Engineering Mechanics
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Module - 06
Lecture - 01
Momentum

In the previous few lectures, we have been looking at the motion of a single particle, and we saw that given a particle.

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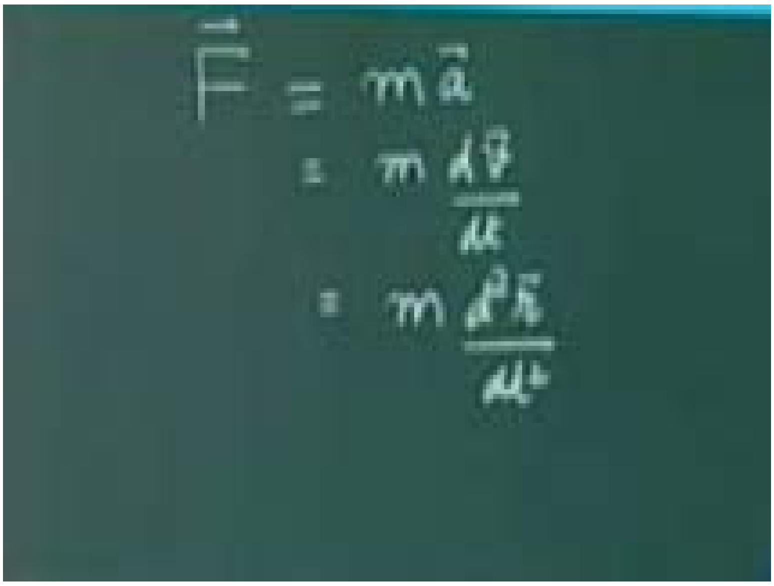

I essentially get its equation of motion and then, solve it to get its velocity or its distance as a function of time. This is essentially what we do, although we looked at constraint motion, motion with friction and things like that. Now, we are going to make the problem slightly more difficult. We are going to ask a question, what happens when I have more than one particle.

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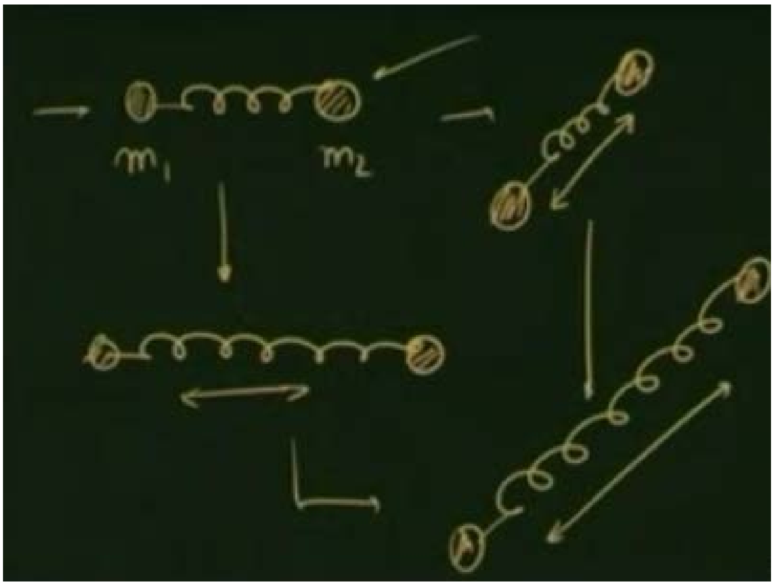

For example, let us take two particles say, of mass m 1 connected by a spring and mass m 2 here. You know from experience, if I apply a force on it, it can do two or three different things. For example, it could either stretch so, that part of this go farther apart, it can change its orientation or can do both.

And mind you, in all this process there is a force that is acting on both the particles through this spring. And I am also applying a force on this, and a force on this. So, how do we go about describing such emotion, and what happens when the number of particles increases? This is what we are going to look at and a quantity that becomes very useful in describing motions when many particles are involved is momentum. Let me motivate that by taking an example.

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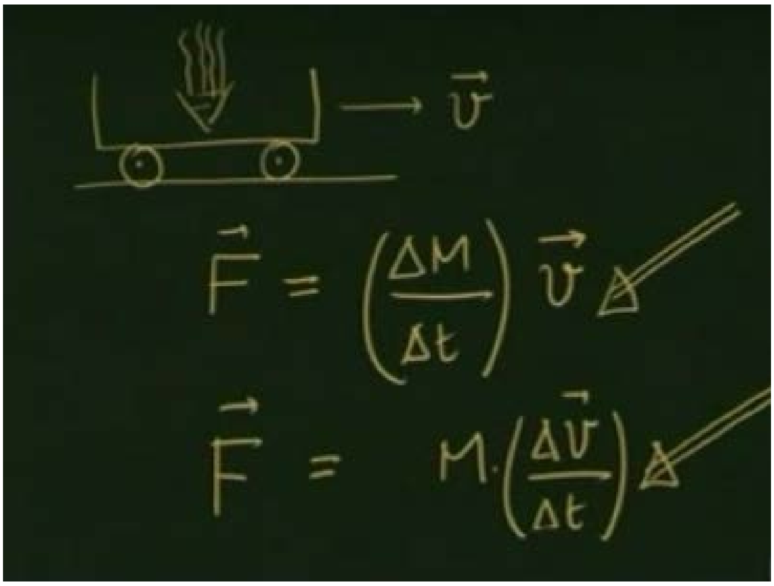

Suppose I have a cart which is moving on a horizontal track without friction, it is moving with velocity v, and I start pouring some sand or some mass into it, either I put it vertically down or slowly put it in. You know from experience that, the cart is going to slow down.

In fact, if you want to keep it moving with the same speed, you would have to apply a force, and that force is going to be proportional to the rate of change of the mass of this cart times v. On top of it, if the velocity changes, I have to apply more force. Compare this formula with the formula that we have been using so far, which is a constant mass particle moving with an acceleration delta v delta t. In general I have to apply a force, if I want to move something with a constant velocity, but its mass is changing or its mass is constant and its velocity is changing. To combine the two things, the net force I have to apply.

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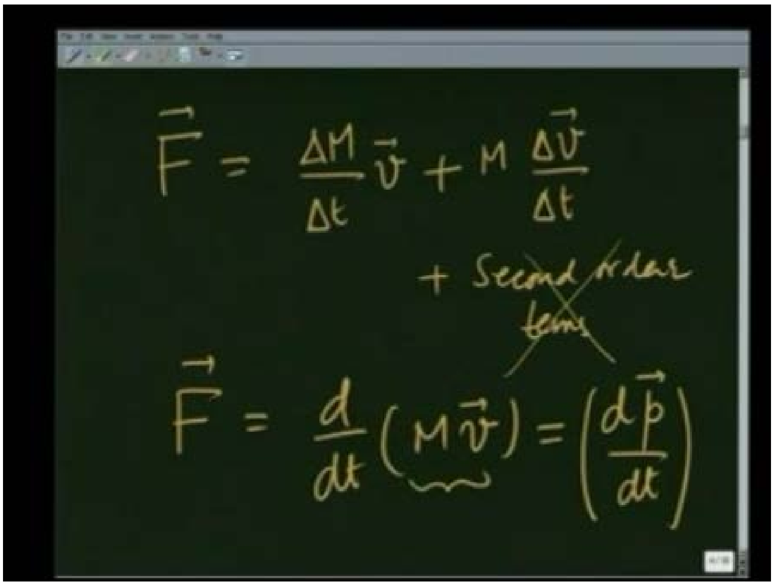

When its mass is changing and its velocity is changing, is going to be this. I have neglected second order terms in mass and velocity as they go to 0 when I take the limit delta t going to 0. So, in general I can write that, the force is equal to d over dt MV. This is the quantity which I define as the momentum of a mass moving with velocity v. So, I general I am going to write F equals dp dt where p stands for momentum. Let us see how this concept helps me in solving problems in slightly more convenient way.

