Numerical Methods and Computation

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Lecture No # 6

Solution of Nonlinear Algebraic Equations (continued)

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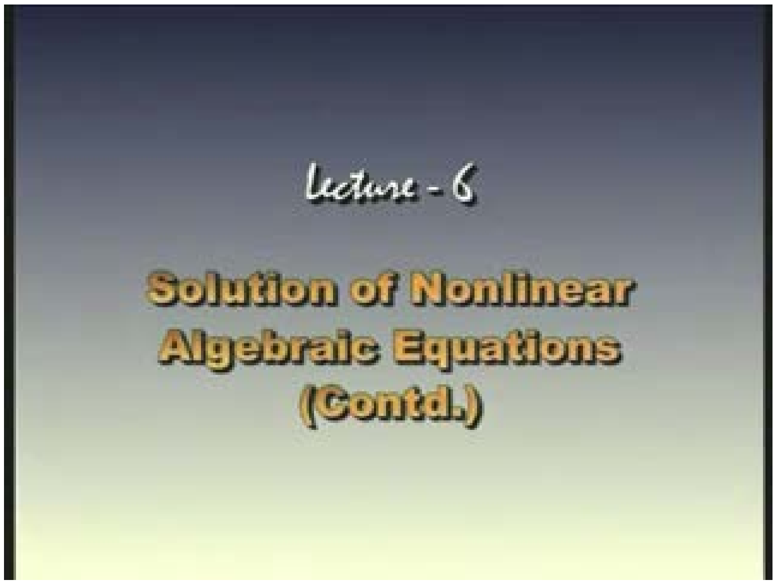

In our last lecture we were discussing about the finding the multiple roots for an equation fx is equal to zero.

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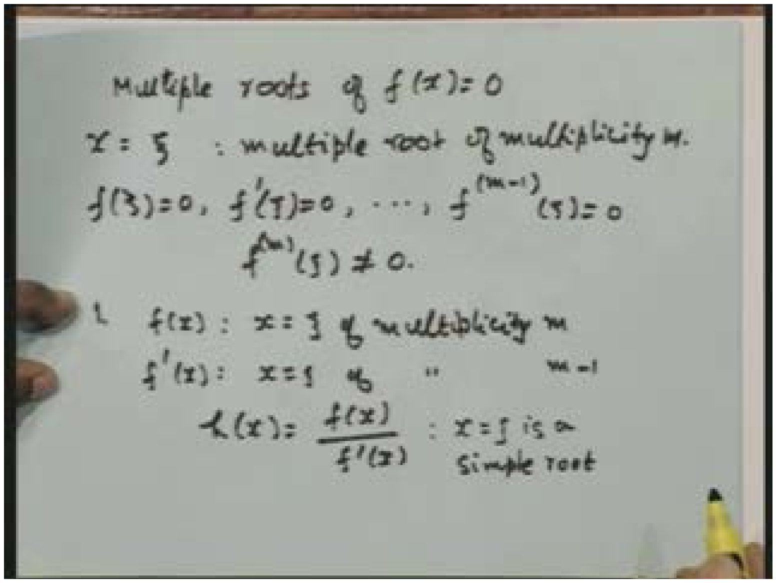

What we are really looking for was the multiple root of fx is equal to zero. We know the definition of multiple root. Let us say x is equal to xi which is a multiple root of multiplicity m. So let us take this multiplicity also. It is a multiple root of multiplicity m. Then we know that f of xi would be zero, and then its derivative is also zero; f prime of xi is equal to zero and so on up to m minus 1^{th} derivative. It is equal to zero and n^{th} derivative is not equal to zero. This is the basic definition of the multiple root of multiplicity m.

Now to find this particular root we shall first of all attempt whether we can use the methods that we have derived for simple roots by some modification or some other criteria; such that we can use those methods without having to make much changes in the methods that we have. Let us first of all attempt in this particular form. We know that if f of x has got a root, x is equal to xi of multiplicity m and then its derivative f prime of x has got the same root x is equal to xi of multiplicity m minus one. Therefore we would like to consider a function which is a ratio of these two and define hx as fx upon f dash x. Then this particular ratio i.e. this new function that we have defined as hx has got a simple root at x is equal to xi. Therefore for this function x is equal to i is a simple root. What does this imply? This implies that since x is equal to xi is a simple root, I can now apply the all the available methods that we have discussed for finding the simple root; to find the simple root of hx is equal to zero and hence the multiple root of fx is equal to zero which is the multiplicity m.

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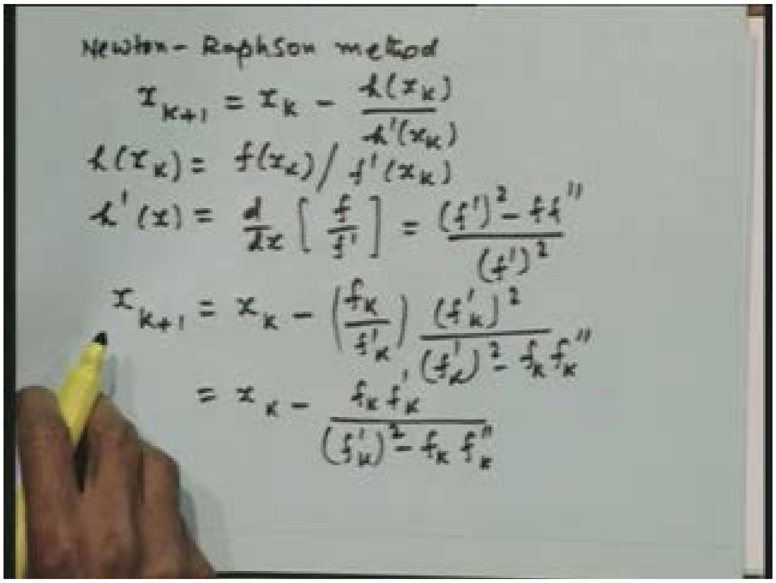

