Engineering Mechanics
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Module - 06
Lecture - 02
Work and Energy – I

In the previous lecture, I introduced the concept of momentum and indicated how it is conservation principle can make it easier to handle certain mechanics problems. We are going to take one more step in the direction of using conservation principles and work with energy conservation in this lecture. You have been hearing about work and energy may be since your tenth eleventh grade, I am going to show you this lecture how these concepts arise naturally when I tried to eliminate the time from the equation of motion.

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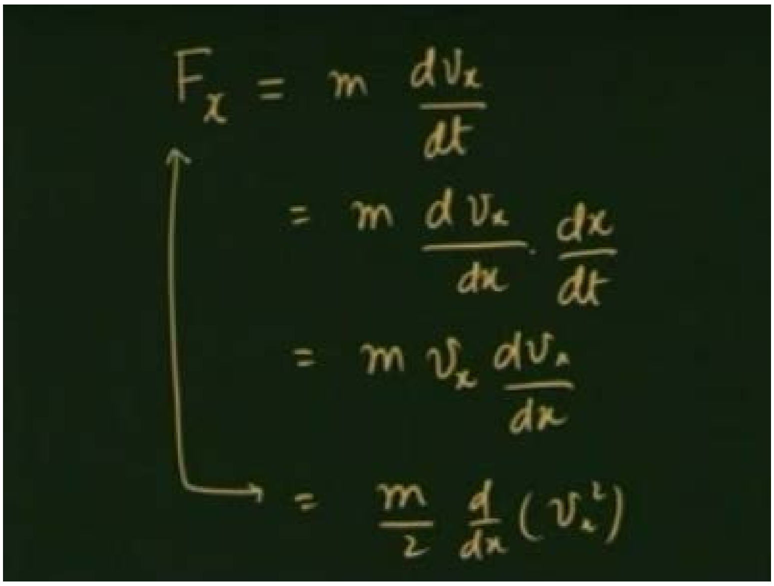

Here, F is equal to m dv over dt, but at times it may so happen that I am not interested in the evolution of the system as a function of time.

Recall that the equation of the motion is for example if a particle is falling in the gravitational field of the earth I may be interested in knowing what this velocity at a certain height from the earth and I do not have to really worry about time. In such situations we eliminate time how do we do that, if I write F equals m dv dt, this is equal to I can write as m dv dx dx dt I am taking only 1 dimension into account right now for three dimension cases, we will see later what happens.

So, F let us take now only x direction is equal to m dvx dt which I wrote as m dv x dx dx dt which is m vx dv x over dx which is m over 2 d over dx v x square. So, you see I have now related the force directly to velocity and there is no need to integrate over time. Let us see how it makes our problem solving slightly easier, let us go back to the example of gravitational fields inside the mentioned that earlier, suppose I want to know the velocity of a particle can gravitational field.

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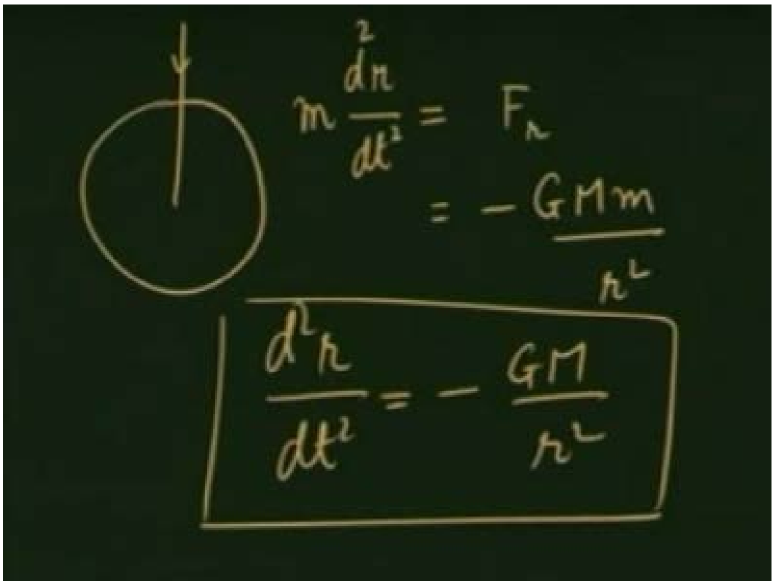

If the particle is moving in the radial direction, then I have dr dt is equal to the force which is in radial direction itself which is minus G M m over r square oh it should be d 2 r by dt square because I am talking about the acceleration and the mass. Therefore, I have d 2 r over dt square as equal to minus G M over r square this equation is very difficult to integrate with respect to time.

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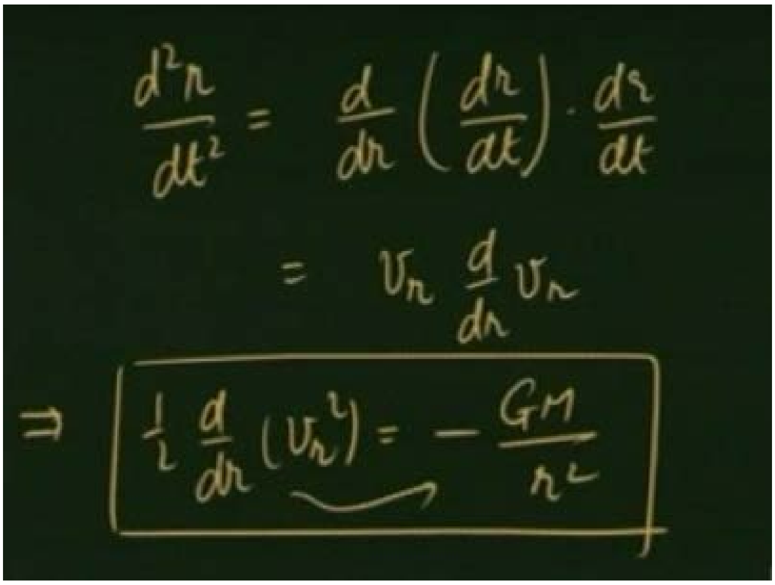

On the other hand, let me apply the trick I used earlier in writing d 2 r dt square as D over dr dr dt times dr dt and that gives me vr d over dr vr and therefore, half d over dr vr square is equal to minus GM over r square. This is an equation which can be easily integrated with respect to r to obtain velocity as a function of the radial distance from the center or the height from the center of the from the surface of the earth. So, the need to eliminate t gives me a slightly different equation, let me look at it slightly more carefully.

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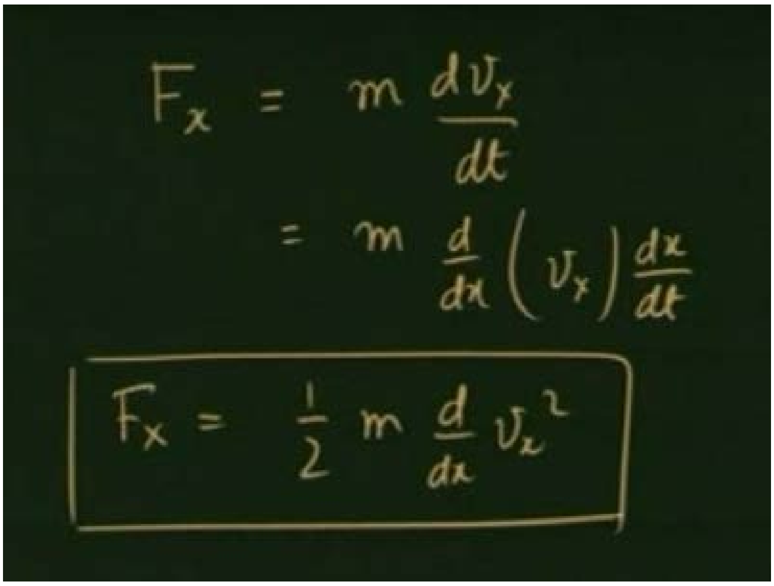

So, I had written earlier Fx equals m dvx over dt by transforming this into the form d over dx vx dx over dt I got that Fx is equal to 1 half m d over dx vx square.

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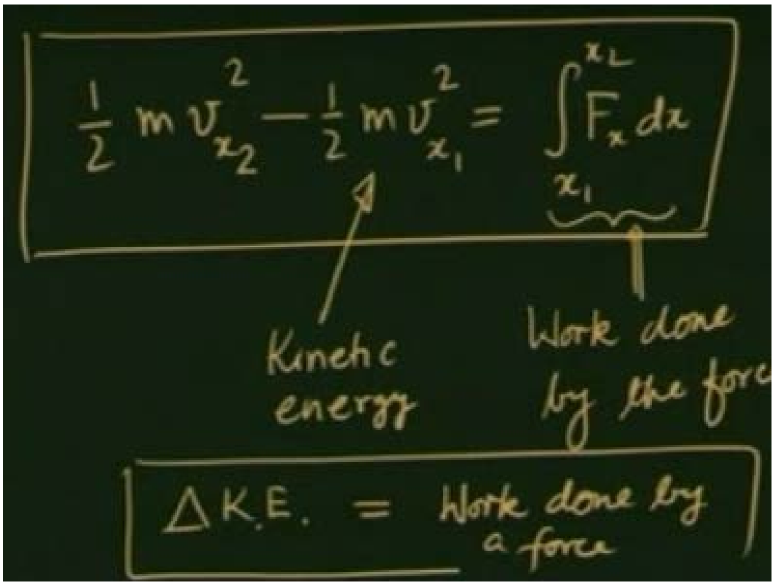

If I integrate this equation, what I get is half m vx at some position x 2 square minus one half m vx at some position vx 1 square is equal to integration Fx dx from position x 1 to x 2. Let us now interpret this equation I am going to call this Fx dx the work done by the force, I am going to call this quantity 1 half mv square the kinetic energy of the particle. So, what it tells me is that the change in kinetic energy is equal to the work done by a force. Not only have I defined the work and kinetic energy for you, I have also obtained a relationship between the 2 in that the change in the kinetic energy of a particle is going to be equal to the work done by a force on it.