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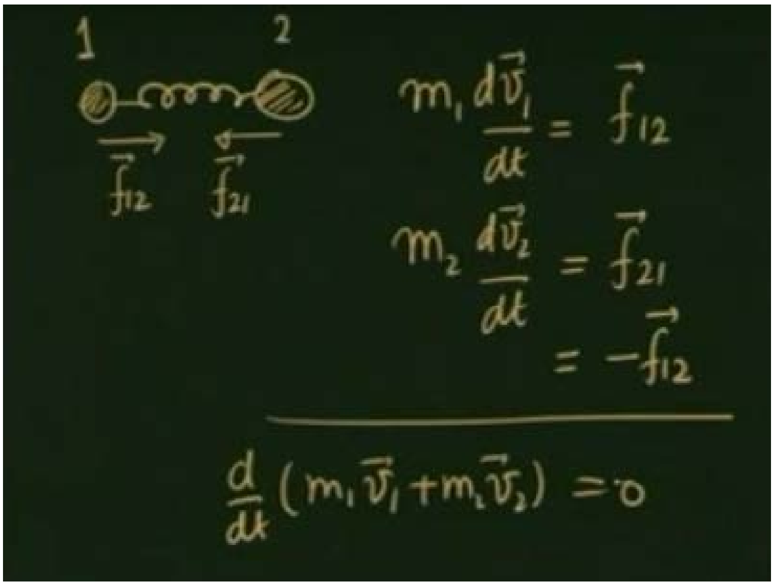

Let us go back to our example of two masses…

Let us go back to our example of two masses which are attached with the spring. Let us assume right now that, we are not applying any force on the two particles. The only force that is acting between them is through the spring. Let the force on mass 1 be f 1 2 in this direction, and let the force on mass 2 be f 2 1. I am using these indices to indicate f 1 2 indicates force on 1 by 2, and f 2 1 indicates force on 2 by 1.

If I write the equations, the Newton's second law equation for each mass, I am going to have m 1 dv 1 over dt is equal to f 1 2 and I am going to have m 2 dv 2 over dt is equal to f 2 1. But by Newton's third law, f 2 1 is going to be opposite and equal to f 1 2. So, the magnitudes are the same, the direction is opposite. If I add the two equations, I get d over dt of m 1 v 1, plus m 2 v 2 is equal to 0.

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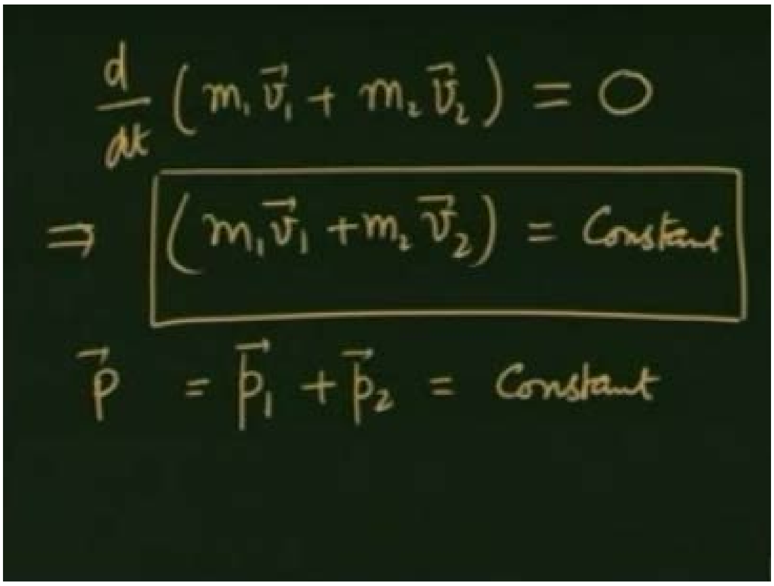

So, we get d over dt of m 1 v 1, plus m 2 v 2 is equal to 0, and what that implies is m 1 v 1, plus m 2 v 2 is a constant. So, what we learn? Is that no matter, no matter what the interaction between the two particles is? I have taken it to be most general f 1 2, f 2 1. As long as Newton's third law is satisfied is obeyed m 1 v 1, plus m 2 v 2, or the momentum of the first particle plus the momentum of the second particle, which I will call the total moment of the system is going to remain a constant. This gives me an inside of the problem, the particles may be doing anything on their own. For example, as we said earlier, they could be stretching they could be rotating.

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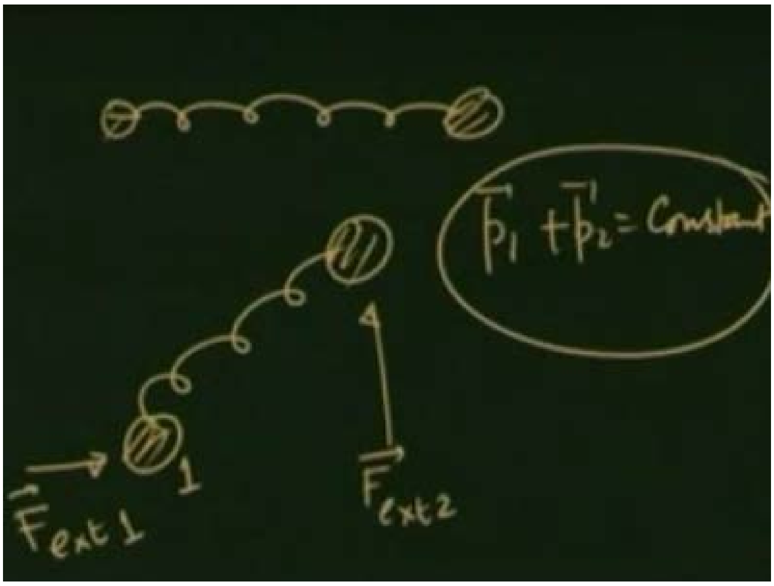

No matter what they do, this quantity is going to remain a constant. This is a statement of conservation of linear momentum in its simplest form. And when combined with other conservation laws like energy conservation, it gives me a great handle to solve mechanics problems. Let us see what happens if the forces were also applied on each of the particles. For example, I could have on particle 1, there is a force and to distinguish it from the internal forces between the forces, I will call it F external on 1. Let me call a force on this, which is F external 2, and see what happens, what the dynamics of the system is?

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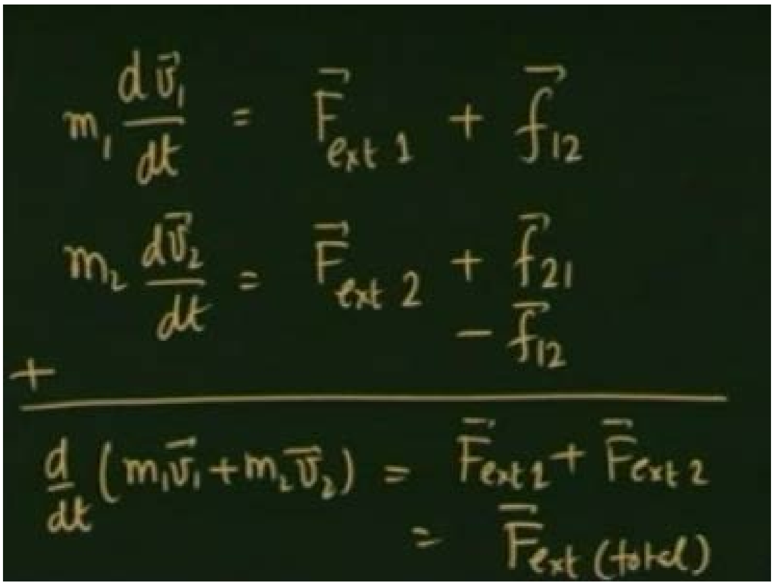

So, now I am going to have dv 1 over dt times m 1 is equal to F external 1, plus f 1 2, and m 2 dv 2 over dt is equal to F external on particle 2, plus f 2 1 which if we recall from the previous slide is minus f 1 2. And if I add the two equations, I again get the d over dt of m 1 v 1, plus m 2 v 2 is equal to F external 1, plus F external 2, which is the total force applied on the system. So, as long as Newton's third law is applicable, what I learn is that given a system of particles.