Let us try to get the Newton Raphson method and its application to find the multiple root. Now what we are saying is, we are applying the Newton Raphson method of hx. Therefore the iterative method would be xk plus one is equal to xk minus h at xk divided by h prime at xk. Now I need these two quantities to put here. So h at xk will be simply equal to f at xk divided by f prime at xk, so that is from the definition of hx. Now differentiate h prime. So let’s get out derivative of this that will be d upon dx of f upon f dash. So let us differentiate it. So if I differentiate you will get f prime into f prime i.e. f prime square minus f into f double prime divided by f prime whole square. So that is the simple derivative of f upon f prime. Therefore I can now evaluate h prime at x is equal to xk and then substitute it over here. So I can put this here and get it as xk plus one is equal to xk minus f of k by f prime of k; that is our value of h at xk. In the denominator we have got h prime. So this will go up as f prime k whole square divided by f prime k whole square minus f of k fk double prime, which we can simplify in one more step i.e. by cancelling one of f prime k. So I can write this as xk minus f evaluated at k, f prime at k from here, and the denominator is the same denominator f prime k whole square minus fk f second prime at k. Therefore I can obtain the simple root of hx is equal to zero using this Newton Raphson method modification and therefore this gives me the multiple root of multiplicity of the original eqation.

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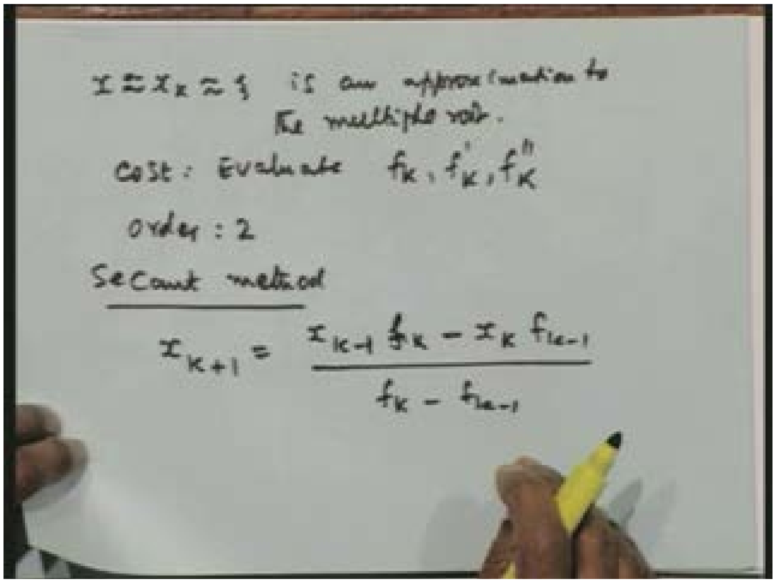

Therefore we can say that x is equal to xk approximation. This is xi. It is an approximation to the multiple root. Therefore we obtain the root without any furher modifications in the Newton Raphson method. Now if this is always so we do not need any other method, but if you look at the cost of the computation; the cost of the computation needs to be evaluated, f at k f prime at k f double prime at k. So we have got this fk f prime k and f double prime k. So I need to evaluate three evaluations to get the Newton Raphson method and the order of the method is same because we have evaluated the simple root of hx is equal to zero. The order of convergence is two. So the order of convergence still remains as two and only thing that we have done is one extra computation. If this extra computation is allowed and we do not have any problem, we can definitely use this particular method; but however we shall give an alternative method wherein we do not need evaluation of three, we need evaluation of only two, provided you know the order or multiplicity in advance.

Now this method can also be applied for secant method. We can apply this secant method. Let us see how you put the same thing into the secant method. Let us take the way in which we had written the computational way of writing the formula. We have written it as xk plus one is equal to xk minus one f at k minus xk fk at minus one divided by fk minus fk minus one.This is the computational form of secant mehod that we have used earlier. So therefore I need to only insert into this the values of fk fk minus one; this is equal to x at k minus one.

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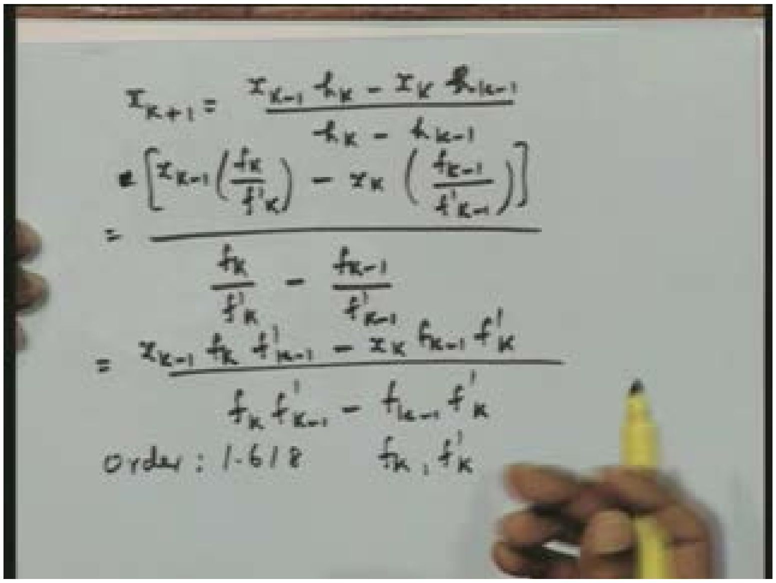

Now I need to write this for the evaluation of the simple root for hx is equal to zero. So I would now use the method as xk plus one is equal to xk minus one h evaluated at xk minus xk, h evaluated at k minus one divided by hk minus hk minus one. Now we have written everything in terms of the function hx. So let us substitute the value of what is hk and what is hk minus one. So that will simply give us xk minus one into f of k minus f prime of k i.e. the numerator minus xk, f k minus one is fk minus one upon f prime at k minus one and then we shall divide the whole thing by hk minus hk minus one, so fk minus f prime at k minus fk minus one f prime at k minus one. Now it is just a matter of simplication over here. We can see that from the denominator first of all we can write this as fk into f prime k minus one minus fk minus one f prime of k. So I first simplified the denomianator; when it goes to the numerator we can see f prime k f prime k minus one cancels with the f prime k from this. Therefore I would not repeat that particular factor. So the numerator would be simply xk minus one from here, fk from here and f prime at k minus one minus xk fk minus one into f prime of k. So that is the simplication of the numerator. So this will be the secant formula as applied to hx is equal to zero. Therefore the order of the method is the same as 1.618. Therefore rate of convergence would not change. In the secant method also we are now using f prime whereas in the original secant method for finding a root of fx is equal to zero we have been using only fk, fk minus one. So we have got only one evaluation at any particular stage but here we need fk. We need f prime at k plus one and f prime k. Therefore at any stage I need to evaluate two evaluations i.e. your fk is required and f prime of k is also required.