Although I have taken a 1 dimension example, I will now generalize to three dimensional example. This is a statement in 1 dimension of the work energy theorem which we use in solving mechanics problems again and again, let us see now if I can generalized to three dimensions.

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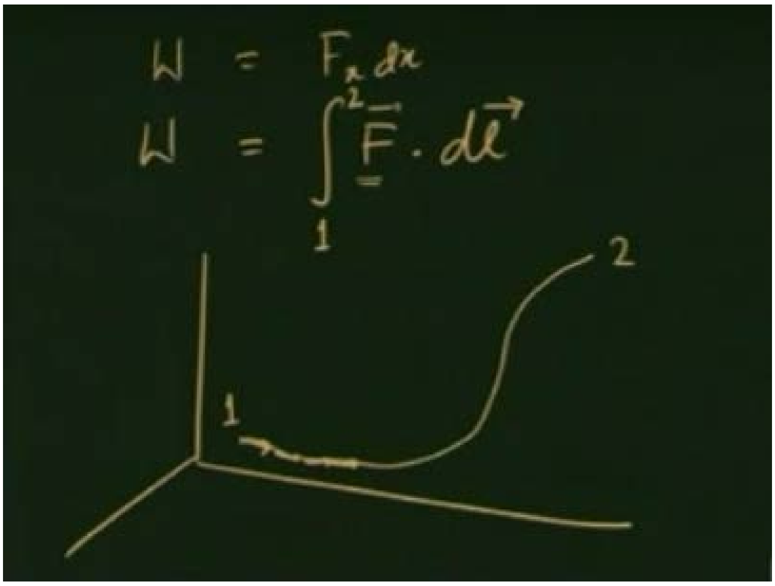

So, work done I had written as Fx dx in 1 dimension in three dimensions the work done is going to be the force taken a dot product with dl integrated from point 1 to point two. Let me explain this suppose a particle is moving from 1 position to some other position from 1 to 2 I can break this path into a small segments of dl each. For each segment, I calculate the work done which is equal to the force in the direction of that segment times the length of that segment and integrate.

So, that gives me the work done by this force notice that I have to then know as to which part I am going over. So, this is the work done, let us see if this can be related to the change in the energy or kinetic energy of the system.

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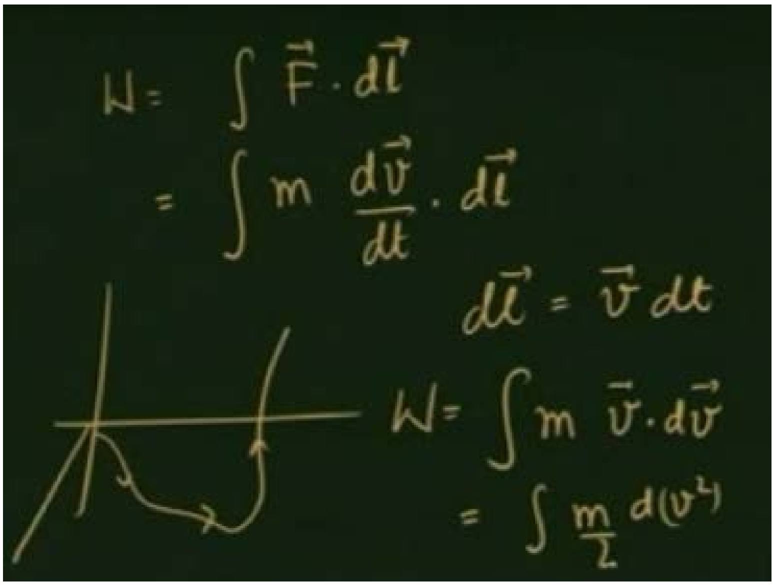

So, work which is equal to F dot dl can be written as m dv dt times dl and the when the particle is moving along a curve like this dl is going to be v dt. Therefore, I have work is equal to integration m v dot dv and this is nothing but integration m dv square over 2.

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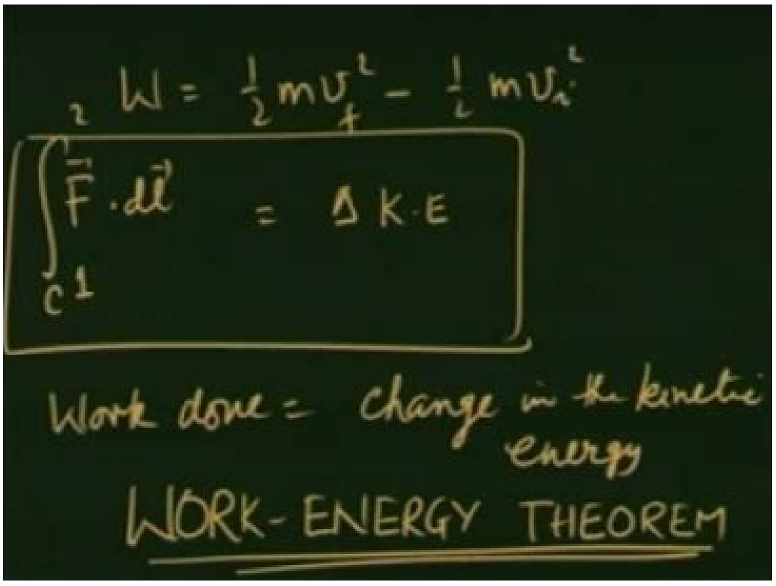

When integrated over, it gives me work is equal to half m v final square minus 1 half m v initial square and work as defined as F dot dl moving over a curve from point curve c from point 1 to point 2 is equal to delta KE.

So, we have in general that the work done by a force is equal to change in the kinetic energy and I am going to call this the work energy theorem I emphasize again when I calculate the work done it is the work done by the force.

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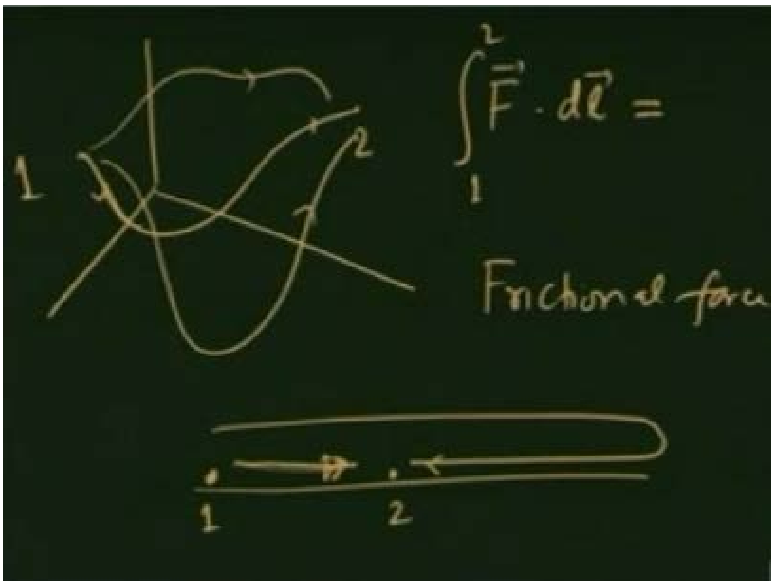

Let me now ask a question since I talked about a particle moving along a curve from one to another point is the work independent of the path or does it depend on which path am I taking. So, if I calculate F dot dl from 0.1 to 0.2 is it the same whether I take this path or some other path or third path or is it different. In general, the answer is that could be different the most simple example for this is a frictional force.

Let us again take a 1 dimensional example, suppose I take a particle from point one to point two directly and I take it to some other point and bring it back here. You can see that the work done by the frictional force in this case is going to be much larger than the work done by the frictional force in this case. So, in general the work done by any force depends on the path taken, let us take another example of this.

