Digital Image Processing

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Numerical Methods and Computation

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Lecture No. # 07

Solution of Nonlinear Algebraic Equations (Contd.) Polynomial Equations

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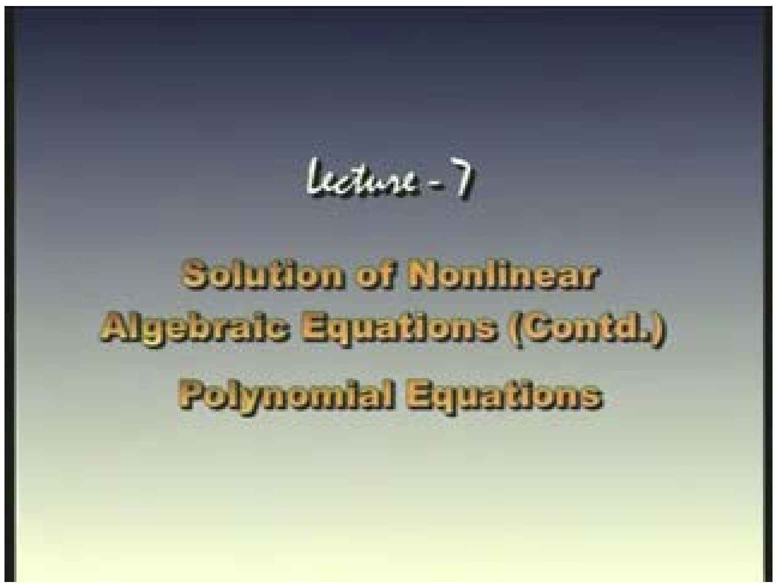

In our previous lecture we have derived methods for finding multiple roots of nonlinear equations. We have also derived the implementation of the Newton Raphson method for a system of nonlinear equations. We have also given what is the importance of a system of nonlinear equations but polynomial equations form an important class of nonlinear equations. They occur in many applications areas. The application of Newton Raphson method can be simplified if you view it in a different way. Of course we can apply directly the Newton Raphson method of polynomial equation; that means if you just write down what is f of xk, f prime of xk we can do it. However the evaluation of a polynomial using its powers as x to the power of n, x to the power of n minus one and so on, when the degree is very high, evaluation of a polynomial at a particular value is very expensive and very prone to round off errors. However if we do the evaluation of a polynomial in nested loop that is the Honors' method, then it is going to be less prone for round off errors. We have the methods for polynomial equations separately besides that, one more reason why we need the methods separately is that we may need all the roots of a polynomial equation; that means if you have polynomial of degree twenty, we may need all the twenty roots of the equations and if you try to find out by Newton Raphson, each method and each root of this equation it may be time consuming and therefore we shall discuss methods specifically for polynomial equations.

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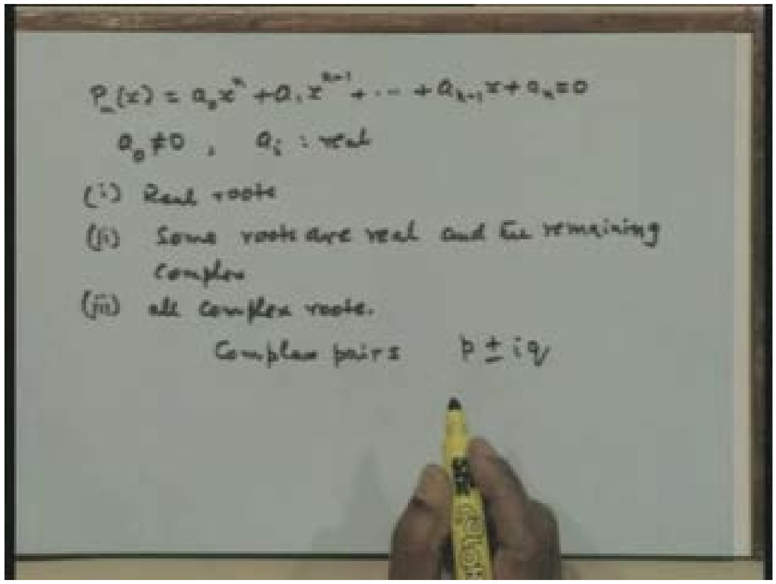

Now let us write down a polynomial equation of a degree n, so we shall denote P for polynomial, n for degree and it is a polynomial function of x, which we will take it as some a_0x to the power of n plus a_1x to the power of n minus one plus so on and a_n minus one of x plus a_n is equal to zero, where the leading coefficient a_0 is not equal to zero; otherwise it would be a polynomial of a lower degree and we will take all a_s as real constants. They are all real constants. Now obviously the roots of this equation may all be real, so there is one possibility that it has got real roots or we may have some roots are real and the remaining complex; and the third case is that it has got all complex roots . For example, if you are finding the roots of x square plus one we know both the roots are complex. So it is a case arise when the roots are all complex. But we all know that if the coefficients of a polynomial are all real then complex roots occur in pairs. So if you are talking of a complex root we are talking of complex pairs. So all these roots that we are talking of are complex pairs. That means if p plus iq is a root, then p minus iq is also the root of this equation. So it will occur as a pair.

