omega cross A. A is a vector. So, it is a given vector at any time you can write A is a function of time delta t and therefore, dA dt in the inertial frame is equal to omega cross A. What if, A was changing in the rotating frame also?

Let us consider that situation. If the, A was changing in rotating frame also. So, at point t it will be something like this and after time delta t it will be something like this. Let me make the rotating frame, this is the way rotating frame was at time t and after time delta t. Let us say, the rotating frame is like this. So, there are two changes that have taken place in A 1. Let me show by black color is this change, which is because the frame the vector A itself has rotated and the other change which is purely a change in the rotating frame which is delta A which i will call rotation.

That is; if I were to look at it from the rotating frame itself the only change I would see is this shown here and the additional change comes because the vector is fixed in the rotating frame. So, the total change A delta A I can write as delta A; in the rotating frame this is in the inertial frame plus omega cross A delta t that was anyway there. So, I can think of this change in 2 steps first I change A without changing it in the rotating frame and then changes in the rotating frame.

So, from here to here, from here to here the vector was fixed in the rotating frame and therefore, the only change observed in the inertial frame was omega cross A delta t and then we made a change delta A rotation plus some terms which are of the order of delta t square that would be: multiplication of delta A rotation which is proportion to delta t plus it is rotation and so, on.

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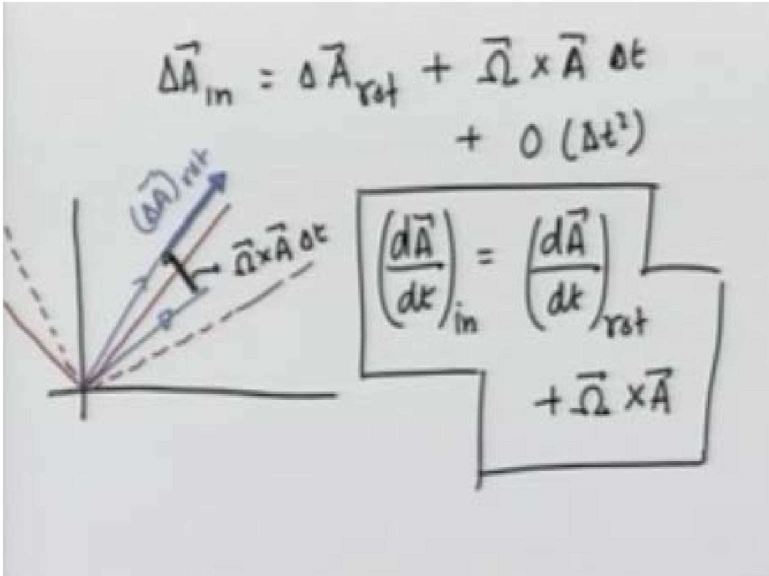

So, to a linear order this is a change first 2 terms and therefore, delta A inertial is equal to delta A as observed in the rotating frame plus omega cross A delta t plus terms of the order of delta t square. So, I hope this point is very clear and this is very crucial to our understanding. I took the vector A and change it in two steps in the first step i just rotated it fixed in the rotating frame. So, this was my rotating frame at time t this is my rotating frame at point t plus delta t.

This change here is given by omega cross A delta t and then in addition, we change it further by this much amount and this is the change which is observed only in the which is the only change observed in the rotating frame and when we add the 2 it gives me the total change and therefore, dA for any vector dt as observed in the inertial frame is equal to for the same vector A dA over dt as observed in the rotating frame plus omega cross A.

This is a very important equation and this is, what we are going to use to derive the pseudo acceleration of pseudo forces; in a rotating frame Although I have derived it in the context of a plane taking XY plane this is true in 3 dimensions also and let me, just indicate that. In a 3 dimensional case.

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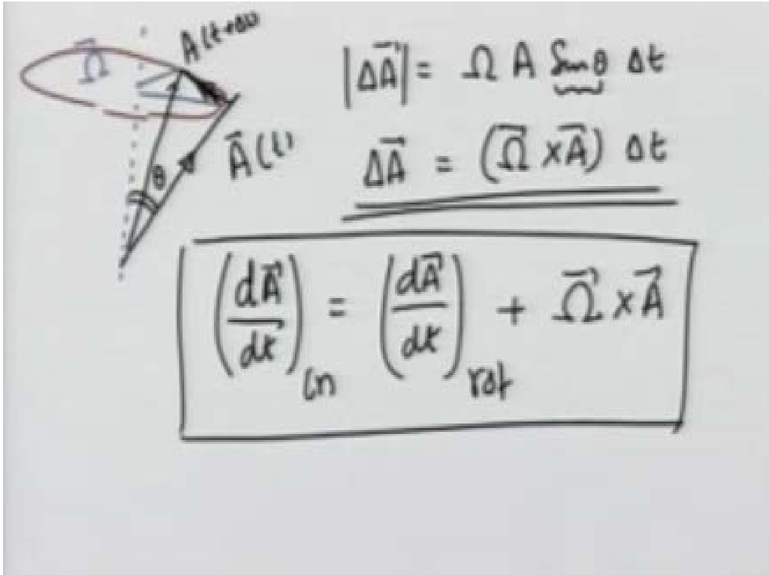

Suppose, there is a vector A and it rotates about an axis with omega. So, this will be called omega when it rotates it rotates like this. So, after time t plus delta t it goes over here this was At and this is A t plus delta t. So, this is the change. If, I look at the center here you can see that, the change in A delta A is going to be this angle is theta omega A sin theta delta t the magnitude of this. And if, I write it with the vectors this comes out to be omega cross A delta t the sin theta being taken care of by the cross product.

So, in 3 dimensional case also in general delta A is if, it is rotating omega cross A and therefore, I can again write in general that dA dt rate of change of a vector observed in an outside frame is equal to dA dt observed in a rotating frame plus omega cross A in general and this is; what we are now going to derive use to derive equations in rotating frame.