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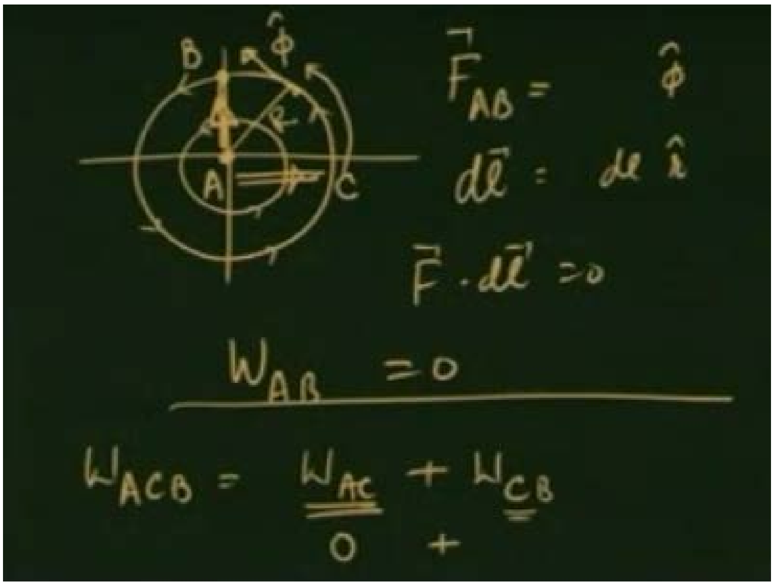

Suppose, there is a force field where a force applied is in circular direction that is if a particle is there, it will experience a force that you know planar polar coordinates in phi direction. Let me then take a particle from point A to point B through 2 different paths path 1 let it be this 1 A to B directly and path 2 let it be like this A to C and C to B like this let this point be C.

In the first case, when I calculate F AB it is in the direction of phi something, but the displacement dl is in direction dl r. So, F dot dl is 0 and therefore work done A to B is equal to 0, how about work when I go from A to C to B, this is going to work from A to C plus work from C to B work from A to C is like work from A to B. Therefore 0, but this is not because here the displacement and the force both are in phi direction, in fact if F depends only on the radial distance.

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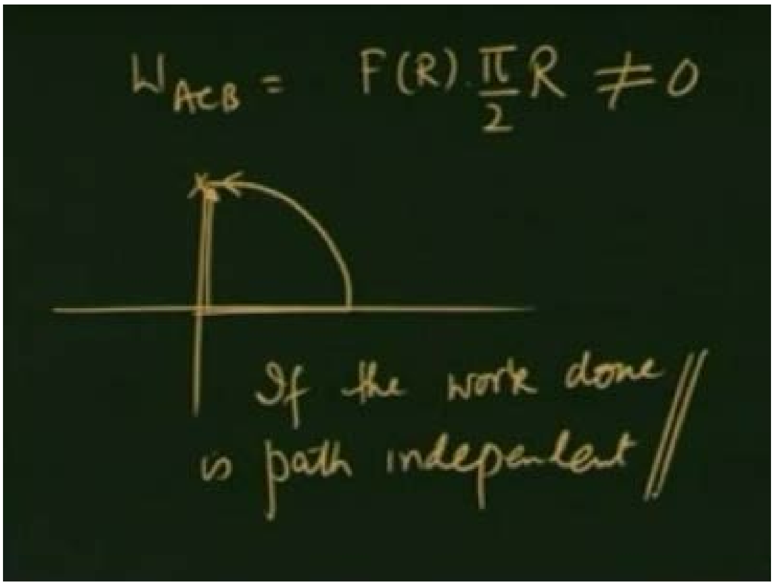

Then, work ACB is going to be the force at radius R times pi over to R which is nonzero, so if the particles moves in this force field from A to B through this path my work energy theorem is going to have a kinetic energy at this point. On the other hand, if it moves along this point there is going to be no change in the kinetic energy.

So, this is another example of the force the force field where the work depends on the path taken, let us see what happens if the work done is path independent. So, we have seen 2 examples of cases where the work depends on the path taken, but, there are many forces in nature for which the work done is path independent.

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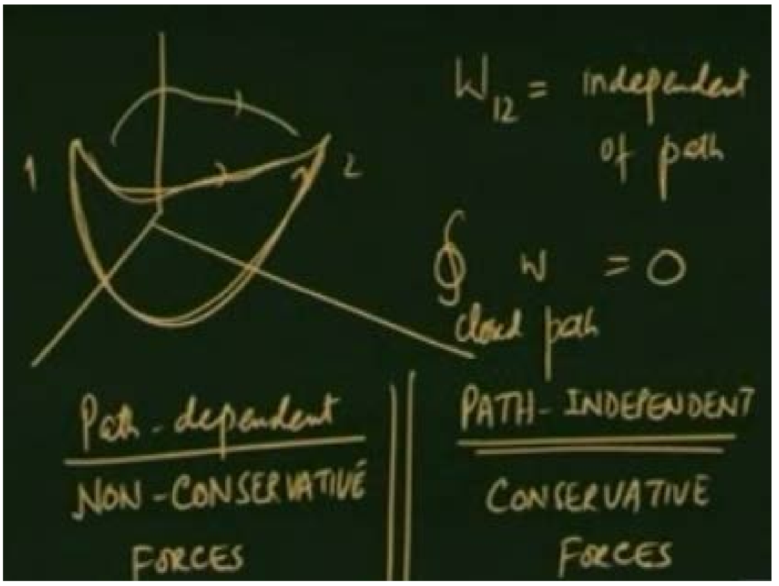

That is no matter which path I take whether I go from 1 to 2 along this path or this path or this path the work done from 1 to 2 is going to be independent of path. An example of this is the gravitational field the work done when 2 chargers move due to their Coulomb interaction. In this case, if I go from 1 to 2 and come back by other path from 2 to 1 work done since the magnitudes are equal it is going to be positive in 1 direction negative in the 1 the other direction and therefore, under closed path work done is 0.

So, we have seen two cases in one case the work may depend on path in the other case the work may not depend on path. For the cases where work is path dependent are known as non conservative forces, whereas when work is path independent these forces are known as conservative forces.

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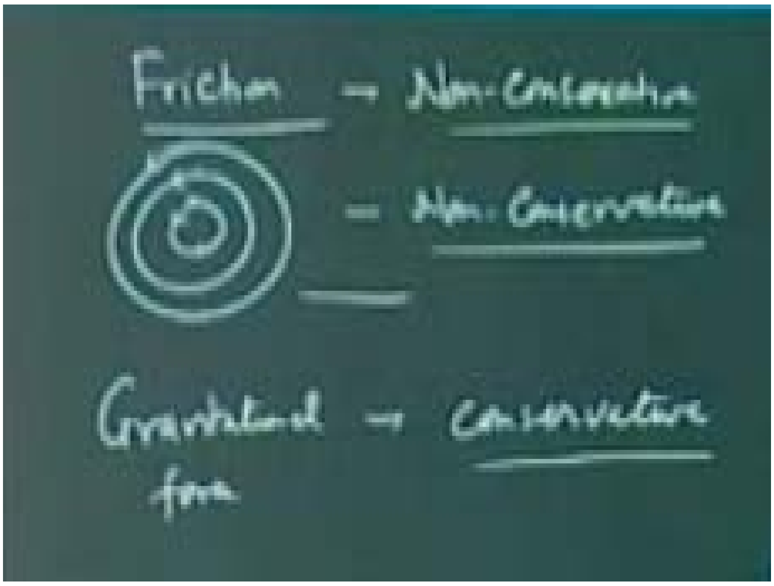

So, we just saw that frictional force is non conservative, the force field like this is non conservative gravitational force is conservative these are important concepts. The certain things I can do for conservative forces is which I cannot do for non conservative forces and that is precisely why I define these 2 classes. You will see later in your course an electromagnetism that this kind of E field or force field arises when magnetic field changes with respect to time, let us now see them.

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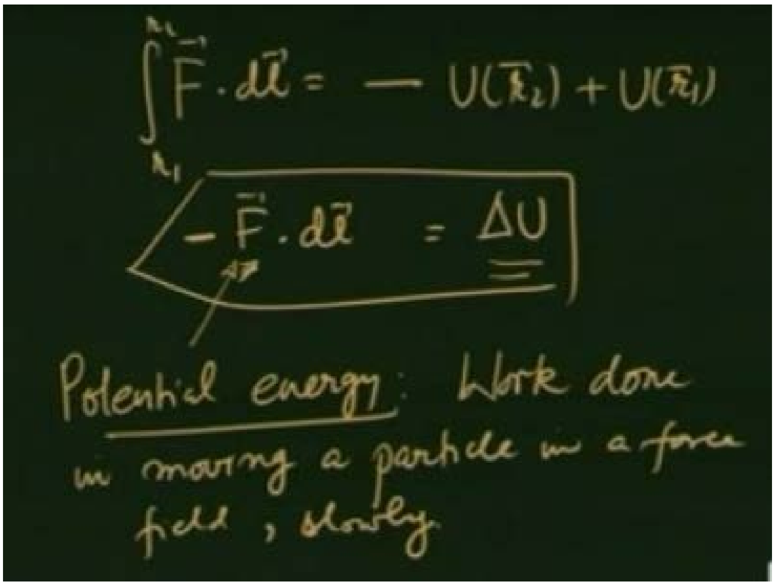

If the work done for conservative forces I am focusing on conservative force if the work done is independent of path 1 to 2 which is equal to integration F dot dl from 1 to 2, then I can write this as the difference between a quantity at these 2 points. I will interpret this in a minute, but for the time being let me call it the potential energy. Recall from your course in thermodynamics or the first law of thermodynamics in your twelfth grade that a similar quantity the internal energy is defined in first law. The difference of the heat and the work done by the gas is independent of the path or the path taken on PV diagram.

