Numerical Method and Computation

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Lecture No # 08

Solution of Nonlinear Equations (Continued)

Polynomial Equations

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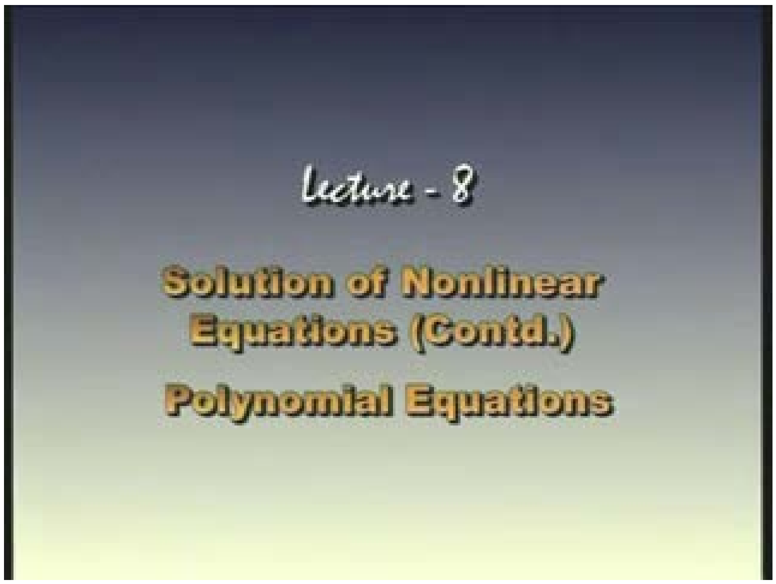

Now in our previous lecture we have discussed the properties of polynomial equations. We have defined the Sturm's sequences and the Sturm's theorem which gives us the exact number of real roots that lie in a given interval ab. We study the number of changes in the signs of the Sturm's sequence. We take the difference between the number of changes in signs at two points a and b, then we decide upon the number of real roots that lie within that particular interval. So what we are considering here is a polynomial equation Pnx.

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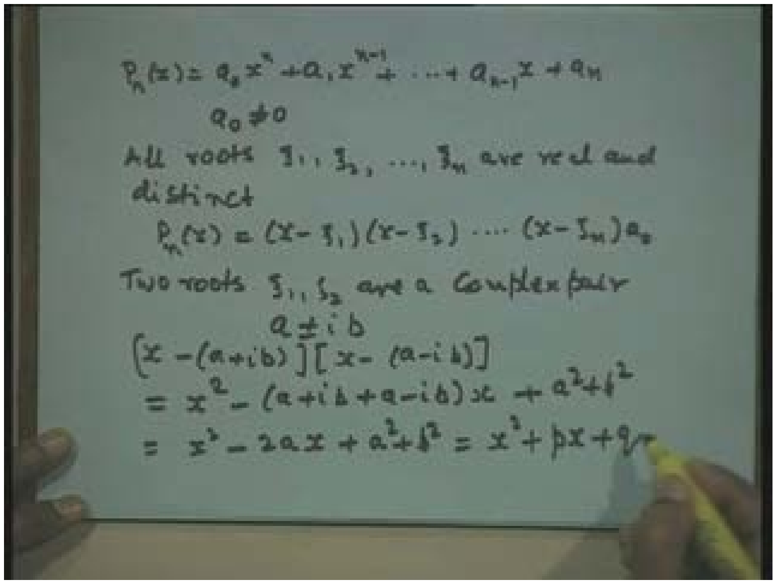

Pnx is of the form a_0x to the power of n plus a_1x to the power of n minus one plus so on a_n minus one into x plus a_n with a_0 not equal to zero. Now if all the roots are real and distinct, so let us take the case of all roots xi_1, xi_2 xi_n are real and distinct. Then I can always write this polynomial Pnx as a product of the linear factors x minus xi_1 into x minus xi_2 two so on x minus xi_n. If the roots of some of them or all of them have multiplicity, we can extend this to the case of the multiple roots also. Therefore a numerical method that we would wish to construct, would try to extract a linear factor out of this set. Then we want the next root to be determined. We shall deflate this polynomial that is divide it by a particular root that has been extracted and we then go to find the next root by using the method on the deflated polynomial.

Now if the roots are a complex pair, let us say two roots are complex pair; let us take the case of two roots say xi_1, xi_2 are a complex pair. So let us take these roots as a plus minus ib. Then let us see what happens to this factor. So I can write this as x minus of a plus ib into x minus of a minus ib; multiply it out, I would then get x square minus of a plus ib plus a minus ib into x and this is plus a square plus b square, that is x square minus two a into x plus a square plus b square or I will write x square plus px plus q. Therefore if I want to look at a complex pair which is of the form a plus minus ib, then the product corresponding to that pair is given by x square plus px plus q where p and q are real. So in this product these two are real, a square by b square is real.

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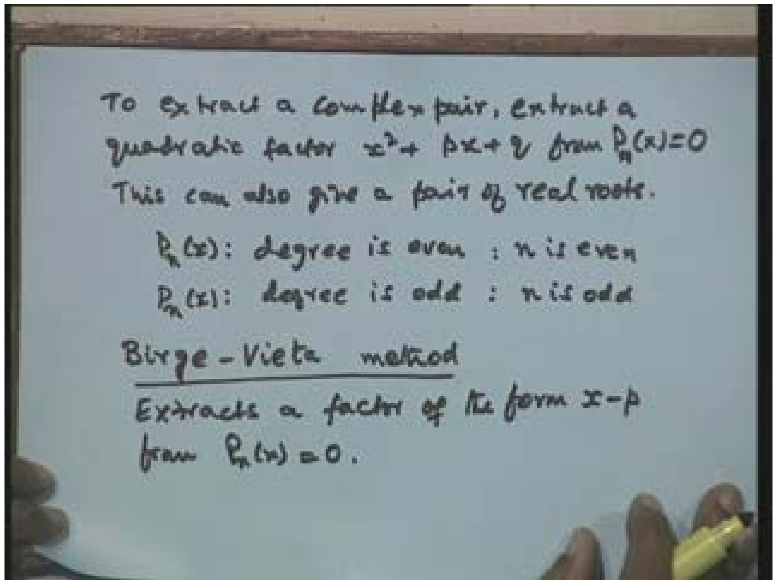