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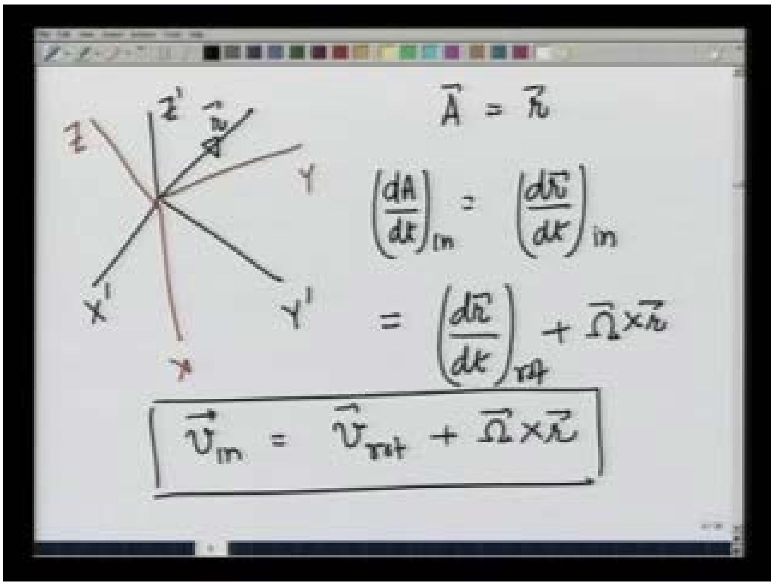

So, consider 2 frames 1 an inertial frame and another 1 a non-inertial frame with the same origin x, y, z X prime, Y prime is a inertial frame Z prime and replace A by the vector r of a particle. Then dA dt in is going to be dr dt observed in the inertial frame and this is going to be equation to dr dt as observed in the rotating frame plus omega cross r, dr dt inertial frame is nothing but, velocity of the particle as observed in the inertial frame. And this is going to be equal to velocity of the particle as observed in the rotating frame plus omega cross r this is 1 very important relationship.

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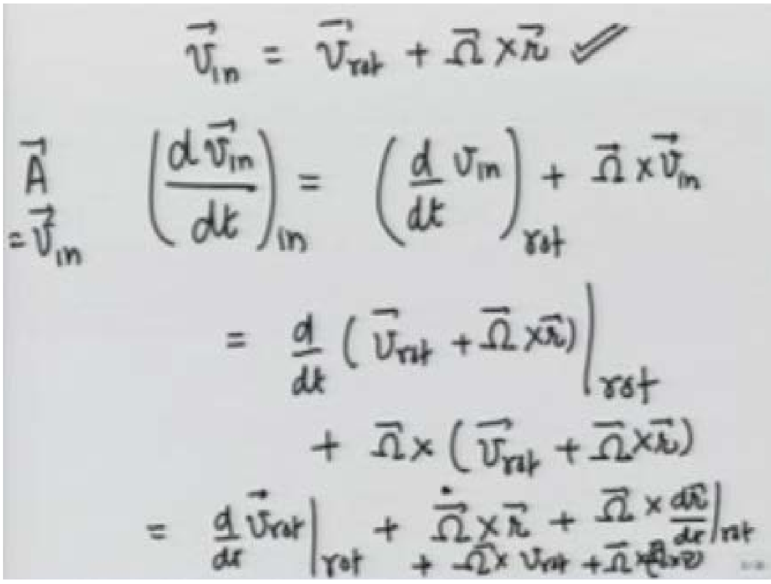

What we are after though is a relationship between the accelerations observed in the 2 frames again we have already derived Vin a in the inertial frame as V rotating frame plus omega cross r. Let us observe; what the acceleration is going to be in the inertial frame and how they related to acceleration in the rotating frame by definition again now we will take, A same as V vector observed in the inertial frame.

So, d by dt of V inertial is going to be d by dt for V inertial observed in the rotating frame plus omega cross V inertial, V inertial we have already seen as this substituting this we get this is equal to d by dt of V rotating frame plus omega cross r as calculated in the rotating frame plus omega cross Vin the inertial frame is v in the rotating frame plus omega cross r. And this gives me d over dt V of velocity in the rotating frame as observed in the rotating frame plus omega dot rate of change of omega cross r plus omega cross dr dt as observed in the rotating frame plus omega cross v rotation plus omega cross omega cross r.

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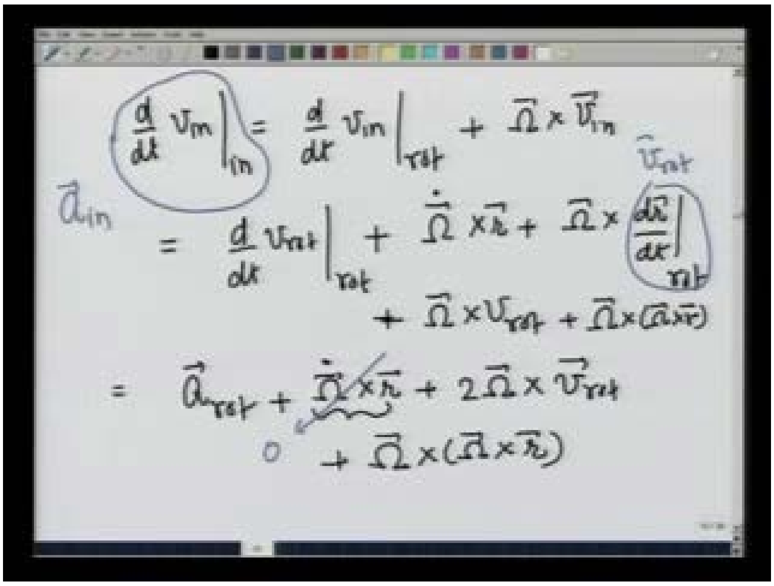

Let me, write it again we are calculating d over dt of V observed in the inertial frame; which is equal to d over dt of V inertial frame as observed in the rotating frame plus omega cross Vin and this comes out to be d over dt of V rotation as observed in the rotating frame plus omega dot cross r plus omega cross dr dt as observed; in the rotating frame plus omega cross V rotating plus omega cross omega cross r notice that this term here is nothing but, V observed; in the rotating frame.

Therefore, this whole thing can be written as acceleration as observed in the rotating frame that is d by dt V rotation in the rotating frame plus omega dot cross r plus 2 omega cross V rotation plus omega cross omega cross r. If we, now restrict to those frames that has a constant angular speed; that means, omega dot is 0 this term drops out and therefore, I can write the acceleration in the inertial frame this is nothing but acceleration in the inertial frame as acceleration in the inertial frame.