Similarly, here if the quantity F dot dl is independent of the path taken it can depend only on the end points and therefore I can write this as a difference if of this quantity I am calling this potential energy. How is it related to the force you just saw that I define it as F dl is equal to minus from r 1 to r 2 as minus U r 2 plus U r 1. Therefore, for a small difference delta U is going to be F dot dl with a minus sign, the way then I interpret this is that the change in potential energy is the work done, if I move a particle by applying a force opposite to the force field in which it is moving.

For example, if I were to move in a gravitational field I will keep it in balance by applying a force opposite to the gravitational field that is what this negative sign integrates. The work I do not the force field, but the work I do in moving the particle is equal to the change in the potential energy that is the interpretation of potential energy. So, potential energy is equal to the work done by the person who is moving it in moving a particle in a force field and I am moving it slowly. So, there is no change in kinetic energy or anything keeping it in balance all the time, so let us see if I apply the work energy theorem to this case what happens.

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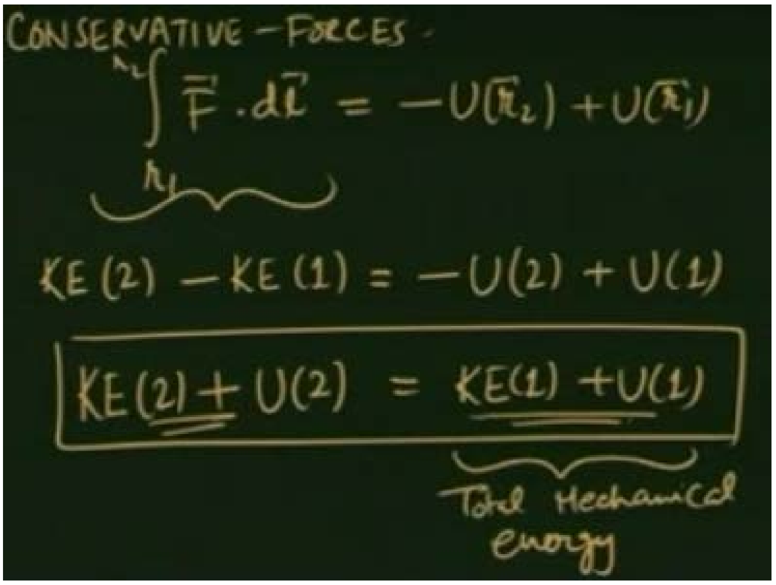

Let me again write I am talking about conservative forces then F dot dl I have defined as minus Ur 2 plus Ur 1 when I am going from to point 2. This by work energy theorem is nothing but KE at point 2 minus KE at point 1 and this is going to be equal to minus U at point 2 plus U at point 1. If I reshuffle, I immediately get that KE at 2 plus U at 2 is equal to KE at 1 plus U at 1. I call this quantity kinetic energy plus the potential energy as the total mechanical energy and as a consequence of the work energy theorem and the conservative nature of the force.

I get it that in conservative force field the sum of the kinetic and potential energy and therefore the total mechanical energy remains a constant you see at 1 point 1 is the same as at point 2. Therefore, it is same as point three point four and so on, this is known as the law of conservation energy, so let us see what we have seen.

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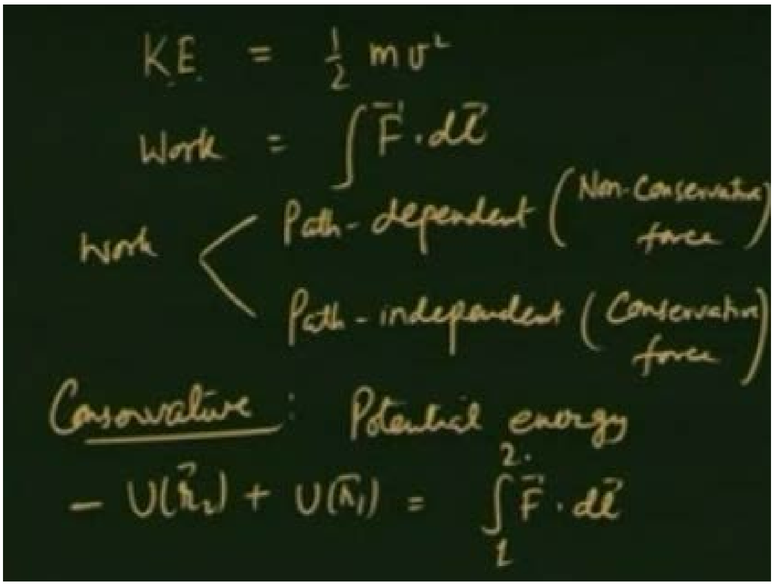

So far, we have seen that I can define the kinetic energy as equal to one half mv square work as F dot dl integrated over work may be path dependent it may be path independent this I will call non conservative force and this is conservative force. If it is conservative, I can define a quantity called potential energy actually I should really say as the difference in potential energy which is defined as Ur 2 plus Ur 1 is equal to F dot dl between 1 and 2.

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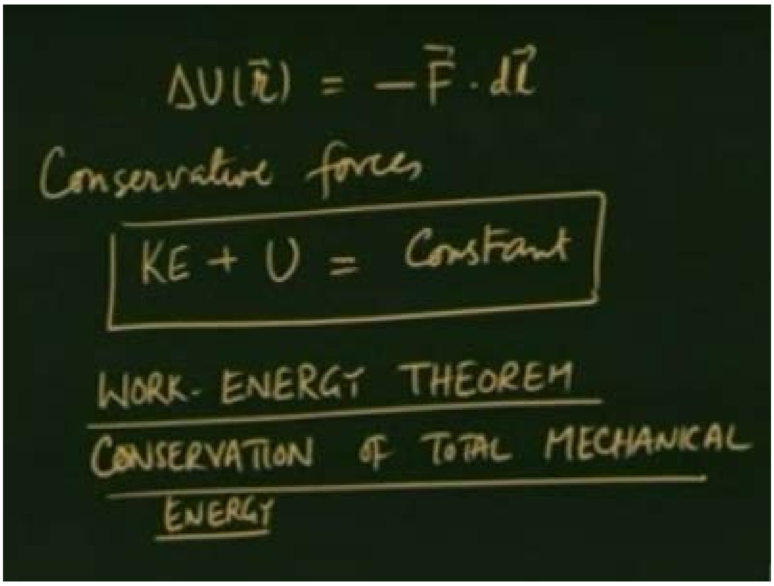

We saw that change in the potential energy at a position is equal to minus F at that point times dot it with dl and for conservative forces KE plus U is a constant. The two key concepts that we have come up and this discussion is work energy theorem and conservation of total mechanical energy before the application and discussing more about them I want to spend some time to get a field for the potential energy.

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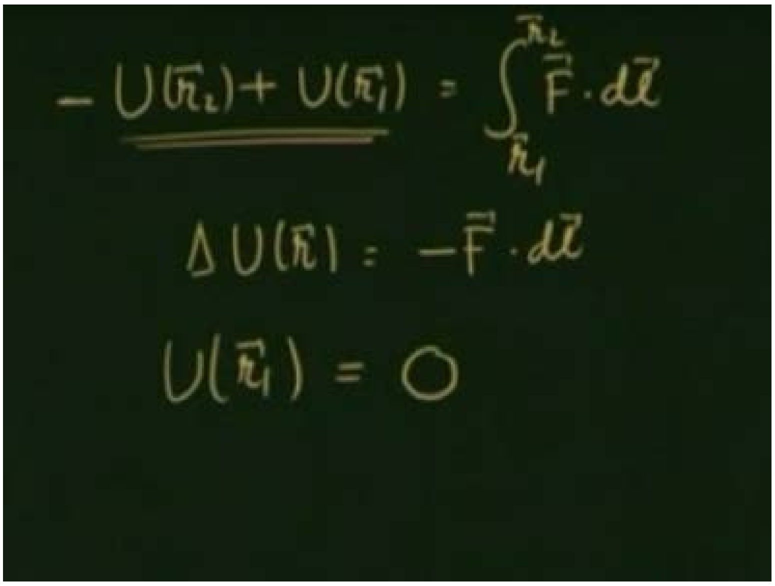

I define the potential energy as Ur 2 minus Ur 2 plus Ur 1 as integral F dot dl from r 1 to r 2 and I am not writing any path in this because this is defined only for those forces for which the work done is path independent. If you write it like this you see that the change over a small distance is equal to minus F dot dl and that gives you the interpretation of potential energy as the work done by U by the external agency to move a particle slowly in the external field. Also, notice that it is the difference in the potential energy that has meaning, the usual practices that you take a reference point r 1. Then, put the potential energy to be 0 there and define the potential energy at all other points with respect to that point, let us then take two examples for this.