Therefore if I want to extract a complex pair from a given polynomial, I would extract a real factor x square plus px plus q. Therefore this implies that it would be sufficient for me to extract a quadratic equation of the form. Therefore we shall say, to extract a complex pair, extract a quadratic factor x square plus px plus q from Pnx is equal to zero. If I want to extract this complex pair, I shall extract a quadratic factor of the form x square plus px plus q from Pnx is equal to zero. Now it is possible that I can extract a quadratic factor and it can give us a real pair also. It is not necessary because when I try to extract x square px plus q, I do not know initially whether I am extracting a complex pair or a real pair. So it is possible therefore to get a pair of real roots also from this. Therefore this can also give a pair of real roots. We take the polynomial Pnx, the degree of this is even. So let us take that the degree is even, that is your n is even; then if I extract the quadratic factor x square plus px plus q, deflate it, I again extract x square plus px plus q; therefore at the end we'll be left out with a quadratic factor because n is even. So each time I extract the quadratic factor, so what will be left out in the end will be a quadratic factor. So I will be able to reduce it finally into a quadratic factor and find the two roots of this. Now when I extract the quadratic factor I will use the usual rule of finding the roots of this quadratic. The rules of this, as minus p plus minus p square minus four ac by two and then find out those roots. Whereas if Pnx degree is odd, if n is odd, then when I extract this quadratic factor x square plus px plus q, then I deflate it and continue the process, I will be left out in the end with a linear polynomial. Therefore I will be left out with a factor of the form some x minus a which will be left out in the end because it is an odd degree polynomial. We will have a discussion on this little later as to whether we should have an even polynomial consider only or odd polynomials can be considered.

Now the first method that we shall discuss is known as the Birge Vieta method which extracts a linear factor from a given polynomial. So we shall call this method as Birge Vieta method. What this method does is, it extracts a linear factor, extracts a factor of the form x minus p from Pnx is equal to zero. So in this method we will try to extract a factor of the form x minus p form Pnx is equal to zero and the procedure or the method would automatically give us the deflated polynomial also. Now the Birge Vieta method is nothing but the implementation of Newton Raphson method to polynomial equations using synthetic division.

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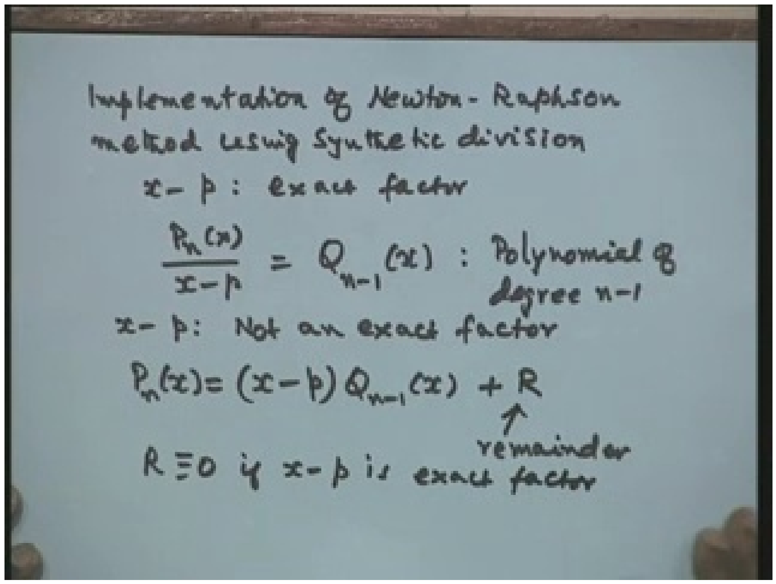

Now we'll define how we are going to do the synthetic division but the Birge Vieta method is an implementation of the Newton Raphson method. It is implementation of Newton Raphson method which we have done earlier for fx is equal to zero. So it is the implementation of Newton Raphson method using synthetic division. Let us suppose x minus p is an exact factor. Then Pnx is divisible by x minus p exactly. So Qn minus one will be simply a polynomial of degree n minus one x. So this will be a polynomial of degree n minus one. If x minus P is not an exact factor and if I divide by x minus P, what I would get here is a polynomial degree n minus one plus a constant which will be a remainder. So let us now take the case x minus p is not an exact factor. Then Pnx can be written as x minus p into Q n minus one x plus R, where R is a remainder. Obviously R is identically zero, if x minus one becomes an exact factor. So R will be identically zero if x minus p is an exact factor. Now it is this particular equation that I would like to use in order to implement the Newton Raphson method by synthetic division. Now let us open it up and write the entire thing here. First of all we will see that as the factor x minus p which is not exact or the value p changes. In Newton Raphson method we start with initial approximation. So if an initial approximation p is taken and every time I change, the remainder is going to change. So as pk approximations tend to exact solution xi, R is going to go to zero. Initially remainder will be large.