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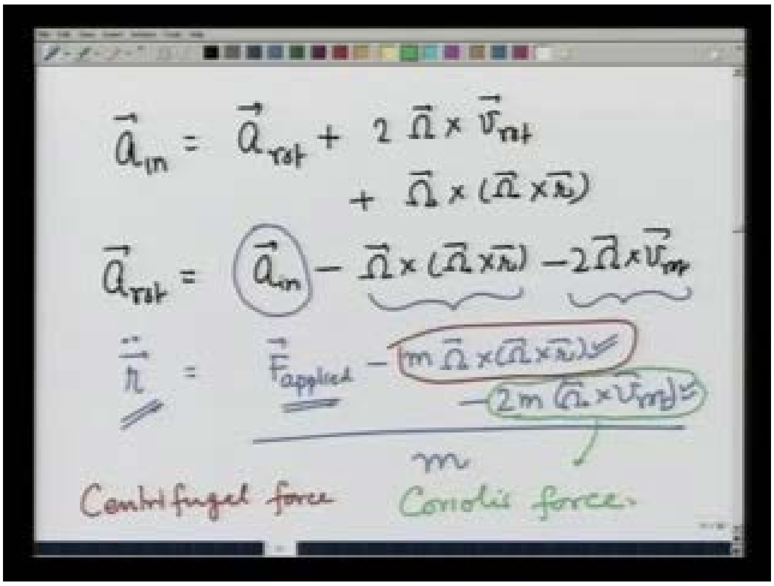

As acceleration in the rotating frame; that means, change of V in the rotating frame with respect as observed in the rotating frame plus 2 omega cross V rotation plus omega cross omega cross r and therefore, a has observed; in the rotating frame is going to be a as observed in the inertial frame minus omega cross omega cross r minus 2 omega cross V rotation. Notice that even if, a inertial is 0 these 2 terms give me non zero acceleration in the rotating frame and that is: what is pseudo acceleration? That is the acceleration we would observe.

If, we are sitting in a rotating frame writing everything in terms of r double dot in the rotating frame I will drop that term now is equal to F applied which gives rise to a inertial minus m omega cross omega cross r minus 2 m omega cross V rotation divided by m. So, the acceleration in the rotating frame depends not only on the applied force, but also in these two terms and these are the pseudo forces. They are known as; this particular term is known as a centrifugal force and the other term the 3 in 1 is known as the Coriolis force.

So, there are 2 pseudo forces that arise when we observe motion in rotating frame 1 is the centrifugal force and 1 is a is the Coriolis force. I emphasize again these are not real forces they arise because we want to take into account; the defect of rotation. So, they appear to be these forces appear to be there because we see some acceleration, but they are not really the real forces being generated by some sort of an agent. This is just to take care of the rotation of our frame.

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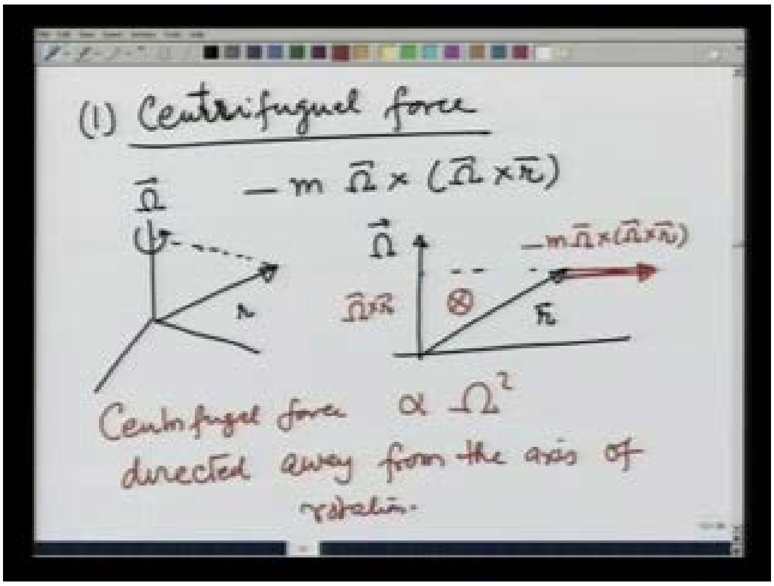

Let us now, try to understand the nature of these forces. So, the first force we consider is a centrifugal. Centrifugal force which is: minus m omega cross omega cross r. So, if I look at a rotating frame which is rotating about an axis and this is r omega cross r is going to be perpendicular to this omega r plane and then omega cross omega cross r is going to be perpendicular to that an omega plane. So, let us see this m omega r plane itself.

So, let me, make r like this and suppose; this is the axis omega then omega cross r is going to be going into the plane this is omega cross r and omega cross omega cross r would be going to the left and therefore, minus m omega cross omega cross r would be going to the right. So, this is the direction of the centrifugal force you can see that;

centrifugal force is proportional to omega square and directed away from the axis of rotation.

So, we always tend to throw things away from the axis of rotation. I emphasize again; it is not a real force we see this effect that; things have been thrown away from the axis of the rotating frame because we are sitting in a rotating frame.

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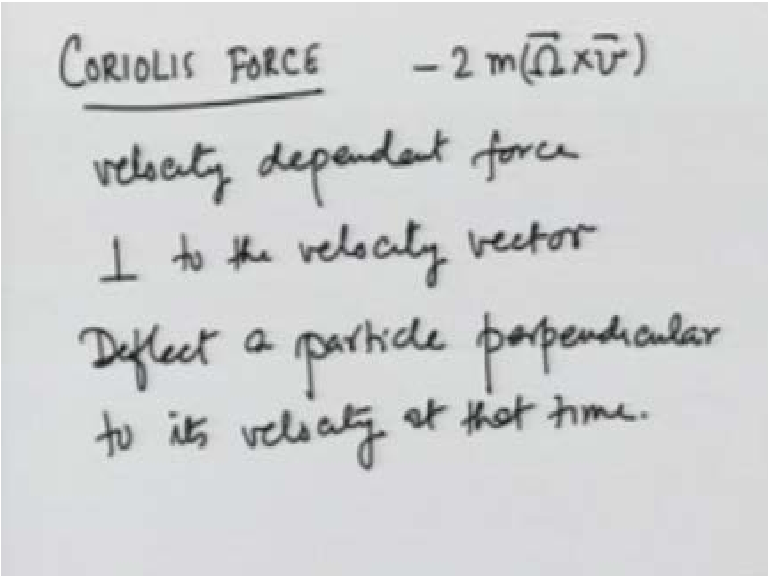

So, something which is going in a straight line if, I rotate I will see it going away from the origin number 2 is the Coriolis force which is minus 2 m omega cross v, v as seen in the rotating frame. This is a velocity dependent force, force and since its direction omega cross v is perpendicular to the velocity vector. So, what this force would do is try to deflect a particle perpendicular to its velocity at that time and this is precisely; what we see the effect when things when we consider things like missiles or bombs on the surface of the earth because of this Coriolis force they tend to deflect from a straight path.

I again emphasize that this force is again a kinematic effect it is because we are observing things in a rotating frame it is not a real force. Now, we will consider the effect of this force in our everyday life and see how motion gets affected because of these forces.