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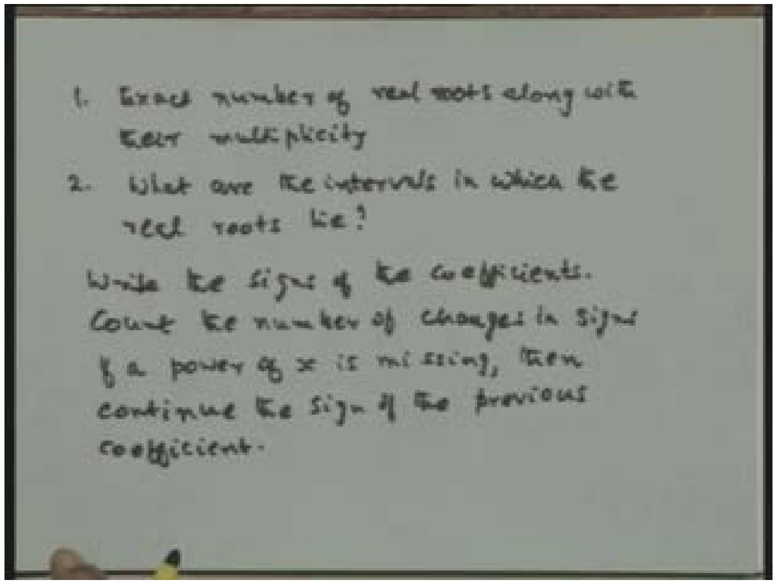

Now in the case of polynomial equation, it will be useful for us to know some information about the polynomial. What you mean by this is; what is the exact number of roots that this polynomial equation has got? How many complex roots it may have? That means the information that we would be interested in the polynomial is; the important thing is what is the exact number of real roots; how many of them are there and along with their multiplicity. So we would also like to know what the multiplicity of their roots is. Now that we know that the number of real roots, we also would like to know what are the intervals in which the roots lies. So we would also like to know what are the intervals in which the real roots lie. Of course one way for the second part is that we know the intermediate value theorem can be used and we can get it. However we would like to have an alternative way where it can give us the exact number of real roots than the polynomial equation can have.

Now we already know something from our school studies, what is known as Descartes rule of signs for finding the number of roots of a polynomial equation, a bound for the number of real roots in the equation? Let us just see what that rule is, so that we will modify it and then give a rule wherein we will be able to say what is the exact number of real roots. What we do is we take a polynomial then we write down all the signs in the coefficients with the plus sign and minus sign. If any particular power of x is missing we continue the sign of the previous coefficient. Then we count the number of changes in signs. We shall say that this polynomial has got this many change of signs. Similarly we can put x is minus x and can consider p_n of minus x and then I can do the same thing; find the number of changes in sign in the coefficients. So write the signs of the coefficients. Now count the number of changes in signs. Now we will say if a power is missing, then continue the sign of the previous coefficient.

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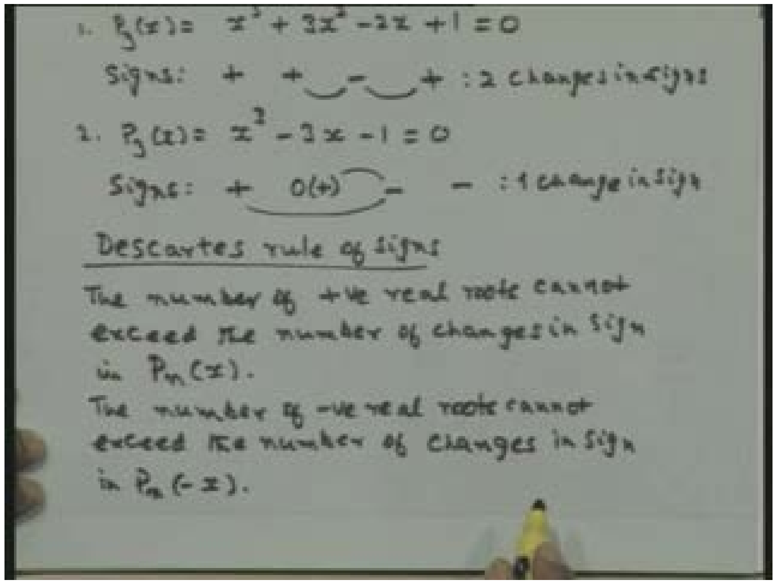

Now let us just take a simple example for this as to how we are going to write it. Let us take a polynomial P_3 of x is equal to x cubed plus three x square minus two x plus one and we are not really looking at the magnitude of the coefficients; we are not looking at the magnitude. We are only looking at the sign of the coefficients. So let us write down what are the signs here. The sign accordingly is; this is plus, this is plus, this is minus, this is plus. So I have got here one change of sign and I have got here one change of sign. So I will say that there are two changes in sign for this equation. Now let us take a different type of example. Again I will take a cubic polynomial x cube minus three x minus one is equal to zero. Now let us again write the signs in this. So I have a plus sign here, but x square is missing. So I can put a zero here and I can put a plus sign so that the sign of the previous coefficient shall continue for counting. The next coefficient is a negative, therefore I have here one change which is from here to here or you can look at it as from here to here. Now based on these signs we have what is now known as the Descartes rule of signs. It is called the Descartes rule of signs. What it states is, the number of positive real roots of a polynomial equation cannot exceed the number of changes in sign. I will explain it again.