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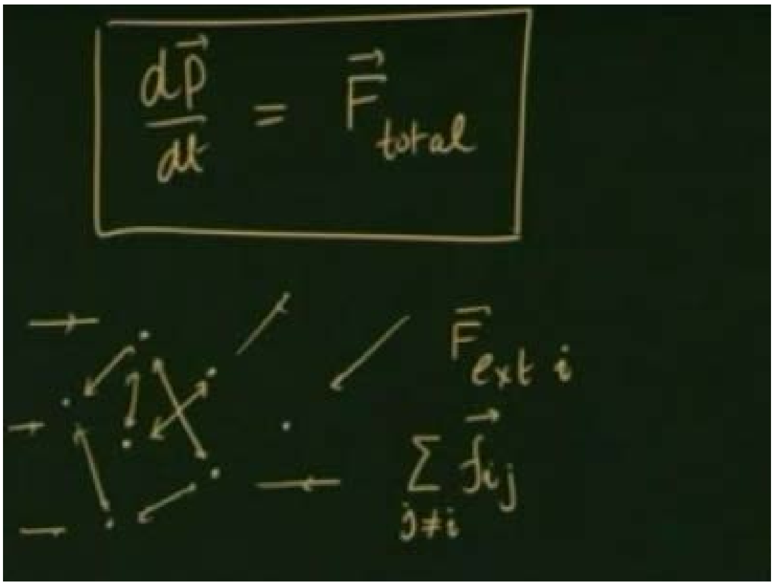

So, what we see is that, the rate of change of total momentum is equal to the net or total force applied from outside, no matter what is happening between the particles. I took an example of a two particle system, is it true in general? Let us see.

So, suppose I have a collection of particles, many of them, and I apply force on each one of them, external force which I will call F external on i th particles. In addition they are also interacting with each other, which I will call the forces on i th particle due to j. So, that the net force on i th particle is going to be sum over j, that is force applied by all other particles, but not i, it cannot apply a force on itself.

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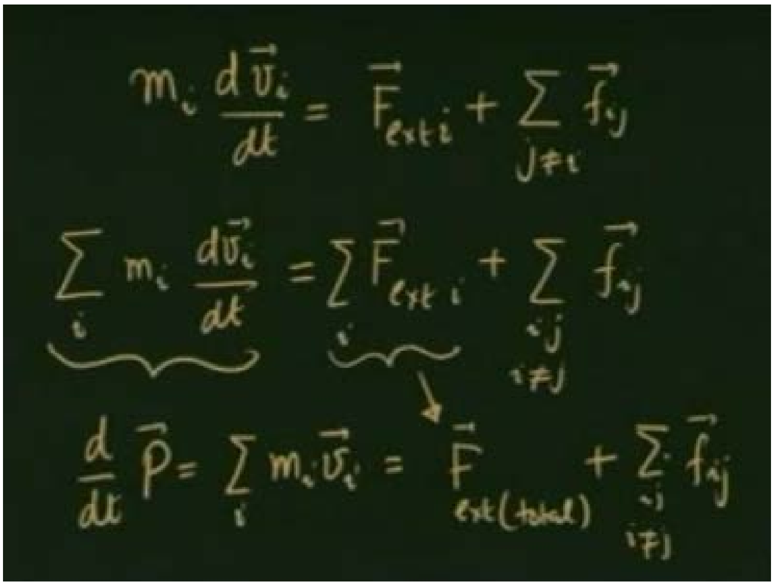

So, that if I write equation for i th particle, it is going to be mi d vi dt is going to be equal to F external on the i th particle plus the forces due to all other particles which are going to be equal to summation j, j not equal to i f ij. To see how the net mass of the, all the particles move together i sum over i. So, that I write this equation as summation over i mi d vi over dt is equal to F external i summation over i, plus summation i j over both i not equal to j f ij summed over.

This is a generalization of the formula previously written for two particle system. This you recognize is the rate of change of total momentum P, which I defined as summation of individual momentum. This should be equal to, this is the net external force. So, F external total, plus this term ij i not equal to j f ij. Let us see what this term adds upto, you can already anticipate, it should add upto 0. How does that happen?

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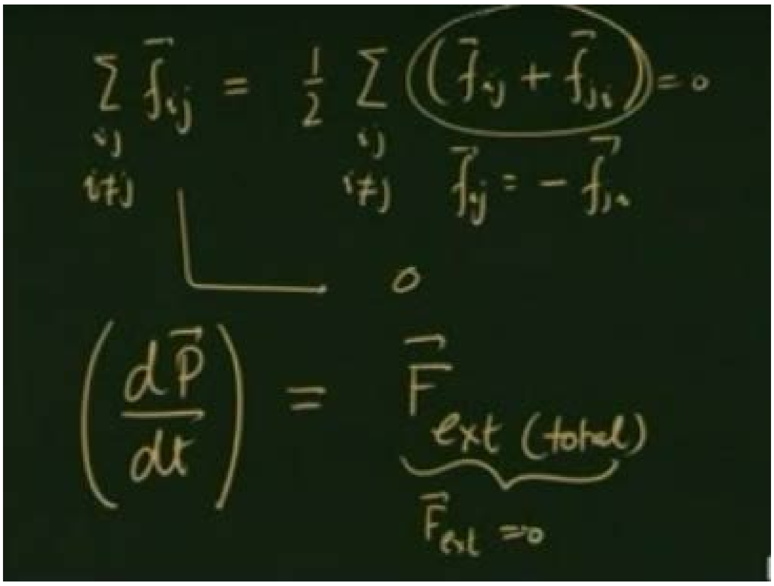

f ij summed over i and j, i not equal to j. I can write as 1 half summation ij, i not equal to j, f ij and just interchange the indices ji because I am summing over i and j completely, it does not really matter. But by Newton's third law f ij is equal to minus f ji. So, this term adds upto 0 and therefore, this term is 0.

And what we learn then, is dP dt for a many-many particle system is also is equal to F external only. Where F external is the total force, it is a sum of individual forces applied on each particle. This is a general statement, and you can right away see that, if F external is 0 then, dP dt is going to be 0 and therefore, net momentum is going to be conserved.

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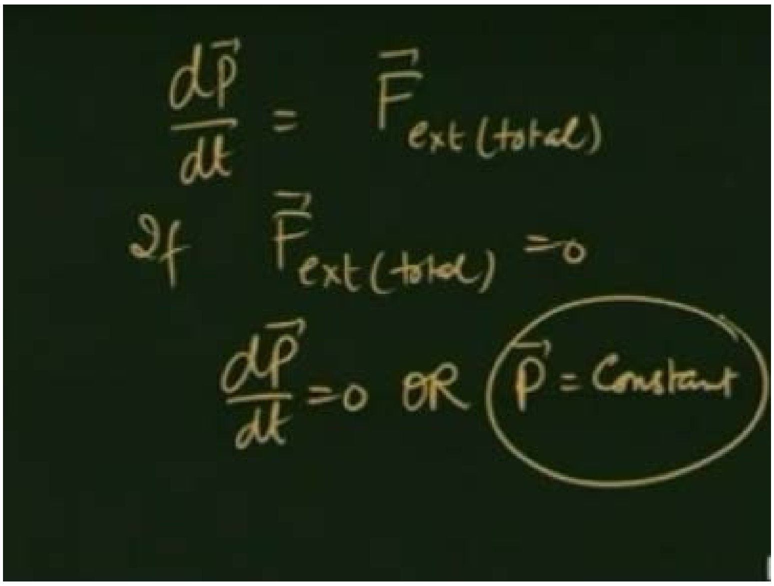

So, let us see this again, dP dt is equal to F external total, and if the total applied force from outside is 0 dP dt is 0 or equivalently P is a constant. So, for a many particle system also, if there is no force applied from outside, the total linear momentum is conserved. And that is a fundamental statement of physics, it is used in conjunction with other conservation laws and makes solution of problems easier at times. Let us then get a feel for how we can visualize the motion, if I look at momentum.

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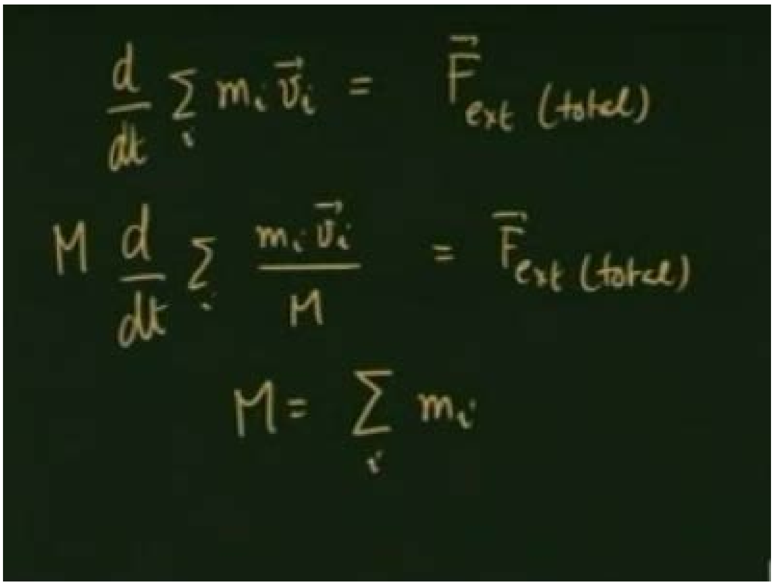

Let me again write this equation d by dt summation i mi vi is equal to F external total. Since, this is a collection of particles, mass is a constant. So, let me multiply this by mass M, I will write in a minute what M is. And write summation i mi vi over M is equal to F external total, where M is the total mass of the system, which does not change with time because I am considering all the particles, no particle is leaving the system or coming into the system.