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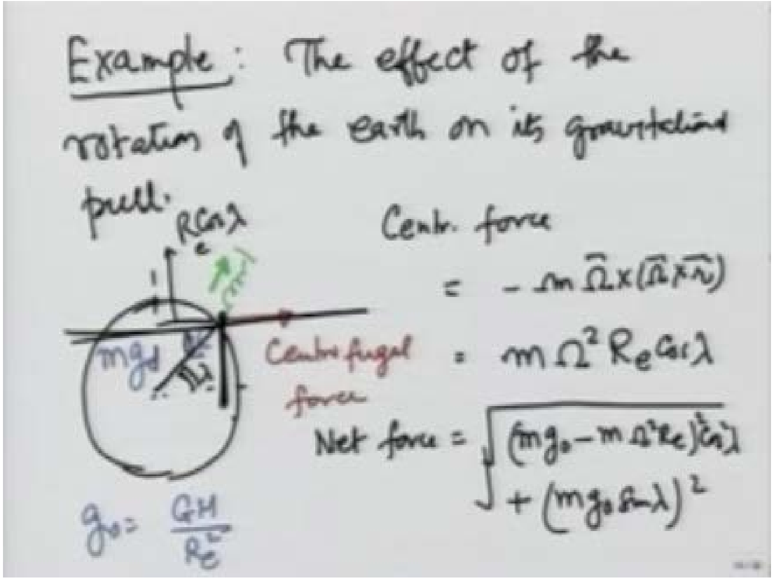

First we consider, the effect of the rotation of the earth on it is gravitational pull. So, you see if I consider, the earth an object at say latitude lambda if, I observe it in this rotating frame is in equilibrium under the forces 1 which is the gravitational pull m. Let me call it g 0 when g 0 is equal to GM over R earth square to it is experiencing a force outward which is centrifugal force and if, I am weighing it is it has a spring balance which is applying a force in this direction and that force is this weight that we measure.

So, it is under the influence of these three forces in equilibrium what should this force be in this spring. It will be given by linear combination by the combination of mg 0 plus the centrifugal force the centrifugal force is minus m omega cross omega cross r and if, you calculate it this distance out here is R cosine lambda this will come out to be the magnitude of it come out to be m omega square R earth cosine of lambda.

So, the net force due to mg 0 and this force is going to be net force is going to be square root of mg 0 minus m omega square Re cosine square lambda square plus mg 0 sin lambda square. I have taken the component along this direction which is mg 0 minus m omega square Re cosine lambda and the component along the perpendicular direction which is mg 0 sin lambda squared them added them.

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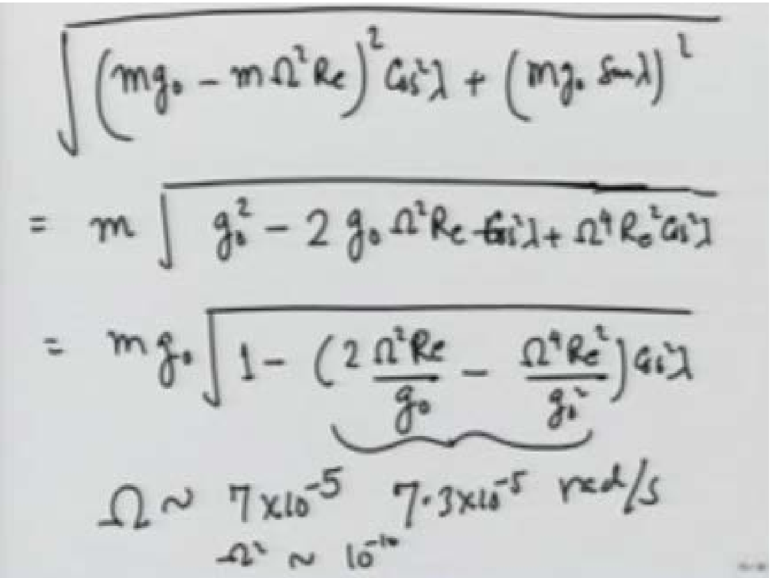

So, the net force that is to be balanced is mg 0 minus m omega square Re square cosine square lambda plus mg 0 sin lambda square, square root which comes out to be m can come out square root of g 0 square minus 2 g 0 omega square Re cosine of square lambda plus omega raise to four Re square cosine square lambda which is nothing but m I can take out g 0 also square root of 1 minus 2 omega square Re over g 0 minus omega raise to 4 Re square over g 0 square cosine square lambda.

So, you see that the gravitational pull is slightly reduced because of the rotation of the earth effect is very small because, omega is of the order of 7 times 10 raise to minus 5 to be precise; actually seven point three times 10 raise to minus 5 radiance per second. So, omega square is of the order of 10 raise to minus 10 nonetheless the effect is there.

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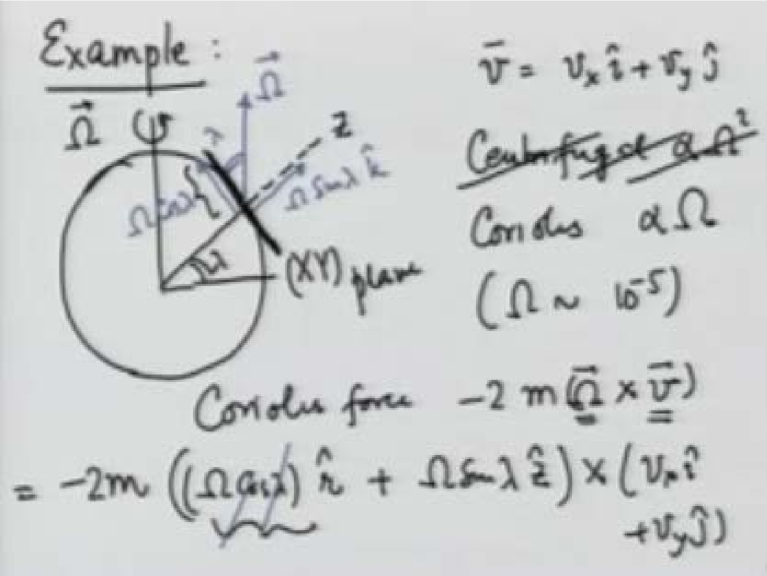

As a second example or before I take this example let me comment on the previous example. The effect of reduction of the gravitational pull is purely because of the centrifugal force. In the second example, I am going to take is motion of a particle along the surface of the earth; say along the surface and in this case the Coriolis force is also going to be important and we will tend to deflect the particle from its intended path.

Let us take, the rotation of the earth it is rotating with omega like this. We are talking about motion of a particle at latitude of lambda and we are considering the motion along the surface of the earth, which I will call my XY plane. In the vertical direction being the Z axis. Omega is like this it has a vertical component here which is omega this is lambda. So, this is also lambda sin lambda in the Z direction and a horizontal and a component towards the north which is omega cosine of lambda.