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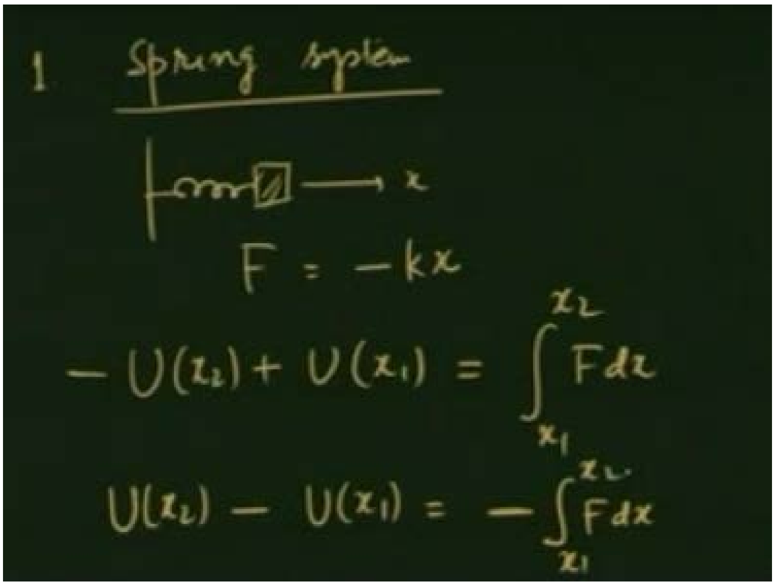

One is a spring system in which if a particle is displaced by distance x from the equilibrium position the force F applied by the spring is minus k x. Therefore, if I were to calculate U x 2 U x 1 minus U x 2 plus U x 1, this should be equal to F dx from x 1 to x 2. According to our interpretation the difference in the potential energy from point 1 to point 2 is going to be force applied by me, so minus sign F dx from x 1 to x 2.

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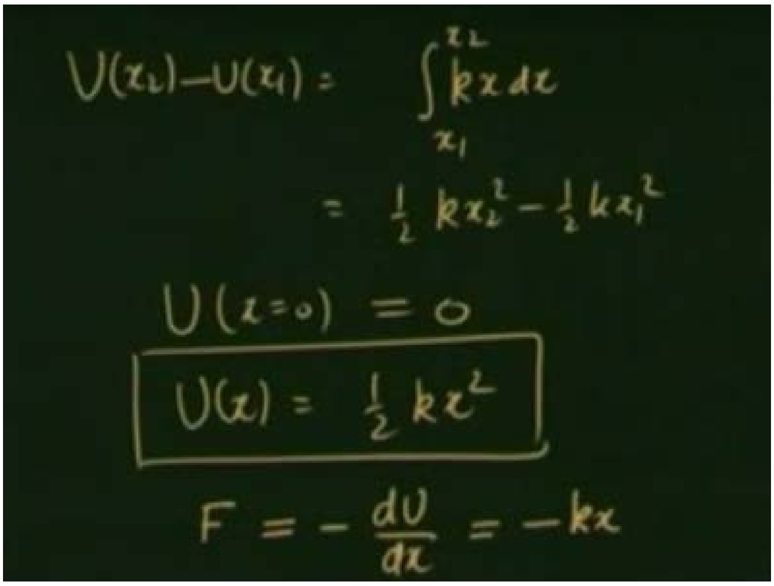

This then becomes U x 2 minus U x 1 is equal to integral k x dx from x 1 to x 2 which is equal to half k x 2 square minus half k x 1 square the formula you are well familiar with.

Also, I take in this case U at x equal to 0 to be 0 and that is my reference point and therefore I write in general U x is equal to 1 half k x square. Also, from this the force is given as du dx by definition and this you see comes out to be minus k x, let me elaborate a bit on that because I did it in three dimensions.

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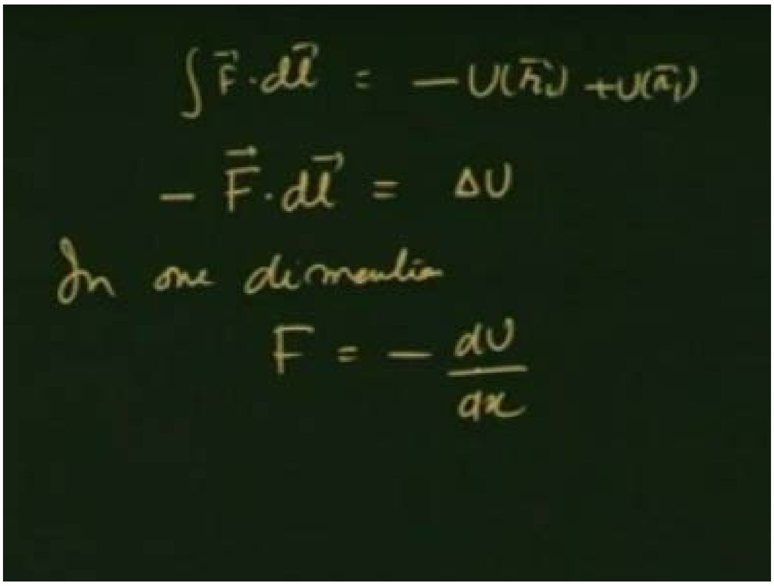

We had F dot dl is equal to minus Ur 2 plus Ur 1 and therefore for a short distance I had F dot dl with a minus sign is equal to delta U and in 1 d that will give me F equals minus du dx.

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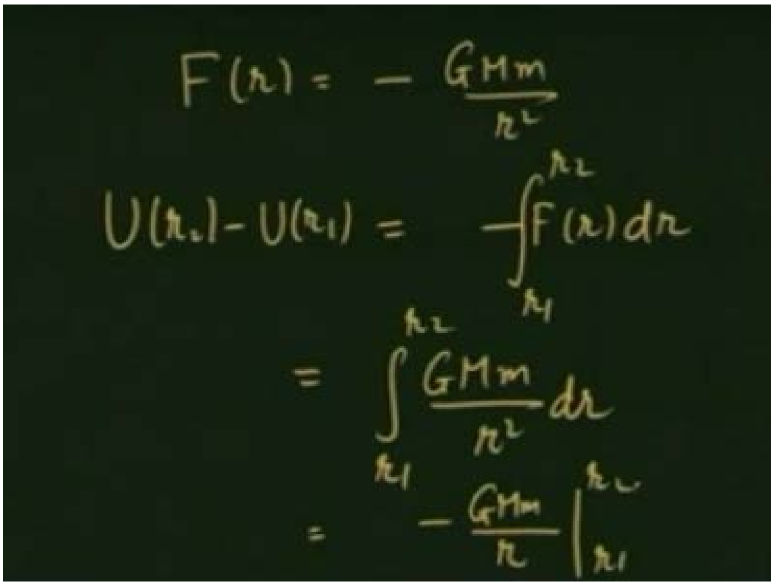

Let us take another example that of gravitational field in which the force F as a function of r is given as minus G M m over r square. So, if I were to calculate the potential energy at a point r 2 with respect to r 1 this is going to be minus Fr and I move the particle along the radial line from r 1 to r 2. This gives me integral G M m over r square dr from r 1 to r 2 which is equal to minus G M m over r r 1 r 2.

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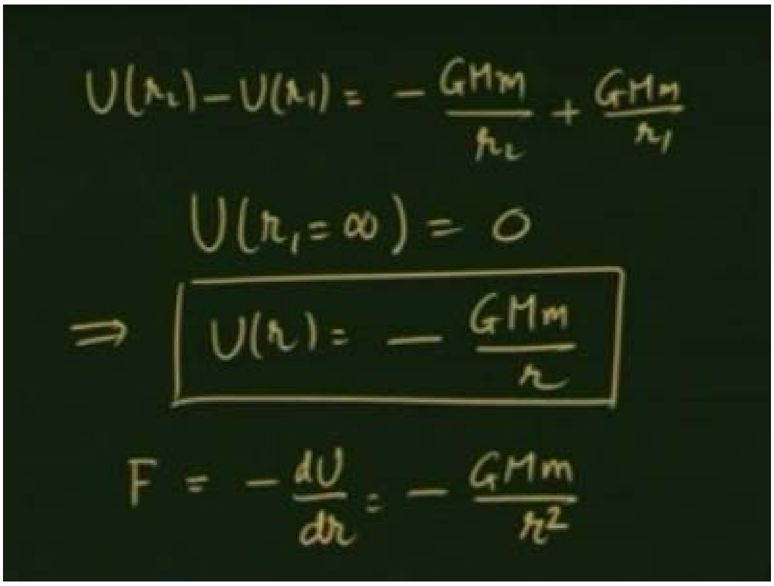

Therefore, I get Ur 2 minus Ur 1 as equal to minus G M m over r 2 plus G M m over r 1. In the case of gravitational field, I take U reference point at infinity. So, Ur 1 at infinity to be 0 and that defines my U uniquely as minus G M m over r. You can you can again see that the force in the radial direction is going to be minus du over d r which is minus G M m over r square. So far, you notice I have been focusing on 1 dimensional cases.

