Engineering Mechanics
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Module – 08
Lecture - 02
Simple Harmonic Motion – II

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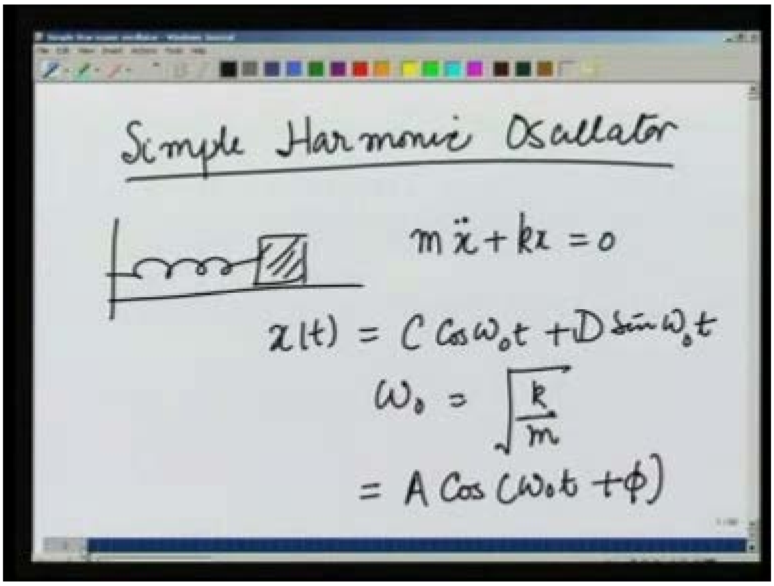

In the previous lecture we looked at general features of Simple harmonic oscillator and I told you that, a prototype system for this is a spring mass system so we keep doing things. With this, I did give 1 or 2 examples so that, we could relate different things or different system. to this spring mass system

The general equation of motion being mx double dot plus kx equal to 0 and the general solution being xt is equal to C cosine omega 0 t plus D sin of omega 0 t where, omega 0 is a square root of k over m and C and D are, determined by the initial conditions. I could also write, the solution can equivalently be written as an amplitude times cosine of omega 0 t plus phi, where the constants are A and phi are determined by the initial condition.

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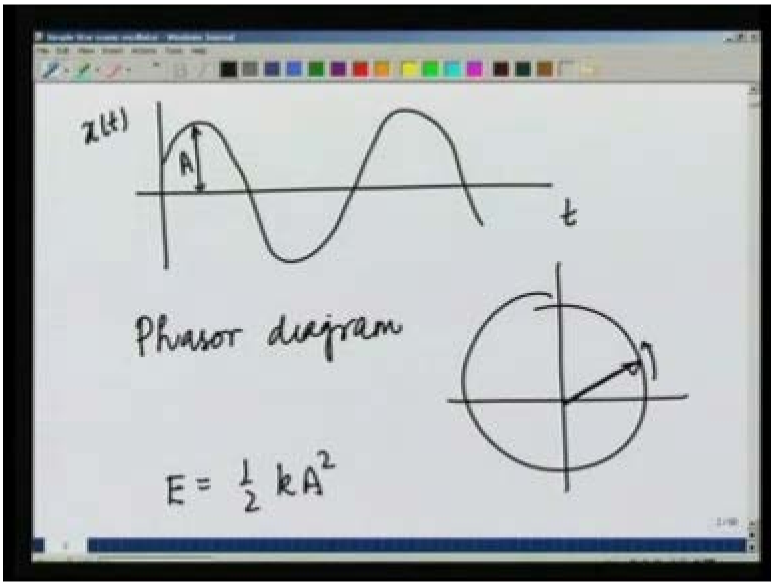

We also looked at, how when I plot xt verses t, it looks like this. That means, the motion is back and forth, x close up to a maximum value, comes down this maximum value is nothing but, the amplitude and we also saw how to represent this geometrically using phasor diagram, where we represented the motion as x component of a vector rotating counter clockwise. Finally we also, saw how the energy of the system is one-half k A square, which is really the sum of the kinetic and the potential energy of the system.

In this lecture little complication to the system and we are going to introduce,

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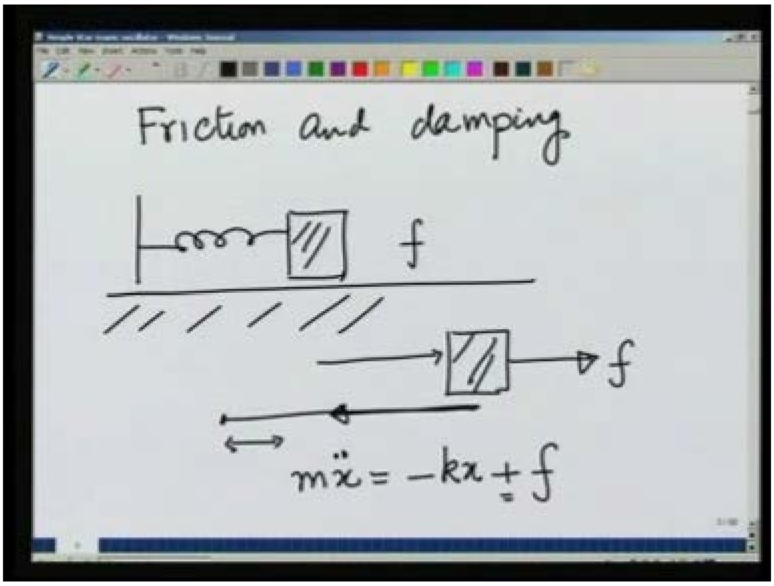

friction and damping. I am writing these 2 terms separately because they are going to mean slightly different things. So, let us see what happens when I take the system and let there be a frictional force on the block, call the magnitude of the frictional force f. Obviously we know from experience, as the block goes back and forth it is going to lose energy due to this frictional force and finally, come to rest. How does that happen? Let us see that.

So, if I write the equation of motion, I will have to be careful because, the frictional force changes direction according to, how the block is moving. Let us first take the case, when I stretch the block out and let it go inside. I will now focus on the motion, as long as it keeps moving this way and comes to a stop after compressing the spring by this amount. In that case, the equation of motion is going to be mx double dot is equal to the force by the spring minus kx plus a frictional force f. I have written this plus sin because; when the block is moving this way the frictional force is going to be in the positive x direction like this. This equation is valid as long as the block is moving to the left.

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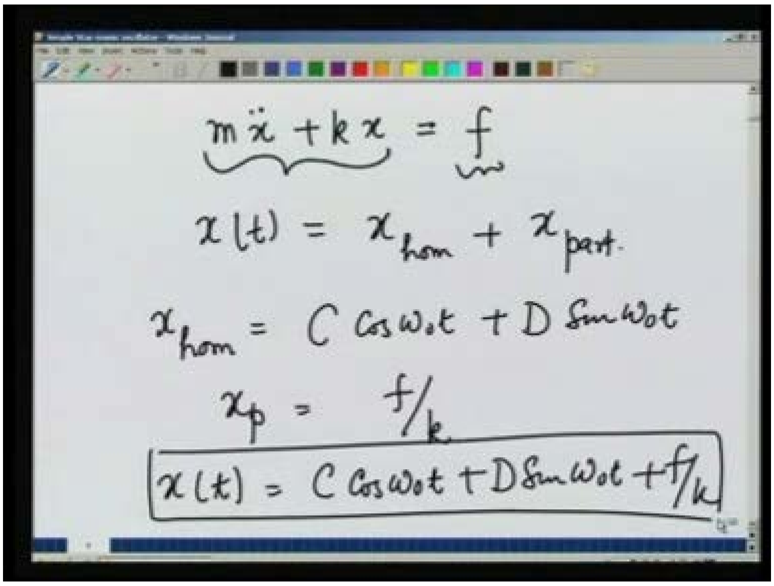

So the equation I have is, mx double dot plus kx is equal to f and we have seen this kind of equation in the past. This part is the homogeneous part of the equation and this part is in homogeneous part. Therefore the solution, general solution x t is going to be the sum of x homogeneous plus x particular due to the in homogeneous part. x homogeneous I already know, is equal to some C cosine of omega 0 t plus D sin of omega 0 t and you can verify that, x particular in this case is going to be f over k and therefore, the solution x t is going to be C cosine of omega 0 t plus D sin of omega 0 t plus f over k

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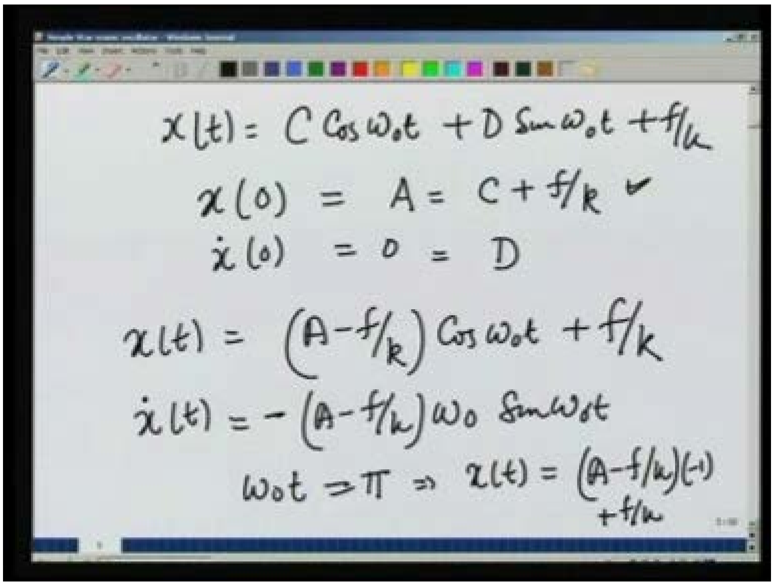

This is a general solution and only after write the general solution; that I impose the initial conditions. So, the initial conditions we have this time is: let me first, write the solution once again, is equal to C cosine of omega 0 t plus D sin of omega 0 t plus f over k. I stretch the block out; that means, I took the block at x equal to 0 to some distance A and released at with no initial velocity. This should be then equal to C plus f over k and this gives me this equal to D and therefore, the general solution x t is going to be, C which is A minus f divided by k from this equation, cosine of omega 0 t plus f divided by k.

When does the block stop? Let us take x dot of t. This comes out to be, a minus f over k times omega 0 sin of 0 omega t with a minus sign. So, when omega 0 t is equal to pi that is; half a cycle, the block comes to the stop and at that time x t is going to be, A minus f over k times cosine pi which is minus 1 plus f over k.