If we are restricting the motion along the plane XY plane then, v is going to be vx i plus vy j. There are 2 pseudo forces: centrifugal which is proportional to omega square and Coriolis which is proportional to omega. We have already seen that omega is of the order of 10 raise to power 5. So, in the first approximation I can neglect terms of omega square order. While calculating the effects of Coriolis force, again we will neglect terms of the order omega square because omega square is much smaller than omega.

So, if I calculate the Coriolis force it is going to be proportional to minus 2 and omega cross v. There is 1 component of omega in the north direction and you can see that will give me a component only along the z direction. Because, omega is along I and j omega towards the north direction this component will give me component along the z direction. So, it will not affect the motion in the XY plane; what affects the motion in the XY plane is only the z component of omega.

So, let me write this omega cosine lambda in some r vector along the XY plane plus omega sin lambda in the Z direction cross v vxi plus vyj and in front I have minus two m. So, as for as the motion in XY plane is concern this term is not required. So, I will just drop it right here because I am considering the motion in the XY plane and therefore, for the motion in XY plane the terms which are important from the Coriolis force are minus 2 m.

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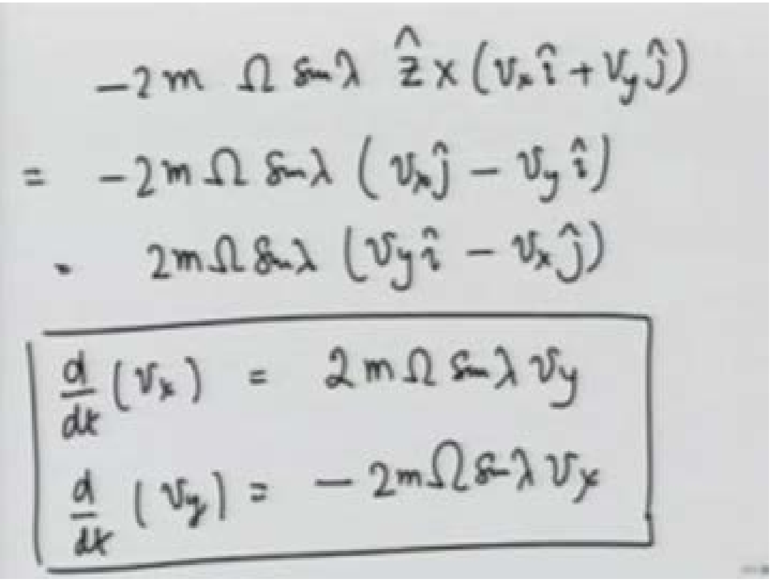

Omega sin lambda z cross vxi plus vyj will gives me minus 2 m omega sin lambda inside z cross i is j. So, vxj minus vyi which is same as, 2 m omega sin lambda vyi minus vxj and therefore, d by dt of vx is going to be equal to 2 m omega sin lambda vy and d by dt of vy is going to be minus 2 m omega sin lambda vx. This is how the deflection is going to be caused. Let us see, how that comes about. So, if I have let me now look at this x y plane and let us call, x east direction and y in the north direction.

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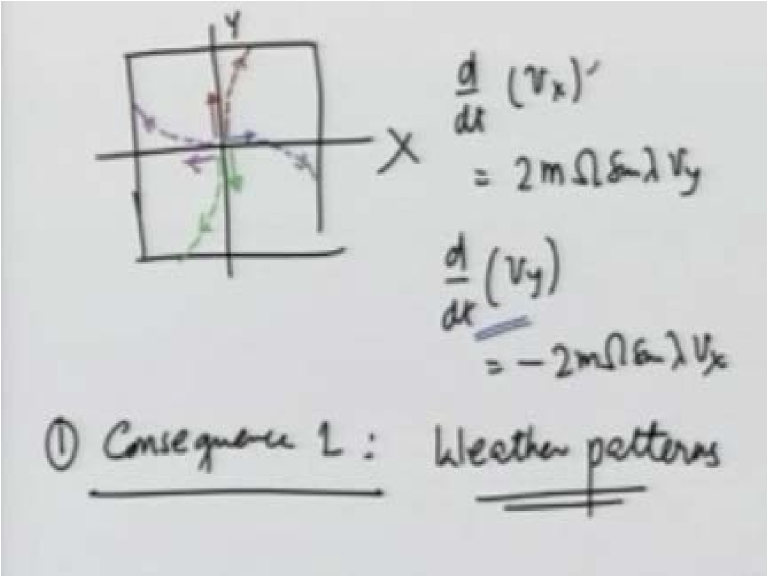

I already have d by dt of vx equals 2 m omega sin lambda vy and d by dt of vy equals minus 2 m omega sin lambda vx. So, suppose we are in the north northern hemisphere and I throw a particle along x direction towards the east so that, initial vy is 0 and vx is finite. Then you will see that v, you slowly start developing a vy because d by dt vy is not 0 in the negative direction. So, this particle would tend to deflect like this.

If I throw a particle to the north; so, initial vy is positive vx is 0 it will tend to develop a vx and therefore, would deflect like this. If I throw a particle towards the south then I have minus vy. So, it will start developing a negative vx and it tend to deflect like this. Similarly, if I throw a particle towards the west it will tend to deflect towards north because of the effect of Coriolis force and this has important consequences. Consequence number 1 weather patterns i will just give you one example which we are very familiar with.

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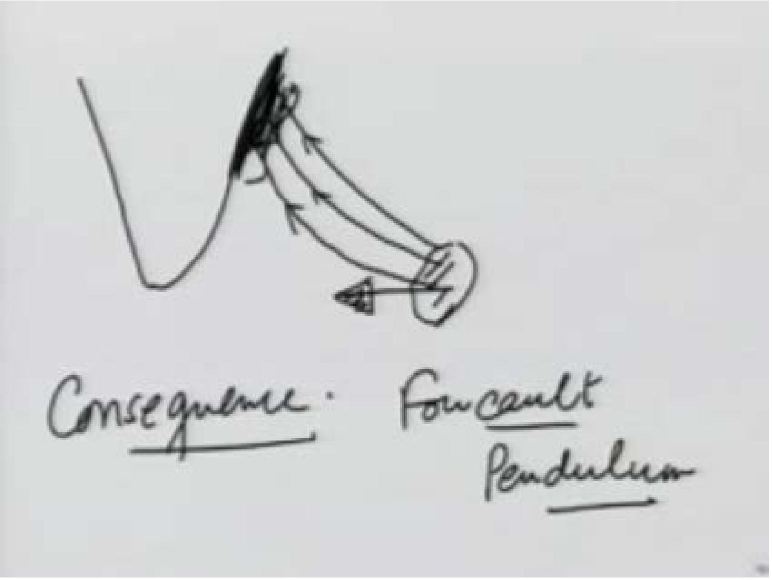

You see every year, the Orissa coast of India is hit by storms how does that happen? Suppose, the storm develop somewhere in the Indian ocean, the storm to starts moving towards the left. If it is moving towards the left we just saw because of the Coriolis force it tends to force deflect north words.