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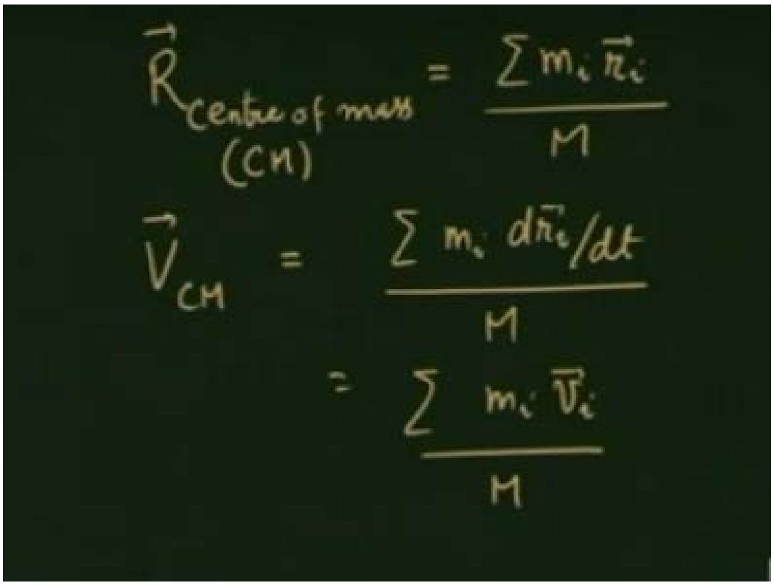

Then, you see if I define a quantity R center of mass and from now on I am going to write it as RCM is equal to summation mi ri over M. Then, the velocity of the center of mass VCM is equal to summation mi dri dt over M, which is equal to summation mi vi over M.

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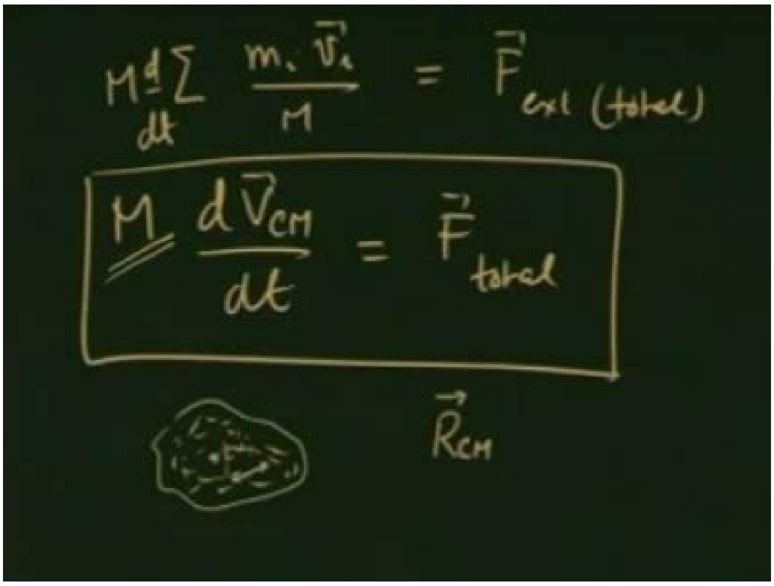

And the equation M summation mi vi over M, d over dt is equal to F external total can then be written as M d VCM dt is equal to F total. I am dropping term external right now. What does this tell me? This tells me that, I have recollection of particles, they may be interacting with each other, they may be doing many-many things with each other, as long as the interaction force between the two particles is equal and opposite. There is going to be a point in the system denoted by RCM, which is going to move as if it is a point particle with total mass M. This gives me a very nice feel about the system.

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So, if I have suppose, a body consisting of many particles, and there is some net force on it F external. This body may get the shaped, it may get different orientation, but if I take a point which is same as the center of mass, this would keep moving according to the equation. F equals M d 2 RCM over dt square, and this gives me a very nice way of looking at the motion. I know one point how it would move, no matter what the body does. To see how the concept of center of mass helps in understanding or solving a problem. Let us take an example, where we drop a bomb vertically down.

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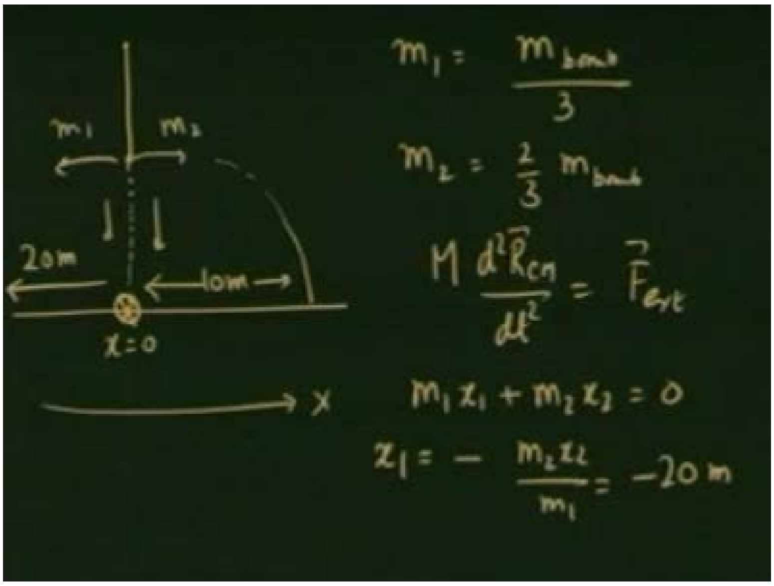

So, that it would have fallen at a place which I will call x equals 0, I am measuring x in this direction. Considering a bomb as a point particle a center of mass is sitting right here, but before hitting the ground, it explodes in middle and breaks into two pieces, one of mass m 1, one of mass m 2. So, that m 1 is mass of the bomb divided by 3 and m 2 is two-thirds the mass of the bomb. So, although the bomb explodes, no matter whether it explode into two pieces, three pieces or four pieces, the center of mass would still keep on moving as if nothing happened. d 2 RCM over dt square is a still F external and F external is only the gravitation of force.

So, center mass would keep on moving this way, and when the bomb pieces hit the ground this would reach here. And that means, as far as the x coordinate is concerned, we are going to have m 1 x 1, plus m 2 x 2 is equal to 0. Suppose, this piece fell 10 meters from where the bomb would have fallen then, x 2 is 10 and therefore, I have x 1 is equal to minus m 2 x 2 over m 1 and that comes out to be minus 20 meters.

So, the other piece is going to fall on this side at a distance of 20 meters. So, what I am trying to show you through this example is that, in a many particle system the concept of center of mass gives me at least one point for which the motion still remains simple. And we are going to take step by step by step how to make motion more and complicated that is, how we take care of deformation, how we take care of orientation, changes, and so on.

But for the time being, we focus on the linear momentum, center of mass motion, and the simplest possible way I can describe the motion of the system. We just saw how the concept of center of mass or the conservation of a linear momentum helps in simplifying the solution of a problem. I will let you think, what if the drag was also there, what the conservation of linear momentum as we applied just now in this bomb problem be applicable. Let us change v as now and go to a slightly different problem, which I would call the problem of momentum transfer.