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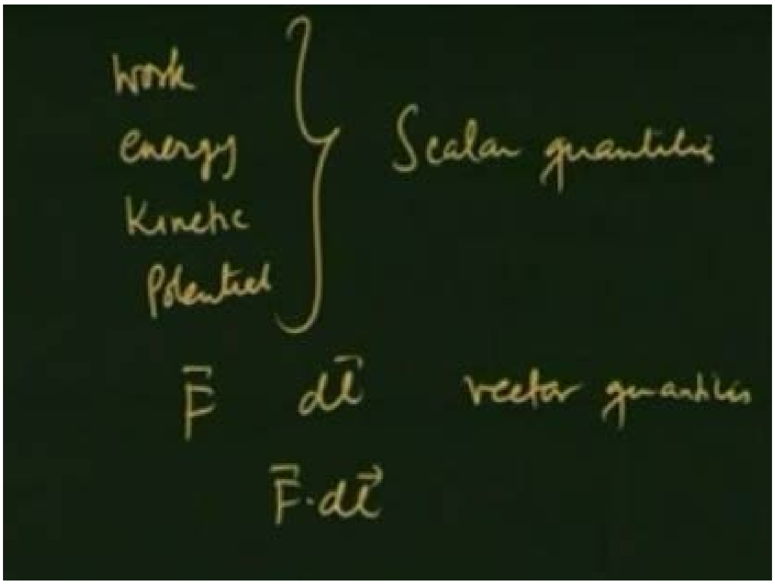

That is because the work done or the energy whether kinetic or potential are all scalar quantities, whereas a force and the displacement are vector quantities and the 2 are related by this dot product which becomes quite easy to work within 1 dimensional case. For three dimension cases, I will talk about later we have special techniques to check things out. So, from now I am going to concentrate on 1 d case and talk about energy conservation in 1 dimension as I had mentioned in the beginning of the lecture.

The motivation to develop the conservation principles of linear momentum and energy is to make solutions of problem easier. Let me now after having developed these concepts take two or three examples, where I show how these principles are applied in solving mechanics problems without really going into integration of the equation is a first example.

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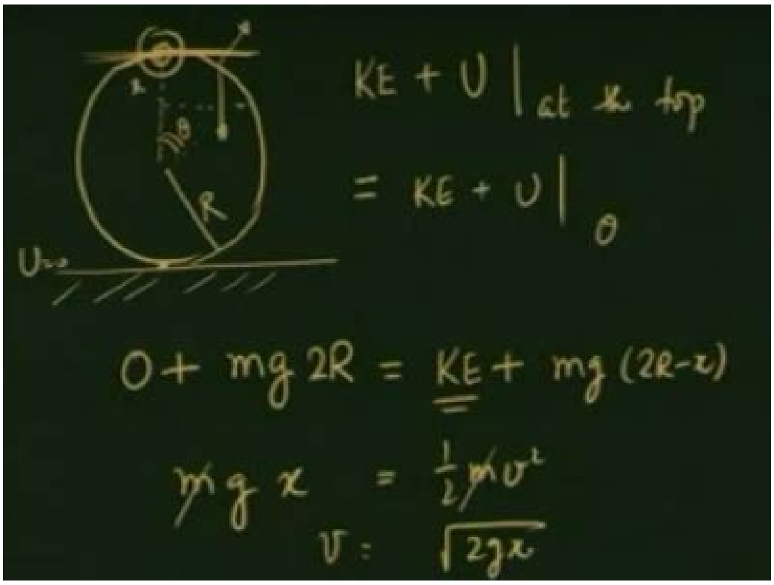

Let us take the case of A bead sliding over a vertical ring of radius R and I want to know what will be its velocity when it is fallen by a distance x or moved by an angle theta from its initial position. If I were to integrate the equation of motion I would have to take care of the force that is applied by the ring the gravitational force on the bead and so on. If I want to apply the energy conservation principle since I know that in this case the gravitational field is conservative. So, the net energy should remain a constant. So, KE plus U at the top that is this position should remain the same as KE plus U at say theta.

KE at top is 0, let me take the U to be 0 here at the ground, so that at the top is going to be mg times 2 R if the height if the mass of the beat is m and this should be equal to KE at K dot or this distance x plus U at this point is going to be mg 2 R minus x. That gives me KE at this point 1 half mv square is equal to mg x or v equals the square root of 2 g x a well known result. I use this simple problem now the slightly more complicated problem where I would ask the question.

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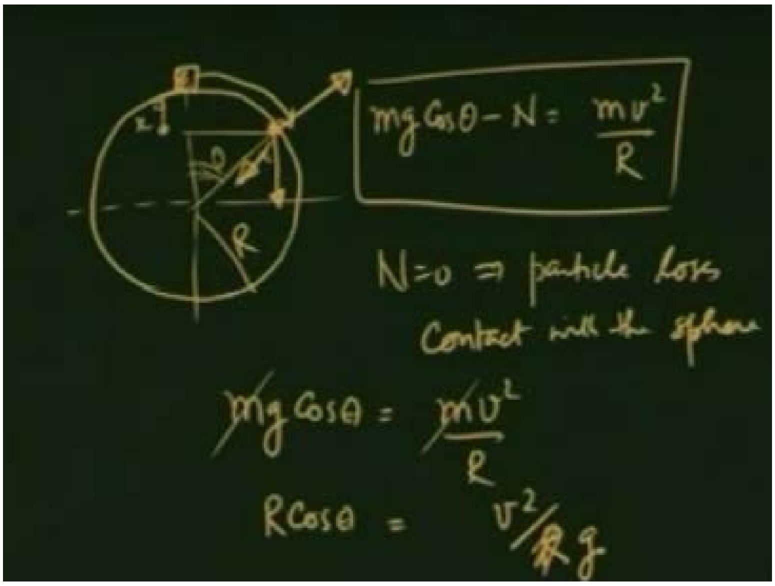

If there was a particle, this is a very standard problem and probably I have solved it in the past in the previous lectures. Now, I want to solve it slightly differently here if there is a particle which is sliding on top of a sphere. I want to know at which point or at what distance from the top x does it fly off this sphere. So, in this case I know at this point it requires a centripetal force to go over this circular radius circular path.

So, at this point the forces on this are mg in this direction and the normal reaction in this direction the component of mg towards center gives it the centripetal force and therefore, if this angle theta. So, is this you have mg cosine of theta minus N is equal to mv square over R. As long as N is nonzero and pointing outwards I know the particle is on this sphere N equals 0 implies particle losses contact with the sphere. So, that at that point I am going to have mg cosine of theta is equal to mv square over R and therefore R cosine of theta is equal to v square over a v square over g, I have already calculated v square.

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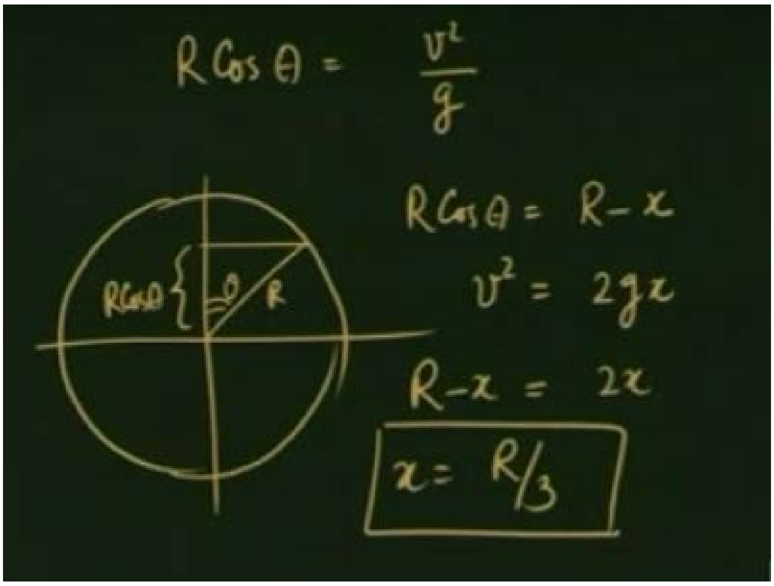

So, we have R cosine of theta equals v square over g and if I look at this problem again I have this R this is theta. So, that this is R cosine theta and you see that R cosine of theta is equal to R minus x and we have already calculated that v square is equal to 2 g x. Therefore, I have R minus x equals 2 x or x equals R over 3, so when it travels down a distance of R by three it is going to fly off the sphere. Another problem where I am going to now combine the principle of momentum conservation and principle of energy conservation is going to be solved now.

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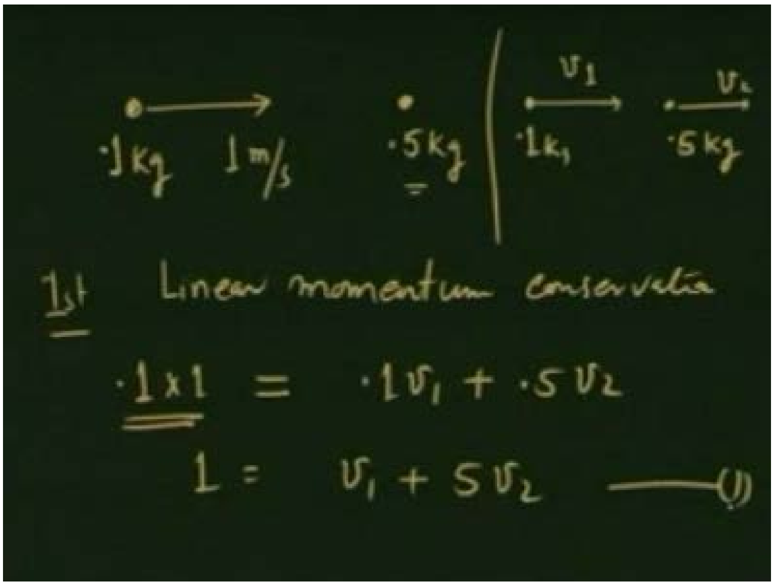

Let us take a particle of mass say point 1 kilogram travelling in this direction with a speed of 1 meters per second it hits another particle of say 0.5 kilogram. After the collision, the 2 particles travel 1 with speed v 1 this is point 1 kg and the other particle with a speed v 2 this is 0.5 kg. We wish to find v 1 and v 2 although you may have seen this problem in the past it does not have to solve a simple problem to start with. So, first I apply linear momentum conservation because there is no external force, these things are moving in horizontal plane.

