Numerical Methods and Computation

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Lecture No # 9

Solution of Nonlinear Equations (Continued)

Polynomial Equations

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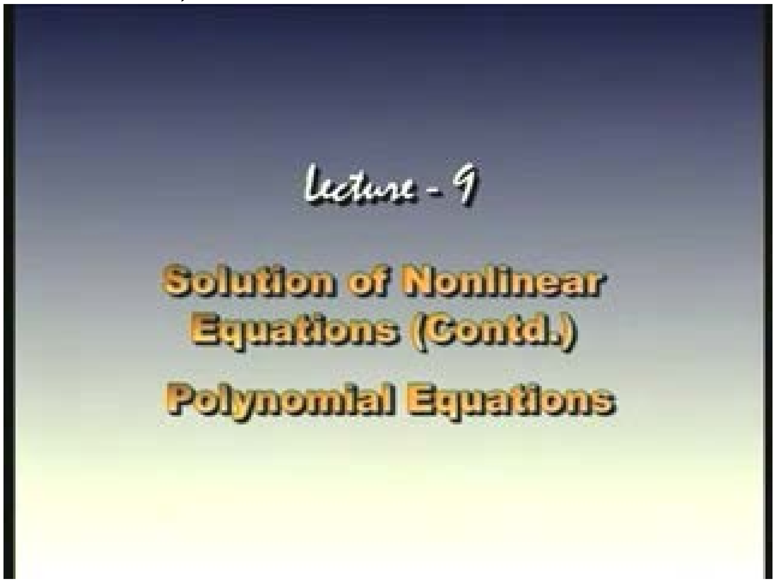

In the previous lecture we had derived the Birge Vieta method for the solution of the polynomial equations. We have shown how we can extract a simple root from the polynomial equation, deflate the polynomial and then obtain the next root if it is required. Now let us just take one more example on this to just illustrate what we should do if a coefficient is missing in the given equation. So let us start with an example and the example which I will take is as follows.

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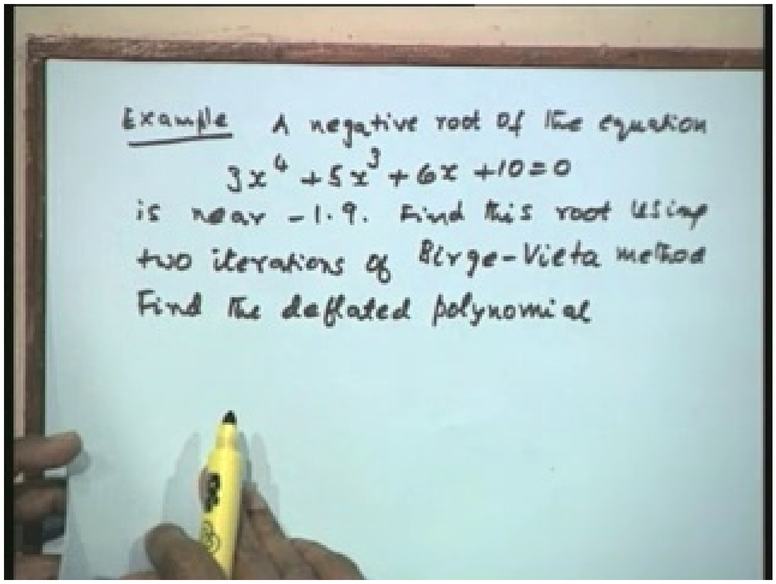

Now I find a negative root of the equation. A negative root of this equation three x to the power four plus five x cubed plus six x plus ten is equal to zero is near – 1.9. Let us take two iterations of Birge Vieta method and also let us find the deflated polynomial. Here we are already given the negative root that the root is near – 1.9. So we have the initial approximation as – 1.9 and we want to use the Birge Vieta method, two iterations and then find the deflated polynomial also. So we can see here that in this polynomial that is given to us the one factor is missing, which is x square. So I just want to illustrate how we are going to include this particular term.

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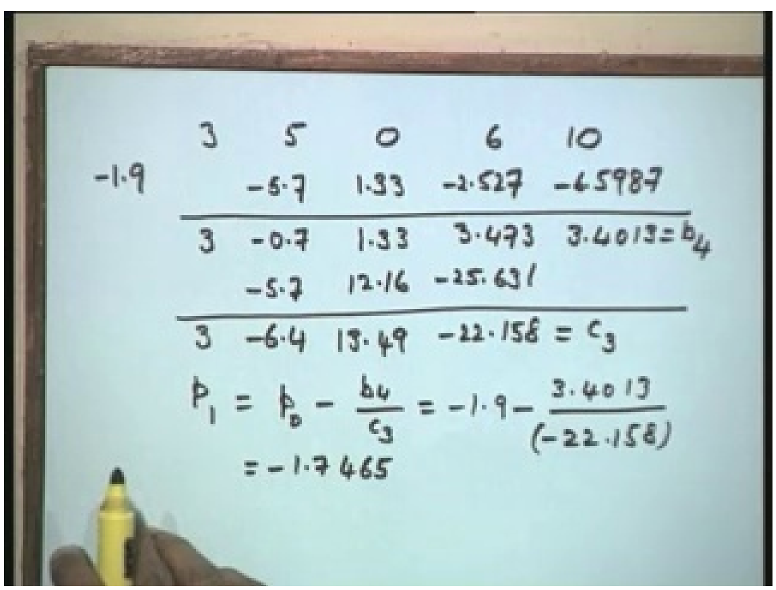

As mentioned we shall use the coefficient as zero in the Birge Vieta method. So we shall therefore write the coefficients of this as 3, 5, 0, 6 and 10. Then we are applying Birge Vieta method with respect to the initial approximation – 1.9. Therefore b_0 naught is a_0 naught. Therefore I will have three. Multiply these two and I will get – 5.7. So I will have here – 0.7. Then I multiply – 0.7 into 1.9. Both are negative sign, I will get here 1.33. Add these two; I will get 1.33; multiply this – 1.9 and 1.33. I would get here – 2.527; subtract it, I will get 3.473. Then again multiply this 3.473 with – 1.41- 1.9; that gives you – 5.5987. So this value is 3.4013 which is the value of b_4. Then I repeat one more step here. So I would get 3; I will get – 5.7; I add up, I will get – 6.4; multiply with – 1.9, I would get 12.16. Add this 13.49, multiply by – 1.9, so I will get – 25.631 which give me – 22.158 and this is equal to c_3. This is your c_0, c_1, c_2, c_3. So this is our coefficient c_3.