So, it will slowly come up and you see how naturally it starts hitting this coast. So, no matter where it is developed if the storm towards the to towards the west it will always deflect up and hit Indian coast along Andhra and Orissa and that is how we get lot of storms there. Another consequence and a beautiful example of this is the Foucault pendulum and let me explain that; that is always given as example of Coriolis force. This was one of the first demonstrations of showing very clearly how the earth rotates.

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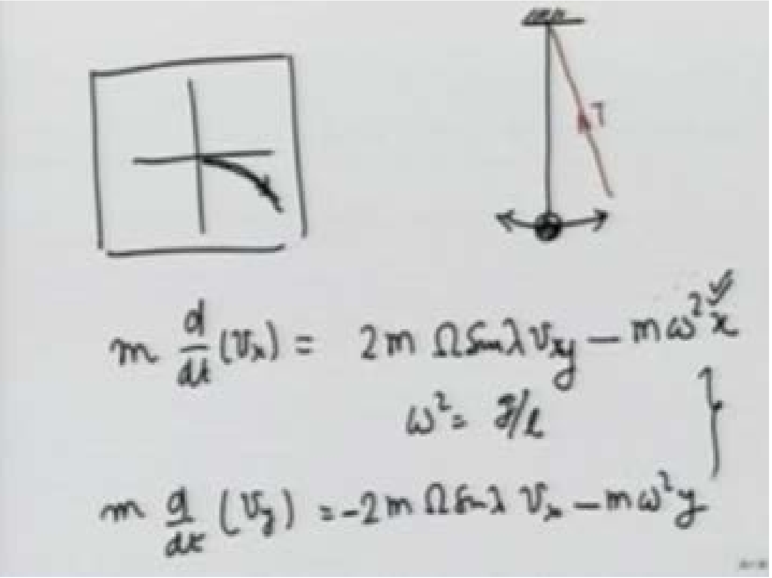

So, when considering this motion along the x y plane suppose instead of letting the particle move along the XY plane, I tie it on a string and make a pendulum out of it. Then also it will keep moving along the surface of the earth; however, now it moves under 2 forces the forces are 1 due to the tension in the string and other due to Coriolis.

So, I am going to write d by dt of vx 1 term we have already seen is 2 m m if if I am considering force then m has to be there. 2 m omega sin lambda vy minus m omega square x, where omega square is nothing but g over l. And similarly m d by dt of vy is going to be 2 minus 2 m omega sin lambda vx minus omega m omega square y.

So, the pendulum is going to swing with under the influence of these 2 forces: 1 the restoration force m omega square x and the other the Coriolis force and you can already see that because along when the Coriolis force x acts, the things the ball tends to deviate. And therefore, as the pendulum swings you expect that that this plane should deviate; should rotate should deviate from the path intended and slowly it should rotate. The calculation of this would become easier if we consider r theta coordinate system.

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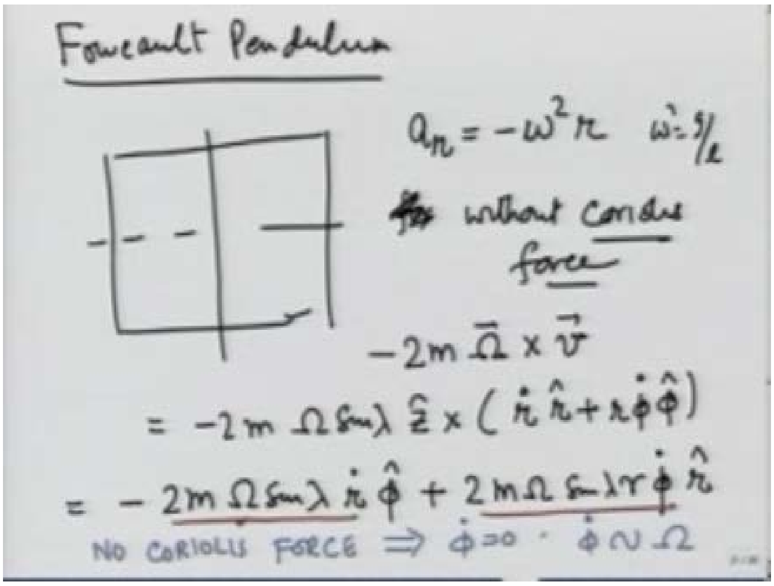

We are continuing with Foucault pendulum. In the xy plane, instead of using xy coordinate system I am using r theta coordinate system so that, I can write that a r is equal to minus omega square r where omega square is g over lambda for without Coriolis force. But with the Coriolis force I have an additional force which is equal to minus 2 m omega cross velocity which, I will write as minus 2 m omega sin lambda in z direction, cross r dot in r direction plus r phi dot in phi direction.

And this gives me minus two m omega sin lambda z cross r is phi so, r dot in phi direction and z cross phi is minus r; so, plus 2 m omega sin lambda r phi dot in r direction. Notice, if there was no Coriolis force, if there is no Coriolis force then phi dot is zero. So, no Coriolis force implies phi dot is equal to 0 and therefore, phi dot we can except to be of the order of omega. Therefore, in these 2 terms omega time’s phi dot is going to be an order smaller than the first term. Let me write it clearly on the next page.

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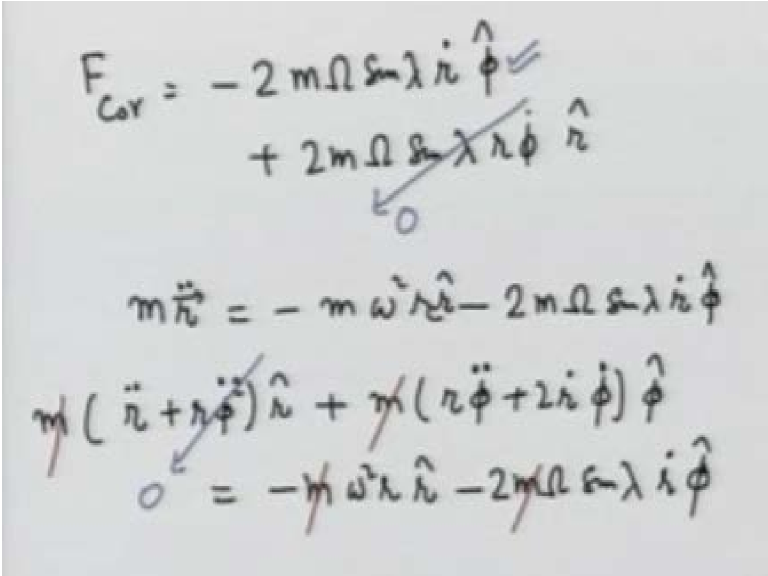

We have F Coriolis equals minus 2 m omega sin lambda r dot phi plus 2 m omega sin lambda r phi dot along r direction. And this we are arguing is an order omega smaller than the first term. So, I am approximating this to be 0; so, effectively this is the force. So, what does my equation including this becomes? It becomes, mr double dot equals minus m omega square r that is the restoration force minus 2 m omega sin lambda r dot in phi direction. So, this should be r vector and this I am going to write as m r vector is r double dot plus r phi dot square in r direction.