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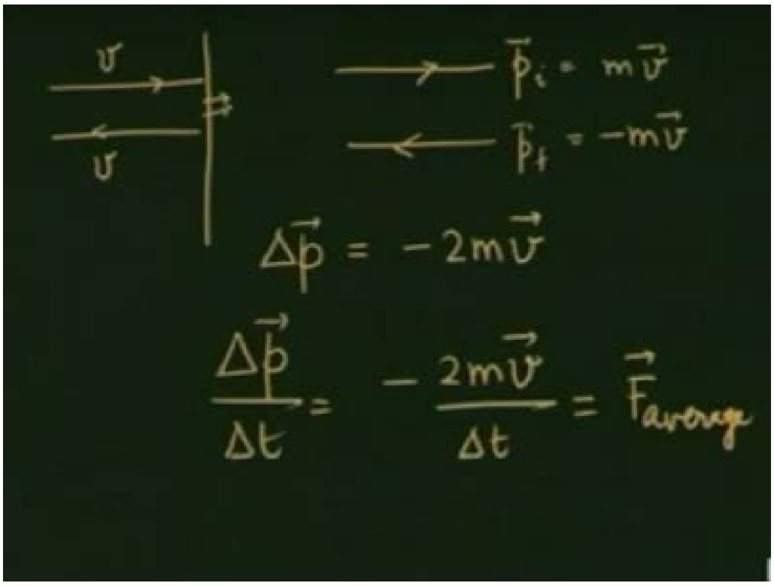

Let us for that look at a ball hitting a wall and bouncing back with the same speed. So, it comes in, comes in with speed v and goes out with the speed v. What has happened to its momentum? Its momentum initially was in this direction, let me call it p initial which is mv, and the final momentum is p final which is minus mv because it has just bounced back.

So, the change in the momentum is going to be minus 2 mv. I know that delta p delta t which is going to be minus 2 mv over delta t, where delta t is the time for which the ball was in contact with the wall is the force that the wall has applied on to the ball. So, this is I will call the average force, I am calling it the average because I do not really know how it change with time. If the wall has applied that much force to the ball, the ball has also has applied equal and opposite force on the wall. So, a ball hitting a wall applies a force on it, let me now ask is this average force?

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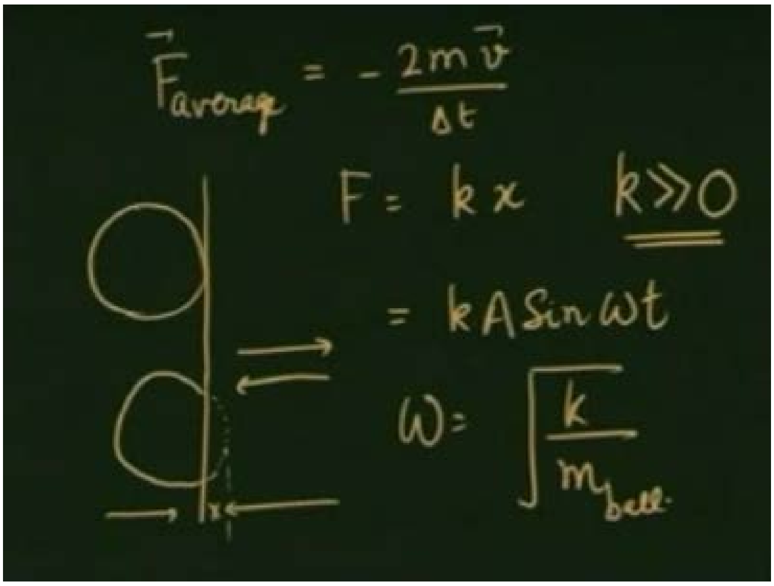

That I have written as 2 mv over delta t with the minus sign here same as the force at different instants. The answer I do not know, but let us try to find out. Let us model the ball hitting, this wall as if it gets squeezed in a simple manner like this, and the force it applies after being squeezed by an amount x, this being x is F equals kx. And since, this is a hard, ball hard ball k is much, much, much greater than 0. It is a very large number. So, it is like a very hard string.

If it hits and the force is this obviously, after hitting the ball is going to form a simple harmonic motion. So, that I can write this as kA sin omega t, where omega would be the characteristic frequency of oscillation with this k and the mass of the ball. So, we see that the force on the ball is of the form roughly as kA sin of omega t.

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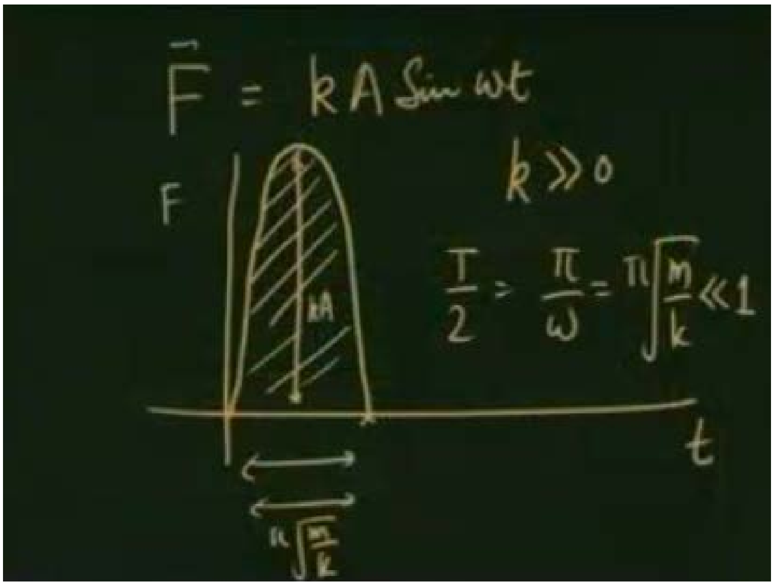

I have chosen sin because initially the displacement is 0. And therefore, if I were plot it with respect to time, it would something like at 0 it starts goes up and comes down, like a sine wave and at this point the ball has left the wall. Since k is much, much, much greater than 0, this peak is going to be very-very high kA. And the time is half the period, which is going to be pi over omega, which is going to be pi square root of m over k, a very-very small number.

So, therefore, what you see is that, there has been a very-very large force spread over a very small time pi square root of m over k because k is very large, this time is very small. So, I may not even observe how the force is varying, all I see is ball hitting the wall and coming back.

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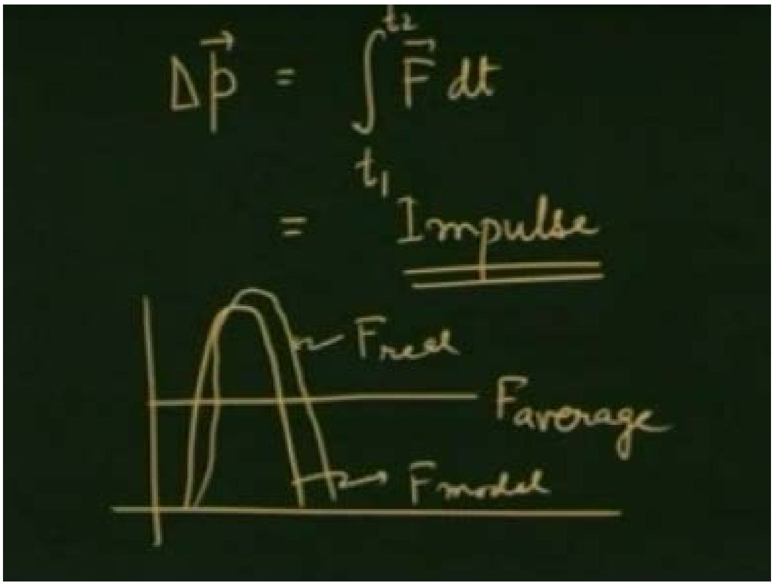

So, in these situations it is much better to talk about the net change in momentum, which is really F dt integrated from time t 1 to t 2 and I call this an impulse. Again, a quantity which is related to the net change in the momentum of the particle. Although I have made the force varying like this, a very neat curve sin omega t, actual may also differ slightly, but the average force was somewhere here. So, this is F average, this is F may be real, and this is F as I have modeled it. But it gives you an idea as to in different situation how do we tackle with momentum changes and forces.

So, a situations where a very large force acts for a very-very short time, we are going to use impulse and use the net change in momentum. You can see that our model is quite okay, take an example of a hammer hitting on the wall. Hammer is made of iron and therefore, it is very hard and the force it imparts therefore, is very-very large. So, what we learnt from this is that, a ball or a particle hitting a wall for a very short time imparts a momentum change delta p to it which is related to the force applied by it on the wall.