Now we have completed the synthetic division. So let us use the Newton Raphson method. This p_1 is equal to p_0 minus b_4 upon c_3. So that will be p_0 naught is – 1.9. I have minus b_4 as 3.4013 divide by – 22.158 and I get this value as – 1.7465. So the first iteration has given the root as – 1.7465. Now I would continue the synthetic division, the second iteration using – 1.765. Now the remaining part is trivial. Therefore I will leave this for you to complete but I will give the solution of this.

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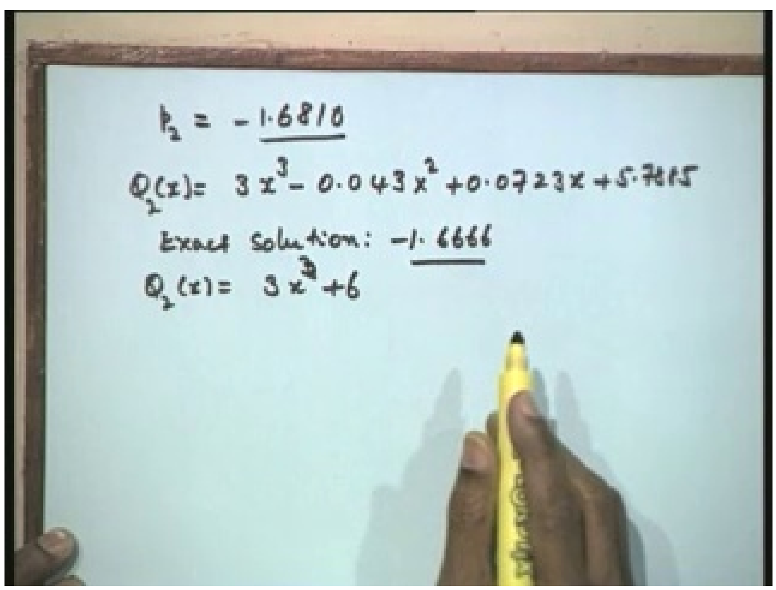

So the second value p_2 two comes out to be – 1.6810. We have been asked to perform only two iterations. So we shall stop the iteration at this particular level and then find the deflated polynomial with respect to this. So I now perform the first step of the synthetic division procedure wherein I will get the Q_2x. The answer for Q_2x comes out to be three x cubed minus zero point zero four three x square plus zero point zero seven two three x plus five point seven eight eight five. This is the deflated polynomial and I will also give the exact solution for this problem. We performed only two iterations. Therefore we have not yet reached the required accuracy. The exact value is 1.666 and the deflated polynomial Q_2, the exact value is three x cubed plus six three x cubed plus six. Therefore you can see that the root we have only got only one decimal place, we still need at least two more iterations to get the few more places accuracy and these two values would go to zero.

Therefore the idea here is to apply the Newton Raphson method to a polynomial equation so that we can extract a simple root from that particular equation. The modification for this will be as for multiple root. We know if the methods we have given earlier for multiple roots, the same can be applied and we can apply the Newton Raphson method for the multiple root also.

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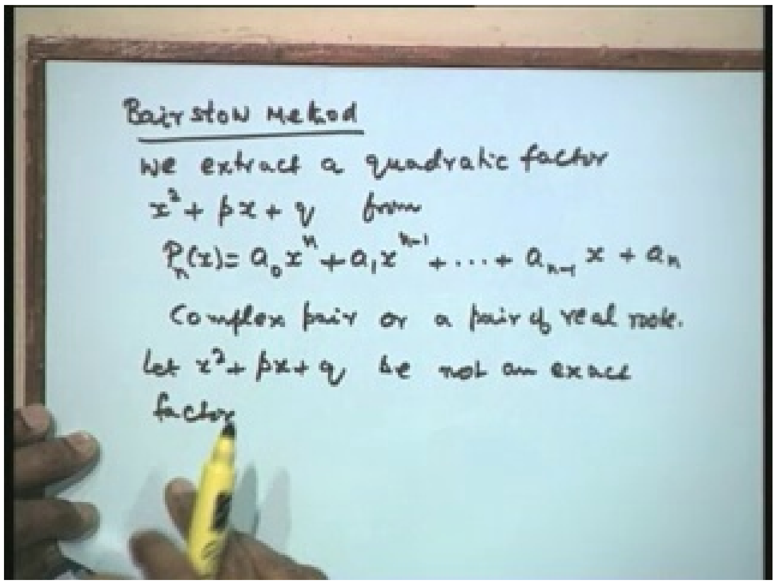

Now we would go to the method known as the Bairstow method. The Bairstow method is that wherein we will extract a quadratic factor and not a simple factor. So we can say we extract a quadratic factor x square plus px plus q from the given polynomial. So let us take the polynomial. Lets' write it again a_0x to the power of n, a_1x to the power of n minus one plus so on a_n minus one of x plus a_n. Now if you extract this quadratic factor, then this can give us a complex pair of roots or it can give a real pair of roots. So we are extracting a pair of real roots or a complex pair. So we would then have a complex pair or a pair of real roots. Now if we divide Pnx by x square plus px plus q, if this was exact I would get a polynomial of degree n minus two. If it is not exact then I will have a remainder. So let x square plus px plus q be not an exact factor. Now if it is not an exact factor then the division by x square plus px plus q would give us a polynomial of degree n minus two and the remainder will be a linear polynomial because we have divided by quadratic.