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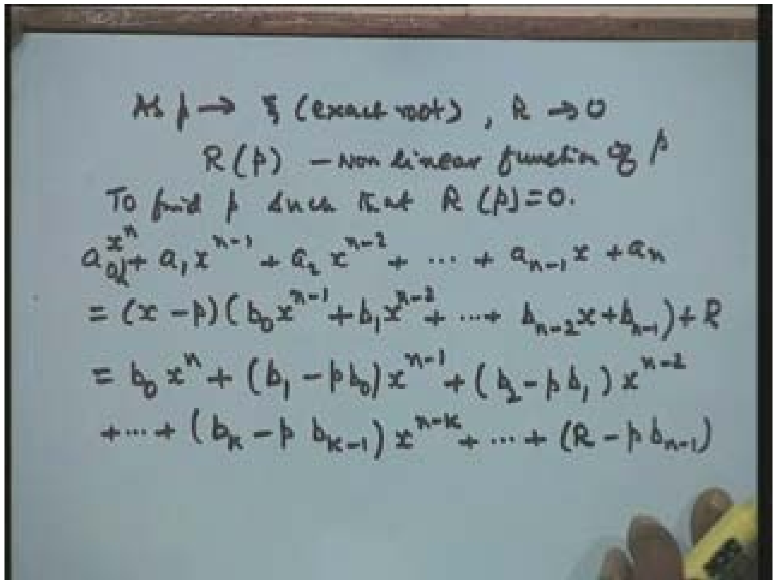

Now as p tends to xi which is an exact root, R tends to zero; R is the remainder, so when once this p becomes exact root, R will become zero or R tends to zero. In other words what we are stating here is that R is a function of p, as p changes R changes. Therefore remainder also changes. Therefore R is a function of p and therefore this is a nonlinear function of p. The problem therefore is to find a p such that R of p is zero that is the remainder should be zero. So the problem is to find p, such that R of p is equal to zero. Now before we write down the value of this that is Newton Raphson method, let us go one step behind and let us just look at the particular equation that we have written here; this Pnx is equal to x minus Pqn minus one x plus R. Let us just write this particular equation again. So I would write this equation as a_0 x to the power of n a_1x to the power of n minus one plus a_2x to the power of n minus two an minus one into x plus an is x minus p, b_0 x to the power of n minus one; this is a polynomial qn minus is n polynomial of degree n minus one. Therefore b_0x to the power of n minus one plus b_1x to the power of n minus two plus so on, bn minus two x plus bn minus one plus R. Now I would like to use the Newton Raphson method to find the value of this p. Now before we do that let us show a procedure which we call as a synthetic division procedure wherein I can determine the coefficients b_0, b_1, bn minus two bn minus one and R from a_0, a_1, a_2 in a simple procedure. To do that let us multiply the right hand side.

The right hand side multiplication would give us b_0x to the power of n. Collect the coefficients, b_1 that is a coefficient of x to the power of n minus one which is the product of these two minus pb_0x to the power of n minus one, then I take the next coefficient b_2 minus p into b_1 that is p into b_1x to the power of n minus two. Then I take the general coefficient x to the power of n minus k, so that will give me bk that is x into bkx to the power of n minus k minus p bk minus one that is minus p into the previous coefficient. This is x to the power of n minus k plus so on. The constant term will be R from here and minus pbn minus one, so the constant will be R minus p into pbn minus one. Now what I do is I compare both sides of this particular equation, then I have here b_0 equal to a_0, b_1 one minus pb naught is a_1 one.

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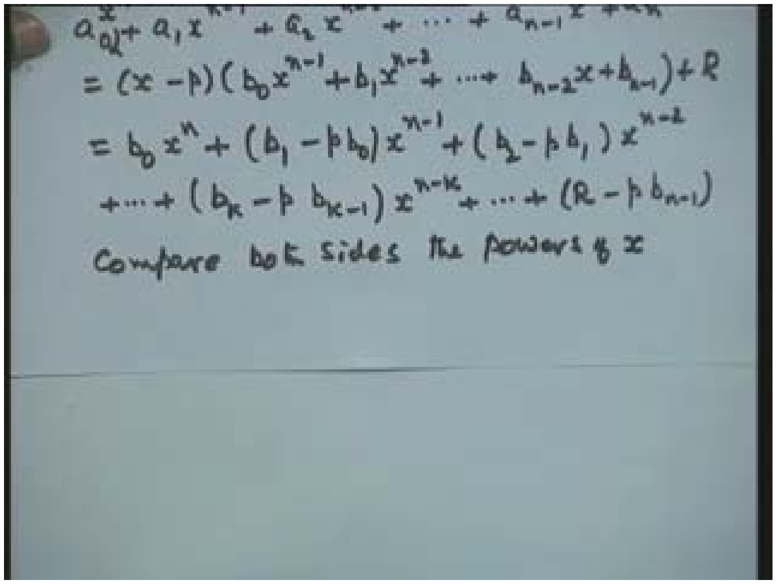

So let us compare both sides. Now we compare the powers of x on both sides.

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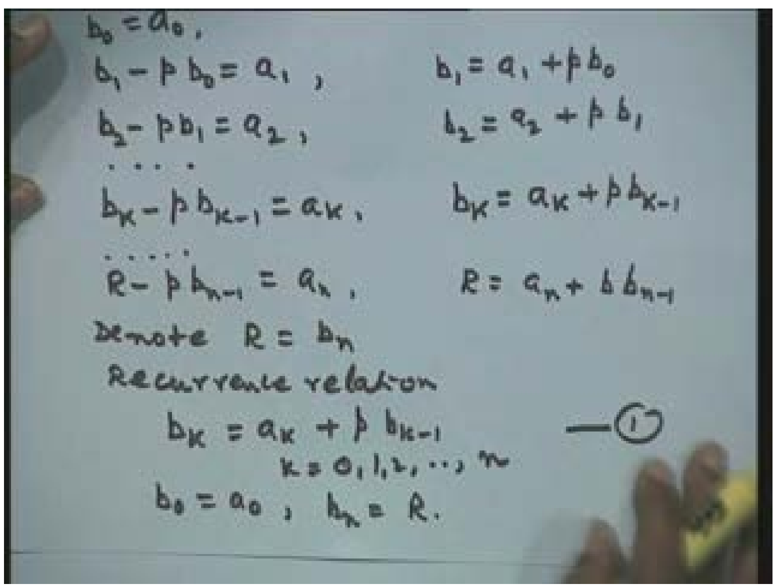

Therefore what I would get here is I would get b_0 is equal to a_0. So b_0 is equal to a_0. Then I get b_1 minus pb_0 is equal to a_1 that is b_1 minus pb_0 coefficient of x to the power of n minus one is equal to a_1. Therefore I get b_1 is equal to a_1 plus pb_0. So I solve from here for b_1 that is a_1 plus pb_0, then I go to the next coefficient that is our b_2 minus pb_1. So I will get b_2 minus pb_1 is equal to a_2, so that is my a_2. I will solve for b_2 again; b_2 is equal to a_2 plus p into b_1. So I take it to this side. Now I compare the general term bk minus pbk minus one and that is equal to ak. Therefore

I can solve for bk as ak plus p times bk minus one. Then I go to the last term, the constant term R minus pb_n minus one is equal to the constant coefficient that is a_n.