Therefore, I must have point 1 times 1 that is a initial momentum must be equal to point 1 v 1 plus 0.5 v 2. This is the momentum to start with this particle has no velocity, so 0 momentum and therefore I have 1 equals v 1 plus 5 v 2 that is my equation 1 since there is no external force. So, potential energy remains the same, the kinetic energy should remain the same if we assume the collision to be elastic.

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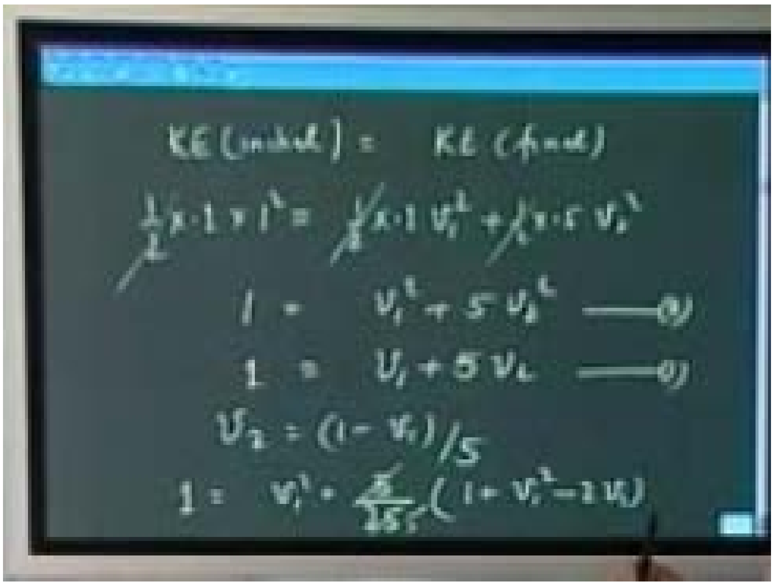

Therefore, KE initial is equal to KE final I have not taken potential energy U into account because it remains the same or on a horizontal surface it does not change. Therefore, I must have 1 half times point 1 times 1 square that is initial kinetic energy, final kinetic energy both particles are moving. Therefore, one half times point 1 v 1 square plus 1 half times point five v 2 square and this gives me half cancels, this gives me 1 equals v 1 square plus 5 v 2 square this is my equation number 2.

Equation number 1, let me rewrite it once more 5 v 2 is equal to 1, solve these 2 equations to get v 1 and v 2. Let us substitute, so I take the value of v 2 which is 1 minus v 1 over five from the first equation and substitute in the second equation to get 1 equals v 1 square plus 5 over 25, 1 plus v 1 square minus 2 v 1 this is 5.

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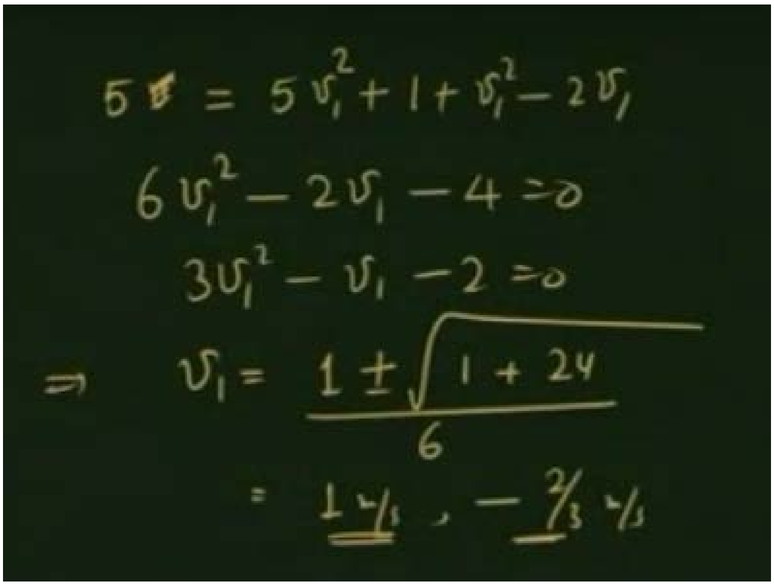

Therefore, I get Five is equal to five v 1 square plus 1 plus v 1 square minus 2 v 1 or six v 1 square minus 2 v 1 minus 4 is equal to 0 or 3 v 1 square minus v 1 minus 2 is equal to 0. This gives v 1 equals 1 plus minus square root 1 plus four times is 24 over 6 and that gives you either 1 meters per second or minus this is 5, two thirds meters per second. This is a trivial solution as if the collision did not take place and this is the solution of interest in which the particle 1 hits particle 2 and goes back with this much velocity.

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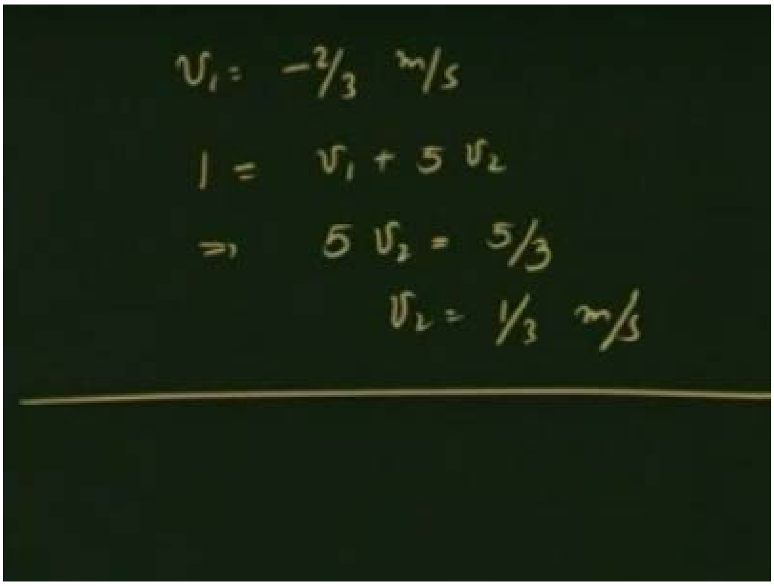

So, I get v 1 is equal to minus 2 thirds meters per second and if you recall I had 1 equals v 1 plus five v 2 and this gives me five v 2 is equal to five thirds or v 2 equals one thirds of meters per second. I gave these 2 simple examples in 1 dimension case to show you how the conservation principals make solution easier. Notice that I did not have to integrate over a force over time, all I did was use the fact that momentum is conserved and energy is conserved.

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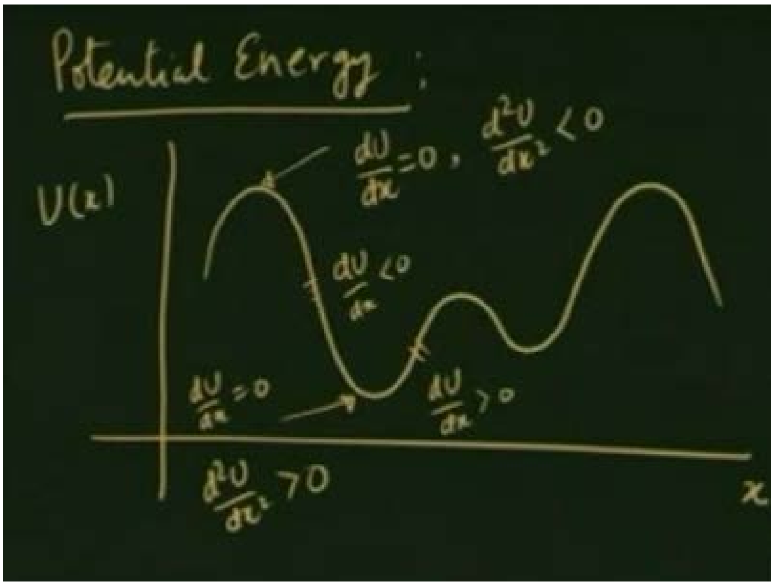

After giving the simple examples, I want to go back to the potential energy And get little more feel for it although I am going to restrict myself in 1 dimension it teaches me a lot. Suppose, there is a potential energy curve like this, so this is U x verses x, I can simulate this by taking a wire in this shape and let a beat move over it because the height would be proportional to the potential energy because of gravitational force. There are different points I want to identify in this is the maximum of potential energy, where I have du dx is equal to 0 and d 2 u over d x square less than 0.