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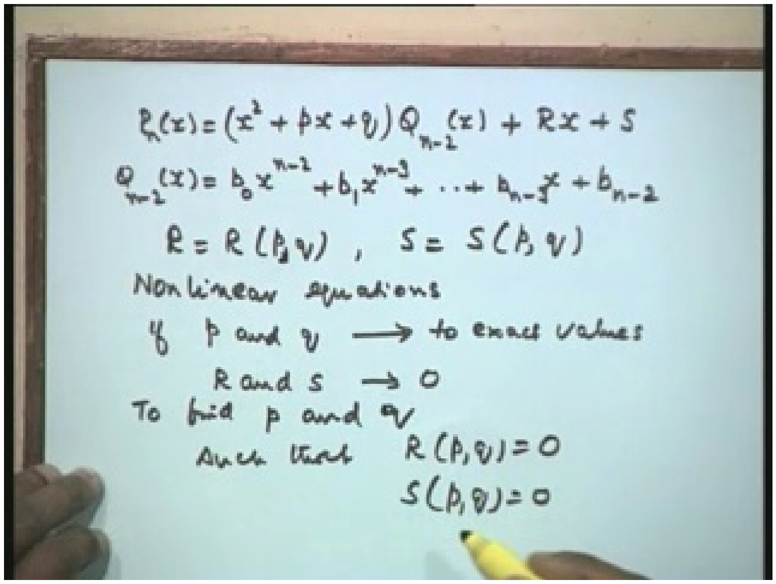

Therefore I can write down Pnx as x square plus px plus q which is the quadratic factor that we want to extract into polynomial of degree n minus two x plus a. Linear polynomial will be the remainder. So it will be of the form some Rx plus s where Qn minus two is a polynomial of degree n minus two. So I can write this as some b_0x to the power of n minus two, b_1x to the power of n minus three plus so on b_n three x plus b_n minus two. So it's a polynomial of degree n minus two. So it will have the coefficients as b_0x to the power of n minus two, b_1x to the power of n minus three and constant term is b_n two. Now p and q are the initial approximations to the exact values of p and q. Therefore as the iteration goes on the values of p and q change, therefore the values of the remainder also changes. In other words R and s, there be functions of p and q, therefore R is a function of pq and s is also a function of p and q. Now as p and q is varying the R and s is varying, therefore they are nothing but non linear equations of p and q. So these give you two nonlinear equations. They are two non linear equations in the two variables p and q. Now we would be getting the exact factor if R and s were identically zero. If R and s were identically zero this is exactly divisible and we have got the exact factor. Therefore the problem is to determine p and q such that R and s tend to zero. Now therefore if p and q tend to exact values, R and s tend to zero. That is because this remainder should be equal to zero. In other words the problem is, if you look back at this, it is to find p and q such that R of pq is equal to zero, s of pq is equal to zero. But we have discussed this problem already. This is a system of two nonlinear equations in two variables p and q.

We have discussed the solution of the nonlinear equation in n variables. So we shall just apply it for the case of two variables and find out what will be the values of p and q from here.

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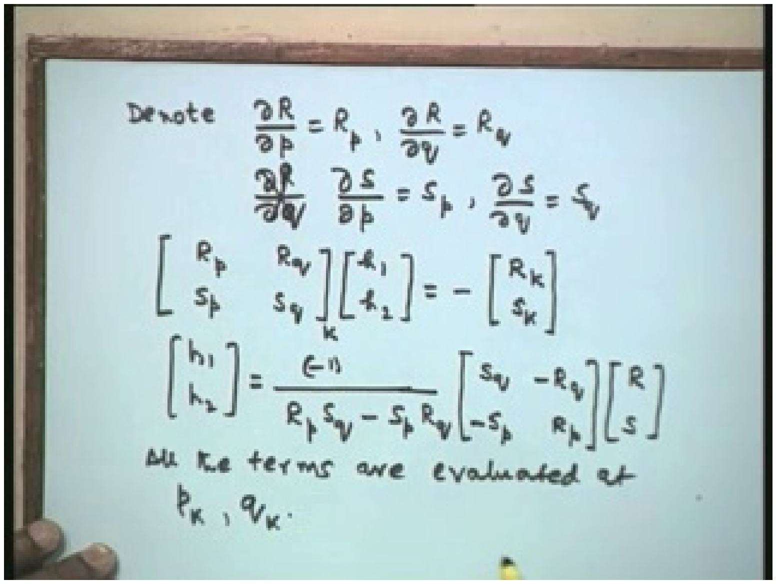

Now let us denote the partial derivatives by simple notations, so that it is easy for us to write down the solution for this. So let us denote the partial derivative of R with respect to p. We are talking about the functions of two variables. Therefore we shall be talking of the partial derivatives of R and s. Let us just denote this by R suffix p, delta R upon delta q by R suffix q. Similarly delta s upon delta p is equal to suffix p, delta s upon delta q is s affix q. We just denoted the partial derivatives by simple notation using the suffix. Now we know that if I am solving this system of two equations, what we have to solve is the Jacobean into the increment is equal to the right hand side evaluated at all these points. The Jacobean of R is delta R upon delta p which is the first element, the second element is delta R upon delta q. So what I will have here is R suffix p, R suffix q, s suffix p, sq; so these are the four partial derivatives that appear in the in the Jacobean and these are the increments that we are solving for. So I will have here h_1, h_2 as the increments. On the right hand side we have the R and s evaluated at the particular iteration. Let us put the iteration as k, now here this will be Rk and sk. So this is the Jacobean, this is the increment, the right hand side with opposite sign.

Now since it is a two by two matrix, find out the inverse and then write down what will be h_1 and h_2. So let us write down what is our h_1 h_2 two. We have this minus sign; let us put it in the numerator. The determinant of the coefficient is Rp into sq minus derivative with respective q of r into sp. So let us write down the denominator first, Rp Sq minus sp Rq. So that is the determinant of the coefficient.

Now let us write down its co factors. So this is sq. The co factor of this is Rp. Then co factor of this is minus sp and then let us take the transpose also. So I will have minus s suffix p minus R suffix q, so this is the inverse of this two by two matrix and this is multiplied by R and s, all evaluated at k. All the terms are evaluated at pk, qk.