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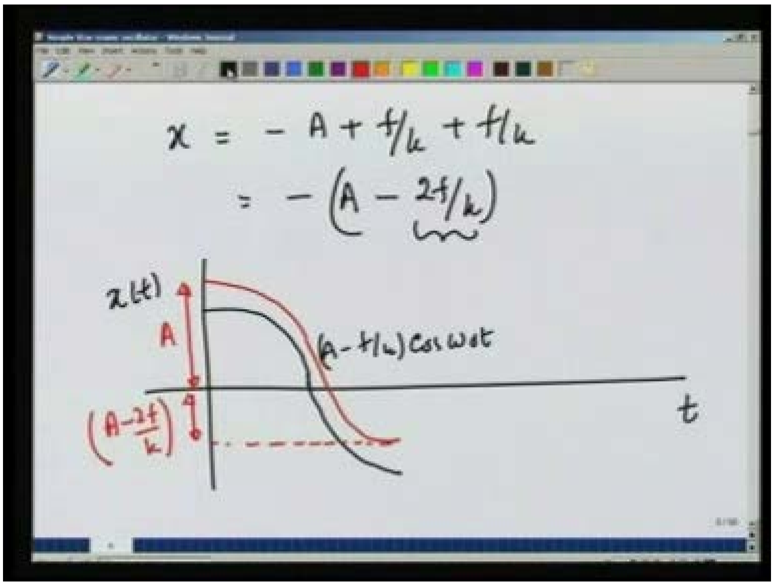

So, after the block started moving left, when it came to a stop, it is stopped at x equals minus A plus f over k plus f over k which is, minus A minus 2f over k. So, when the frictional force f is present, by the time stops for the first time, the block has lost amplitude of the amount 2f over k. When it goes back, it will again lose this amount. I can later, derive the same thing using energy consideration, but right now let us focus on, how the motion looks.

So, if I would plot in the first half cycle, x t verses t, this curve shows A minus f over k cosine of omega 0 t. What I got to do to this now, let me complete this, it comes up to this point; is added to f over k. So, the solution is going to be, I will shift it up by f over k. So, this distance is A and this distance here is A minus 2f over k. So you see, it has lost that much amplitude by the time comes here, after it reaches its point here. It starts its journey back towards right again. And let us see what the equation that case looks like.

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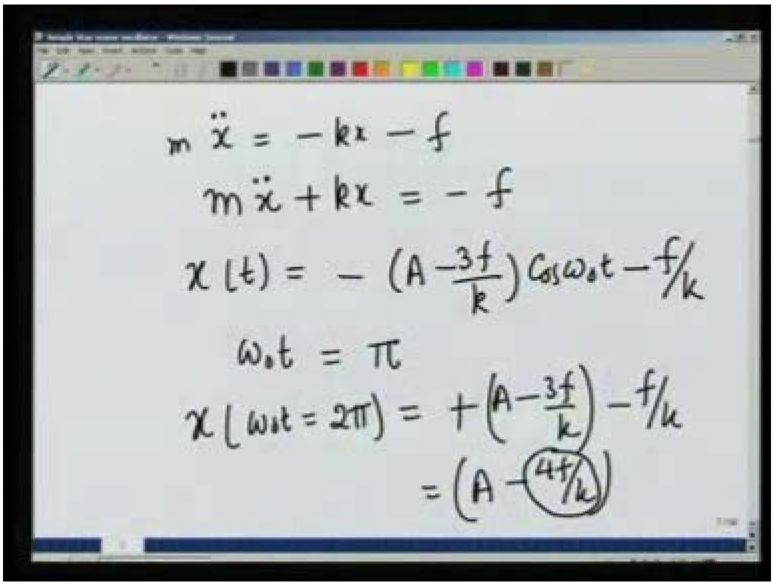

Equation in that case is going to be, mx double dot is equal to minus kx minus f, minus f because, now the frictional force is going to work towards the left and therefore, I can write the homogeneous part like this, it is going to be equal to minus f. I will leave the solution for you to work out, with the initial condition that, the distance to the left is now A minus 2f over k; you will find that while moving to the right, x t is described as minus A minus 3f over k cosine of omega 0 t minus f over k. Again it comes to a stop, after time omega 0 t equals pi. Therefore, x after times such that, omega 0 t is equal to 2 pi total after it started the joining right in the beginning, it is going to be, minus A minus 3f over k times cosine pi that will make it plus minus f over k, which is equal to A minus 4f over k.

So, during the entire cycle that the particle came this way and went back, it has lost the amplitude of 4f over k. How does the solution look?

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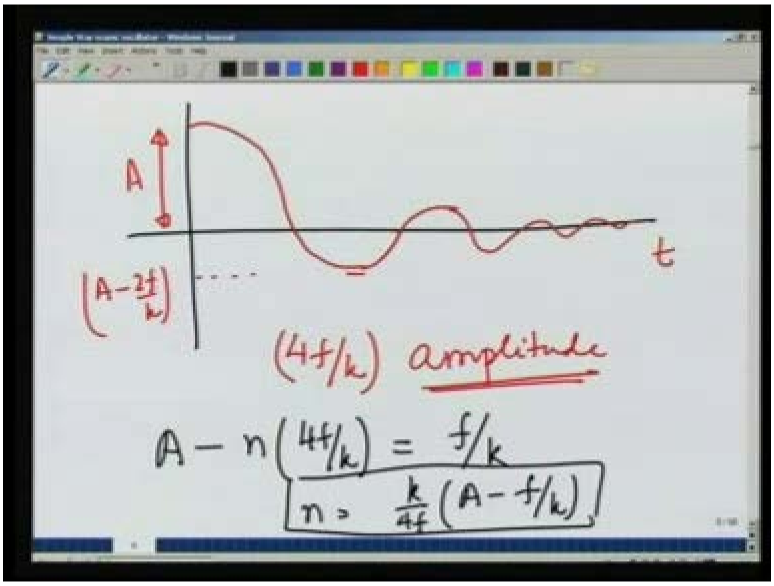

I already showed you, that when I plot x t verses t, it started from a distance A and came up to A minus 2f over k, then again it loose to f over k. So, when it goes back, it is going to go down and then its journey starts all over again, going to lose 4 over k again over the entire circle and therefore, the motion raised on like this. How many cycles does it complete? Let us see that.

So, in each entire cycle it is losing 4f over k amplitude. So, this is a distance it is losing every time. Let us say it completes n cycles. So, it would have lost from the initial amplitude A, n times 4f over k distance and it finally, comes to a stop where does it come to a stop? At a distance f over k because, their friction and spring force, they balance each other and this gives you, n is equal to A minus f over k times k over 4f that is, a number of cycles that the particle would perform before it comes to a stop.

As I said earlier, the same thing can be derived using energy methods and let us see how.

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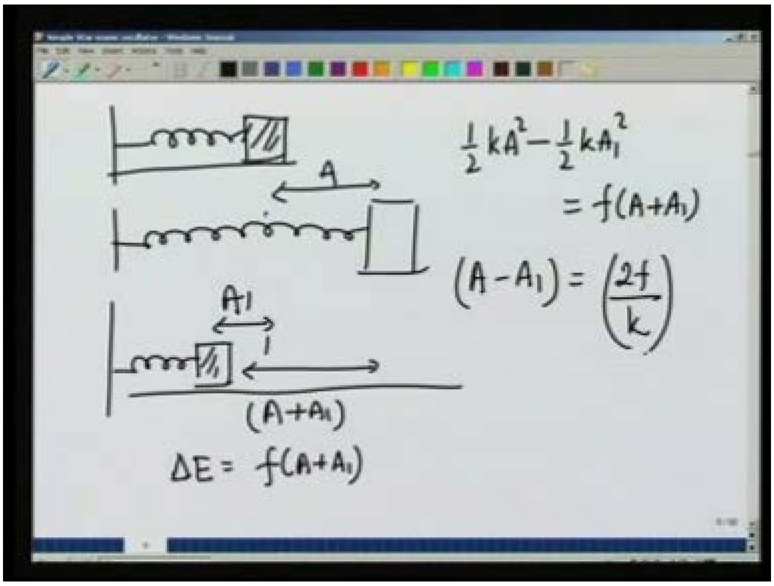

When I took this spring mass system and stretched it to a distance A and released it. By the time it stops, it moves a distance up to A1 from its equilibrium position. So, after releasing it would have travelled a distance of A plus A1 and during this time, the energy it will lose due to friction is delta E which is equal to, f times A plus A1.

Therefore, the initial energy which is, one-half k A square minus the final energy which is one half k A1 square should be equal to f A plus A1 and that gives you, that A minus A1 you can work out the algebra is going to be, 2f over k. So, by the time the mass completed half cycle it has lost amplitude of 2f over k. In the entire cycle, it will lose 4f over k and then, you can work out things as we did earlier. So, this is a simple example of how friction affects the motion.

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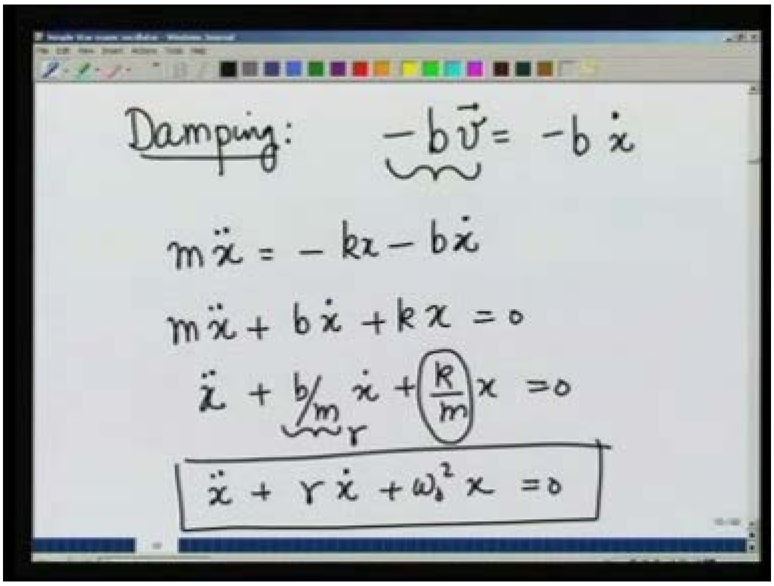

Now, we go to the next level of retardation and that I am going to call, in general damping and usually when I talk about damping in oscillations; I mean a term which is proportional to velocity. So, there is a retardation or damping force which is proportional to the velocity and; obviously, opposite to the velocity. Let me write this in 1 dimension as minus bx dot and therefore, the equation of motion in this case is going to be, mx double dot is equal to minus kx minus bx dot, which I can write as mx double dot bringing all the terms, x terms to the left, bx dot plus kx is equal to 0, dividing by m, I am going to write this as x double dot plus b over m x dot plus k over m x is equal to 0. This term we have already identified as omega 0 square, I am going to call this term gamma or, the damping coefficient and therefore, write this equation as x double dot plus gamma x plus gamma x dot plus omega 0 square x is equal to 0.

This is the equation for a damped harmonic oscillator where, the damping is propositional to the velocity. We will see how we will get different solutions for this and different kinds of motion under separate circumstances.