There is this point where I have du dx is equal to 0 and this is a minimum and therefore d 2 u dx square is greater than 0. At this place, du dx is less than 0 on this place du dx is greater than 0, so let us look at this four points and see what I can say about that.

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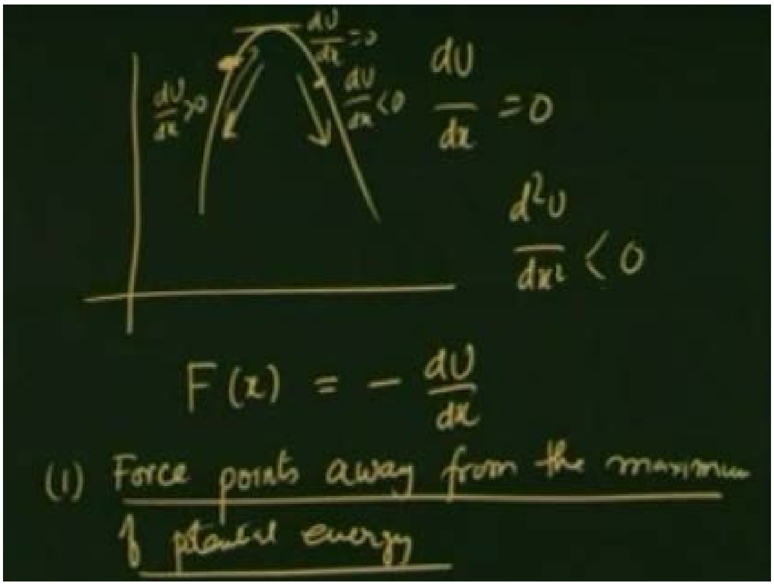

Let us start with the maximum of the potential energy, where I had du dx equals 0 and d 2 u over dx square is less than 0. So, you notice since it is 0 here du dx on this side d 2 u du dx is 0 and as x goes up since d 2 u dx square is less than 0 du dx on this side must be greater than 0 as is also evident from the graph. Similarly, on this side du dx is less than 0, so if I were to look at the force F x, which is equal to minus du dx you can see that on both the sides the force would be pointing away from the maximum. So, looking at the maximum I see that the force points away from the maximum of potential energy, so any particle at the maximum of the potential energy is going to be running away from it.

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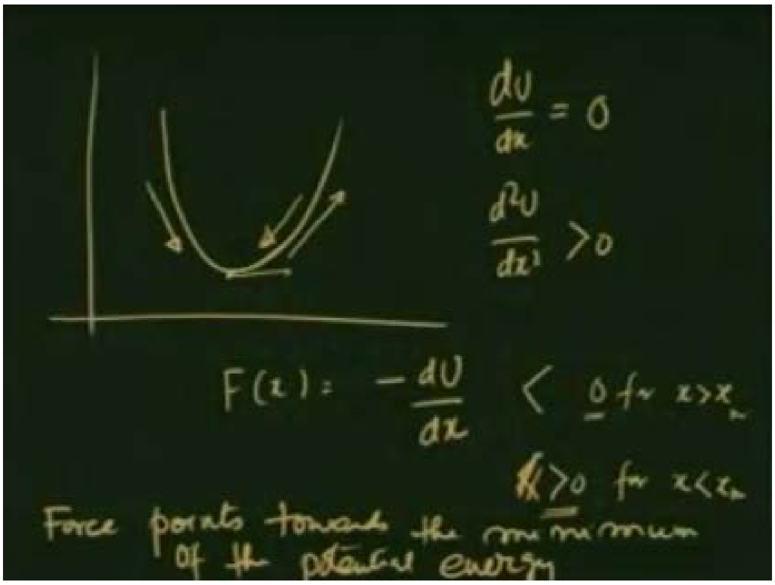

Let us look at the minimum or a minimum at a minimum du dx is again 0 and d 2 u dx square is greater than 0 and therefore, from as you go towards the positive xi slope is positive on this side slope is negative. Therefore, F x which is minus du dx is going to give you a force on this side which is going to be towards this is going to be less than 0 for x greater than x min and it is going to be less than greater than 0 for x less than x min.

So, you see the force points in the positive direction if you go to the left of minimum and in negative direction, if you go to the right of the minimum. Therefore, the force always points towards minimum, so let us writer a force points towards a minimum of the potential energy. So, any particle at the minimum who tend to move towards the minimum if displaced from it there.

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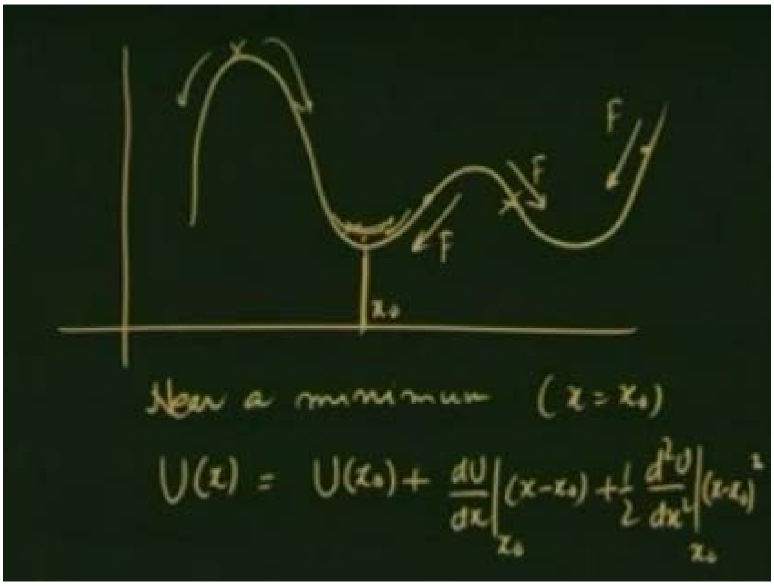

So, let us see now if I had a curve like I plotted earlier what would happen if I had a particle here, it will just come down this way or this way. If I had a particle here, if I displace it will come down back to the minimum at these points the force is in this direction at this point the force is in this direction at this point the force is in this direction this is F, F, F, F is always opposite to the slope. One interesting thing that comes about then is if you are at a minimum and particle is displaced from, it tends to come back. In fact, if the displacement is very small, you will see that it will start performing simple harmonic motion, let me explain that.

So, near a minimum and let us call that minimum x equals x 0, let us take this one, I have U x slightly away from the minimum. By Taylor series expansion U x 0 plus du dx at x 0 x minus x 0 plus 1 half d 2 U dx square at x 0 x minus x 0 square since this is a minimum dU dx at this point vanishes.

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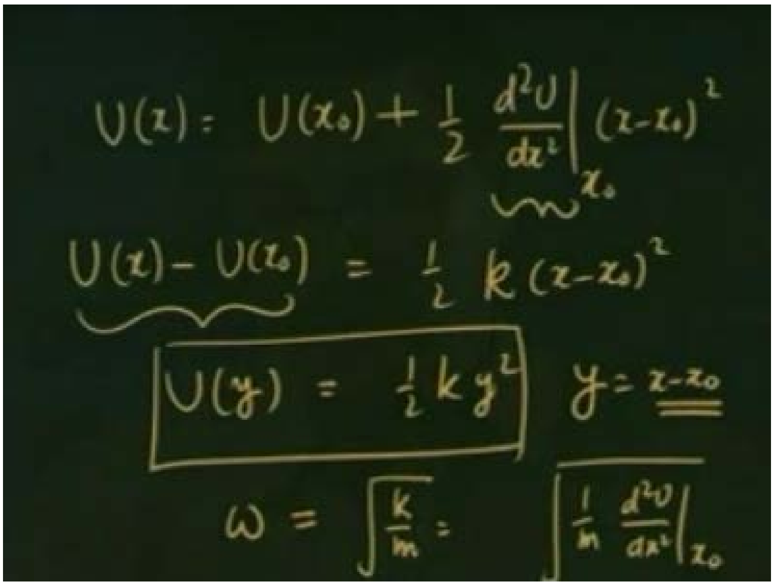

Therefore, I have U x as equal to U x 0 plus 1 half d 2 u dx square at x 0 x minus x 0 square and therefore the difference in the potential energy U x minus U x 0 is equal to 1 half this being a minimum is a positive quantity. So, let me write this as kx minus x 0 square for small displacements, if I call this U y by taking U at x 0 to be 0 because as I said earlier you always define potential energy with respect to a particular point is equal to 1 half k y square. Here, y I take to be x minus x 0 or the displacement from the minimum, you can see that the potential energy changes like a spring. Therefore, any particle displaced from here would execute simple harmonic motion with the frequency of k over m where k is given by d 2 U dx or the second derivative of the potential energy.

This is going to be square root of 1 over m d 2 U dx square at x 0, so what we have done in this lecture is tried to eliminate the time from the equation of motion. Consequently, we found that we can relate certain quantities which depends on the position these quantities are energy kinetic energy or the potential energy to evaluate the velocities at different positions. We also discussed in short very briefly what the conservative and non conservative forces are; we have restricted ourselves to 1 dimensional motion in this case. In the next lecture, I am going to generalize the three dimensions motion and see how really we can answer the question, whether force field is going to be conservative or non-conservative.