Let us write; the number of positive real roots cannot exceed the number of changes in sign in P_n x. Now I can do the same thing with P_n of minus x, then I will have the conclusion that the number of negative real roots cannot exceed the number of changes in sign in P_n of minus x. So I can go for the negative roots. So the number of negative real roots cannot exceed number of changes in sign in P_n of minus x. In other words this rule gives us an upper bound for the number of real roots. For example, if the polynomial has got five changes in sign it will say that in P_n x may have fine five real roots. Since the complex roots can occur in pairs, we can say that all the five are real or one is real and four are complex i.e. two complex pairs or three real and one complex pair. That is because complex has to come as pair so you will have to reduce by two. So either it will have five real roots or three real roots or two complex and one real root or four complex. So that way we can give the upper bound for the number of positive real roots as well as for negative real roots.

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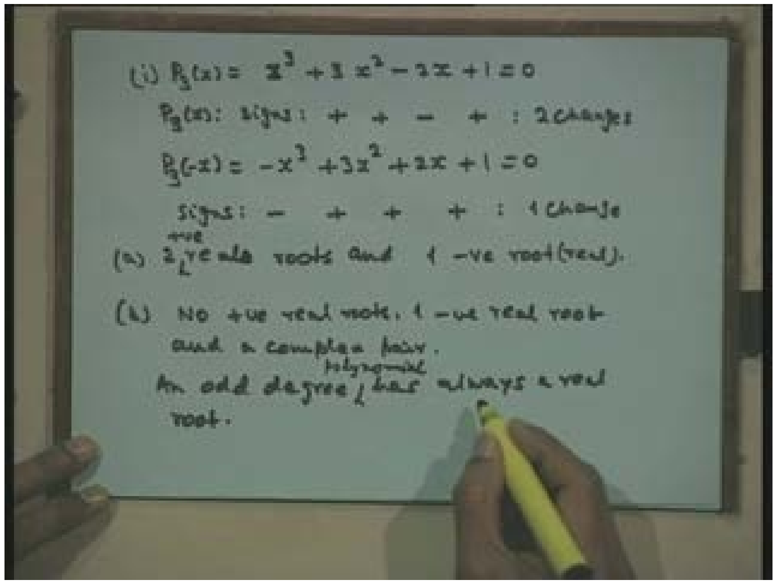

I will just repeat this example. Let us have a look at the same example again. So let me just write that example again. Let me take this P_3 of x. This is your x cube plus three x square minus two x plus one is equal to zero. Therefore I have for P_3x, let's write down the signs of this; so we have plus plus minus plus, so we have got here two changes. Now let us also write down what you have; P_3 of minus x. Now P_3 of minus x gives minus x cube, put x as minus x here, I will have plus three x square, I will have plus two x here and we have plus one here. Now I will look at the signs of the coefficients of this particular polynomial. So I have a negative sign, plus sign here, plus sign here, plus sign here; so I have got one change of sign here. Therefore we conclude that this polynomial has got two positive real roots or zero real roots or real positive roots and one complex pair and definitely one negative root; because if you have an odd degree polynomial, an odd degree polynomial will always have a real root because the complex roots have to comes as pairs.

Therefore if there is one change of sign there cannot be a complex root because a pair has to come. Therefore one change is means surely that this equation has got one negative root. We are guaranteed of one negative real root. Whereas in this case we have two changes of sign therefore it could be two positive real roots or two minus two, that is zero real roots and a complex pair. Therefore we conclude, for this particular polynomial it has got two real roots. It has got two real roots or we will also call it positive and one negative real root. That is one conclusion. Alternatively it has got no positive real root, one negative real root and a complex pair.

Now as mentioned earlier an odd degree polynomial will always have a real root. So an odd degree polynomial has always a real root. Whether the real root is positive or negative that we can decide upon by testing the changes in signs but an odd degree polynomial always has got definitely a real root. Now to understand it better let us repeat the second example also which we have discussed earlier. So let us write that example.

Student question: "Sir, odd degree polynomial has real coefficient?"
Yes, we are we are talking of only real coefficient problem. Once the coefficients become complex, then complex pairs need not occur and once the coefficients are complex, then the complex roots need not be pairs. It can be alone. So therefore the description which we are making, that the number of real roots will reduce by factor of two will not come into picture. So we will be only considering the real coefficients.

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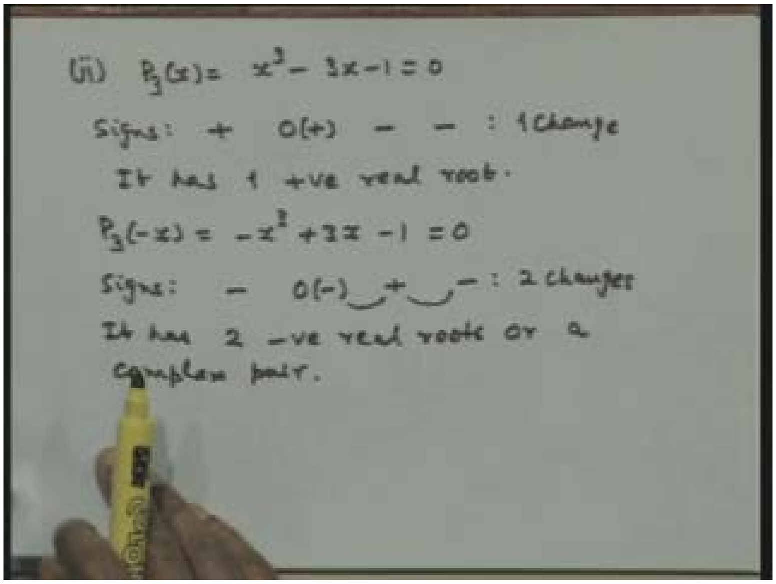

So let us take the second equation P_3 x is x cubed minus three x minus one is zero. So again let us take the signs in this particular one which we have done earlier. This is a plus sign, a zero which continues into a plus sign here, we have a negative sign, a negative sign. Therefore I have got one change in sign in this; I have got one change in sign. Therefore we are sure that this equation has got one positive real root. Now we consider whether there is any negative root. So I will put x is equal to minus x in this. So I will get minus x cube plus three x minus one is equal to zero.