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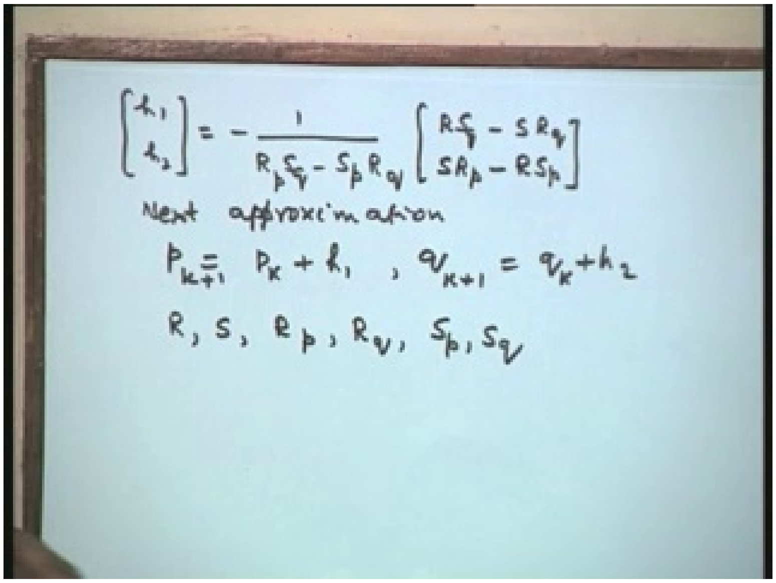

Now let us write down one more step from here. So let us simplify it and write the one more step; h_1 h_2 will be equal to as follows; let us multiply it out, I will keep the denominator as it is. Now I am multiplying this, therefore R into sq minus s into Rq; that is a product of this first row and this vector R and s. Then the second one; I will write this first one because of the sign that I have here; positive sign minus R into sp. As we have discussed earlier all are evaluated at the current iterated value that is pk and qk. Once I determine this values of h_1 h_2 two, I have the next approximation. So the next approximation is equal to p_1 is equal to or pk plus one is equal to pk plus h_1 qk plus one is equal to qk plus h_2.

So we are talking of the current iterate as a pk qk. Therefore the next iteration will be pk plus one and qk plus one and I have this particular result. This is the required result for our next iterate. Now I need to compute this. There are six quantities that are required R, s and the four partial derivatives. So I need computation of R, I need of computation s, I need the two derivatives of R and I need the two derivatives of s. So I need to evaluate six quantities at pk, qk and these six values, I would like to obtain using the synthetic division.

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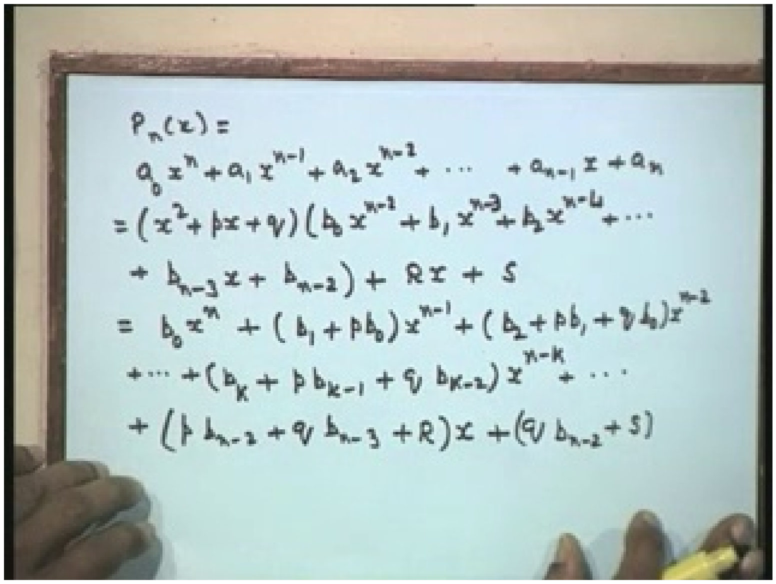

The synthetic division procedure that we have done for Birge Vieta can be extended in the direct manner as we have done there for extracting the quadratic factor also. I would start from here; we will write down Pnx and let us start from here; that is a_0x to the power of n, a_1x to the power n minus one. Let us write down one more term a_2x to the power of n minus two, a_n minus one of x plus a_n and this is your quadratic factor x square plus px plus q and we are multiplying it by qn minus two. Therefore b_0x to the power of n minus two, b_1x to the power n minus one; let us write down one more term bx to the power n minus four plus so on, plus bn minus three of x, bn minus two plus Rx plus s; so this is our remainder. Now let us multiply the right hand side and collect the coefficients. So I will have here b_0x to the power n, the product of x square into this is b_0x to the power of n plus; let us take x to the power of n minus one, x square into this product will give you x to the power of n minus one. So I will have b_1 and then the product of these two will give you x to the power of n minus one. So I have here b_1 plus p into b_0 that is coefficient of x to the power of n minus one. Now the next coefficient is x to the power of n minus two. So x square will multiply with this x to the power of n minus two. So I will have b_2 here, then this px will multiply with this to give me x to the power of n minus two that is pb_1; then q will multiply with this x to the power of n minus two to give me qb_0x to the power of n minus two. Now let us take the general term; the general term is x to the power of n minus k. So x square will be multiplying bk x to the power of n minus k. So I will have bk over here. Then p will multiply with the previous term; the previous term is bk minus one, so p into bk into minus one and q will multiply with the previous term that is q bk minus two into x to the power of n minus k. Then let us come to the term x. Now the when you come to the term x I will have here x square which will not contribute, because I need only x. Therefore px into bn minus two, it will contribute; that is pbn minus two, these two will contribute and then this will also contribute that is q into bn minus three; but we already have R here, so we shall now add R here plus into x.