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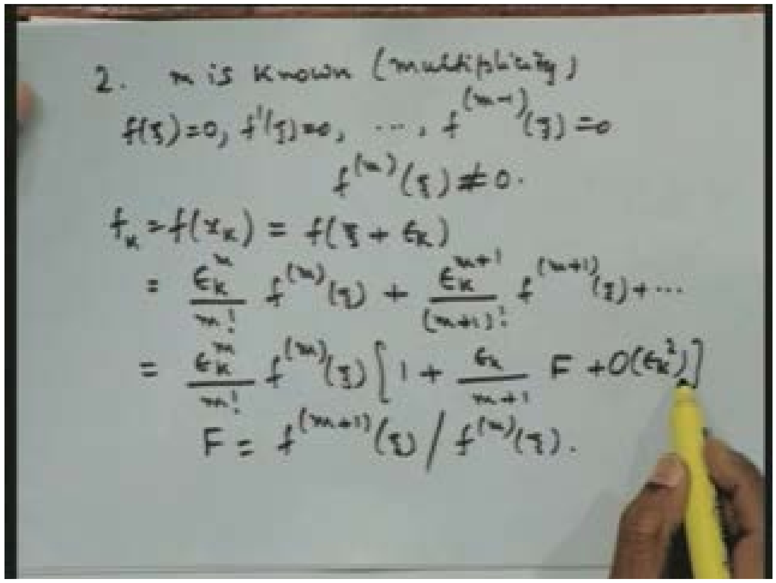

So we have now again increased one function evaluation as the cost. Therefore it will be more expensive than the normal secant method. As I said earlier if the evaluation of f prime k is not very costly, we can definitely go about using the secant and the Newtan Raphson methods. However if the muliplicity is known; let me asume that the multiplicity m is known. Now I would like to use the definition of multiplicity as we have written earlier. Let us repeat it once more; the f of xi is equal to zero, f prime of xi is equal to zero upto m minus 1^{th} derivative; all of them are zero and n^{th} derivative at xi is not equal to zero. So this is the definition of the multiplicity at the root x is equal to xi. I would like to have a look at what happens in the error analysis as we have done in the Newtan Raphson method. Let us start with the what is the definition of fk. Definition of xk is, f of fx and xk is the exact solution plus the error. Now I would expand this in Taylor series. We are just following the steps that we have done for showing the error analysis of the Newtan Raphson method. When I expand it, I will use this information. Therefore when I write f of xhi the first term is zero and second term is epsilon k f prime of xi i.e. zero and so on. All the terms, the first m terms are zero because of this information. The first non vanishing term in the expansion is containing f of m nth derivative of xi. Now if I write that one, the first non vashing term is epsilon k to the power of m divided by factorial m, nth derivative at xi plus (the next term is) epsilon k m plus one by m one plus derivative f of m plus one of xi. Before I add the denominator let us first simplify this expression. Take out whatever is possible for us. This the common factor here. So I will take out the factor epsilon k to the power of m by factorial m; fm xi also I will take it out and write this as one plus (I have taken out epsilon k to the power of m) I have epsilon k left out here; denominator is m plus one and this ratio let us just call it as f. Let us call the ratio F as the ratio of f m plus one of xi diveded by nth derivative with respect to xi. So I have just taken the common factor here and then I have written this remaining part as one plus epsilon k by m plus one; f is this ratio and whatever we are writing here are second order terms and higer order terms.

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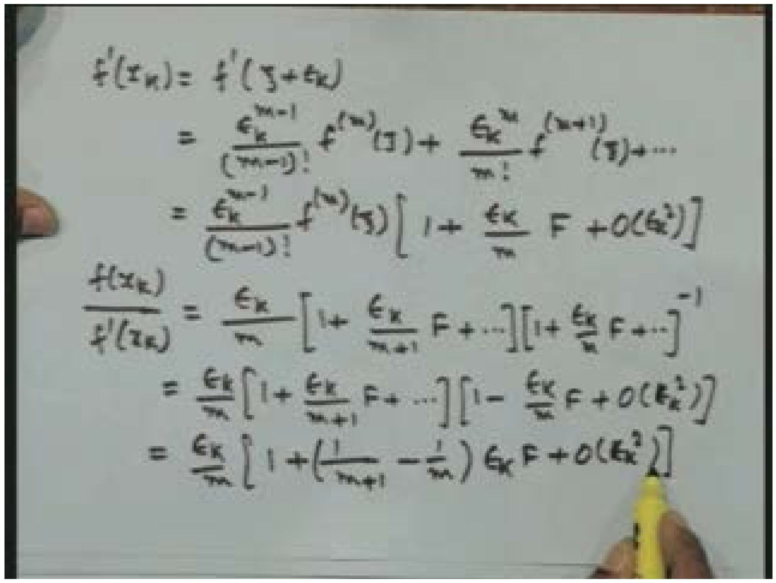

Now the same thing we we would like to do for the denominator also. Now if I take the denominator, the denominator will contain f prime of xk. So this is f prime of xi plus epsilon k. Now again expand this by Taylor series. Again I would be using the information that all these derivatives are zero upto m minus 1th derivative and this will be the non vashing term. Therefore the first non vanishing term will be containing f nth derivative but this will be epsilon k to the power of m minus one by m minus one factorial and f of m xi plus the next term is epsilon k to the power of m by factorial m fm plus one xi plus so on. Now as we have done in the previous case let us now take out the common factor here also; epsilon k to the power of m minus one by m minus one factorial, fm of xi is again a common factor. Now whatever is left out is, one epsilon k here, m here; this ratio is exactly the same f; fm plus one divided by fm xi is same as f and order of epsilon k square. Now I will take the ratio of f of xk upon f prime of xk. Now let me just put this slide back once more; we have here fk, the factor outside is epsilon k to the power m by factorial m fm of xi and the factor here is epsilon k to the power of m minus one m minus one factorial m^{th} derivative of xi.