Now I will write down the signs again. This is negative, now x square is missing, so that will be zero and negative sign will continue. Then I have a plus sign, have negative sign so I have one change of sign here and have one change of sign here, so I have got two changes in sign here. Therefore I conclude that the polynomial has two negative real roots or a complex pair. Now in totality therefore we would be able to say that this polynomial has got one positive real root and two negative real roots or one positive real root and a complex pair. Now therefore the Descartes rule of signs gives us only a bound for the number of real roots. If a polynomial has got ten changes in sign it will at the most say that they may have ten real roots, but it will not have. It will not say that it will have exactly ten real roots or it has got exactly six real roots and this particular aspect of the exact number of real roots can be obtained by forming what is known as Sturm sequences and using words known as Sturm theorem. It is a very simple application.

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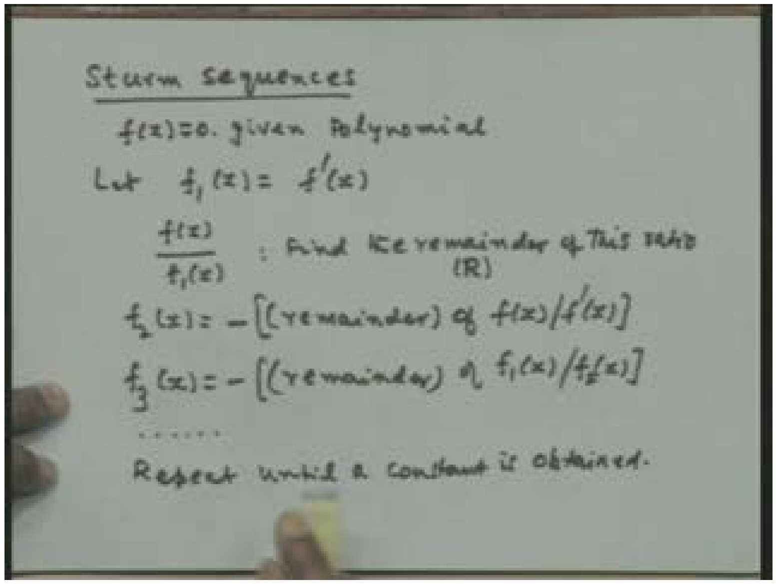

So let us define what a Sturm sequence is. Now we start with a given polynomial. So fx is a given polynomial. Now let us differentiate fx and denote it by f_1. So we will differentiate fx and write it as f_1 is equal to f prime of x. Then I will divide fx by f_1x and find it's reminder, find the remainder of this particular ratio. Put an opposite sign to this reminder and call it as f_2x. So let us call this remainder as R. Then I will define f_2x as minus. This particular remainder I will take this as minus remainder of fx upon f prime x. Now I repeat the procedure. I will take f_1x, f_2x; I will divide f_1x by f_2x, this division is an ordinary division; find the remainder, put the opposite sign to it and call it as f_3x. So I would define f_3x as opposite sign of the remainder of f_1x by f_2x. Therefore I will continue on. You can see that fx is a polynomial of degree n, f_1x is f prime. Therefore polynomial of degree n minus one, divided it out and we have got a remainder. Each time I am dividing by polynomial of one degree lower, therefore after some stage, after n stage I am going to get a constant. So we will repeat until a constant is obtained.

Now this gives us a sequence of functions. So let us write down the sequence of functions fx, f_1x, f_2x…. f_nx. This sequence is called a Sturm sequence. We call this as the Sturm sequence. We can see that right through we are interested only in the sign of the remainder. Therefore when I get f_2x as a rational coefficient in the leading term, since we are interested only in the sign I can throw away or remove that particular coefficient from there. Let us suppose I have got sixteen x plus sixty four, sixteen x square plus sixty four x. I can take common factor sixteen and write this as x square plus four x and in the next step I can throw away this sixteen because I don't need to use it; because I need only signs. Therefore any particular factor that is there to me can be dropped in the next computation, so that the computation will become easier for had purposes also.

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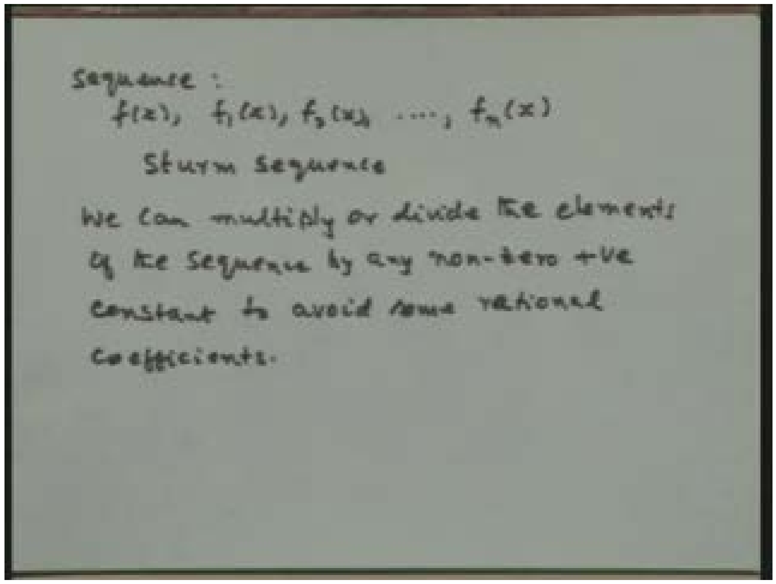