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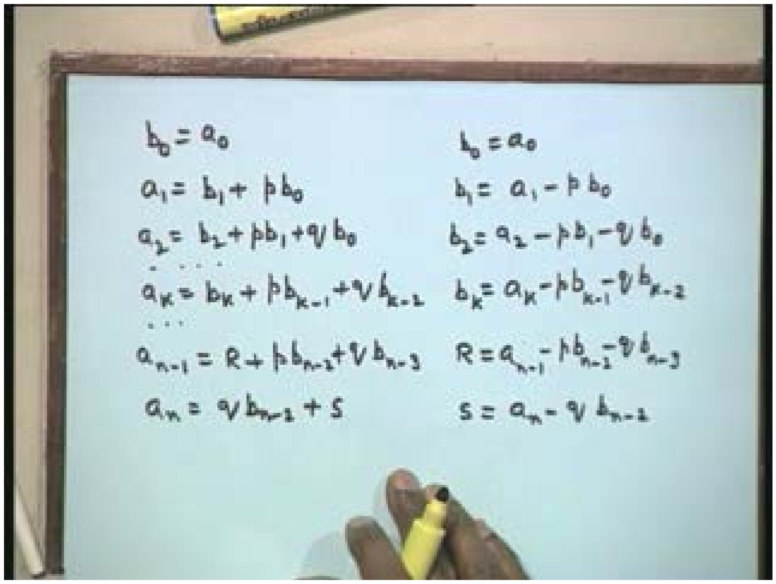

Now the constant term; there is only one constant term here, q into b minus two and s over here. So I will have here qbn minus two plus s. Now let us compare the coefficients as we have done in the previous case. We will have here b_0 is equal to a_0. I will have here b_0 is equal to a_0 zero. I will write on this side again, b_0 is equal to a_0. Then I compare coefficient of x to the power of n minus one. This is a_1 and this is b_1 plus pb_1. So I will have here a_1 is equal to b_1 plus p times b_0. Now I solve for b_1; b_1 is equal to a_1 minus pb_0. Now the contribution of all the terms has not yet come. Now if you compare x to the power n minus two, the contributions of all the three coefficients here would come in to picture. So let us try the next coefficient which is b_2 plus pb_1 plus qb_0 that is b_2 plus pb_1 plus qb_0 is equal to a_2. So i solve for b_2; b_2 is a_2 minus pb_1 minus qb_0. So I have taken these two terms to the right hand side. Now let us go to the general term. In the general term the left hand side is ak, the right hand side is bk plus p times bk minus one (I am writing this term) plus qbk minus two, so this will be my ak. Therefore the solution of this will be bk is equal to ak minus pbk minus one minus qbk minus two. So we have solved for bk.

Now let us compare the coefficient of x. The coefficient of x on the left hand side is a_n minus one. So the coefficient of x on the left hand side is a_n minus one. Therefore from this I would get here a_n minus one. We will first write R. Let us write R plus p times b_n minus two q_n minus three. So I am writing from this term that is R plus pb_n minus two plus qb_n minus three. So let us now therefore solve for R that is a_n minus one minus pb_n minus two minus qb_n minus three and finally the constant coefficient is a_n on the left hand side and we have here this term, therefore we will have a_n is equal to qb_n minus two plus s. Therefore I solve it for s from here; s will be equal to a_n minus qb_n minus two. Therefore it is possible for me to generate the entire set of the equations from this particular generalized formula, wherein we define correctly what the remaining values are.

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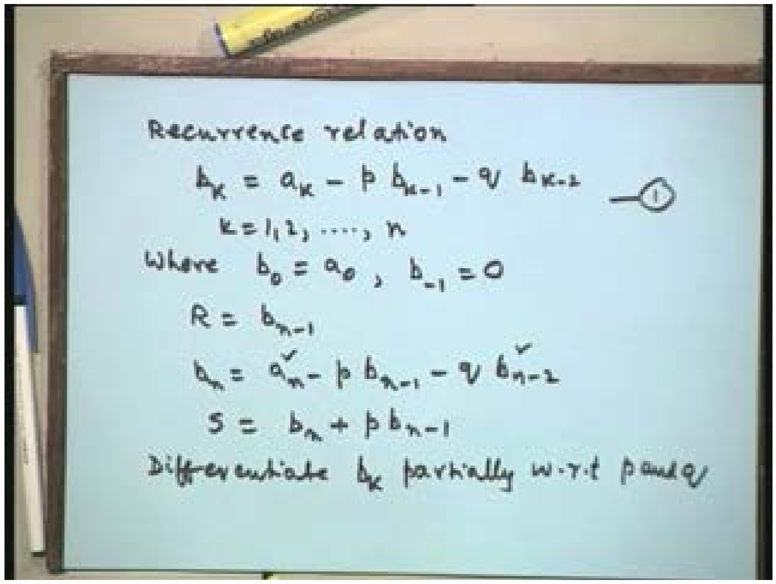

The recurrence relation for the solution will be as follows. It will be simply bk is equal to ak minus pb_k minus one minus qb minus two. Now would like to use the k from 1, 2, and 3… and n. So we will use this. Therefore we have some quantities which are to be defined here. So we know that b_0 is a_0. So we will set, where b_0 is a_0 zero that is from the coefficient, in order that we can write everything in terms of the recurrence. We will set b minus one; when you put k is equal to one here, I have got b minus one, so b minus one is zero. So that is how we define b_0 is a_0; b minus one is equal to this. Let us just look at this particular coefficient, which we have written for R. The right hand side, if I set k is equal to n minus one here, a_n minus one minus pb n minus two minus q n minus three, therefore out of this recurrence relation whatever I get for b_n minus one will be simply equal to R. So therefore from this we will get R is equal to b_n minus one, R is equal to b_n minus one by just comparing this particular term that we have with this one.