Now if I take the ratio of these two you can see that fm xi cancels, epsilon k to the power of m minus one cancels and m minus one factorial cancels. So whatever is left out is simply epsilon k in the numerator here and m in the denominator, so remaining parts get cancelled out of this particular ratio and what we have here is one plus (I am writing this particular term) epsilon k upon m plus one into f plus higher order terms; in the denominator is this particular factor which i will take it up and write this as epsilon k by m to the F to the power of minus one. So I have taken the denominator up.

Now let us open it up or simplify it further. I will retain the first term as it is, epsilon k by m plus one of F. Then I expand it out by binomial expansion; one minus epsilon k by m into F plus order of epsilon k square times. Now let us multiply it out and simplify it, so what I will have here is epsilon k by m into one and epsilon k term is one upon m plus one from here, minus one upon m from here, epsilon k into f plus order of epsilon k square. This is epsilon k square; this is epsilon k square and this is epsilon k square. So whatever is left out are all the remaining terms of order of epsilon k square.

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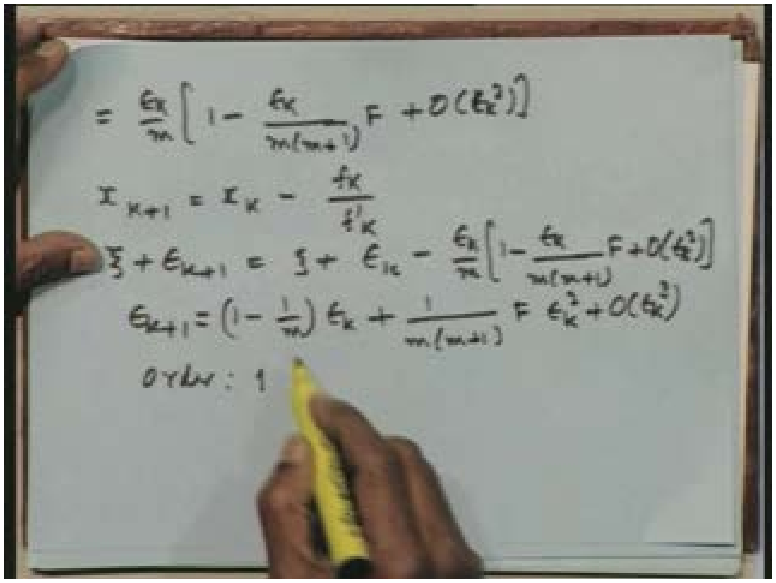

Now let us simplify it further. So I would therefore have this as epsilon k by m into (now here I am simplifying this, therefore I will have here m minus m minus one). So I will have one minus epsilon k by m into m plus one into f plus order of epsilon k square. So this is what I have from the simplification. May be if you just want to have both the terms together; I will bring this slide down once and we can just have a look at what we have. This is one minus epsilon k by m into m plus one into fk square.

Now let us put this into our Newton Raphson method and see why the difficulty was arising. So the method was xk plus one is equal to xk minus fk by f prime k.
Therefore we have the error as xi plus epsilon k plus one is xi plus epsilon k minus (now we have just now obtained the ratio - this and this; this is ratio) therefore this is epsilon k by m into one minus epsilon k by m into m plus one into f plus order of epsilon k square. Now let us simplify; xi cancels off from here, so the left hand side is epsilon k plus one and on the right hand side I will connect epsilon k from here, so one minus one by m into epsilon k. This is plus, we can write down this quantity; one upon m, m plus one f epsilon k square plus order of epsilon k cubed. So we just collected the terms and wrote this as one minus one upon m into epsilon k; this is plus one upon m into m plus one f into epsilon k square plus one.

Now the error for the Newton Raphson method; the right side is epsilon k. Therefore order is one. Therefore Newton Raphson method as applied to finding a multiple root would drop down the order by one from quadratic convergence to linear convergence only. As I said this can easily be experienced when you are actually doing the computation. When you know that the method should converge fast, after few iterations what would happen in your computation when look at the results is, you find that after even large number of iterations the error is reducing it very slowly. If you know the exact solution of the problem and then see the error, the error is reducing by only a factor of one decimal place or so. Therefore this slow convergence will imply that you are possibly going at a multiple root. Now we have shown that for a multiple root Newton Raphson method has only order one; then what do we do? We can do a very simple modification to the Newton Raphson method.

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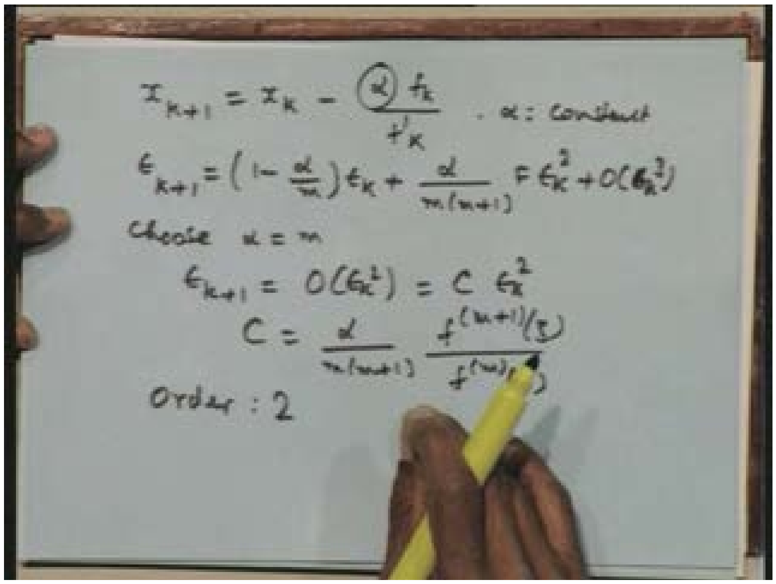

That simple modification is; suppose I start the Newton Raphson method with xk plus one is equal to xk minus alpha. Alpha is a constant. So I would try to start a method like this. Now let us just go back to this slide once. Why I have introduced an? I have introduced an alpha for this, because this ratio is fk by prime k. So let me go to slides behind; we have taken the ratio of f by prime k at this quantity which you have finally brought it to the form of this as fk by f prime k. Now if I put an alpha over here, what I am doing here is; I am bringing alpha over here. If I am bringing an alpha over here this is going to be an alpha here, that means this error expression would therefore look like, (I would just write the same thing straight away because it is trivial quantity) one minus alpha by m epsilon k plus alpha m into m plus one F epsilon k square plus order of epsilon, epsilon k cubed. Therefore if I introduce this alpha over here, I am introducing an alpha here in this bracket, one minus alpha by m and I am multiplying a by alpha in this one.