Therefore from here I get R is equal to a_n plus pb_n minus one. As you can see that the entire set of equations can be given by single recurrence relation out of this set and that recurrence relation is this recurrence relation and where I define b and sr. So let us just denote R as $b_{n,}$ and then I can get all these coefficient $b_{1,}$ b_2 from recurrence relation. I can now define a recurrence relation bk is equal to ak plus pbk minus one and k going from 0, 1, 2, 3, n of which the first and last coefficients are to be defined. We have b_0 is equal a_0; b_0 is equal to a_0 is defined from this top and b_n is equal to R is equal to b_n; so b_n is equal to R.

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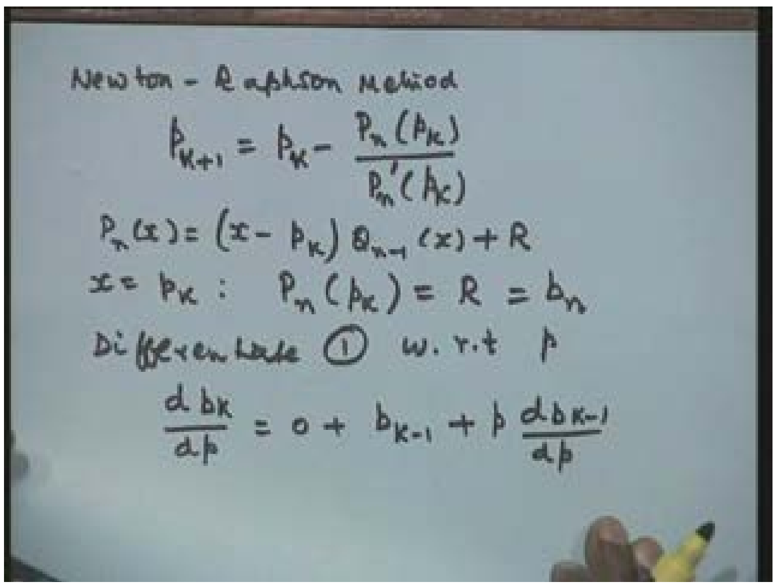

Now every coefficient bi can be obtained from ai by this simple recurrence relation. Now we will just put it in a f format which I shall call it as synthetic division. Let us write down what is our Newton Raphson method. Let's write down the Newton Raphson method for finding this particular root, P_n plus one is equal to P_n minus P_n of pk by P_n prime of pk. So this is our Newton Raphson method. Now in this I need the values of the polynomial at pk; the value of the derivative of the polynomial at pk. Now what I would do from here, I would get the value of the derivatives also using the same recurrence relation but in a different format. Now first of all if you just go back to one sheet earlier and look at this particular starting point of Pnx is equal to x minus pq_n minus one of x here, at any particular stage if I put x is equal to that particular approximation pk here, this term goes off and I will get R; therefore P_n at pk is equal to R. What we have here is P_n of x is x minus pk; at any particular stage this is the equation n minus one x plus R. So as the initial approximation changes the first equation looks like this. Now when I substitute x is equal to pk here, what I would get here is P_n of pk is equal to R. This term cancels and I will get R. So what we are stating here is that the numerator in the Newton Raphson method P_n of xk would be nothing but R that is equal to b_n. So here we have denoted R is equal to b_n and therefore P_n of pk is equal to R.

Now we are stating that R is equal to b_n. Therefore in the recurrence relation that I have got, the last term of this that gives me b_n from this set, I that set the last value itself will give me the value of the numerator P_n at pk. Now I want the denominator, that P_n prime of pk. Now to get this it is a very simple and interesting procedure. I would take one and differentiate this with respect to p. So what I would do is differentiate the equation one with respect to p, then it will simply be dbk upon dp is (ak is independent of p) zero plus product of these two terms. So let us differentiate it; bk minus one plus p dbk minus one upon dp, so I have differentiated this product to get this particular quantity here. Now we will use a notation to simplify this.

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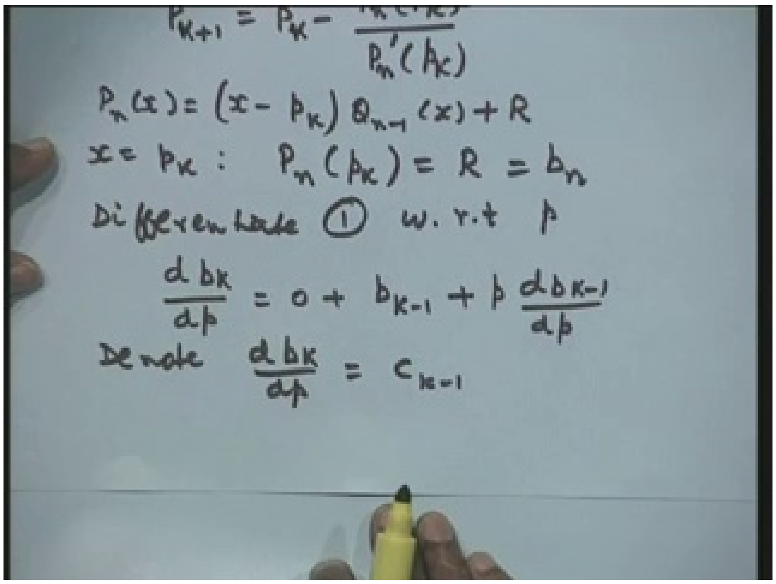

What I would do, I would say denote dbk upon dp; this quantity as c k minus one. It is just a notation, there is no special meaning about it; we are just denoting dbk is equal to ck minus one. If I do this, then this equation will look like this is; ck minus one, put k is equal to k minus one; this will be ck minus two.