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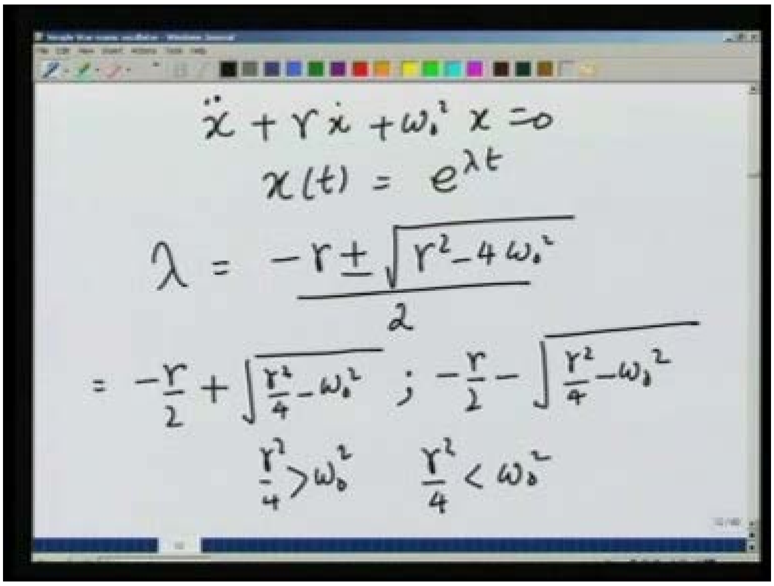

So, the 2 roots for this equation, x double dot plus gamma x dot plus omega 0 square x is equal to 0. When I assume my solution of the form e raise to lambda t, lambda that I get is minus gamma plus or minus square root of gamma square minus 4 omega 0 square over 2, which I can write as; minus gamma over 2 plus square root of gamma square by 4 minus omega 0 square or the other root is minus gamma over 2 minus the square root of gamma square over 4 minus omega 0 square.

Depending on the relationship between gamma and omega, the motion is going to be different. For example, if gamma square by 4 is greater than omega 0 square, in that case I am going to have no imaginary part in the roots and therefore, solution is not going to be of the oscillatory nature. On the other hand, if gamma square over 4 is less than omega 0 square, I am going to have a i imaginary parts in the roots and that is going to lead to oscillatory motion, as we saw in the undamped harmonic oscillator case. So, let us examine these cases 1 by 1.

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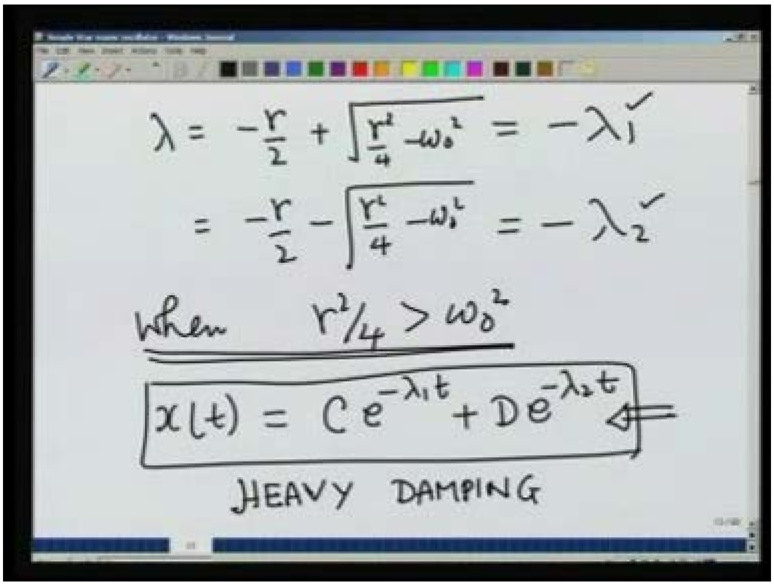

So, I got 2 roots lambda equals minus gamma over 2 plus square root of gamma square over 4 minus omega 0 square or also minus gamma over 2 minus gamma square over 4 minus omega 0 square square root, let me call this minus lambda 1, let me call this minus lambda 2 so that, the solution in the case when, gamma square over 4 is greater than omega 0 square and this is important, now I am going to focus on this. When this is so, lambda 1 and lambda 2 are real and the general solution x t is going to be of the form C e raise to minus lambda 1 t plus D e raise to minus lambda 2 t.

When gamma square over 4 is greater than omega 0 square, this case is known as the heavy damping case. You can see from this expression, that the solution is not oscillatory any more. So, right now let us focus on heavy damping. Later we will see, when gamma square over 4 is equal to omega 0 square, both lambda 1 and lambda 2 become equal. In that case they seem to be only 1 solution. We will obtain the other solution, by taking the limit lambda 1 going to lambda 2 and that is known as critical damping. We will discuss that also and finally, when gamma square over 4 is less than omega 0 square that is known as light damping case, in that case we will obtain oscillatory solution.

So, right now let us focus on heavy damping case.

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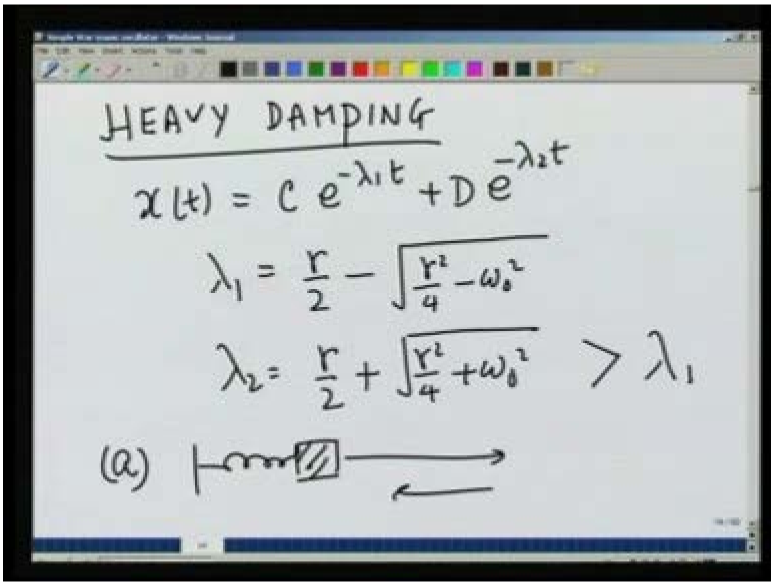

In that case, the solution x t is of the form C e raise to minus lambda 1 t plus D e raise to minus lambda 2 t, where lambda 1 if you recall, is gamma over 2 minus the square root of gamma square over 4 minus omega 0 square and lambda 2 is gamma over 2 plus a square root of gamma square over 4 plus omega 0 square and this is the way it is defined, is greater than lambda 1. I will study 3 cases in this case and see how the motion looks like. So, let me take case 1, where I take this spring mass system, stretch it and leave it and let us see, how the motion looks like in the that case.

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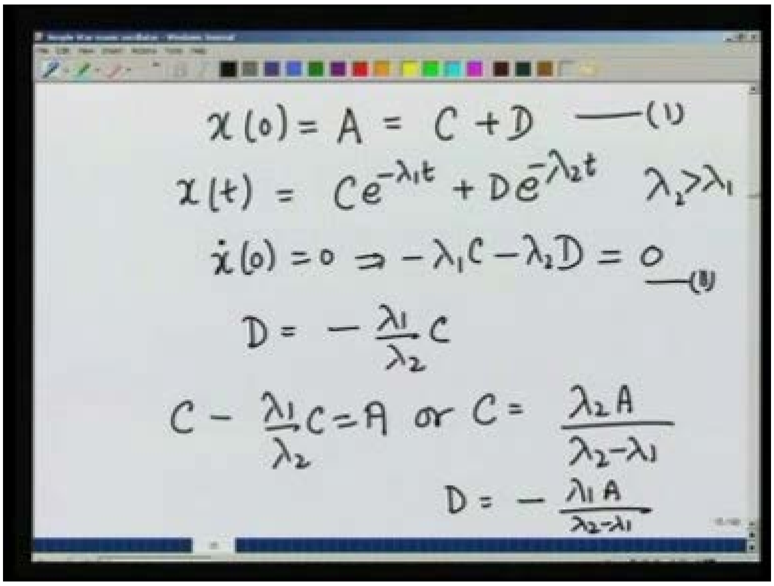

So, what I am given in this case is, that x at 0 is equal to some value A which, from the general solution is going to be C plus D. Recall, that my general solution x t is equal to C e raise to minus lambda 1 t plus D e raise to minus lambda 2 t, lambda 2 greater than lambda 1. And since, x dot at 0 time t equal to 0 is 0 this tells me, that minus lambda 1 C minus lambda 2 D is going to be equal to 0. These are two equations that give me, the coefficients C and D. From equation 2 I have, D is equal to minus lambda 1 over lambda 2 C. Substituting this in equation 1 I get, C minus lambda 1 over lambda 2 C is equal to A or C equals lambda 2 A over lambda 2 minus lambda 1 and therefore, D equals minus lambda 1 A over lambda 2 minus lambda 1. I have obtained both the coefficient, in the case when I pulled the mass out and let go.

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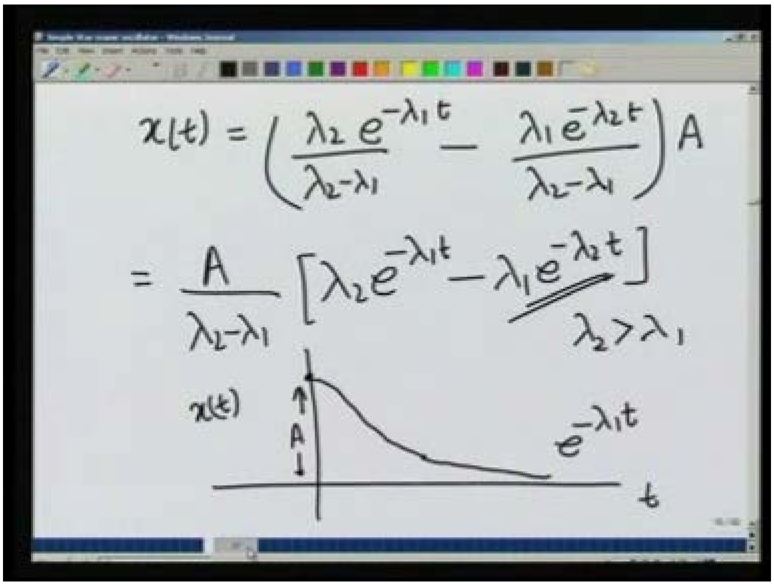

And therefore, the general solution in this case is going to be of the form, C was lambda 2 over lambda 2 minus lambda 1 e raise to minus lambda 1 t minus lambda 1 e raise to minus lambda 2 t over lambda 2 minus lambda 1 times A. You can substitute C and D and see this is a solution, which is equal to A over lambda 2 minus lambda 1 lambda 2 e raise to minus lambda 1 t minus lambda 1 e raise to minus lambda 2 t. And keep in mind that lambda 2 is greater than lambda 1

This motion if plotted, xt verses t is going to look like, I start with value A at time t equal to 0 and then it slowly decays and how does it decay? Since, lambda 2 is greater than lambda 1, as time goes becomes larger and larger time goes towards infinity, this term is going to die down much faster than this term. So, it is going to be decay at larger time as e raise to minus lambda 1 t. So, this is the solution.