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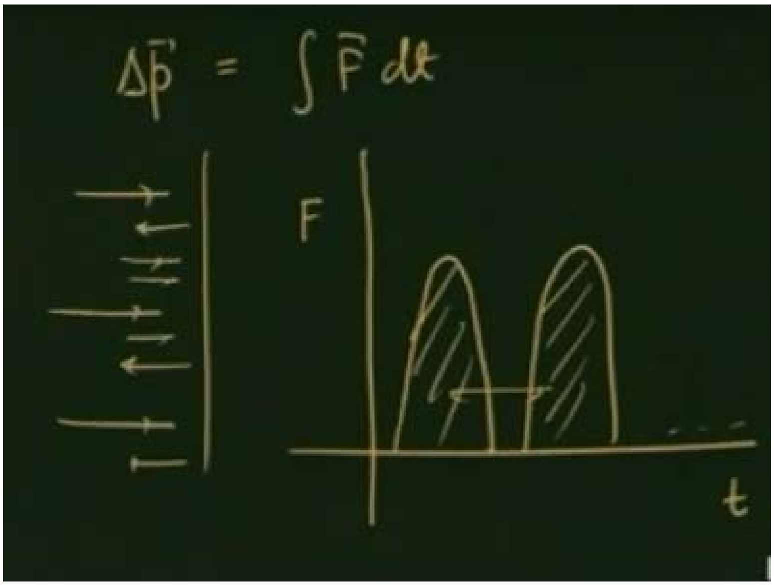

Let me now ask, what happens if there were many-many balls hitting it. One ball comes hits it goes back, second ball comes hits it goes back, third ball comes hits it goes back. In that case, if I were to plot the force on the wall with time, is going to be the first ball hits goes back, second ball comes hits goes back and so on. I made this process more rapid now so, that these curves starts overlapping.

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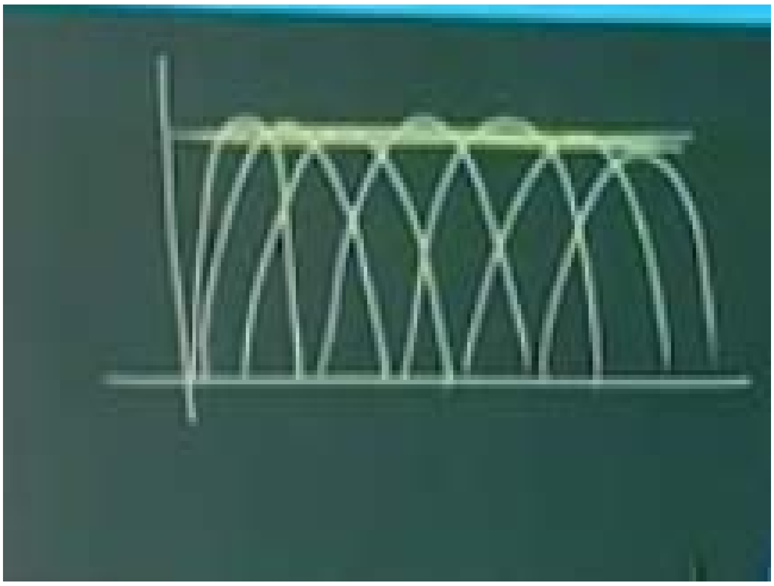

In that case what I would observe is that one ball hits gives this curve, the next ball hits the wall even before the first ball has not come back, third ball hits again, fourth ball hits again, fifth ball hits again, and so on randomly. So, the net force may be somewhat like this, let we make it slightly thicker, which is a sum of all these forces. So, if there are many particles hitting randomly continuously on a wall, we see that there is a net force that is applied on the wall, and this is roughly a constant, how do I calculate this? This is another application of momentum and force relationship.

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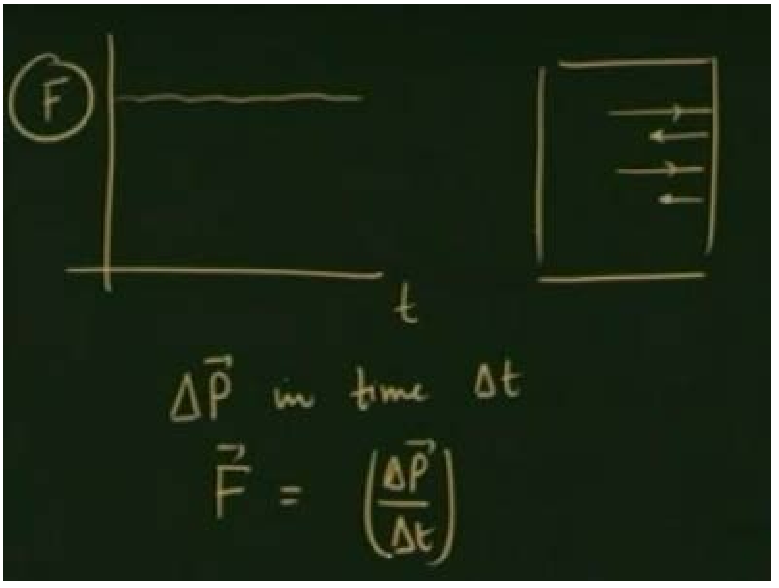

So, what is happening is that these balls are hitting at different times, but randomly an overlapping forces or net forces something like this, almost a constant with time, and I wish to calculate this force. In a box or on the wall, where the balls are hitting and going back or like this. We again go back to how much momentum are these balls transferring to the wall per unit time. So, I will calculate the momentum transfer delta P, in time delta t, and this force that I have written here, the left hand side top is going to be delta P over delta t. Let us calculate that as an application of what we learnt so far.

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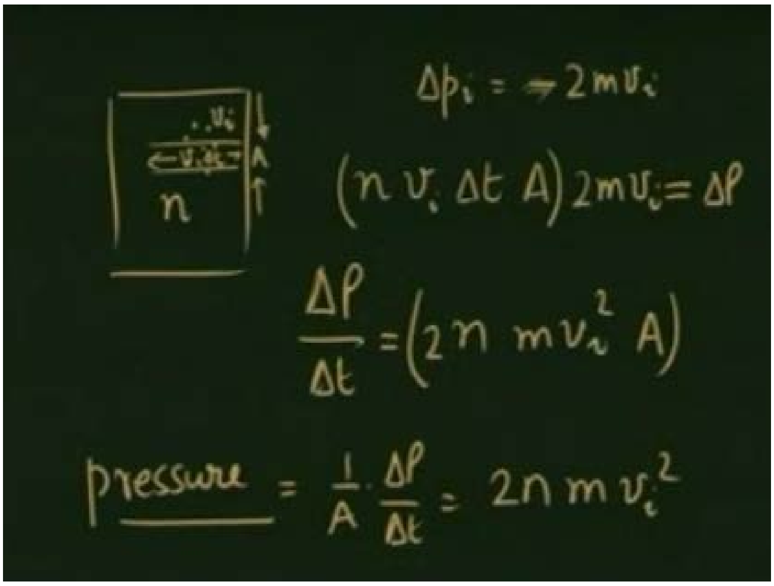

So, let us take a box where the small particles or small balls of density n are there, and they are hitting the wall with speed vi. So, net momentum change for each particle is going to be delta pi is going to be minus 2 m vi. Since, I am not worried about the sign right now, only the magnitude I will just remove this.

How many particles are hitting, number of particles hitting is in time delta t is going to be n times vi delta t times the area. This is the area, this is the length vi delta t, and in volume A times vi delta t, I have n times these many particles. These many particles are hitting in time delta t, and they are each imparting momentum vi. So, net moment of transfer is going to be this and therefore, I get delta P over delta t, which is n m vi square times 2 times the area.

This is a net moment of transfer and therefore, this is the force applied by these particles on the wall. If I were to calculate the pressure, this would be equal to 1 over area times the forces, which is going to be 2 times the density m vi square. You are familiar with such a calculation from your previous study of kinetic theory of gases, but here we look at it from a slightly more advanced point of view. In addition this example also tells us, how if the flow of these particles is continuous, like a water stream hitting a wall or some object, how much force would it apply. As a final example of the application of concept of momentum, let me look at the variable mass problem.