Now you can see that we started with alpha as a constant. If I now chose alpha is equal to m, then epsilon k vanishes. Now we shall say, choose alpha is equal to m. If I choose alpha is equal to m, then epsilon k square is order of epsilon k square or we can call it as C epsilon k square, where now C is this quantity alpha by m into m plus one; if you open it out (what we had written it earlier) fm plus one xi by fm of xi. So the order has immediately gone up to two and now we have got the rate of convergence of this method as two. That means again we have got back our quadratic convergence by simply using the order of multiplicity that is known to us in advance in the Newton Raphson method here. If we are able to know it therefore we are now avoiding evaluation of one more function. As we have seen earlier we wanted evaluation of f at xk, f prime at xk and f double prime at xk for the Newton Raphson method to have the required second order convergence. So here I don’t need any extra evaluation. The cost is the same;

identically same as the cost as the earlier method for simple root except that we know in advance the order of multiplicity. So that factor can be brought in here and then use that as the required Newton Raphson method, the order of convergence will be remain same. Of course if you do not know the multiplicity of the root, we have to follow the other way of getting the root.

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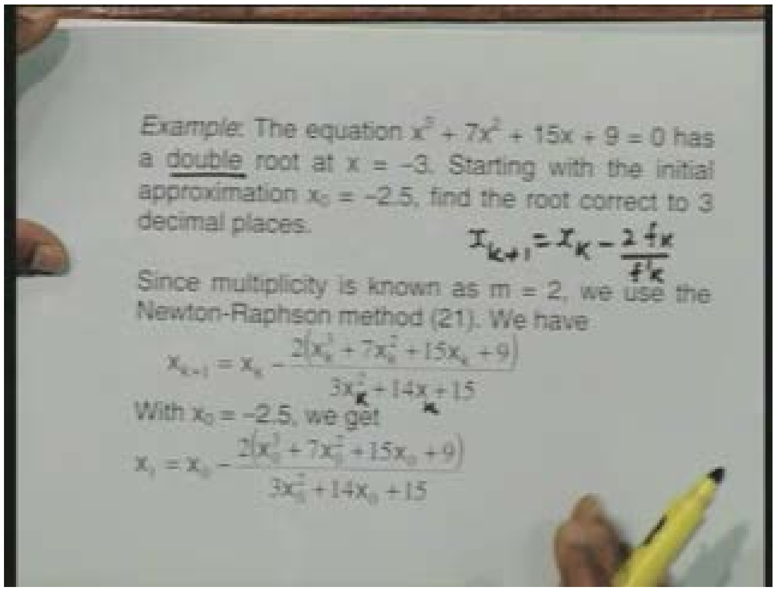

Now let us take a simple example of this. Now I take this equation x cubed plus seven x square plus fifteen x plus nine is equal to zero. It is a double root. It has double root at x is equal to minus three. I start the initial approximation x_0 is equal to minus 2.5 and the find the root corrected to three decimal places and I would be using the Newton Raphson method.

Now the first thing is, the multiplicity given to us; it is a double root. Therefore m is equal to two. Now since m is equal to two is given to us I would now write down my method as xk plus one is equal to xk minus two times fk by f prime k. So I would now be using that xk plus one is equal to xk minus two times fk upon f prime k. Now let us substitute the values of f of xk, f prime xk i.e. xk plus one is xk, two f of k is from here, that is xk cube plus seven times xk square plus fifteen times xk plus nine. We are differentiating with respect to x. So I will have three x square plus fourteen xk plus fifteen. I have differentiated this; this is three x square plus fourteen xk plus fifteen in the denominator.

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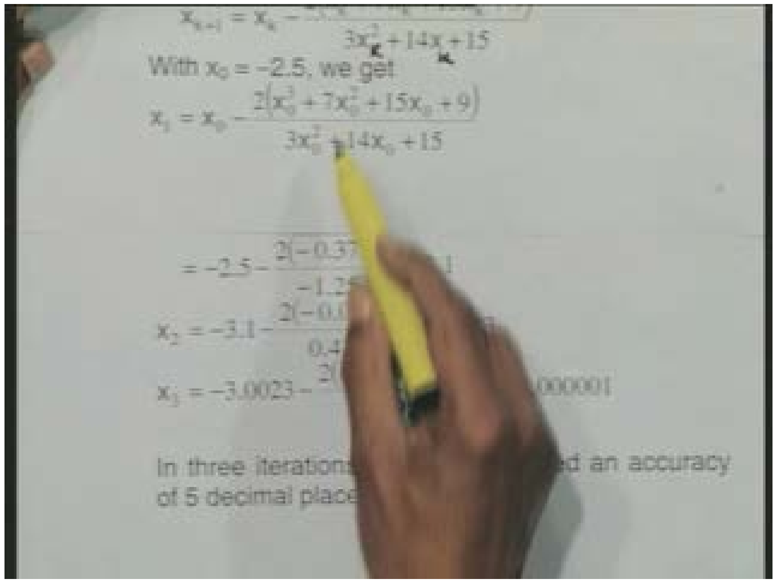

Now we start the iteration. It is given to us the initial approximation as minus 2.5.
So I can substitute it here; k is equal to zero, I have the right hand side as x_0, two times x_0 whole cube, seven time x_0 square, fifteen times x at zero plus nine divided by three x_0 square plus fourteen x_0 plus fifteen. So we just substitute the value of minus 2.5 in the expression and then evaluate it from here. I substitute it over; this is minus 2.5 and I evaluate the numerator, two times minus 0.375, the denominator is minus 1.5 and that is equal to the value minus 3.1. So minus 2.5; the iteration has gone to minus 3.1; I substitute x_2 is equal to x_1 minus two times everything evaluated at x_1 numerator and denominator is also evaluated at x_1. So I have here minus 3.1 minus two times of what I evaluated; I evaluate this at x_1 i.e. at the value minus 3.1, I get minus 0.021 and the denominator is 0.43. I simplify the whole thing. I get minus 3.0023. I repeat the same thing by taking this as x_2 two and numerator evaluated at x_2; denominator evaluated at x_2. So I have got here minus 3.0023 minus two times the numerator by denominator and this is minus 3.5000001.