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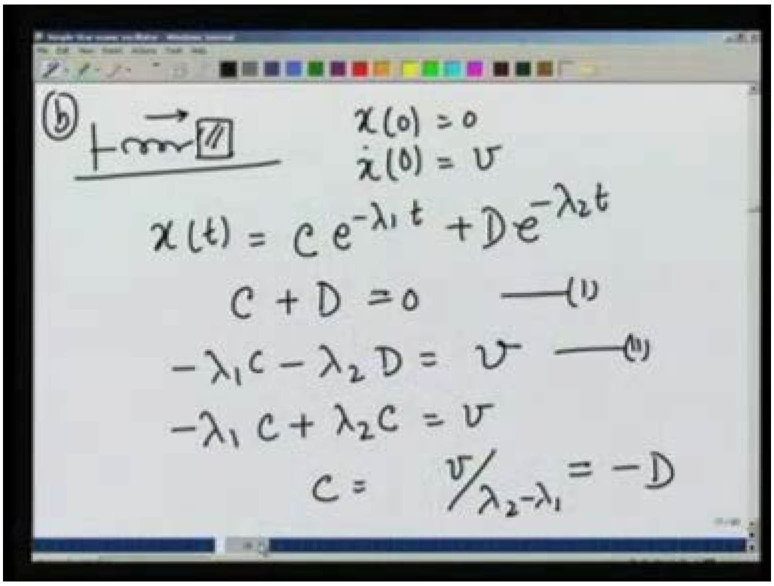

Let us now take second situation in which, I take this mass and give it an impulse so that, at x 0 equal to 0, it starts with a velocity x dot is equal to v in the positive direction. Again looking at the general solution x t equals C so this is my situation be; e raise to minus lambda 1 t plus D e raise to minus lambda 2 t. I have C plus D is equal to 0 and x dot which is minus lambda 1 C minus lambda 2 D is equal to v. This is my equation 2. Substitute D equals minus C from here, I get minus lambda 1 C plus lambda 2 c is equal to v or C equals v over lambda 2 minus lambda 1 is equal to minus D. And therefore, the solution in this case, when I am hitting the mass with an impulse, giving it an impulse of velocity v in the beginning right at the equilibrium point; is going t.

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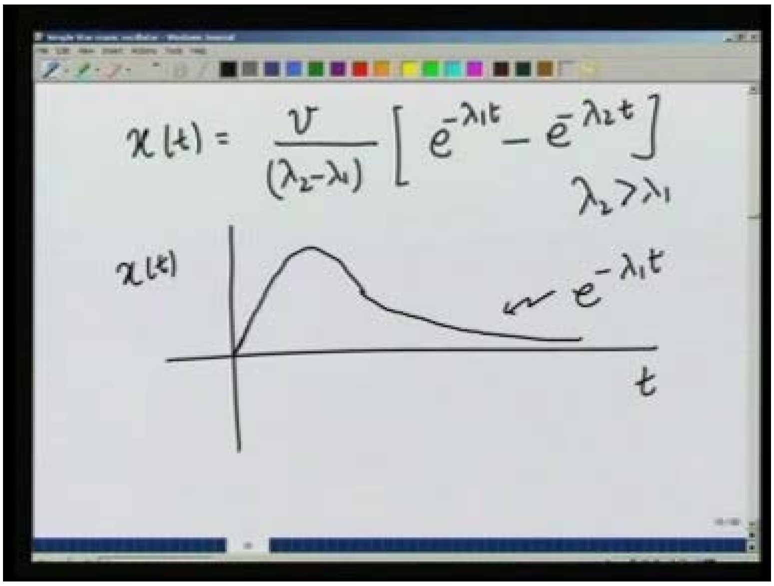

So, be x t is equal to v over lambda 2 minus lambda 1 e raise to minus lambda 1 t minus e raise to minus lambda 2 t. By plotting it, you can see that the solution is going to look like; initially the particle goes out and then, it decays this distance of decays exponentially. Again far away since lambda 2 is greater than lambda 1, it decays as e raise to minus lambda minus 1 t. This is the second situation.

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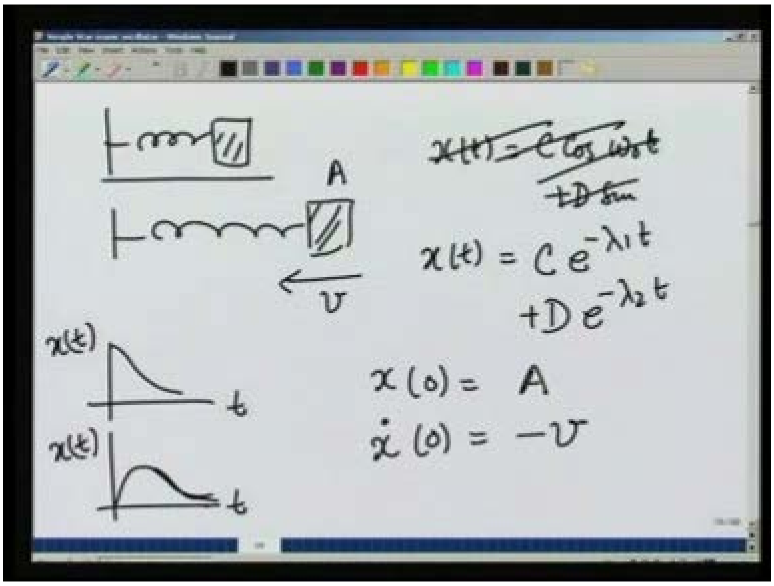

Now, let us take the most general situation in which, I take the mass, stretch it out and also initially give it away velocity going nv.

Why, I am doing that is; I want you to note, that in the previous two situations the solution were looking like this. So that, the particle really never cross the equilibrium point once it releases yet and the first situation it does, just slowly when towards the equilibrium point, in the second case it went out and slowly started coming towards equilibrium point. I want to see whether, I can really cross the equilibrium point. So, I am taking it out and pushing it in.

Again the general solution, x t is C cosine of omega 0 t plus D, in this case the solution, I went back to undamped oscillator x t heavily damped oscillator solution is, x t equals C e raise to minus lambda 1 t plus D e raise to minus lambda 2 t. And what I am doing is, at time t equal to 0 I am displacing it out and also giving it a velocity minus v and let us see what happens.

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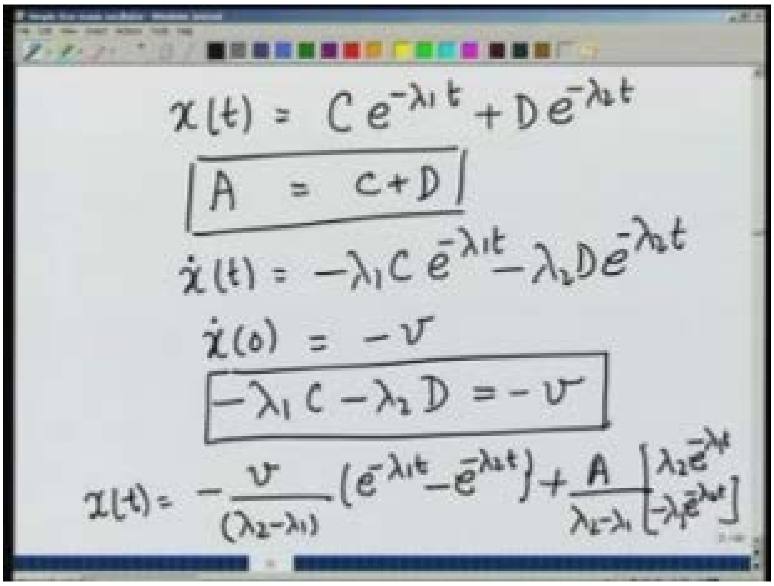

x t is equal to C e raise to minus lambda 1 t plus D e raise to minus lambda 2 t and therefore, I am going to have A is equal to C plus D. Similarly, x dot t is equal to minus lambda 1 c e raise to minus lambda 1 t minus lambda 2 D e raise to minus lambda 2 t and when I take, x dot 0 to be minus v I will get minus lambda 1 C minus lambda 2 D is equal to minus v. I have 2 equations and 2 unknowns. I can solve them. I leave it for you to work out and you see that the general solution in this case comes out to be, minus v over lambda 2 minus lambda 1 e raise to minus lambda 1 t minus e raise to minus lambda 2 t plus A over lambda 2 minus lambda 1 times lambda 2 e raise to minus lambda 1 t minus lambda 1 e raise to minus 2 t. Let me write this solution neatly on the next page.

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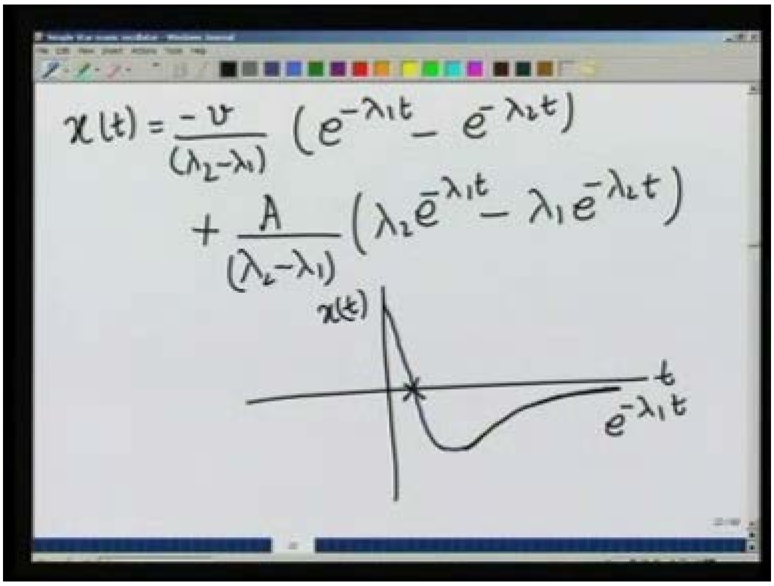

x t in this case, when I have taken the block out and then pushed it angular velocity comes out to be, minus v, minus because I am pushing it n divided by lambda 2 minus lambda 1 e raise to minus lambda 1 t minus e raise to minus lambda 2 t plus A over lambda 2 minus lambda 1 lambda 2 e raise to minus lambda 1 t minus lambda 1 e raise to minus lambda 2 t. And when I plot the solution in general, it is going to look like this. It starts with a value A, goes down, goes to the negative side and then after that decays the solution decays like this. So this 1 case, where the mass thus cross the equilibrium point, but once it reaches the left hand side maximum stretched, it again the solution decays exponentially.

So, we conclude that in heavy damping case the particle can cross the equilibrium point at most once and then far away, the solution decays as e raise to minus lambda 1 t. So, when time becomes large, slowly the part will comes towards the equilibrium. This is the case of heavy damping. Let us summarize this.

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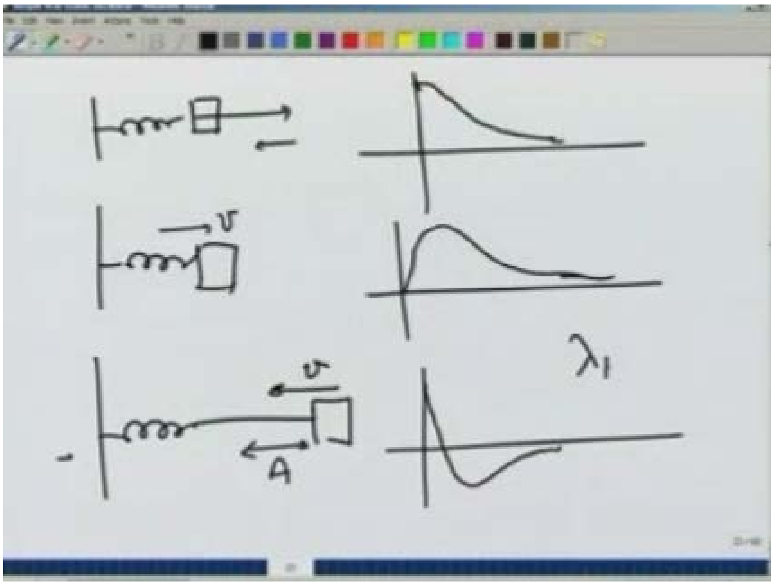

This in the case, when I stretched it and let go, the solution looked like, something like this, x t slowly decayed towards zero; that means, the mass slowly came towards the equilibrium point. The second case, when I took the mass and gave it an impulse, the mass went out and then again slowly it started approaching the equilibrium point and the third case, when I stretched it and then, I gave it a velocity also in that case, in general the mass cross the equilibrium point and then finally, slowly approach towards the equilibrium point. These are the cases where lambda 1 was not equal to lambda 2 and what we called, heavy damping.