Now let us set b_n; b_n will be equal to a_n minus pb_n minus one minus qb_n minus two. I am setting k is equal to n so that I have b_n, a_n here, b_n minus one b_n minus two. Now if you just look back into this particular coefficient we have written, s is equal to a_n minus qb_n minus two. So we have this a_n. We both these terms, therefore I can take this to the left hand side and define this as my s. So I can define s as b_n plus pb_n minus one. So I will take it to the left hand side and define this quantity s as a_n minus qb_n minus two. We are going to write it as a synthetic division procedure. When the first step is complete I am getting all my bks using the values of previous aks. Now after the first step is complete I am able to determine two quantities out of the six we want; R is determined immediately as b_n minus one; s is determined as b_n plus pb_n minus one. Now I need the partial derivatives. Now just as we have done in Birge Vieta we will take this equation and differentiate it partially with respect to p and q and recognize them as the quantities which we can pick up from the second next step of the synthetic division procedure. So let us differentiate partially. So let us say differentiate bk partially with respect to p and q.

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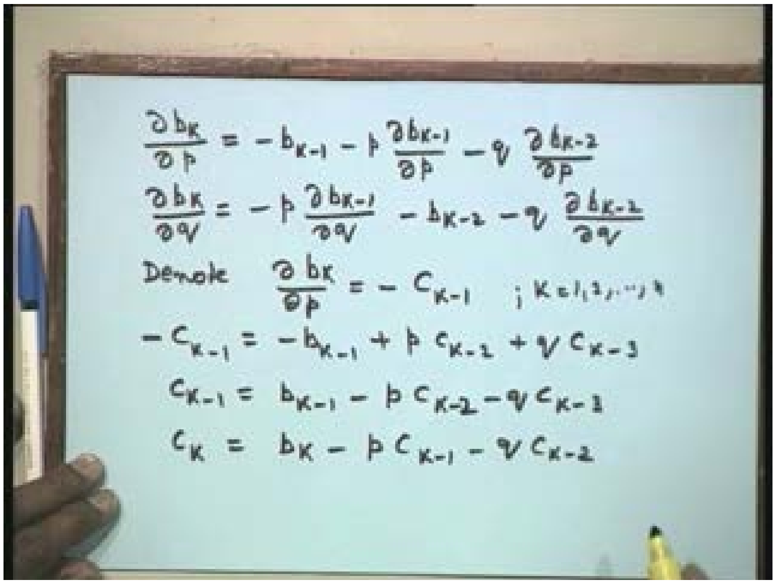

So I will have here derivative with respect to p. Now ak is independent of p, so there will be no contribution from here. There will be contribution from this product. It is a product of two terms. So I will have minus bk minus one minus p, partial derivative of bk minus one with respect to p, that is a product of these two; terms q is independent of p, so I will have only one term from here minus q partial derivative of bk minus two upon delta p. Similarly differentiate with respect to q. So partial derivative with respect to q will be now the contribution of ak is zero the p is independent of q. So I will have only one term from here that is minus p partial derivative with respect to k minus one of q, which is the contribution from here. This is the product of two terms. So I will have two terms coming from here, minus bk minus two and the contribution is minus q partial derivative of bk minus two with q. These are the two partial derivatives.

Now we want to show that I can write this exact form as we have done for bks. So let us therefore write some notation. So we will denote partial derivative of this with respect to p as minus ck minus one that is k running from 1, 2, 3… n, it is just a notation. Now let us insert it. If I insert it here, we can see that there is a minus sign appearing. Both sides there is a minus sign. We will take one more step; minus ck minus one is minus bk minus one. This is plus p, this is bk minus one. So put k is k minus one here. I will get here ck minus two plus q. This is bk minus two. So I can set k is k minus two. So I will have k minus three or remove the minus sign. I can write k minus one is bk minus one minus pck minus two minus qck minus three.

Now we shall suitably define these quantities that are occurring. We will put the value of k. What I do in order to show that this is same as the previous recurrence relation? Let us write k is equal to k plus one; replace k by k plus one. Then this will read ck is bk minus pck minus one minus qck minus two. Now if we just go back and see what we had written for the coefficients bks, we can see that both of them are identically the same. See bk is equal to ak minus pbk minus one qbk minus two. Now I am replacing a by b and b by c. I get exactly the same thing. In other words it would be possible for me to obtain the derivative with respect to p by using the synthetic division on bk and get our ck's from there. Now we will recognize what are the quantities but let us do for the second one also now.

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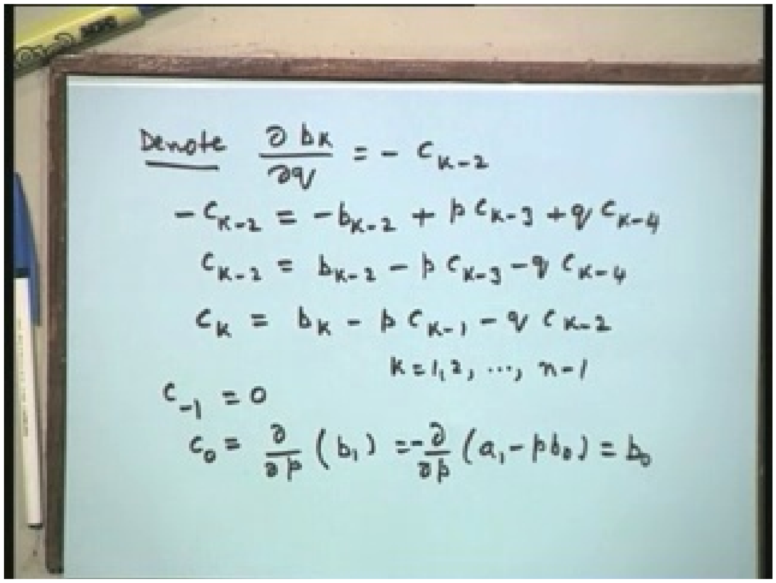

Similarly to obtain such a relation from this, I will now again denote the derivative with respect to q; partial derivative with respect to bk by q is equal to minus ck minus two. Remember we are not equating anything, it is just a notation. There is no relationship; I mean we cannot combine these two. You say you are using the both; say k is equal to k minus one here; are they equal? No it is not so; because we are just using this as a notation, so that I can bring this particular formula into suitable form. So therefore it is only a notation. Now if I write this as minus ck minus two here, I would get here minus ck minus two on the left hand side; on the right hand side, let us start with minus bk minus two, middle term. So let me write bk minus two, this middle term, then I will write this as plus p of ck minus three. Now this is bk minus one partial with respect to q. So I put k is equal to k minus one and similarly I will have qck minus four. This is k minus two, therefore put k minus two here, I will get ck minus four. Now let us remove the minus sign again on both sides, so I can write ck minus two is bk minus two minus pck minus three minus qck minus four. Again let us replace k by k plus two. So then it will read ck is equal to bk minus pck minus one minus qck minus two. This equation is identically the same as the equation that we had written in the previous step.