Therefore this equation can be written as ck minus one is equal to bk minus one plus p times ck minus two. So this equation can be rewritten in the form of ck minus one is equal to bk minus one plus ck minus two. Now if I put k is equal to k plus one, I could rewrite this as simply ck is equal to bk plus pck minus one. Here we have to define the first coefficient. Let us define what c0 is. If you look at c_0, when you take k is equal to one here that is your c_0, that will be db_1 by dp. So this is by definition db_1 upon dp. Let us go backward one step and see what is our b_1; b_1 one was a_1 plus pb_0. So we will substitute further that is equal to d upon dp of a_1 plus pb_0 and therefore this is equal to this; this is zero plus b_0 plus zero. Therefore this is equal to b_0; b_0 is equal to a_0. So if you look at that, just b_0 is equal to a_0 zero. Therefore derivative with respect to p will be zero. Therefore out of this product only b_0 will survive and everything else will go. Therefore c_0 is defined as b_0 and the previous step b_0 is also defined as a_0.

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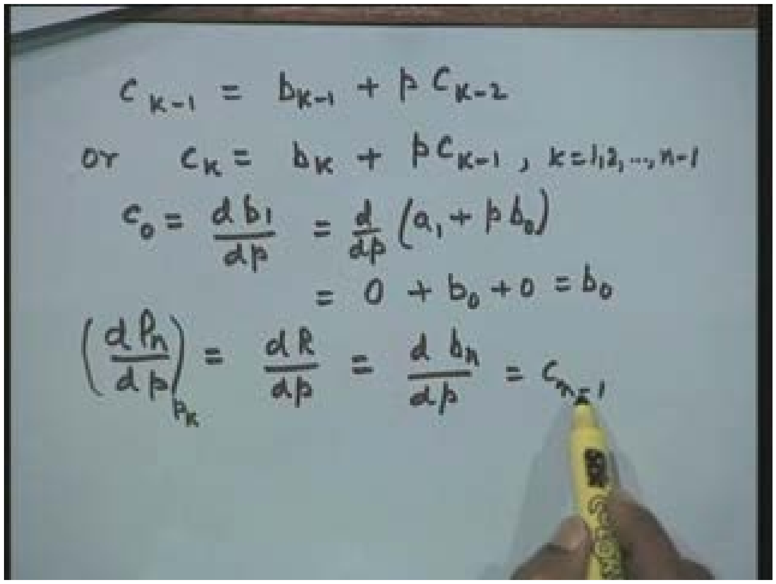

Now with this recurrence relation, I can set k is equal to 1, 2 upto n minus one. Now what we were asking for here is, I need the derivative P_n prime at pk. So let us write down what is dPn upon dp, so that is what I need in the denominator; Pn prime of pk that is dPn upon dp. But we have shown earlier that P_n of pk is equal to R. Therefore this is equal to dR upon dp. We are showing from this P_n of p, this is equal to R. So this is of course at pk. We are talking at an approximation pk; dR upon dp means (we have shown R is equal to bn) dbn upon dp. Let's go to the definition of c. So if I set here, k is equal to n then this is cn minus one. Therefore this is equal to cn minus one. Now what we are therefore stating is in this recurrence relation we are going only up to n minus one; the last coefficient ck, when k is n minus one, cn minus one will give me the derivative of Pn. Therefore the numerator is given by the last coefficient in the previous set recurrence relation and the last coefficient in this recurrence relation gives me the derivative. This is what is required for the new application Newton Raphson method but if you look at the two recurrence relations. If you just look at the two recurrence relations, this is your bk is equal to ak plus pbk minus one, ck is equal to bk plus pck minus one. Both are identically the same except b is replaced by c, a is replaced by b. In other words the procedure that I use for getting bk from ak is identically same as how we derive c from b; that means as I proceed on with the synthetic division if I obtain bk from aks, I can then obtain from ck by identical procedure from bk.

Now let us see what I mean by synthetic procedure, how we can get computationally a very simple procedure to obtain all these coefficients and the application of the Newton Raphson method, therefore it becomes extremely simple for polynomial equations. Now let us write what is known as the synthetic division procedure. So what I would do here is I would write down all the coefficients of the polynomial in the order.

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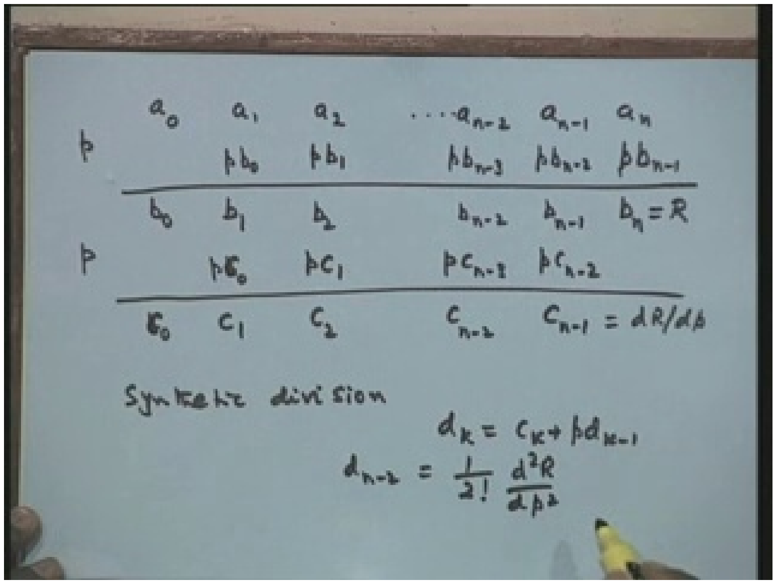