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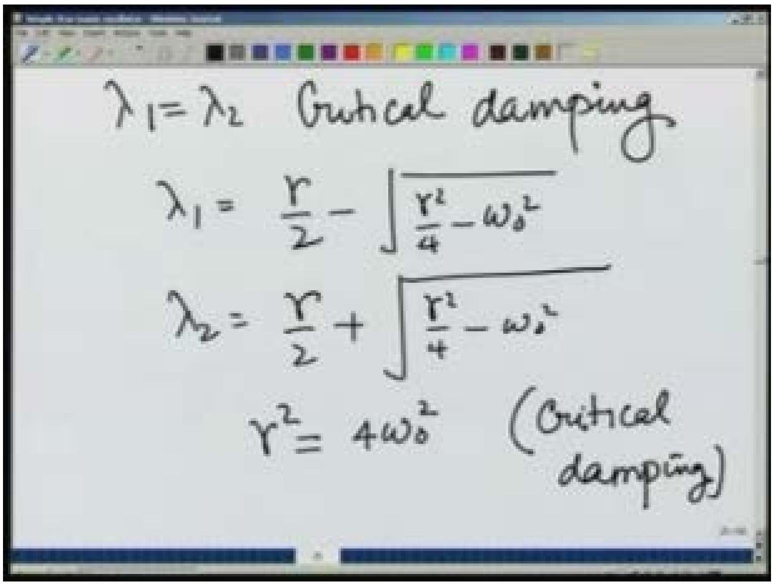

Let us now take the case, when lambda 1 is equal to lambda 2 this is known as the case of, critical damping and what it means is, you remember lambda 1 was gamma over 2 minus the square root of gamma square over 4 minus omega 0 square and lambda 2 was gamma over 2 minus plus square root of gamma square over 4 minus omega 0 square. What it means is; that gamma square is equal to 4 omega 0 square, critical damping.

Why, I am emphasizing critical damping separately from heavy damping is because, it is its motion is qualitatively different from that of heavy damping and that is useful sometimes, when I want to stop something from within a minimum distance when it is given an impulse. So, will we will see that. Since, there is only 1 lambda, I find that the solution x t is of the form e raise to minus lambda 1 t.

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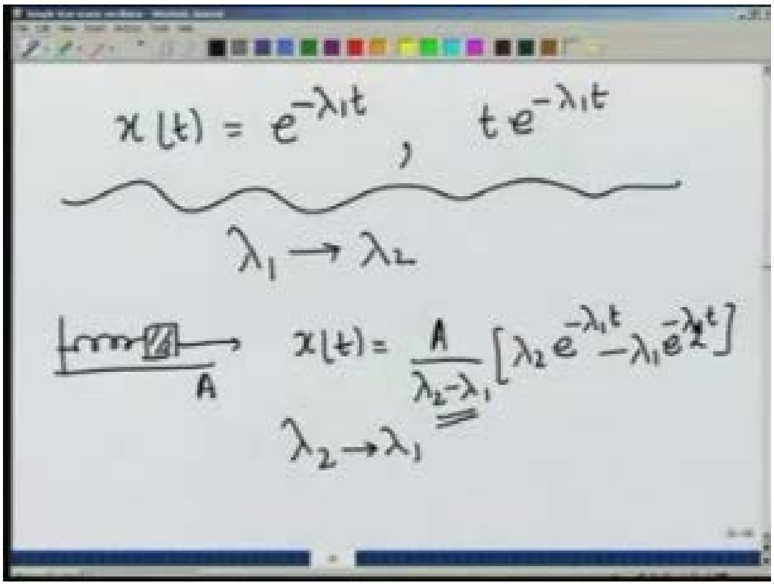

However, mathematically I know that, for a second order differential equation there should be 1 more solution. By standard techniques 1 finds the solution to be e t times e raise to minus lambda 1 t. However, I am not going to take this approach. What I am going to do is, let us take limit lambda 1 going to lambda 2 In all the 3 cases that we studied in the heavy damping and then see what the solution looks like.

Recall, when I had taken a mass, stretched it out and released it; solution in the that case for heavy damping case was x t equal to A over lambda 2 minus lambda 1 lambda 2 e raise to minus lambda 1 t minus lambda 1 e raise minus lambda 1 t. Now, I am going to lambda 2 t. Now, I am going to let lambda 2 go to lambda 1 and find the limit of that solution. Notice that, I cannot put lambda 2 equals to lambda 1 directly because, I am dividing by lambda 2 minus lambda 1 and therefore, I have to take appropriate limit.

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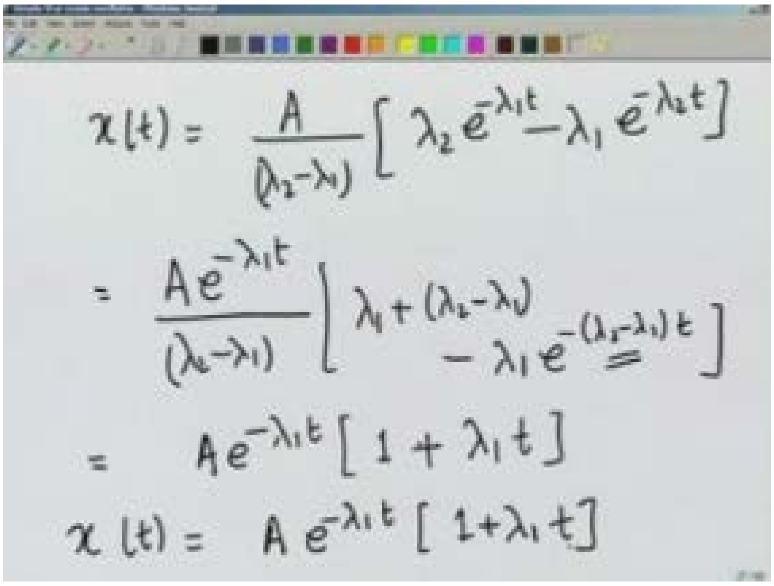

So, let us see what happens in this case. I have, x t equals A divided by lambda 2 minus lambda 1 lambda 2 e raise to minus lambda 1 t minus lambda 1 e raise to minus lambda 2 t. Let me take e raise to lambda 1 out, write this as A e raise to minus lambda 1 t divided by lambda 2 minus lambda 1. Inside I have, let me write lambda 2 as lambda 1 plus lambda 2 minus lambda 1. I am writing it in this form because, I am dividing by lambda 2 minus lambda 1 e raise to minus lambda 2 minus lambda 1 t.

Since, lambda 2 approaching lambda 1, I am going to expand this and keep it only up to the first order. The solution comes out to be A e raise to minus lambda 1 t 1 plus lambda 1 t. You can check it for yourself and therefore, in this case x t is equal to A e raise to minus lambda 1 t 1 plus lambda 1 t and what is lambda 1 in this case? It is nothing but, gamma.

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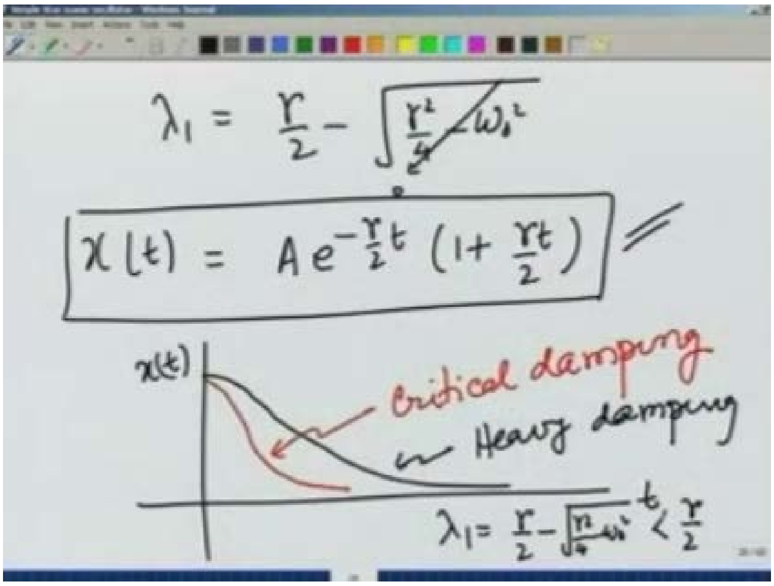

Remember lambda 1 is equal to gamma over 2 minus the square root of gamma square over 4 minus omega 0 square and this term is 0 in this case and therefore, the general solution in critical damping case is, when I stretch the spring out, A e raise to minus gamma over 2 t 1 plus gamma t over 2. You notice 1 difference right away compared to heavy damping. In heavy damping, the solution decay, it is also going to do the same thing in this case. However, it is going to decay much faster. This is critical damping and this is heavy damping.

Why does this happen? Recall that in heavy damping lambda 1 is equal to gamma over 2 minus a square root of a positive number and this is smaller than gamma over 2 and therefore, the solution decays slower. So, compared to what you would have thought intuitively, that in heavy damping the motion will get damped very fast what happens is, quite the opposite, it is in the critical damping that the motion gets damped very fast.

So, we have given you the solution for critical damping case, when I stretch this spring out or the mass out and let it go

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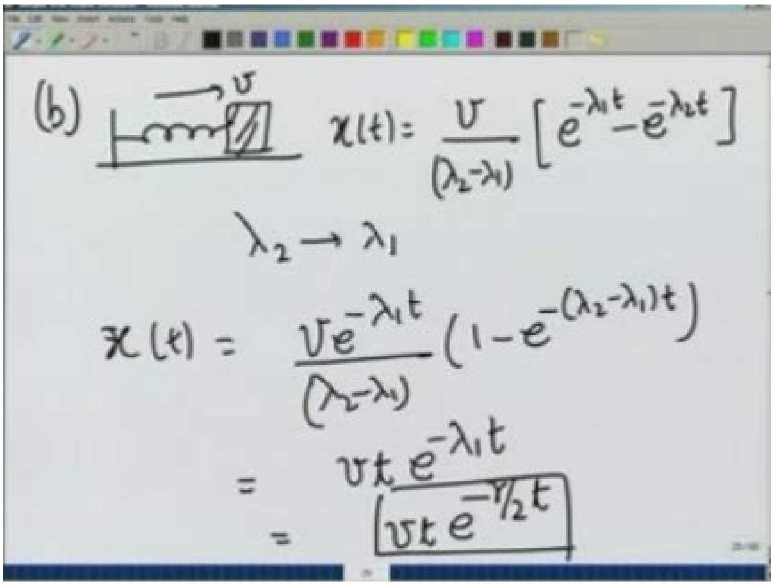

Let us examine the other case. Case b, in which case, I take this mass and give it an impulse in this direction. We call that in heavy damping the solution was x t is equal to v over lambda 2 minus lambda 1 times e raise to minus lambda 1 t minus e raise to minus lambda 2 t. Again, I am going to take the limit lambda 2 going to lambda 1. So, I will have to expand this. So, write x t as v e raise to minus lambda 1 t over lambda 2 minus lambda 1 1 minus e raise to minus lambda 2 minus lambda 1 t and I find the solution in this cases, vt e raise to minus lambda 1 t. Lambda 1 in critical damping cases gamma by 2. So, this is vt e raise to minus gamma over 2 t.