If you want to do by hand, we will be able to do by hand also. So at any particular stage we can multiply or divide by a positive constant so that the rational coefficient can be removed there. For example, you have got the leading coefficient as two by three. So to do the computation next step would be difficult. So we can throw away one by three. So if I remove or throw one by three, I am multiplying by three throughout; that means I will take it as two x square plus three x. So it is easier for me for dividing in the next stage. Therefore we can multiply or divide by any positive real constant to remove some of the rational coefficients. So we can multiply or divide the elements of the sequence by any non-zero positive constant to avoid some rational coefficients. So the main idea here is to remove the rational coefficient i.e. simplify our divisions. Now we give what is known as the Sturm's theorem. So let us write it in the next sheet.

Let us define what is the Sturm's theorem. Now let me just give the result. The result is so simple. We start with your polynomial fx. We construct the Sturm sequence. I take any value x is equal to a; any number x is equal to a; substitute x is equal to a in all these functions and then I study the number of changes of sign in this Sturm sequence. Let us call these number of changes as some v of a, of course if f of a is zero, then it is a root. We are assuming f of a is not zero. I take another value x is equal to b, substitute and then again study the changes in sign. The difference in these two numbers that is v of a minus v of b in magnitude will give me the exact number of real roots in the interval ab. This will give you the exact number of real roots. If the number of changes of sign at x is equal to a is five, if the number of changes of sign at x is equal to b is three, then the number of real roots between this is five minus three, that is two exact real roots are there. Now I can further subdivide this and then find the exact number of real roots in that particular thing. So I can go on subdivision and then find the number of real roots until I get a required small interval. So this will give you the way of finding the exact number of real roots.

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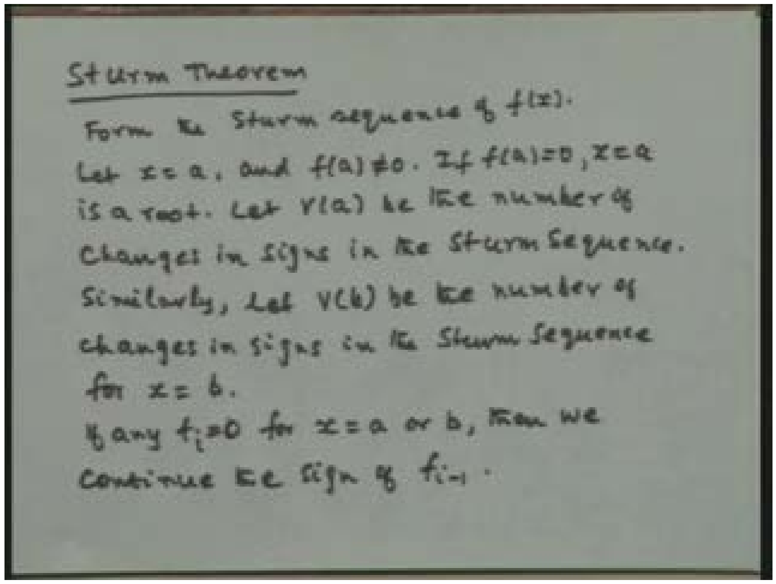

So for this we first of all form the Sturm sequence, and then let us take any number. So let x is equal to a and of course f of a is not equal to z; what we are saying is f of a is equal to zero, x is equal to a is a root. So therefore we need not construct this particular case. Now let va be the number of changes of signs in the Sturm sequence. Similarly let v of b be the number of changes of signs in the Sturm sequence for x is equal to b. Now we have already mentioned that if any function is missing in between, that means a function turns out to be zero at x is equal to b, we continue the sign of the previous function just as we are counting the signs in the coefficients. Here also we say that if any function fi is equal to zero, for x is equal to a or b then we continue the sign of fi minus one. That means in the Sturm sequence that we are following f, f_1, f_2, f_3 we have got, any fi is equal to zero. We just take the sign of the previous coefficient, previous function and then continue counting in the sign of this one.

Now once you are able to find the number of exact roots, then we would able to find the number of complex roots. It is possible for us to modify it slightly and find the multiple roots also. I will just explain what would happen if there is a multiple root; what happens to the coefficient, there will be change in coefficients. Once you decide the roots, then the total numbers of roots for the polynomial equation are n; so I can subtract these exact roots that we have real roots and then I have the number of complex roots. So using this term I will be able to tell exactly the number of positive real roots, exact number of negative real roots and the exact number of complex roots. So that is the importance of the Sturm terms.