I will write down all the coefficients, a_0, a_1, a_2 … and a_n minus two, a_n minus one, a_n. Then I will take the fact that the root I am extracting p, p is an approximate root. Now on this line I will get my values of bs. All my bs will come here. So let us just look at this and go about this one. Let me put this sequence here; b_0 is a_0. So there is no change in b_0. Now multiply p with b_0 and put it here; b_1 is equal to a_1 plus pb_0. So multiply pb_0 here, so I will insert this p into b_0, the current b that is available to me, add these two and that will give me b_1 a_1 plus pb_0 is equal to b_1. Now multiply b with the current b_1 available that is your pb_1; I get p into b_1, add these two and I will get b_2. Now we proceed on like this, at this moment I have got here p into b_n minus three, I will get here b_n minus two, then p into b_n minus two will give me b_n minus one; p into b_n minus one will give me b_n. Now we have just proved that b_n is equal to P_n pk that is equal to R; that means the last coefficient that I have got here is R. The last coefficient in this one is given as R. You have started with b_0 multiplied by p_0 a_1.What is currently available on your screen is p_1; multiply by p, add a_2, what is currently available to you on the screen is b_2. So all these coefficients will be so trivially done on the calculator. Once you start the procedure whatever is available on the screen is currently available data for you to manipulate on. So pb_1, add it a_2, we have got b_2; multiply by b_2, add this number, I will get b_3. So you get this, compute and then finally land into b_n and this turns out to be value of the numerator or Pn at pk.

Now we said that the next synthetic division will give me ck but ck will be obtained exactly in the same way as we obtained bk. So just as we obtained bk from ak I can now obtain ck from bk that means I should continue this procedure one more step down. I can assume that p is available to me here. Now we are shown that c_0 is b_0, again c_0 is b_0; there is no change. Once we have defined c_0 is equal to b_0 then c_1 will be b_1 plus pc_0.Therefore the same procedure will be obtained here. I should just start with c_0. Let us take c_0. So this will be my c_1. I multiply this p with c_1, I add, I will get c_2. I go on doing it, this will be pc_n minus three, it will give me c_n minus two and I multiply this by pc_n minus two and I will get here c_n minus one.

Now we have shown in the previous step that our derivative is simply c_n minus one. Therefore this last coefficient is your dR upon dp. Now you can see that these coefficients are obtained from this, as we have done in the previous step. This procedure is called the synthetic division. It is something like the Horner's process for evaluating a polynomial. As I mentioned earlier the worst way of evaluating a polynomial is to substitute the value in it and take all the powers and write it. Suppose you want to get a polynomial degree twenty x to the power of twenty plus coefficient x to the power of nineteen and you want to get the value of this as 1.5653. Now if you substitute it here, take the twentieth degree of this, substitute it there, nineteenth degree and this, that is a worst way of evaluating a polynomial because the maximum possible round off error would be there in such an evaluation. The best way of evaluating a polynomial is the nested looping of the Horner's procedure. So this is almost the same as that particular procedure which we are doing. In fact if you open it from backwards, we are now multiplying by the previous coefficient. So it is like a loop. We are doing it similar to Horner's procedure for evaluating a polynomial and this is the best way of evaluating a polynomial.

This is called the synthetic division procedure. The interesting thing that you will notice here is this coefficient turned out to be the value of the function at pk; this turned out to be the value of the first derivative at pk. Now the question that comes is, can I find the second derivative and third derivative from here? If you want it, then the answer is yes. Just proceed with the synthetic division procedure one more step. I get d, I can write down dk is equal to ck plus p dk ck, dk minus one. So I can write down a new relation like dk is equal to ck plus pdk minus one. I would stop at one more less; here it is n, I stop at n minus one. I will stop at n minus two, and then this dn minus two would be interestingly one upon factorial two of d square R upon dp square. Therefore I can find any derivative of the given polynomial by continuing this synthetic procedure except that we will have factorials here.

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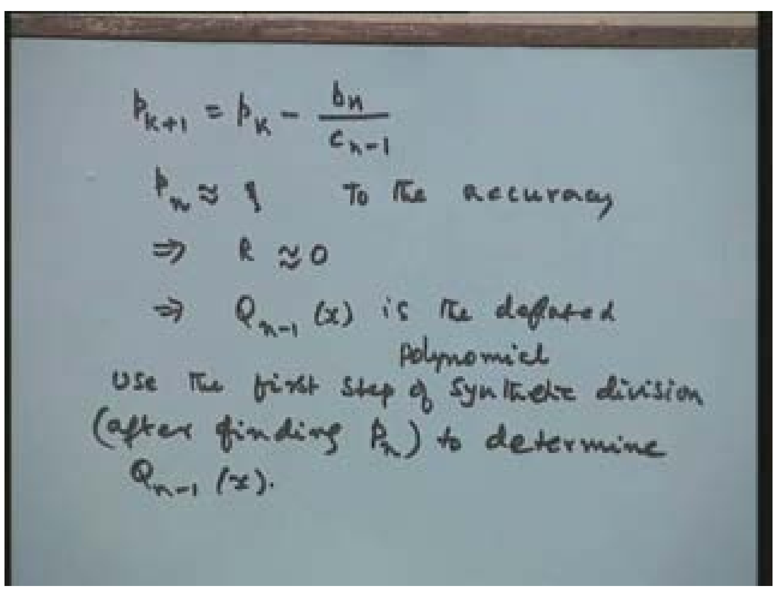

Now once these two values are obtained, I will now substitute in the Newton Raphson method that we have. So with these values that are available for us, I will now do pk plus one is equal to pk minus b_n by c_n minus one. Now this is the next iteration that has been obtained. Let us for each value of pk we are doing synthetic division procedure once and then computing. Now let us assume that a root has been found. Let us say that some pn is approximately xi so to the accuracy. So that means you wanted six decimal places of accuracy, so you have computed the number of iterations using that many iterations to get the root approximating the exact root. Then I want the deflated polynomial; so that I can go to the next step. Now if you look at the first step of the procedure that we have written here. If I have obtained a root to the required accuracy then R is approximately zero. So this implies that R is approximately zero because you have obtained the required accuracy. Now when R is approximately zero this itself is the deflated polynomial, because I am dividing this x minus p; what is left out is the deflated polynomial. Therefore this implies that whatever I am getting, Qn minus one x in that synthetic division is the deflated polynomial.