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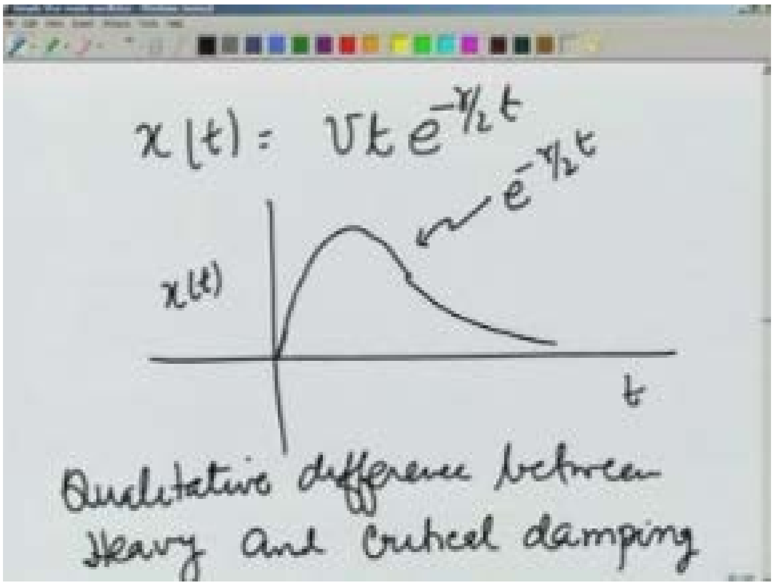

How does this solution look? X t equals vt e raise to minus to gamma over 2 t. When plotted against t, x t is going to increase initially as vt and then decay towards the equilibrium, again with the coefficient exponential decaying as e raise to minus gamma 1 gamma over 2 t. In this case again, the decay is faster than the heavy damping case. The third case I will leave for you to work out.

I came up to this point because, now I want to show you the qualitative difference between heavy and critical damping. To look at the qualitative difference between critically damped motion and heavy heavily damped motion.

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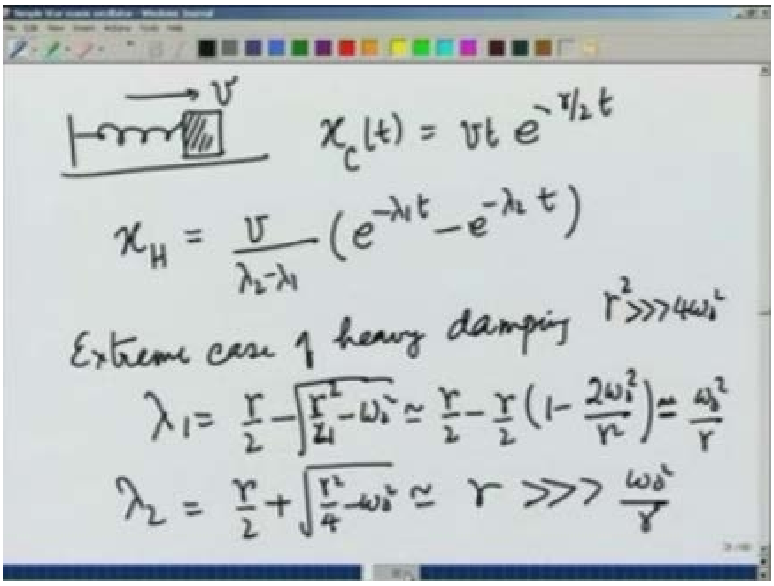

Let us take the case, when a block is given an impulse so that, it starts with the velocity v from its equilibrium point. In fact, critical damping is used in such situations for practical purposes, where I do not want a body to move very far from its equilibrium point, when it is given an impulse. I have already seen that, critical solution is vt e raise to minus gamma over 2 t and heavily damped solution, let me write x heavy is equal to v over lambda 2 minus lambda 1 e raise to minus lambda 1 t minus e raise to minus lambda 2 t.

To compare, I will take the extreme case of heavy damping where gamma square is much larger than 4 omega 0 square so that, lambda 1 which was equal to gamma over 2 minus gamma square over 4 minus omega 0 square its square root becomes, approximately equal to gamma over 2 minus gamma over 2 1 minus 2 omega 0 square over gamma square where I have used, the binominal theorem and this is roughly and this is equal to omega 0 square over gamma and lambda 2 is equal to gamma over 2 plus a square root of gamma square over 4 minus omega 0 square which is approximately equal to gamma itself, much greater than omega 0 over gamma. So, in this case the heavily damped solution becomes xH

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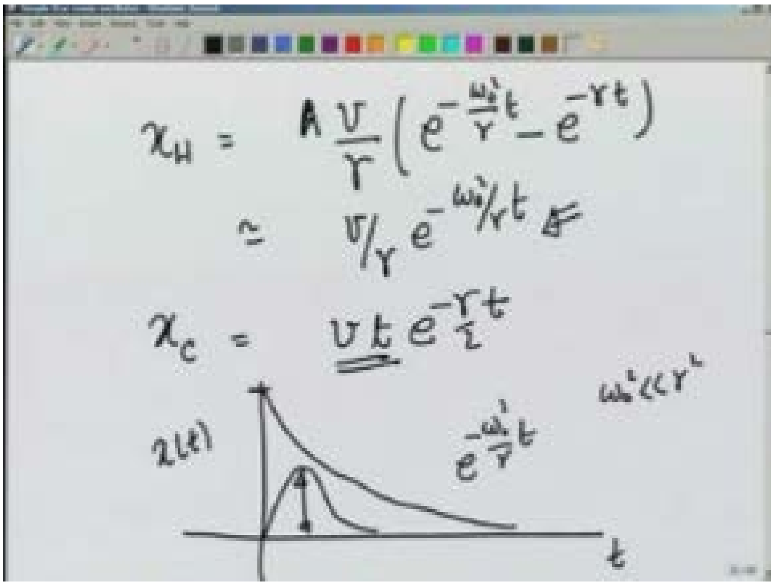

Becomes v over gamma e raise to minus omega 0 square over gamma t minus e raise to minus gamma t which is approximately equal to v over gamma e raise to minus omega 0 square over gamma t. This is extremely heavily damped solution. On the other hand, its critical was vt e raise to minus gamma t. There is a qualitative difference as I already pointed out. If, I want to plot x t verses t for the 2 cases I will find, that the heavily damped case, it starts roughly at v over gamma and decays very slowly as e raise to minus omega 0 square over gamma t. Remember omega 0 square is much less than gamma square. So, this quantity is really very small and the solution decays very slowly.

On the other hand, for the critically damped case the solution would go up and then the decays very fast as e raise to minus gamma t. How far does it go up? Is it above this axis or so on? Let us check that. So, in the case of heavily damped

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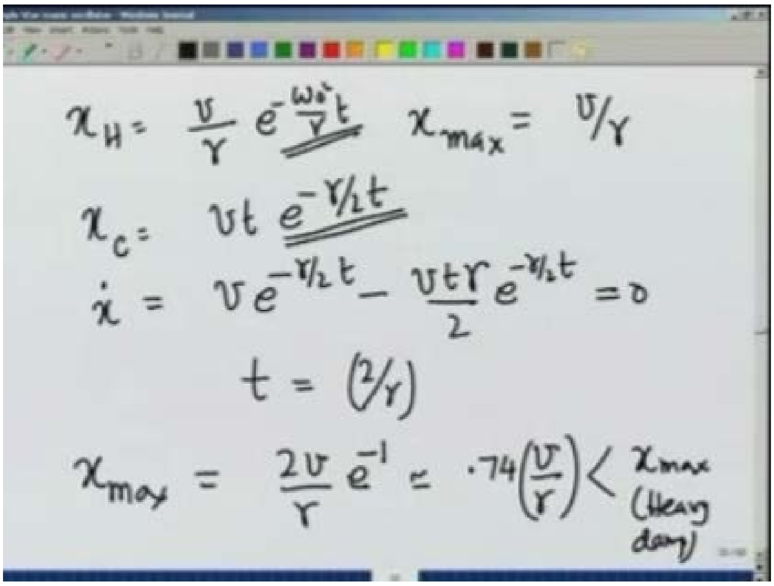

v over gamma e raise to minus omega 0 square over gamma t, x max is at equal to 0 which is v over gamma. So, in the heavily damped case, right away it starts decaying from a very large value, x critical is equal to vt e raise to minus gamma over 2 t; that means, if I had that factor of too earlier, I did not, it should be gamma over 2 gamma over 2. So, let us find what the maximum value; that is achieved is maximum value of x that is achieved in this case.

So, let us take x dot which is equal to v e raise to minus gamma over 2 t minus vt times gamma over 2 e raise to minus gamma over 2 t is equal to 0 and that gives me value of t to be equal to 2 over gamma and therefore, x max in the case of critically damped oscillator, is going to be 2v over gamma e raise to minus 1. e raise to minus 1 is roughly 0.37. So, this is the roughly 0.74 v over gamma which is, less than x max heavy damping.

So, in a critically damped case when I give an impulse, a particle does not go out as far as, it would go in the case of heavily damped oscillator. And through the comparison of these decay coefficients, I also know that, in the critically damped case the particle the decay of the motion is much faster than heavily damped case.

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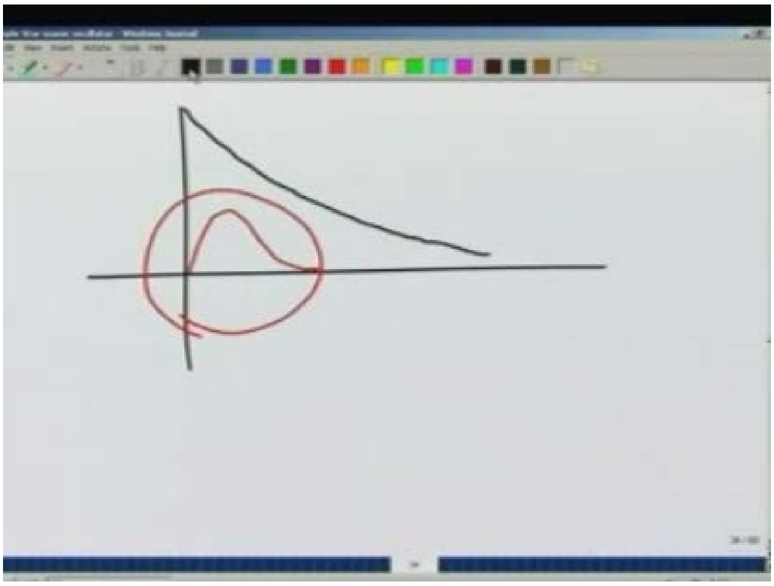

So, if I go to compare again, this would be heavily damped oscillator after it is give an impulse and this would be critically damped oscillator after it is give an impulse. So, critically damped oscillators are used to damp out motion in minimum distance when, something receives an impulse and that is the importance of critical damping.