Now we can see that in three iterations we have got an accuracy of five decimal places. Indeed it would be very interesting for you to just take it as an exercise and solve the same problem by Newton Raphson method without inserting the factor two. You will be able to know just after three iterations with your computations that we are converging very slowly in this one. So hence if I do not know the multiplicity that is given to us, I would go to the alternative way of finding the second derivative also and then using the Newton Raphson method.

Now very important extension or application of the solution of a single nonlinear equation is a system of nonlinear equations. Why it is very important here is, in most of the practical applications the mathematical model of the physical system is either ordinary differential equation or a partial differential equation and in most cases the process are nonlinear processes. Therefore the differential equations that are produced are nonlinear ordinary differential equations or nonlinear partial differential equations. Now we use some numerical methods; finite difference method or finitely methods to solve this equation and we produce a system of nonlinear algebraic equations which are the solutions of nodal points; the points of intersection of the mesh. Therefore what we have is thousands of nonlinear algebraic equations and how do we solve it? It is possible for us to extend directly the methods that we have discussed now for solving the system of nonlinear equations.

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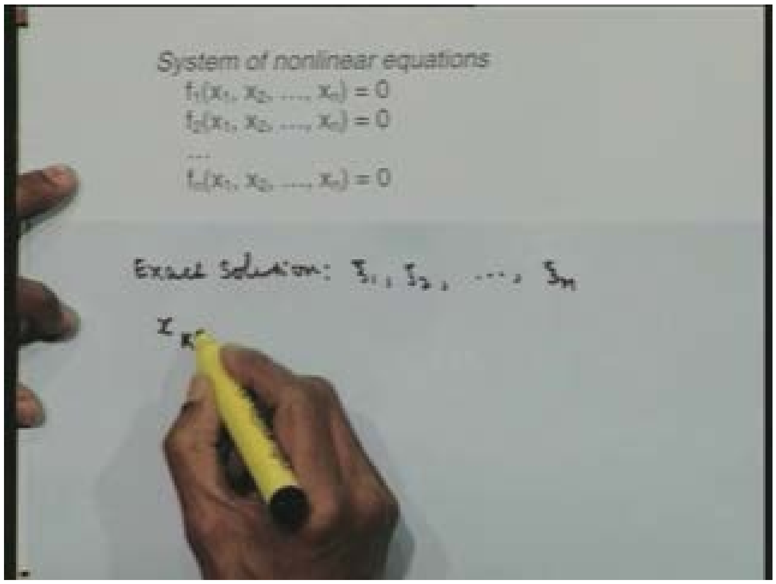

So let us take the system of nonlinear equations as this. So I have got n equations as unknowns. Now as I said it is not unusual to get few lakhs of equations, particularly in finite difference methods. If you have got a three dimensional problem, in the three dimensional problem it is not difficult at all to get the thousands or even few lakhs of nonlinear equations. Now one comment before we give the method is that, we have already seen that if Newton Raphson method is to converge it is necessary to have some good initial approximation; we cannot be very far away from the exact solution of the problem. Now the case becomes much more important or discussion becomes more important in the case of system of nonlinear equations. Wherein let us say you have got ten variables and if one variable or two variables are given very accurately but other variables given are very far away from the exact solution. By the time those have been modified the solutions which are close enough could be spoiled. Therefore it is necessary that all these values, through some other procedure or from physical considerations, one must have an idea of what could be the order of magnitude or what the type of solution could be, so that it can be used as your starting solution.

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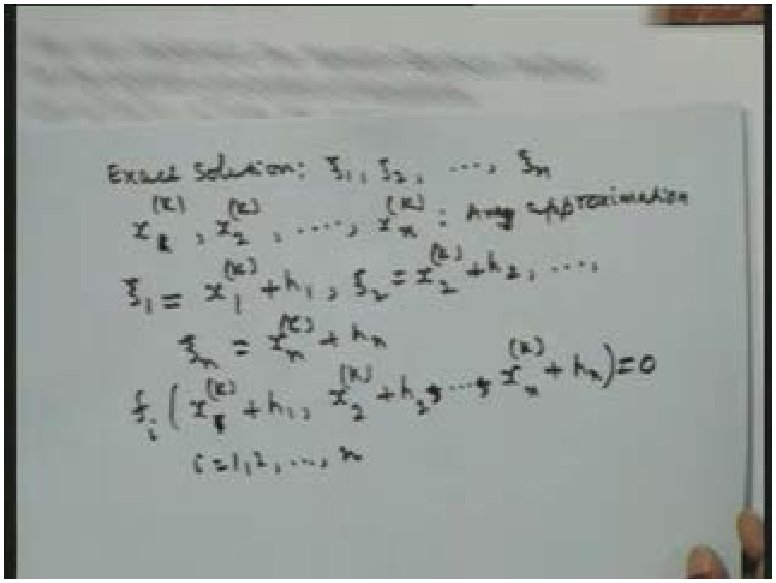

We shall follow the same procedure as used in deriving the Newton Raphson method earlier in one of our lectures. So let us take the exact solution of this particular problem. Let us say the exact solution is xi_1, xi_2 and $xi_{n.}$ So this is exact solution of this problem. We have used the Taylor series expansion to get the Newton Raphson method. One of the ways of deriving is using the Taylor series method. We shall use that method to derive the method for the nonlinear equations. Now let us take any initial approximation; x_1k, x_2 k and x_nk. So this is any approximation. I would now add to these suitable quantities, so that they become the exact solution, which means I will add xi_1 is equal to x_1 k plus some h_1, xi_2 is equal to x_2k plus h_2 and so on. I will put xi_nk plus h_n is equal to xi_n. So I am adding the suitable quantities so that I get the exact solutions. Therefore if I insert this here, xi_1, xi_2 and xi_n are exact solutions. Therefore when I substitute, it is satisfied identically. So if I take any equation, any i_{th} equation, this will read as x_1k plus h_1 x_2k plus h_2 plus x_nk plus h_n is equal to zero. I have substituted these values in this.
So i is going from one two three n.