Therefore it is possible for me to get from the same synthetic division level to get the values of the partial with respect to q also, except that we recognize correctly what is this coefficient that are obtained from there. If this is the common relation, now I will take k is equal to 1, 2, 3...n minus one, that one less than the given the polynomial, where I will define c of minus one which is undefined quantities starting them as zero; because you know as k is equal to one I will have here c_1, b_1, c_0 and c minus one; c minus one is zero.

Now let us define what c_0 is. Now c_0; if you put here k is equal to one I will here partial with respect to qb_0. So I will have here partial derivative with respect to p of b_0. Let us just go two slides behind and let us just write down what is our b_0; b_0 is equal to a_0. This is equal to b_1. You are putting here c_0 that is equal to partial derivatives of b_1 upon p. Now let us write down what is the value of b_1; b_1 is a_1 minus pb_0. So I would now substitute this as a_1 minus p times b_0 and this will be simply equal to the a_1, it is independent of p. This is zero; b_0 is equal to a_0. Therefore its derivative is also zero. Therefore what is left out is the derivative of this product only and that is simply equal to b_0. I should put a minus sign here. Therefore c_0 zero is also equal to b_0.

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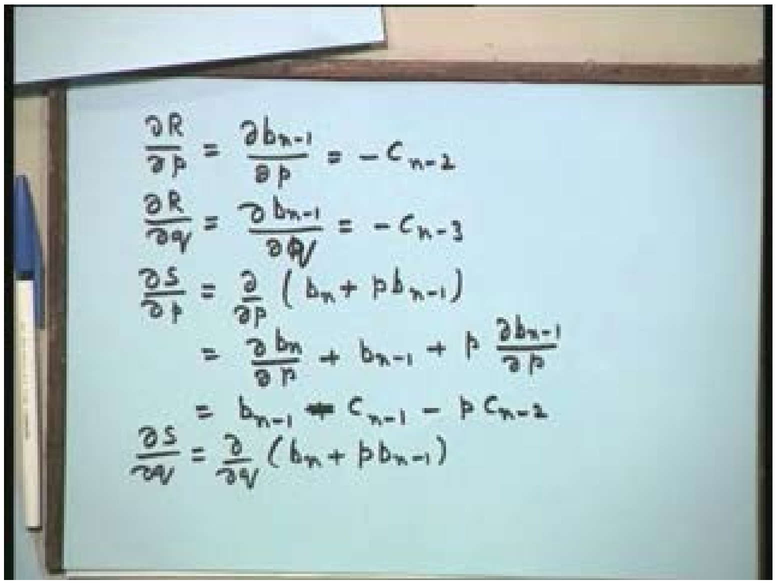

I need the derivative of R with respect to p. So I have to differentiate this partially with respect to p. So let us just retain it here. So I have to differentiate this with respect to p partially so i will have here this. Let us look at this definition; k is n minus one, k is n minus one means minus c_n minus two. So the value of this is simply minus c_n minus two. Now let us also write down what is our derivative with respect to q. Now derivative with respect to q will be this. Now let us go to the definition of this; put k is n minus one, I will get here cn minus three. So this value is minus cn minus three. Then we need derivative of s with the respect to p. Now look at s here; s is b_n plus pb_n minus one. So we shall differentiate this quantity with respect to p. Now let us differentiate it. This is b_n by delta p, plus this is a product, so I will have b_n minus one plus pb_n minus one by p.

Now let us again go to the definition of the derivative. So let us first of all write this b_n minus one minus cn minus one I (because I want the derivative delta by delta p of b_n; so if you look at this, I have to set k is equal to n. So this is minus cn minus one) and this is minus pc_n minus two. We have got this derivative here; delta b_n minus delta p is minus c_n minus two, which is minus c_n minus two. Now I want the last one that is derivative with respect to q of bn plus pbn minus one. Yes it is a notation that we have used, we have denoted partial with respect to p as minus ck minus one and derived that all these terms can be obtained from the single recurrence relation ck and ck is equal to bk minus that particular term. Then we denote derivative with respect to q as minus ck minus two. Then we found that the recurrence relation is identically the same thing that means I will be able to get the derivative with respect to q also, provided I will use the correct definition of what I have started with.

The definition I started with is derivative of R with respect to q is minus ck is minus two. Here derivate with respect to p is the notation we have used. So from the same line of our synthetic division c_0, c_1, c_2, c_3, c_n minus two by correctly recognizing them as partial derivatives, I will be able to partial derivate with respect to q as well as partial derivatives with respect to p also. So it is only the correct notation that we have to imply. That is why when we are writing the derivative with respect to q here, for example we are now looking into the definition of derivative of q that we have used. Therefore I am picking the correct coefficient that should be used in order to find the derivative with respect to q.
Therefore this is how we are able to manage that all the partial derivatives will be obtained from the single line of synthetic division procedure that we obtained over there.

Now I differentiate this and then substitute it, so this will be simply equal to minus of delta bn by delta q that is cn minus two. I have got here in the previous step, I can use the same thing and this is plus c_n minus three. Therefore we are now in a position to get all the six coefficients from the single set of synthetic division procedure. There are these are four quantities that we have obtained and in the previous step we have obtained two quantities R and s here. R and s were coming from the first level of the synthetic division and these four quantities are coming from the second step of the synthetic division procedure. Now let us illustrate how the synthetic division procedure comes from here.