Now what we would therefore say is once you have got your required solutions to the accuracy, do the synthetic division procedure only upto the first step of it. So we will say use the first step of synthetic division procedure. Use the first step of the synthetic division (after finding P_n) to determine Qn minus one of x. So we would repeat this procedure of synthetic division that we have done here. This synthetic division procedure, one step of it we will do and then stop it and then pick that as my polynomial Qn minus one which is deflated polynomial. Now I will obtain the next root if it is required to apply on Qn minus one. Therefore as we proceed the degree of the polynomial will go down each time by one. So if you have started the polynomial degree at twenty, we have extracted to the root six decimal places, now I have a polynomial of degree nineteen; now again I extract a next root, then it goes on reducing. Now one very important thing that happens here is that the root that has been extracted that means it has been removed, therefore there is no chance that when you write an approximation to the next root, it is going to go back to the original root. As we mentioned earlier suppose you are using Pnx itself and you are using the next root again, there is a possibility that, if the roots are sufficiently close, you may be going again to the root that have already been determined. Now therefore once you extract it or remove from that particular root there is no chance that the root will again go back to the original root. So that is the advantage. Now let us take an example on this.

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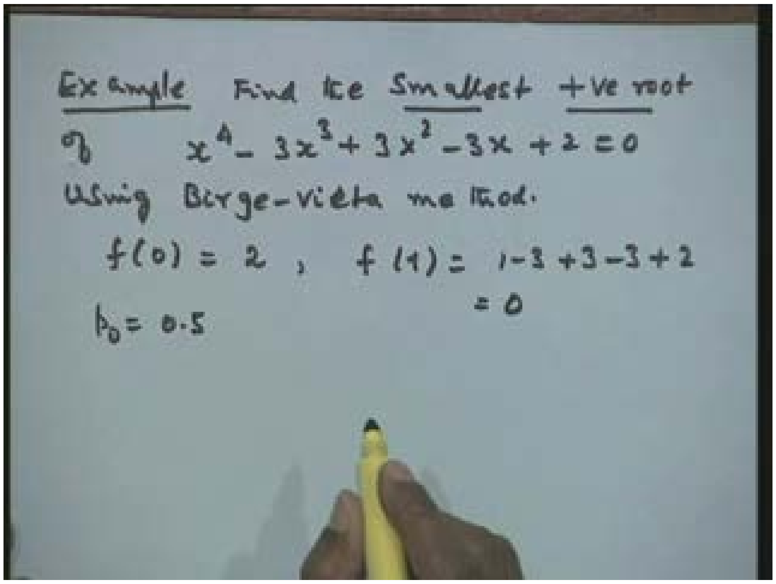

So let us take this example. Let us find the smallest positive root x four minus three x cube plus three x square minus three x plus two using Birge Vieta method. Let us fix up the smallest positive root required. So I would consider what my f of zero is; that is equal to two, f of one is one minus three plus three minus three plus two. For this the root turns out to be one. So this is equal to zero. Let us take an approximate value and then let us take the root of the value p_0 is equal to some 0.5. Then try to extract the root from this one.

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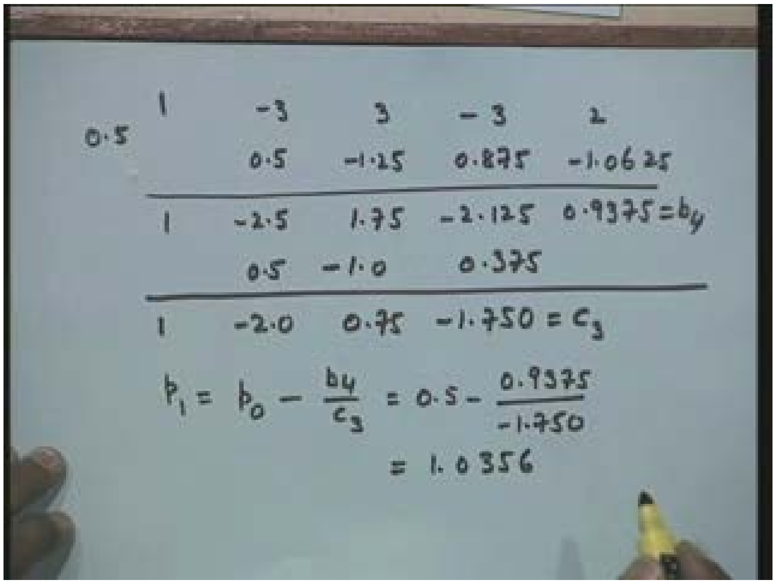

So let us write down the coefficients; the coefficients are one, minus three, three, minus three, two. So we take this p here. I will write this as 0.05. This is your b_0; that is one; multiply these two, I will get here 0.5. Sum them up, that gives - 2.5; 0.5 into 1.25 gives – 0.25 that is – 1.25; sum them up, I will get 1.75; 0.5 into this is 0.875; add the two, you get 2.125; add these two. Now multiply by 0.5 that is – 1.0625, add up and I will get here 0.9375 and that is equal to b_4; this is b_0, b_1, b_2, b_3, b_4. Then I need one more step; 1, that is c_0 is equal to b_0; 1 into 0.5 that is – 2, that is equal to 1.5 into – 2, that is – 1, that is equal to 0.75; we multiply by 0.5, I will get 3.75; - 1.750 that is equal to c_3. Now first iteration is complete. I would now find out what is my p_1; p_1 is equal to p_0 minus b_4 by c_3. So we can substitute the value 0.5 minus 0.9375 by – 1.750. I will give you the value of this; this value is 1.0356. Now the first iteration is complete. Now I will repeat this synthetic division procedure using this particular value. So let us do that.