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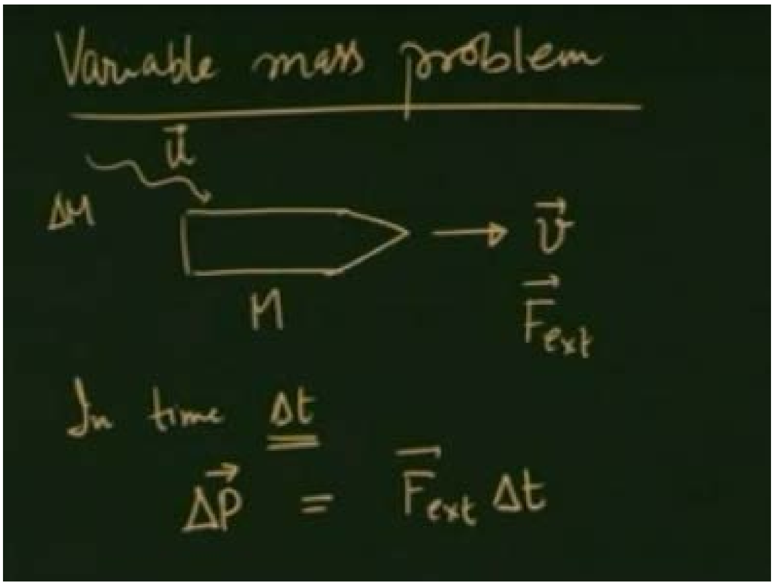

Problem where on my system of interest either the mass comes and adds on, or my system of interest is dropping some mass. A familiar example of this is the rocket propulsion, where the gases are exhausted. So, to formulate this, let me take a mass M which is initially moving with velocity v and on to this, I add on a mass delta M, which is coming in with velocity u. If this is an outer space, there will be no external force on the system. If this is in on earth or some other planet, there will be gravitational force on the system. So, let us assume there is a net force F external acting on the system. In time delta t therefore, the momentum change of the entire system is going to be equal to F external times delta t. Let us see, what the momentum change of entire system is.

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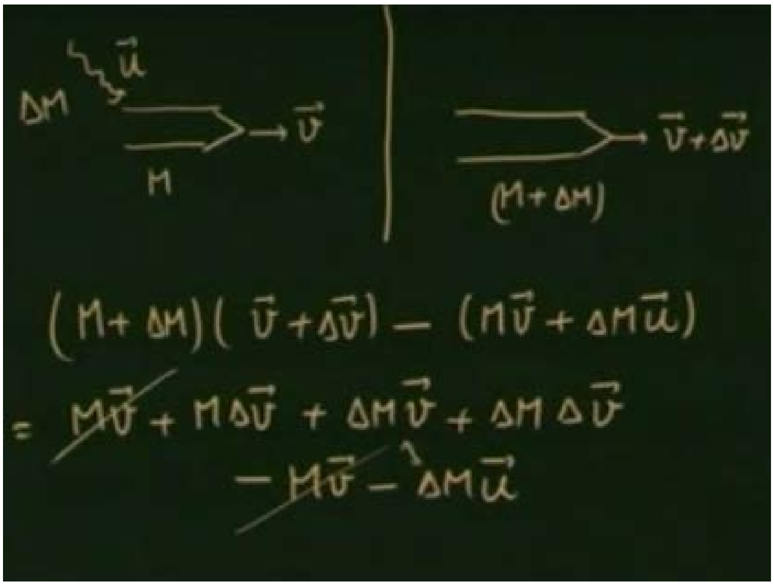

So, initially this mass is moving with velocity v, mass is M, mass delta u and finally, it goes to becomes a system of mass M plus delta M, moving with velocity v plus delta v. Notice that, I have taken all quantities to be positive, and that is to keep my calculations simple. I do not have to worry about any minus signs appearing anywhere.

So, net change in the momentum is going to be the momentum, which is M plus delta M, v plus delta v, minus Mv plus delta Mu, which is Mv plus M delta v, plus delta Mv, plus delta M delta v, minus Mv minus delta Mu. This cancels with this, I combine these two so, that I get the net momentum change to be.

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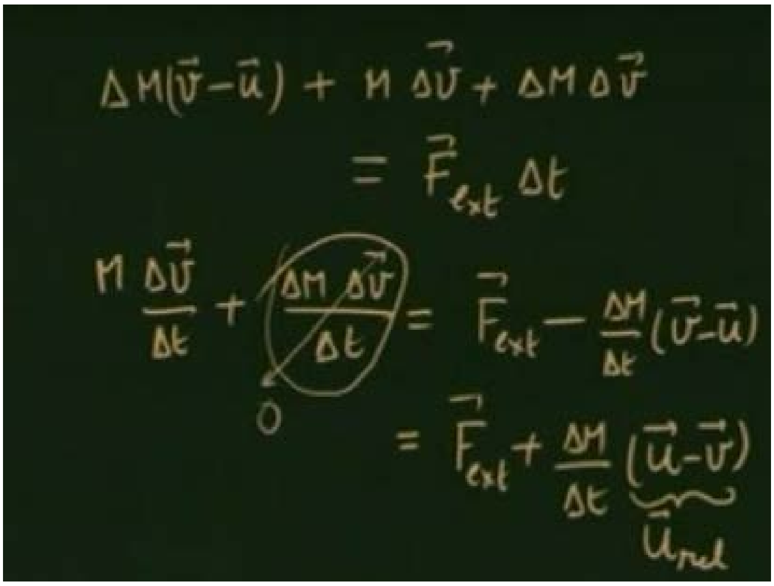

Delta Mv minus u, plus M delta v, plus delta M delta v, and this must be equal to the vector F external delta t. And therefore, I have M delta v over delta t, plus delta M delta v over delta t is equal to F external, minus delta M over delta t v, minus u, which I am going to rewrite as F external, plus delta M over delta t u, minus v. I do so, because this quantity I then recognize is nothing but u relative of mass delta M with respect to my initial mass. So, the system that I am focusing on satisfies this equation, when I take limit delta t going to 0, this term can be dropped because it is going to have a delta t on top and therefore, neglect it.

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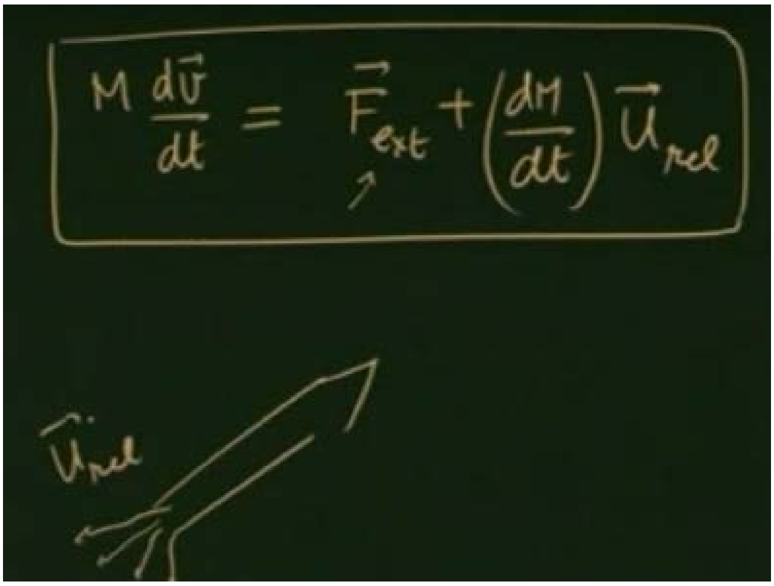

So, final equation that I get that from this is M dv dt is equal to F external, plus dM dt u relative. Mind you, that the note, that the total momentum is still changing out according to this, but I am focus on one particular system on which the mass is adding on, or from which the mass is going out.

Sometimes we get confused as to what sign should be here, the best way to remember this is, in the rocket problem, the velocity always goes up, dM dt is negative, u relative is negative with respect to rocket and therefore, this term is positive. And that is what we want if the velocity should go up. Let us try to apply this to the rocket problem. In my rocket problem, what we have is a rocket that exhaust gases and this exhausts at a constant relative velocity with respect to the rocket. So, suppose I fire a rocket from the earth surface vertically up as a sample problem, let me take this direction to be y.