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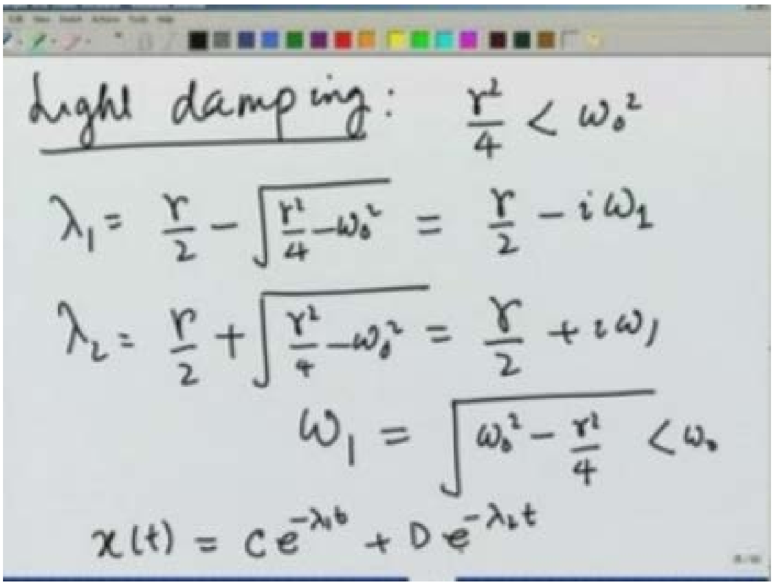

Having discuss the cases of heavy and critically damping, let us now consider. Light damping and if you recall from the beginning of the lecture, this is a case when gamma square over 4 is less than omega 0 square so that, lambda 1 which was gamma over 2 minus square root of gamma square over 4 minus omega 0 square; can be written as gamma over 2 minus i omega 1 and lambda 2 which is gamma over 2 plus square root of gamma square over 4 minus omega 0 square, can be written as gamma over 2 plus i omega 1 where, omega 1 is equal to square root of omega 0 square minus gamma square over 4 less than omega 0.

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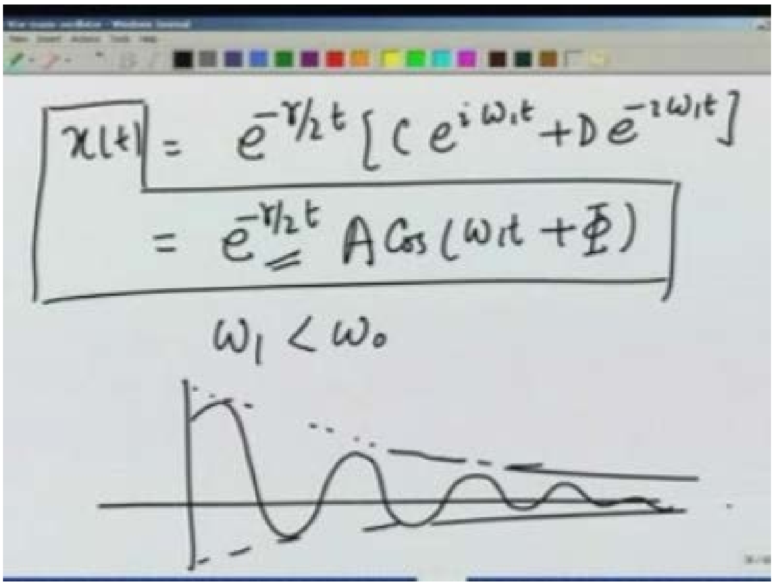

So, the general solution x t, which I have been writing as C e raise to minus lambda 1 t plus D e raise to minus lambda 2 t is going to be, in this case x t is equal to e raise to minus gamma over 2t C e raise to i omega 1 t plus D e raise to minus i omega 1 t which is equivalent to writing this as, e raise to minus gamma over 2t some amplitude cosine of omega 1 t plus a phase constant phi.

This is the general solution, in the case of a lightly damped oscillator. It is oscillating with the frequency omega 1 which is less than omega 0 which make sense, because it is being slowed down all the time. So, it takes longer and its motion is decaying. So, a general solution may look something like, this is the exponential like this, the distance keeps on decreasing with time.

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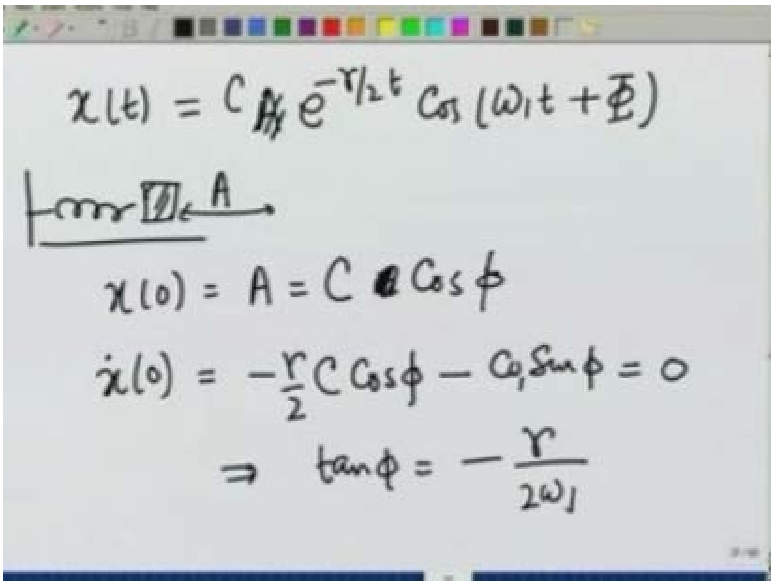

Let us analyze this more carefully. So, if I would write x t is equal to A e raise to minus gamma over 2 t cosine of omega 1 t plus phi and take the case of, where I took this oscillator, is stretch it out by a distance let us say A and then let go. How would the solution look? So, since x 0 is equal to A, this is will be equal to and let me use a different constant here C. C e raise to minus gamma over 2t is 1. So, C equals, C cosine of phi and x dot 0 is equal to minus gamma over 2 C cosine of phi minus C sin of phi and this is equal to 0. This tells me that, tangent of phi or they should be in omega 1 here, tangent of phi would be equal to minus gamma over 2 omega 1 and from that I can also calculate what coefficient C should be since,

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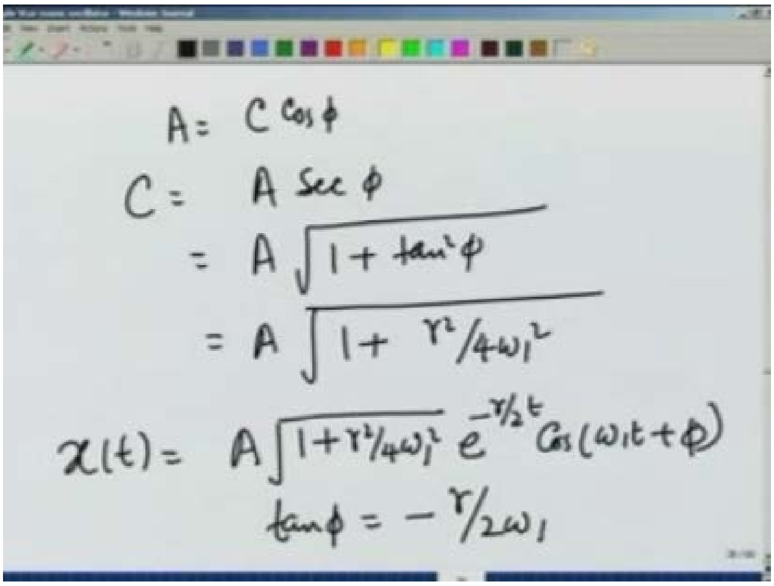

A is equal to C cosine of phi, C is equal to A secant phi which is A square root of 1 plus tangent square phi which will be equal to A. I have already calculated tangent plus. Let us see what the value of tangent is. It is gamma over 2 omega 1 gamma square over 4 omega 1 square.

So, the motion when I take the particle out and leave it is going to look like, x t is equal to A, the distance of which I pulled it out is A gamma square over 4 omega 1 square e raise to minus gamma over 2t cosine of omega 1 t plus phi with tangent phi being minus gamma over 2 omega 1.

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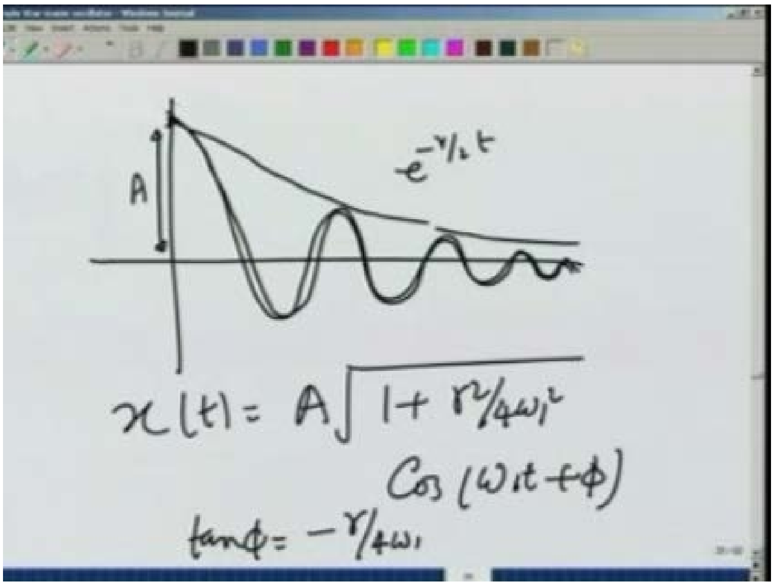

If I go to plot this, it starts with the distance A here and the motion go something like this because, there is a factor of e raise to minus gamma over 2t multiplying it. So, this is how the motion looks like in lightly damped case x t equals A square root of 1 plus gamma square over 4 omega 1 square cosine of omega 1 t plus phi with phi being, tangent phi being minus gamma over four omega 1.

So, slowly the particle starts losing its amplitude. Most of the time when we talk about the lightly damped oscillator, it is the case where we do not want damping to be there.