So, in r direction plus n r phi double dot plus 2 r dot phi dot in phi direction and that should be equal to minus m omega square r in r direction minus 2 m omega sin lambda r dot in phi direction. Again as I argued earlier, phi dot square is of the order of omega square so I am going to neglect it here and therefore, my equations of motion in the r and phi directions become; this m is also going to drop out.

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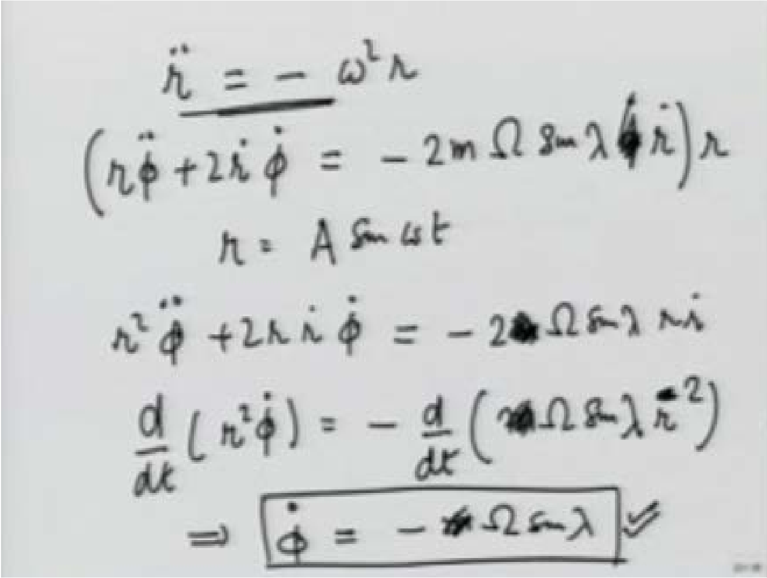

So, my equations become r double dot equals minus omega square r and r phi double dot plus two r dot phi dot equals minus 2 m omega sin lambda r dot. First equation gives me the usual oscillatory behaviour with r equals some amplitude sin omega t To solve the second equation, I multiplied this whole thing by r to get r square phi double dot plus 2 r r dot phi this is phi dot is equal to minus 2 m omega sin lambda r r dot. And this I can write as d by dt of r square phi dot equals minus d by dt of m omega sin lambda r square.

A comparison gives me phi dot equals minus m omega sin lambda. And therefore, as the pendulum swings its plane of oscillation keeps rotating backwards with a a phi dot which is minus m omega sin lambda.

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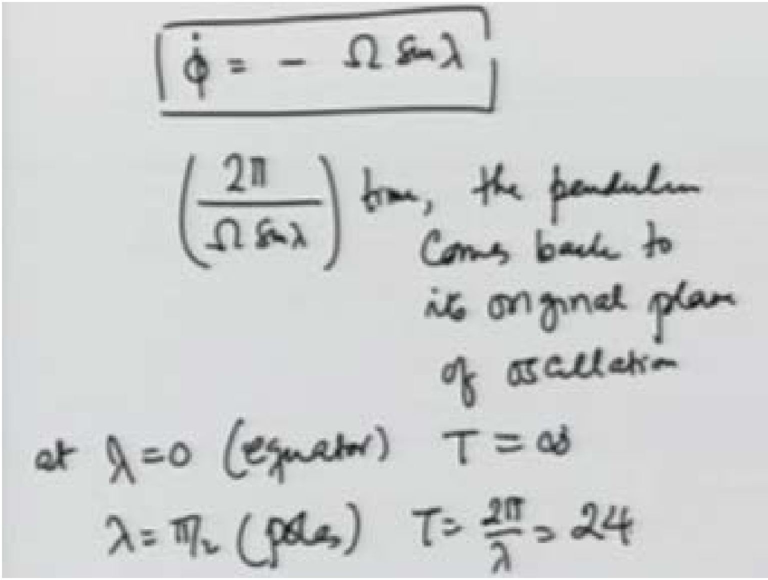

Therefore this m has dropped out; this m is not there this m is not there this m is not there and therefore, phi dot is equal to minus omega sin lambda. So, it completes a entire cycle and 2 pi over omega sin lambda time the pendulum comes back to its original plane of oscillation. At lambda equals 0 that is equator, this time is infinity and therefore, the plane never rotates at lambda equals pi by 2, that is the poles t equals 2 pi over lambda equal 24 hours.

So, as within 24 hours the plane comes back to its own you can physically visualize this, if this is the plane of rotation and the earth is rotating beneath it at the north pole the earth comes back to its original position, in 24 hours and therefore, we see sitting on earth that the plane has come back to its original position in 24 hours. And as you go further away from the poles the time increases.

So, I have given you two examples of deflection of a particle moving along the surface of the earth: 1 where I showed that x and y when I gave vx it tends to deflect in y direction and another example which is the Foucault pendulum. A consequence of this deflection is that when you fire say a missile or a bomb over long distances they tend to deflect on their intended target. And the correction for that has to be made; that correction has to be made because there is Coriolis force or the earth moving underneath that moving target so that, it deflects on a straight line as far as seeing from the earth is concerned.