What I now do here is, I expand this in Taylor series; i.e. the function of n variables and retain only the first order terms, that means I would now write this as f_i evaluated as x_1k, x2k, x_nk plus; from earlier expansion this will be the h_1 into partial derivative with respect to x_1 evaluated at the point, plus h_2 partial derivative with respected to x_2 evaluated at the point and so on. So this will be having h_1 into delta f_i upon by delta x_1. All of them are evaluated at k, so you can put it as "at k" if you like. Then we will have this as h_2 partial derivative of f_i with respected to x_2 evaluated at xk plus so on and h_n partial derivative of fi by x_n evaluated at k is equal to zero.

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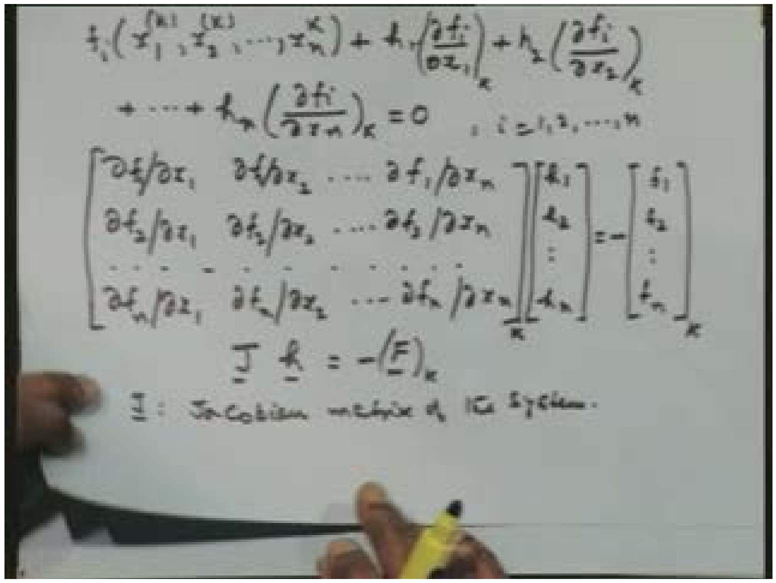

Now I have a collection of these equations one, two, three, n all these equations. I take this. This is a quantity that is known to us; x_1k, x_2k, x_nk is a previous iteration. So the computed values are all known to us. So I can take this to the right hand side and then I will write this in the matrix format for each one of them. So let us take i is equal to one, two, three, so on. So I will have delta f upon delta x_1, delta f of upon delta x_2, delta f_1 by delta $x_{n,}$ f_2 by delta x_1, delta f_2 by delta $x_{2,}$ delta x_n. So we are putting in the matrix format. So this will be delta f_n by delta x_1 delta, f_n by delta x_2. This is delta f_n by delta x_n. Multiply this by h_1, h_2, h_n. Now I write this particular thing; this is taken to the right hand side, therefore I will have this as minus f_1, f_2, $f_{n,}$ all of them evaluated at the iterate k i.e. k^{th} iterate. This matrix if you denote this by J; this by vector h, this is a matrix h; this is the right hand side, let us call it has f; f is evaluated at k. So I can put it as f evaluated at k. This J is called the Jacobian matrix of the system. So it has a name called the Jacobian matrix of the system. Now I think we better put a suffix here also, because delta f_i by delta x_1 all of them are evaluated at xk. So these are all evaluated at x_k. Therefore these are all numbers, these are all constants. Therefore given at any iteration $x_{k,}$ we have a constant coefficient matrix multiplied by the vector and the right hand side is known; therefore what this has produced us is a system of linear of algebraic equations. So this is a system of linear algebraic equations. Now therefore we need methods for solving the system of linear algebraic equations, which is a topic which we shall be taking in the later lectures as to how to solve a linear system of algebraic equations.

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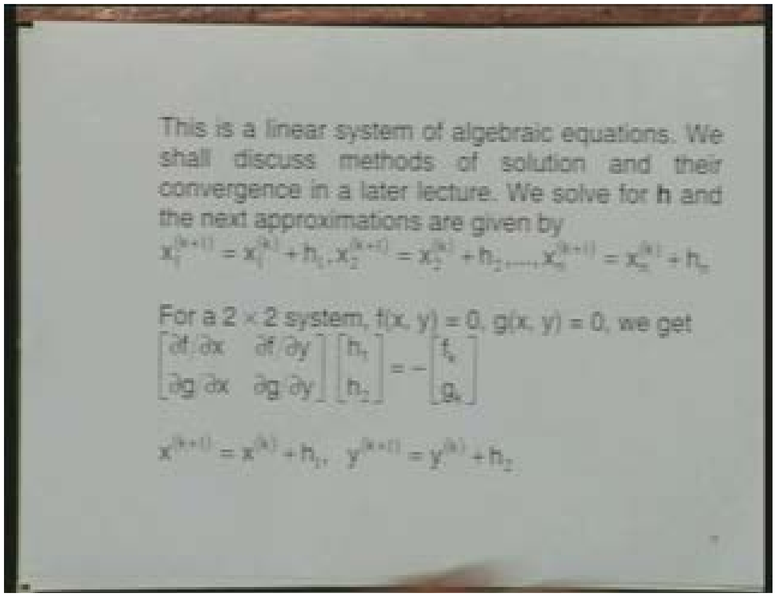