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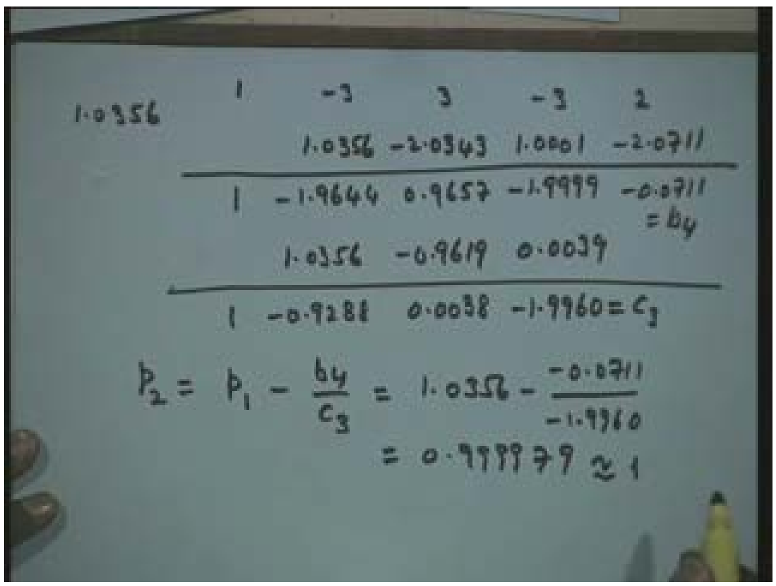

So let us again write this here as 1.0356 and then write down these coefficients, 1, - 3, 3, -3 and 2. I have 1 here, multiply 1.0356. I will give the values of this, if it is slightly difficult for you to do this. So this is – 1.9644. Now I multiply these two and put it over here and I will get – 2.0343 that is this product and I sum them up and I will get 0.9657. Now I am multiplying this and this and putting it here. It comes out to be 1.001, - 1.9999; I multiply these two and put it here – 2.0711 and this is -0.0711that's equal to your b_4. Indeed this b_4 is an indicator for us whether this solution is good enough of or not. Now let us proceed with the next step. I have 1 here; multiply this 1.0356 therefore this gives us – 0.9288. I multiply these two, - 0.9619 and this gives me 0.0038 and finally I have this as 0.0039, - 1.9960. Now the value c_3 is available. I can compute my next iteration that is p_2 is equal to p_1 minus b_4 upon c_3 and that is p_1 is 1.0356 minus – 0.0711 divided by – 1.9960 and this value is 0.999979, so it is approximately one.

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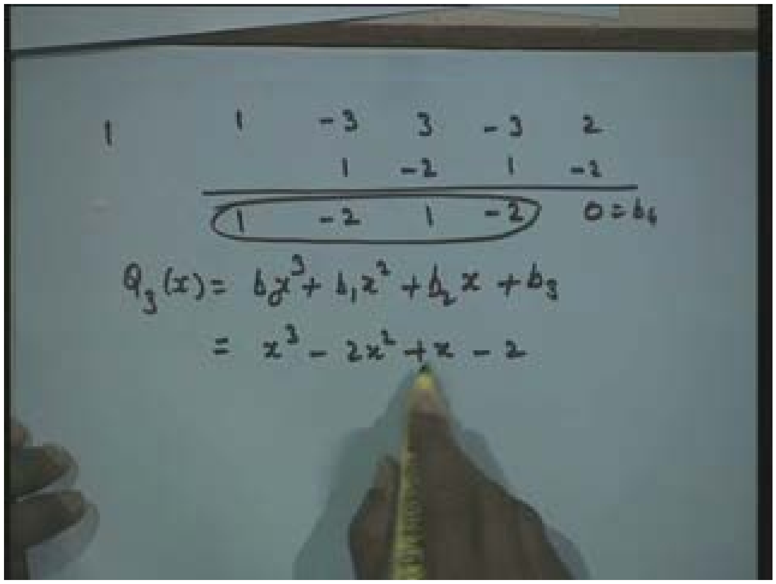

Now we have extracted this root. This is approximation with required accuracy. We have extracted the root, now I want the deflated polynomial. To get the deflated polynomial I must use the last approximation and do the first step for the synthetic division procedure. So I now take this 1 as a root and then write down the coefficients; 1, -3, 3, -3, 2. So I have 1 here, - 2, this is – 2, 1, 1, - 2. Now I want to compute this because I do not need it here, it turns out to be that is zero because that is turned out to be an exact solution. So if you just try to write down; this is zero, this is your b_4. Now these should give me my q_n minus one x, therefore the polynomial of degree q_3, the deflated polynomial, is b_0x^3 plus b_1x^2 plus b_2x plus b_3.So that is our b_0x^3, b_1x^2, b_2x plus b_3 that is x cubed minus two x square plus x minus two; therefore we have now obtained the next lower factor that is x cube minus two x square plus x minus two. Now if I want the next root I will now apply the Newton Raphson method on this deflated polynomial. So therefore in the synthetic division procedure your deflated polynomial is automatically available for us in the first step itself.

This procedure is possible only if we are finding the real roots, if it is a complex root then it is a complex pair. So I will not be able to use this to extract a single factor from a given polynomial to extract a complex root. Now if I want the complex root the only way is that I have to extract a pair; if I extract a pair as x square plus px plus q then I can extract the complex pair as well as a real pair of equations also.