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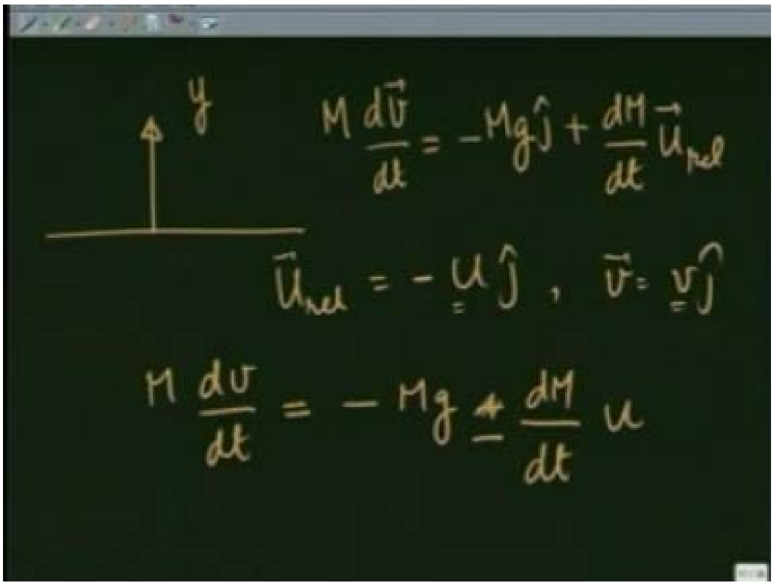

And therefore, I am going to have for the rocket M dv dt is equal to F external, which is only the gravitational force in the negative direction, plus dM dt u relative. U relative is uj in the negative direction and dM dt is also negative, but that I do not have to write explicitly so, and v is some vj. So, when I transform this vector equation in terms of these quantities, I have M dv dt is equal to minus Mg, plus dM dt times u, I took a minus sign here so, this is actually going to be minus.

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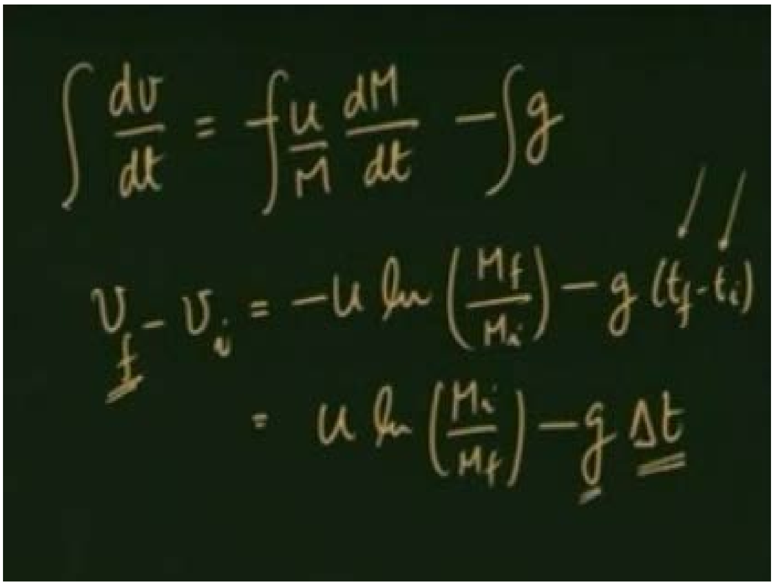

And therefore, I have, if I divide it by M all over I get dv dt is equal to minus u dM dt 1 over M, minus g integrating. Since, u is a constant, I get v final, minus v initial is equal to minus u log of M final over M initial, minus g t final minus t initial. Where the rocket was fired at initial time ti and went upto tf which is same as u log of M initial over M final, minus g delta t, where delta t is the time for which the rocket was fired.

So, you can see that the final speed that it gains, depends on the ratio of the initial mass to the final mass. The larger it is, more it will gain. And if larger delta t is there, the gravitational force slows it down. And therefore, you want to fire the rocket in as short a time as possible, and that is precise the y when the rocket is fired which you may have seen a PSLV going up or ASLV going up, there is a lot of, lot of fuel is burnt right in the beginning and as short a time possible as they can.

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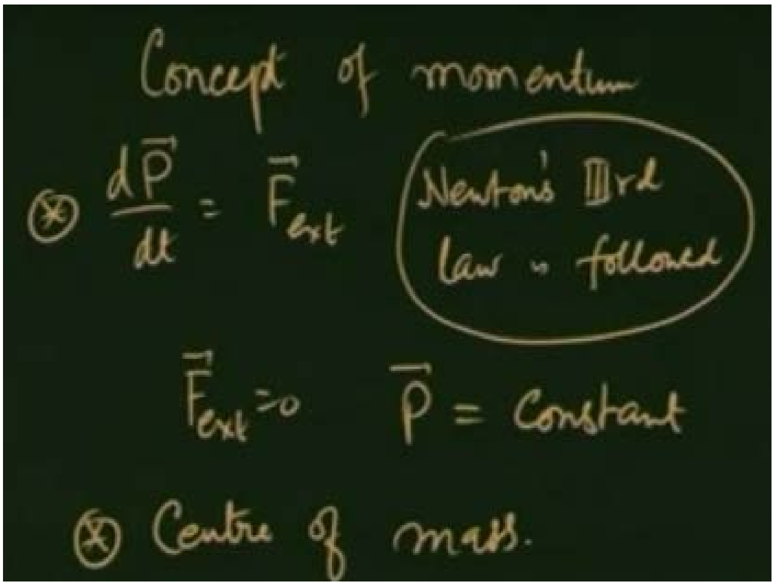

So, what we have seen so far is that we have introduced the concept of momentum. We have seen that the momentum satisfies the equation dP dt is equal to F external only, irrespective of what the nature of force between the particles in a many particle system is, the only requirement is that Newton's third law is followed. This is the requirement, there are examples where it is not followed and I let you think about it.

Third we saw therefore, that if F external is 0, the total momentum is a constant and that helps in solving problems as we will see in the coming few lectures. We also introduced the concept of center, let me just bullet them, of mass and saw that this is the point which keeps on moving as if, there was the total mass M sitting at this point. And finally, we looked at the variable mass problem. I would erg because I will be using conservation laws in the coming few lectures, that whenever you apply conservation laws, you should also try to look at what is going on.

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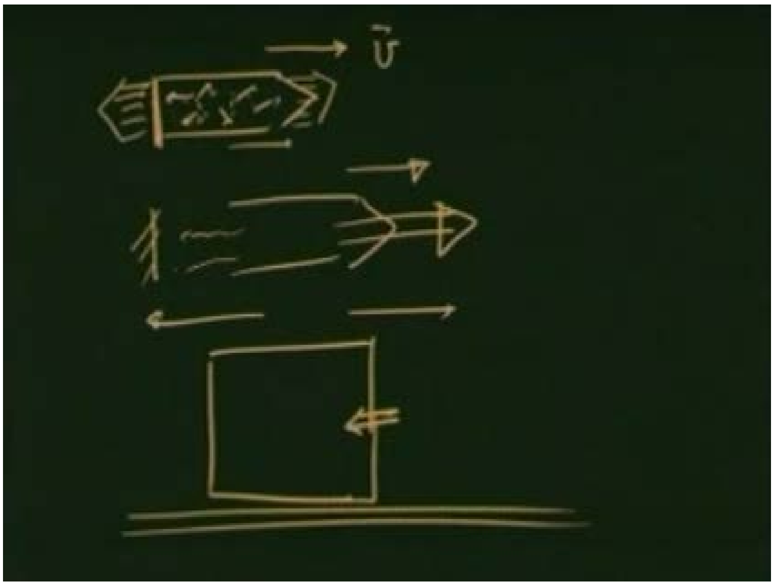

For example, in the rocket problem after all rocket gets propelled in this direction because there is a force that is pushing it. So, you should ask yourself where is this coming from, although I get my answer, I can solve problem in a very easy manner if I will apply conservations of linear momentum. But you should look for, how does this force arise. Let me look at the rocket problem and see what is happening is, this gas inside which is burnt, or the fuel which is burnt applies pressure all over.

As long as this side is closed, the pressure on this side, whatever forces applies, balances the force on this side. The moment I open this side, this part is removed the gas is coming out. So, the force on this side is not balanced, and consequently rocket starts moving in this direction. Let me then end this lecture by leaving you with the similar problem. Suppose, I take a box which is evacuated, there is nothing inside it. And I punch a hole here, I let you think which way should the box move, should it move to the right, should it move to the left?