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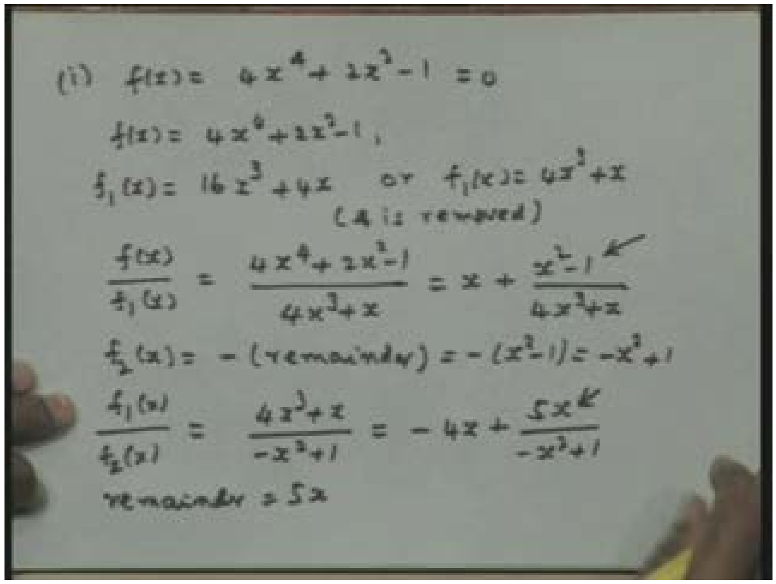

Let me take this example four x four two x squared minus one is equal to zero. So I want to state the numbers of roots for this particular equation four x four plus two x square minus one is equal to zero. So I just have the first function fx is equal to four x four plus two x square minus one. I differentiate this and put it as f_1. So this is sixteen x cubed plus four x. Now we are saying that there is a common factor here which I can throw away. So I will write it as or I will take it as f_1x as four x cubed plus x. So here four is removed. So this is what we said that we need not consider that factor at all. We can just remove that factor from f_1x. Of course we are removing only the positive constant. Now I will write down my f_2. Now I will divide fx by f_1x that is four x four plus two x squared minus one divided by four x cubed plus x and this division which we are talking of is the manual division or ordinary division that we can do. I can see that I will have multiplied this by x plus, subtract it x squared minus one divided by four x cube plus x. So this is nothing but you can just check it up; four x four x to the power of four x square plus x square two x square minus one. Now what we are stating is that the remainder of this is x square minus one, so the remainder of this is this quantity. Therefore I will denote f_2x the opposite sign of x square minus one, therefore I will write f_2x is equal to minus sign of remainder that is minus sign of x squared minus one; that's equal to minus x squared plus one. Now I determined f_2. Now I will determine by f_3; now I will write down f_1x divided by f_2x. So I have to divide f_1x divided by f_2x. Now f_1x is; we have here four x cubed plus x f two x is minus x squared plus one.

So I can just write this as minus four times x plus five x by x squared plus one. So this is four x cubed minus four x plus five x i.e. plus x minus one. Therefore the remainder for this would be this quantity that is your five x.

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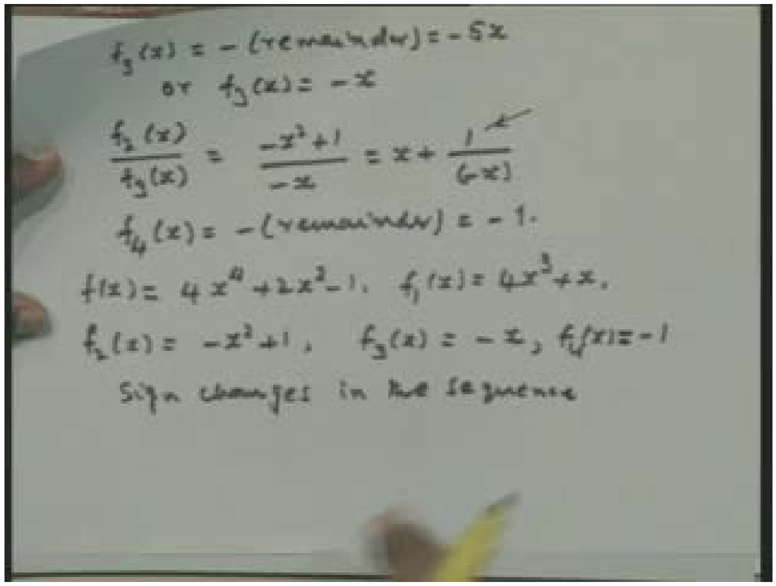

Therefore I would define my next function f_3x is equal to minus of the remainder, that is equal to minus five x. I can again drop five. I can throw away the constant five. So I can take this as f_3x is equal to simply minus x. I can remove the positive constant five from here, then divide f_2x by f_3x, f_2x is minus x squared plus one divided by minus x. This I can write it as x plus one upon minus x. Remember I would like to retain this f_3x as minus x in the denominator, I will not bring minus sign here because the denominator should be our function that we are tackling; that is your $f_{1,}$ f_2 or f_3 ,whatever the function. So the denominator I will keep them as minus x minus x itself and consider the remainder as the numerator of this part. Therefore f_4x is equal to minus the remainder of this is equal to minus one. In general we will reach f_1x when we produce a constant. We started with a polynomial degree four, therefore we continue up to f_4 to produce a constant over here. Therefore we now have got the Sturm sequence.

Let us now write down the Sturm sequence. Here the Sturm sequence will be fx. That is a given function, four x four plus two x squared minus one; f_1x is the next of this function four x cubed plus x; f_2x is the next function that is minus x squared plus one; f_3x is equal to minus x and f four x is minus one. Now what I need to do here is I need to study the changes of sign in this one and then locate the roots. So let us form a table. So let us find out what are the sign changes in this sequence.