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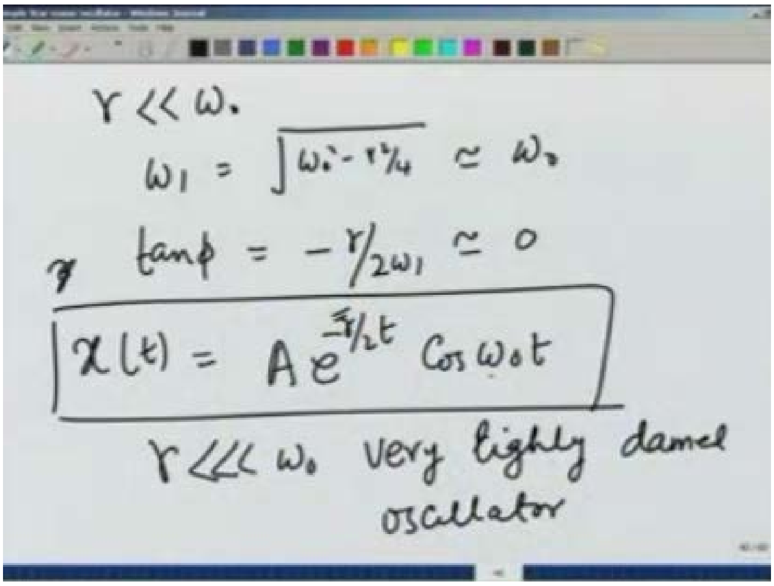

So, we want gamma to be much less than omega 0. In that case, we will see that omega 1 which is equal to omega 0 square minus gamma square by 4 is roughly equal to omega 0 and I can write a tangent phi that we looked at, minus gamma over 2 omega 1 is also roughly equal to 0. So, I can write the general solution as, amplitude e raise to minus gamma over 2t cosine of omega 0 t.

Let me call this the case, when gamma is much less than omega 0, very lightly damped oscillator. It so happens that, in these cases there are 2 time scales: 1 related to gamma and 1 related to omega 0. Therefore, I can talk about averages of a system, energy and so on, over a few cycles. Let me explain further.

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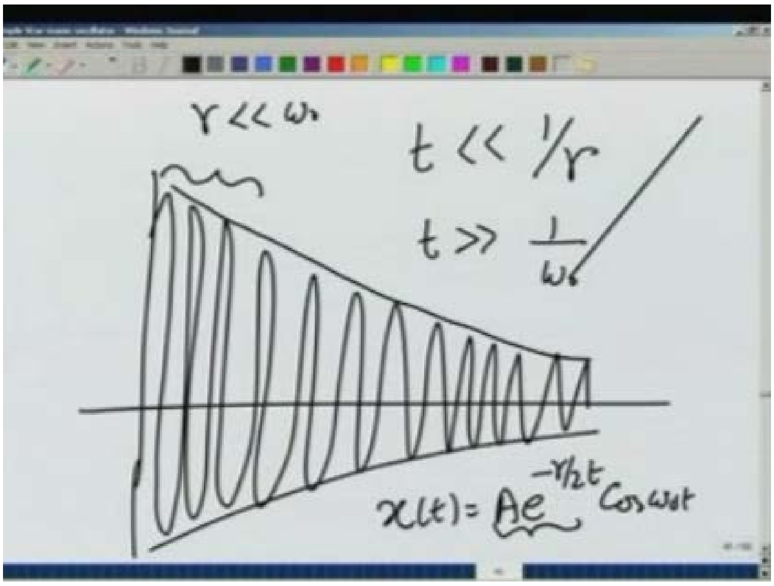

If gamma is much less than omega 0, you will see that by the time, the solution decays significantly. There have been many oscillations of the system. So, if I observe the system over a short period of time, that time, which is much smaller than 1 over gamma, over that I will find, that system moves roughly with the same amplitude and t is much greater than 1 over omega 0.

So, within this time there have been many oscillations. But, at the same time it is much less than 1 over gamma so and amplitude has not changed significantly. So, in that case when I write the solution, xt equals A e raise to minus gamma over 2t cosine of omega 0 t, I can really think of the system having a time dependent amplitude

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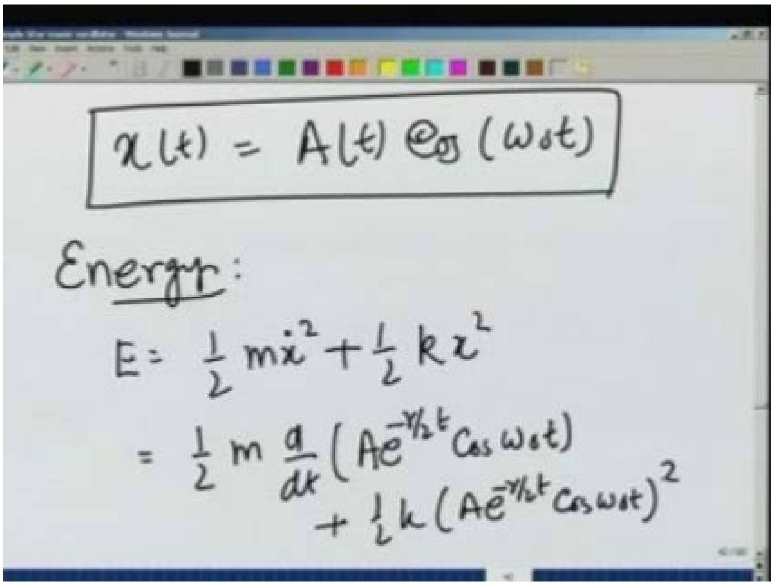

A t and then, oscillating with this time dependent amplitude as cosine omega 0 t. And then I can talk about averages over many cycles. Let me explain now. Let us talk about energy E is one-half mx dot square plus one-half kx square. In this case, I can roughly write this as one-half m d over dt of A e raise to minus gamma over 2t cosine omega 0 t plus one-half k A e raise to minus gamma over 2t cosine of omega 0 t square.

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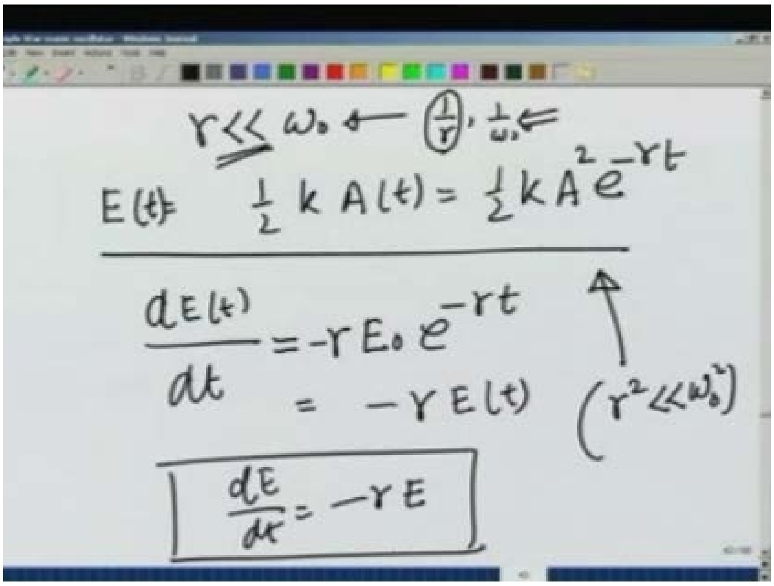

And taking the approximation that, gamma is much less than omega 0, you can write this energy as one-half k A t square which is equal to, one-half k A initial square times e raise to minus gamma t. So, the energy can be written in this form. Let us see how it decays; E t over t, you will see as E0 e raise to minus gamma t times minus gamma, when I take the derivative. So, this is really as minus gamma E t. In writing this form of the energy, I have neglected gamma square in comparison with omega 0 square. This you should keep in mind.

So, the energy really decays as dE dt equals minus gamma energy present at that time. Why I am able to talk about energy or amplitude at a given time is; precisely what I told you earlier. Because gamma is much less than omega 0, there are 2 time scales 1 related to 1 over gamma and 1 related to 1 over omega 0. So, within this time there are many oscillations. So, for a short period I can think of this, as an oscillator moving with time dependent amplitude.

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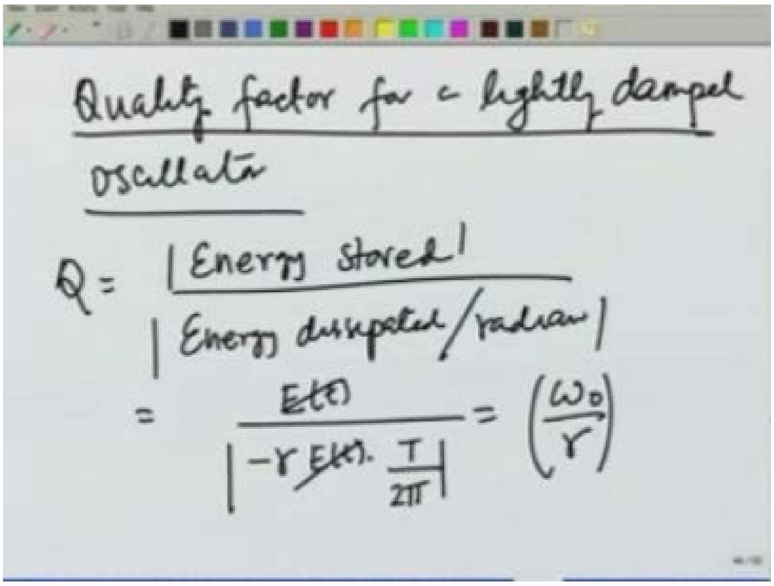

Then, I define something called the quality factor for a lightly damped oscillator. As I said earlier, most of the time when I am dealing with lightly damped oscillators, I want the leakage or the decay of the energy to be as small as possible. So, quality factor would be high when the energy stored is really large and leakage or dissipation of energy is very low.

So, this is defined as energy stored divided by energy dissipated per radian of cycle. Energy is stored at any time is Et and we have seen that dE dt is minus gamma Et and the time that it takes for 2 complete 1 radian is going to be, t over 2 pi. So, this comes out be omega 0 over gamma. I take the magnitude. So, there is no question of minus sign. This is known as the quality factor of an oscillator. Higher the quality factor less the decay of energy.

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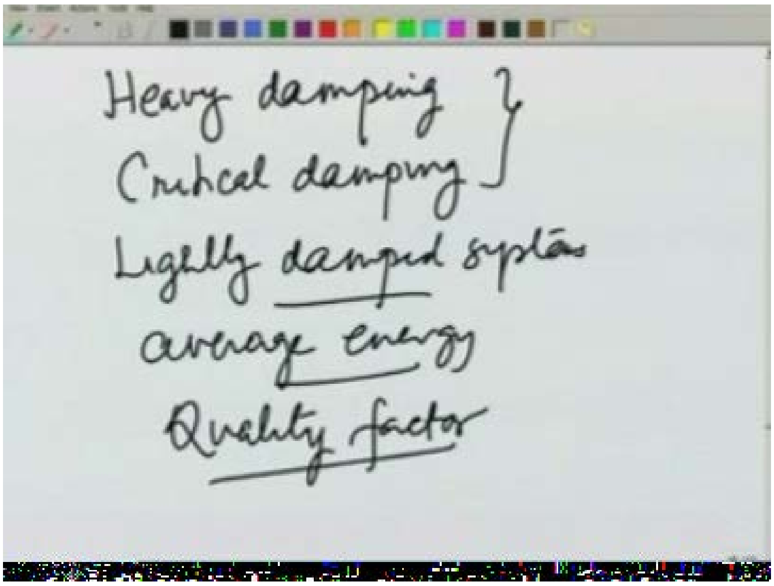

So, let us conclude this lecture by saying that, we learnt about heavy damping, we learnt about critical damping and understood the difference between critical and heavy damping and realize how critical damping is important when, we want to damp out certain impulses within a very short distance and short time and then we learnt about lightly damped systems, their average energy and their quality factor.

In the next lecture, we are now going to introduce a force that drives the system and see how, it response to that force.