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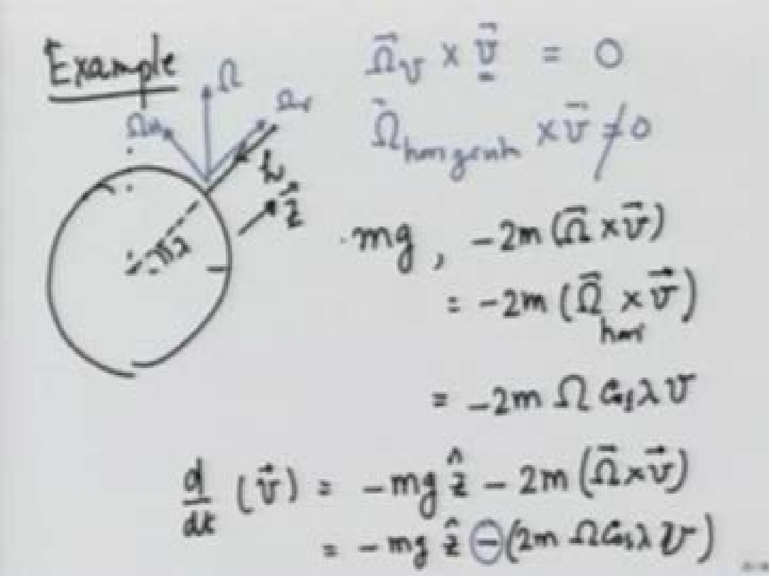

As the final example of the effect of Coriolis force let us take, a particle which is dropped from a height h and see how it deflects. In this case this is omega it has 2 components: 1 the vertical component, 1 component going towards the north and you will see that omega v; the vertical component cross v because of particle is dropping straight down is 0.

So, the deflection would come because of the omega horizontal. Let me call this omega horizontal, omega vertical cross v is not 0. So, the forces that are acting on the particle are the mg vertically down and minus 2 m omega cross v and that we have already seen is nothing, but minus 2 m omega horizontal cross v. Which is going to be if I am at latitude lambda minus 2 m omega cosine lambda times v. And this is going to be perpendicular to north direction and the vertical direction.

So, now if I were to write the equation of motion, I am going to have let me call this direction, the z direction. I am going to have d by dt of v is equal to minus mg z minus 2 m omega cross v. And up to the first order we have already seen if the particle is coming down, this is going to be minus mg z minus 2 m omega cosine lambda v. This sin I am right now not writing clearly because, I will talk in terms of west north east north south west east.

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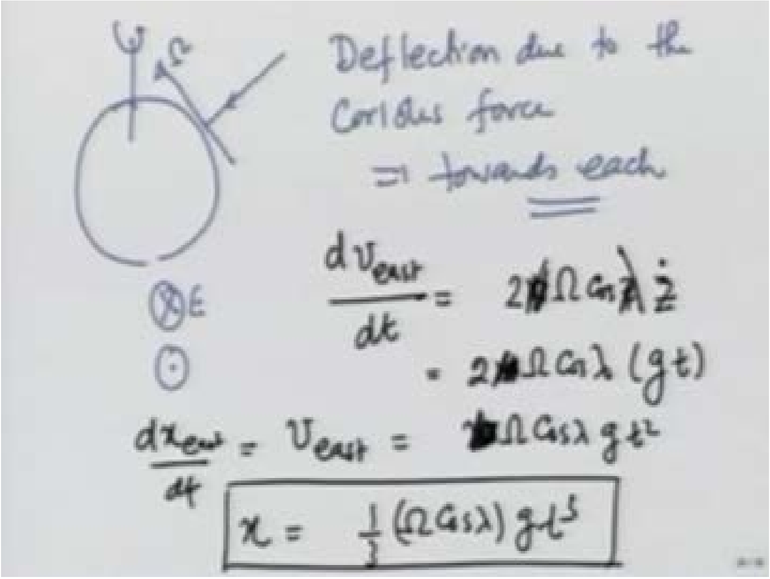

So, what we have figured is that if I have this Earth and the particle is coming down only this component of omega matters and omega cross v is coming. The earth is rotating like this; east is going into the board. So, this is east west is coming out so omega cross v is towards west minus omega cross v is towards east. So, the deflection due to the Coriolis force is going to be towards east. What is the equation for that? We have d v towards east divided by dt is equal to 2 m omega cosine lambda z dot z dot is the velocity. Which is nothing but, 2 m omega cosine lambda and to first approximation; z dot is nothing but, gt.

Integrating this we get v east to be equal to m omega cosine lambda gt square and therefore, this is nothing, but dx towards east divided by dt x comes out to be one-third. Again this m is not going to be there because, now we are concerned only with the acceleration. One-third omega cosine of lambda gt cubed. So, a particle dropped from a height tends to deflect towards east on the northern hemisphere.

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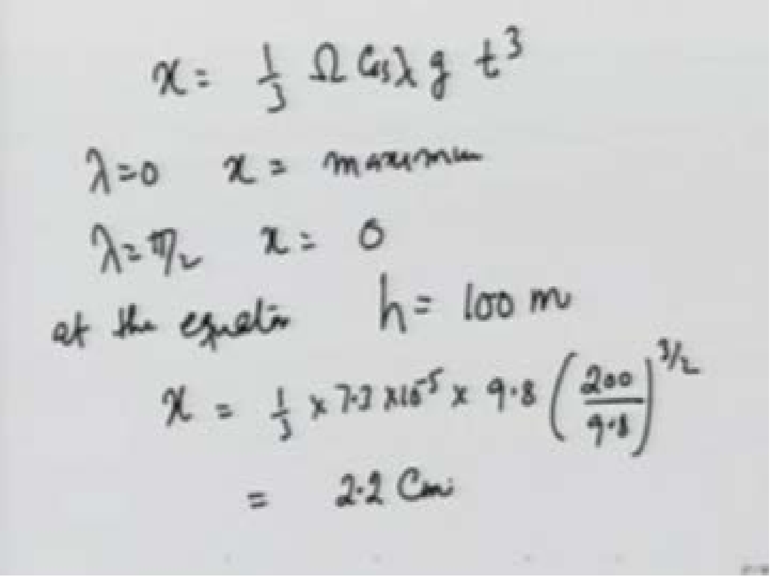

How much is the magnitude? So, x equals one-third omega cosine lambda g t cubed. Lambda equals 0, x is maximum lambda equals pi by 2 that is at north or South Pole x is 0. So, at the equator if a particle is dropped from a height of 10 meters i can calculate x which is one-third omega is 7 point 3 times 10 raise to minus 5 times cosine lambda is 1 g is 9 point 8. This is going to be 200 divided by 9 point 8 raise to 3 halves that is a time cubed and this comes out to be 2 point 2 centimeter's.

So, we can see the effect is small, but significant. If we go to the southern pole omega cross v would change direction and therefore, the deflection would be westward. This concludes our lecture on the rotating frames and also our 2 lectures on motion in non-inertial frames. What i have tried to do in these 2 lectures is given you an idea as to how to with the motion in non-inertial frames. We can do many more complicated problem the idea here was to just convey an idea give you a feeling for non inertial and pseudo forces.

Thank you.