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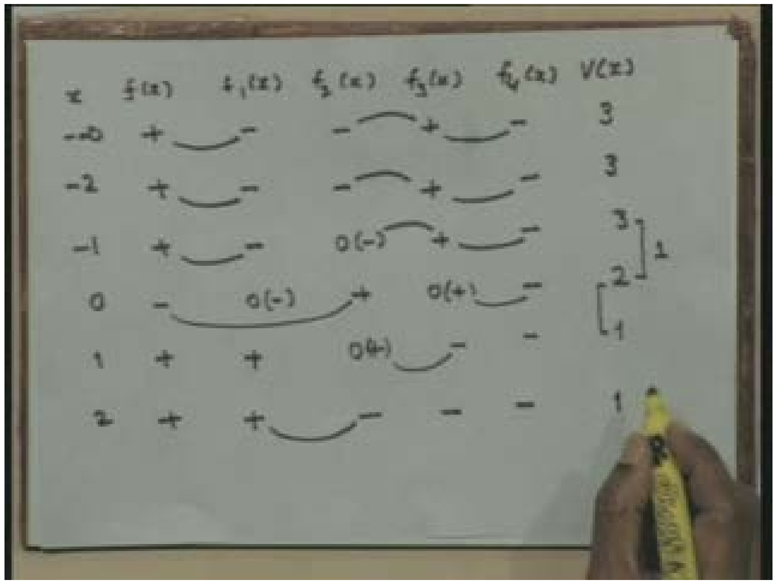

Now let us write down fx here; f_1x, f_2x, f_3x, f_4 of x and v of x; v will be the number of changes in the sign in this one. Let us put x also here. Let us start with minus infinity. When I put minus infinity in fx, this is minus in to the power of four, so this will have the positive sign. So I will have a positive sign in fx. When x is minus infinity in f_1, it is all negative. So it is going to be a negative sign. This is minus x square, where it is always going to be minus infinity. This is minus x, therefore this will be plus infinity. So they will have plus sign. F_4 x is negative constant, therefore I have one change of sign here; I have another change of sign here, I have one more change of sign over here. Therefore number of changes of sign with in minus infinity is three.

Now I will reduce the amount of work. We can have a look at this closely for modx greater than one this is always going to be negative. When x is negative, this is always going to be negative and when x is negative this is always going to be positive and this is always negative. So if modx is greater than one the signs of all this have been fixed and in fact for this also it is fixed, because this is four times x four. So this is always going to be positive. Therefore when modx is greater than one the signs are always almost fixed. So we don't have to see what happens at minus ten, minus five, minus three and so on. We just pick up a number greater than magnitude of one on the negative side and then study what happens. For example I can take minus two. This we have already discussed, so if we put minus two I will have a plus sign here; minus two I will have a minus sign here, and if I put minus two in this, this will be again minus sign; this will be a plus sign and this is a minus sign. So I have chosen a number which is in magnitude greater than one and therefore the number of changes of sign in this, is again three which shows that there is no negative real root beyond minus two.

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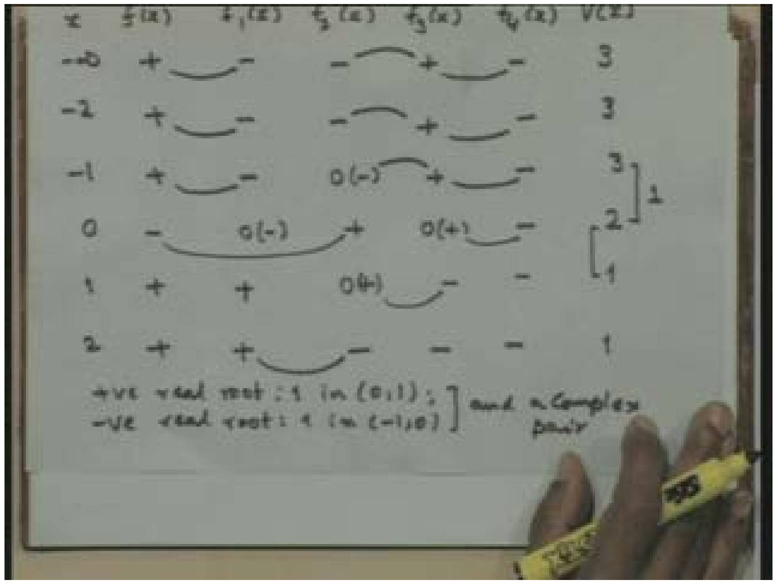

Let us take minus one. If I take minus one, fx is still positive and this is negative and this is a zero, x is minus one; So I would continue the sign of the previous coefficient; f_3 is positive, f_4 is negative. Therefore it still has one negative sign here and one change of sign, one change of sign and one change of sign. There are three. Therefore still there is no negative root between minus one and minus two. Also so let us take zero. At zero f_4 is negative, f_1 is zero. So I will continue the negative sign of the previous coefficient. At zero this is positive. This is zero. Therefore I continue the previous sign, f_4 is negative. Now I have a change in sign, minus to plus; I have a change in sign from plus to minus here. I have got two changes in sign, therefore now I have got v of minus one minus v zero in magnitude is one. Therefore this is one change in sign over here. Therefore we will conclude a little later. Therefore there is a negative root between zero and minus one.