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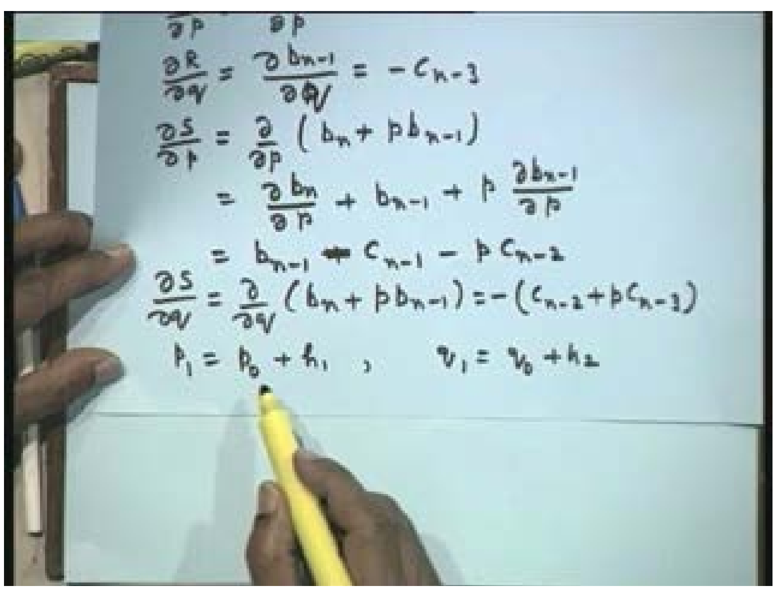

Now let us write down as we have done Birge Vieta. Let us write down but before that maybe we should see what the next approximation is. Now we are computing h_1, so p_1 will be equal to p_0 plus h_1 and q_1 will be equal to q_0 plus h_2. So we are substituting these six values in the earlier ones. We have derived the values for h_1 and h_2 two, we will substitute in that and get the values at this.

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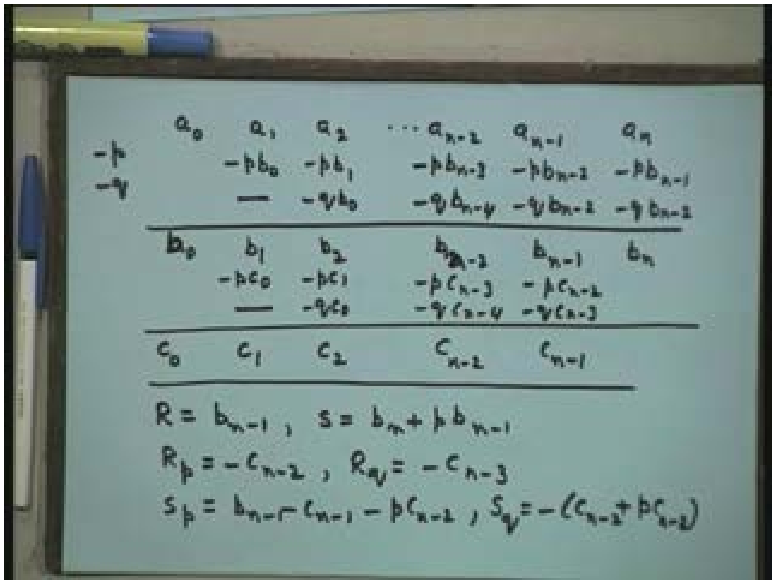

Now let us write down the coefficients a_0, a_1, a_2, a_n minus two, a_n minus one and a_n.
Unlike in the Birge Vieta, here you have got minus p, minus q, the factor that we are multiplying with minus p minus q. So I will be performing the synthetic division with respect to minus p, minus q; there we have taken it as p because all are positive multiple factors are coming, but here we have got negative sign. So we will insert minus sign right over here; b_0 is a_0 that has been found. The first value that is available, we are multiplying it by p. So that comes from b_1 is equal to b_0. See b_1 is equal to a_1 minus pb_1; so whatever value is available here is multiplied by this, so a_0 is equal to $b_{0,}$ so let me replace it by b_0. So I multiply this and this, so I will put here minus pb_0 and there is no previous value available for me to multiply by q, so there will be nothing here. I add it, I will get here b_1. I will get b_1 now; b_1 is multiplied by the current value that is minus pb_1. The previous value available will be multiplied by q that is minus qb_0; add it, I will get here b_2. When I reach this position b_n minus three is available to me, so I will be multiplying b_n minus three with minus p. The previous value is b_n minus four that will be multiplied by this. So I will have minus qb_n minus four, add it, I will simply get b_n minus two. So I multiply this by p minus pb_n minus two minus qb_n minus three, I will get here b_n minus one and finally minus pb_n minus one minus qb_n minus two and I will get here $b_{n.}$ Now the first level of synthetic division is complete, so I repeat now one more level of synthetic division.

This is c_0; c_0 is equal to b_0. We have proved c_0 is b_0; b_0 is a_0; so we will have simply c_0. So the procedure is immediate, so I will have here c_1, this is minus c_1, minus qc_0 that is equal c_2 and when we have come to b_n minus two I have c_n minus three available to me. So I will have minus pc_n minus three minus qc_n minus four that is c_n minus two; then I will multiply pc_n minus two minus qc_n minus three and that gives me c_n minus one. Now the synthetic division is complete. Now these values shall be used in the six quantities that we have obtained for R, s derivates with the respect to this. For example we had written R is equal to b_n minus one. So this value is simply equal to R and we have also shown s is equal to b_n plus pb_n minus one. So I add to this bn, p times b_n minus one; that will be my value of s. There are other values that we have just now done that are partial derivative with respect to p is minus c_n minus two. So this particular coefficient c_n minus two itself is a derivative with respect to p. Similarly we are shown the derivate with respect to q is simply minus c_n minus three. So this previous coefficient is simply derivative with respect to q.

Similarly we can obtain the remaining two coefficients with derivate respect to p that is equal to b_n minus one minus c_n minus one minus pc_n minus two and finally derivative with respect to q is minus c_n minus two plus pc_n minus three. Therefore again to conclude one synthetic division using the factors minus p, minus q would give us all the six quantities that are required over here and then we can obtain the values of h_1 and h_2, obtain the next iterated value p_1, q_1 and then repeat the synthetic division with p_1, q_1 and then produce the next result.