Then let us take one. If I take one, this is positive, this is positive; f_1 is positive. Then if I take f_2, this is zero. So I continue the plus sign. I continue the plus sign, when one this is negative and this is also negative. Therefore I have one change in sign here; again there is a change in the value of v of x. Therefore there is a root between zero and one. I have already checked that mod x is greater than one. These are not changing the signs. Therefore I can immediately go ahead and have the same result that we have over here. I can put two here now and then write down the signs that we have earlier. This is going to be plus sign, this is going to be plus sign, and this is a negative sign, a negative sign and a negative sign. So we can see that when x is equal to two, this is a positive sign; this is a positive sign; this is a negative sign; a negative sign and a negative sign. So I now want one change of sign only here. We can conclude positively from this result that there is one negative real root between minus one and zero; one positive real root from zero to one; the other two must be a complex pair. Therefore we have got here positive real root. There is one in (0, 1) there is negative real root; there is one in (minus 1, 0) and a complex pair. Therefore from this example we have seen it is possible for us to tell exactly the number of the real roots and its term sequence would then automatically give us the interval in which the root lies.

Now in a general problem maybe you may not take that closely; you may take a larger interval, say between zero and ten depending on this or zero and five and you will find that between zero and five there are three real roots. Now if there are three real roots then we can reduce the interval further and to locate the interval in a particular required interval; but it will give us the exact number of real roots.

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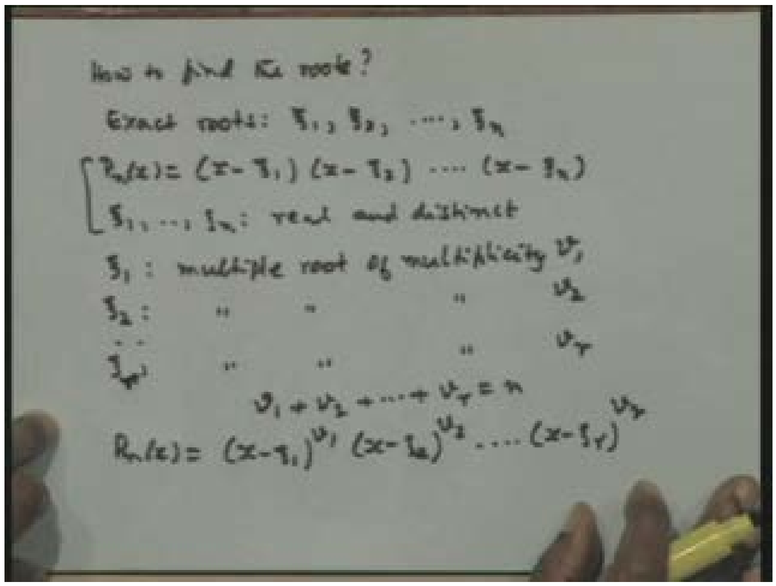

We know now that the roots lie in an interval and the exact number of real roots is known. Now how to find these real roots, then the problem is to now how to find these roots. Now I would explain the philosophy that is used in deriving some of these methods. Now let us assume that the exact roots are say xi_1, xi_2, xi_n. Let us say that these are the roots of given polynomial xi_1, xi_2, xi_n. If these are the exact roots, then the polynomial can be written as x minus xi_1, x minus xi_2, x minus xi_n. So polynomial can be factorized because these are n real roots. Let us take these take them as xi_1, xi_2, xi_n. We have taken this as a real and distinct. Once I take these roots as real and distinct, then I can always factorize them and write it as x minus xi_1, x minus xi_2, x minus xi_n. If they are real and have multiplicity, I can also write as follows. Let us suppose that xi_1 is a multiple root of multiplicity, let us say $v_{1;}$ and let us also take xi_2 as a multiple root of multiplicity v_2 and let us take all of them as multiple roots. Let us take as a multiple root. Of course the sum of this has to be n because the total number of roots is n. So we must have the condition that the sum of all this is equal to n, the total number of roots are n. That means I can write down my polynomial as equal to x minus xi_1 to the power of v_1; x minus xi_2 to the power of v_2 and so on; x minus xi_r to the power of v_r. What we would therefore do is, if I want a real root I will try to extract a real root from that particular equation. Now I am using the word extract in a slightly difference sense from what you are using. In Newton Raphson method we are saying find the real roots of this particular thing; but what we want to talk here is we want to extract a real root from the polynomial equation and then produce automatically, what will be the polynomial that will be obtained by removing that root; that means the deflated polynomial.

Suppose we have a polynomial of degree ten. If I extract a factor from there, a linear factor, then what is left is a polynomial of degree nine. So therefore we shall say extraction of a root would imply automatically, we are talking of the deflated polynomial. So that will come out of this particular method that we are constructing. Therefore it is possible for us to extract one root or a root which is a multiplicity v_1 starting with any particular procedure and the method would automatically take care of giving us the deflated polynomial, so that we should work on the deflated polynomial. If you have a polynomial of degree fifty, if I want all the roots of this polynomial, even though I know the location of the fifty roots I would not attempt to use the method on the polynomial of degree fifty times. I would extract a root, deflate it, get a polynomial degree forty nine, get the next root, deflate it, and get a polynomial of degree forty eight. So each time deflation will reduce the degree of polynomial. It will also of course reduce the amount of computation that we are performing and another very important thing that it does is that it will not allow the root to go back to that original root. Suppose x is equal to say 0.5 was a root, I have extracted the root. If I had repeated the method on the original polynomial it may be possible that you're again going to the same root 0.5 but not to the next root. But here once you deflate it, you remove that particular root from the pnx we have got the next degree polynomial, we are definitely going to approach another root for which we are taking a starting value from a given point. Therefore this reduces the computation also. It is called the process of deflating the polynomial and then getting all the roots and finally after some stage it will come down to a quadratic factor and that quadratic factor we can find out two roots as a quadratic factors and get all the roots.