Let us now consider a particular case of a two into two system and show how we can obtain the solution; we shall then apply it to get the solution in an example. Now let us take simply a two by two system and see how we can solve a simple problem. Let us take a two by two system, take the system as f(x, y) is 0; g(x, y) is 0. For this what we will have here is delta f by delta x, delta f by delta y and the second row is delta g by delta x, delta g by delta y; the vector is h_1 and h_2; right hand side is f_k, g_k; f is evaluated at $x_{k,}$ y_k and g is evaluated at x_k. $y_{k.}$ What will be the next approximation? We should have mentioned there. The next approximation will be, as you can see x_1k plus one is equal to x_1k plus h_1; x_2k plus one is x_2k plus h_2; x_nk plus one is x_nk plus h_n. So the remaining part is the same as the Newton's method. So this is really an extension of Newton's method to a system of nonlinear equations. So once we determine this h_1 and h_2 from here we can go back, evaluate these partial derivatives again with respect to k; I can as well put this as a suffix with k. I will evaluate everything at the current iterate again, solve the system, get the increment and then again repeat the same thing. So that means for each iteration, we shall be solving one linear algebraic system of equations. The order of the system depends on the number of the equation that is given to us. Here we are solving two by two system; in the general case we are solving the n by n linear algebraic system where we have one particular system for each iteration.

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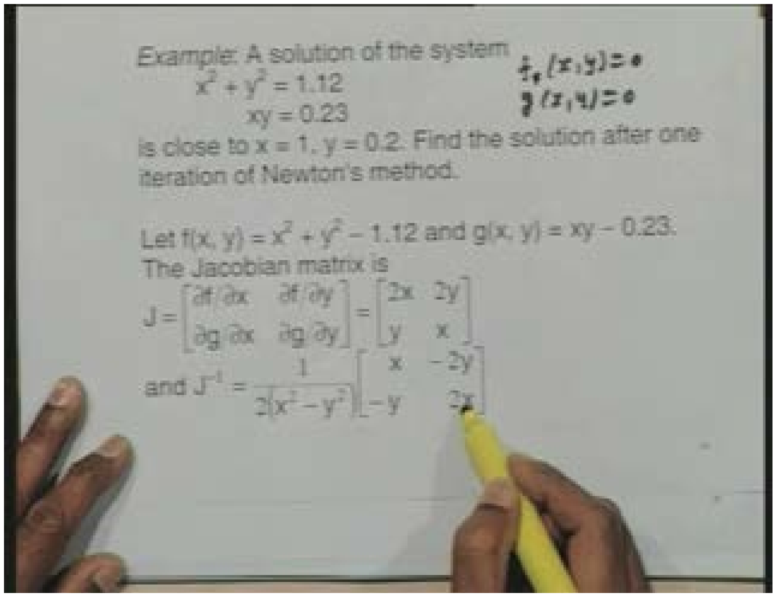

Now let us take a simple example on this. We are given x square plus y square is 1.12. I have taken a simple example in which you will be able to eliminate it, but we will not eliminate it. Just to illustrate it we have a simple example; x square plus y square is 1.12, xy is 0.23. The solution of this problem is close to 1 and 0.2, so that means we are given initial approximation to the problem. Find the solution after one iteration in the Newton's method. So what I need to build up is the Jacobian's matrix of this first. So I build up the Jacobian. Now we will write this as f(x, y) as the first equation, so we take everything to left hand side. Even when we are writing this, we will have to write this as f_1 of (x, y) is equal to zero or g of (x, y) is equal to zero. So these are the two equations; therefore I will bring this right hand side to the left hand side and define it as f. For the second equation I bring 0.23 to the left hand side, define it as g (x, y), therefore f (x, y) will be x square plus y square minus 1.2 and g (x, y) is equal to (x, y) minus 0.23. Then the Jacobian matrix will differentiate this partially with respect to x; differentiate this partially with respect to y; then differentiate g also partially with respect to x and y. So I have here two x, I have here two y and the derivative of this will be y and x. Therefore the Jacobian matrix is this.

Now since it is two by two system what I have done here is, I have inverted it; because once I invert it, I do not have to repeat the solution of the system of linear equations. I just have to evaluate what it is. So this is a two by two system. So I will find the determinant of this i.e. two x square minus two y square that goes into the denominator. I will write down the co-factors for this. Co-factor for this is x; co-factor for this two x; co factor for this and its transpose also, I take it as minus two y and minus y here. So I get this as the inverse of this particular matrix J. Then once I write this as the inverse of the matrix, I can now write down my system of equations.

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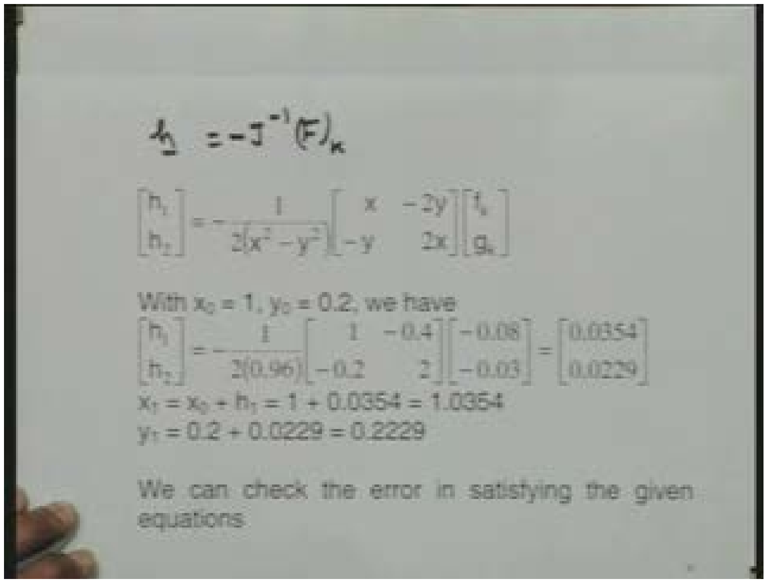

So h_1, h_2 are there. Now what we are writing here is the next step. We are writing this vector h is equal to J inverts of $(F)_k$. So this means h is equal to minus J inverts of $(F)_{k.}$ Now we have h is equal to h_1 h_2. We have just evaluated the J inverts. So I substitute it here; one upon two x minus x square minus y square and the Jacobian is this, we have just written it. Now I will substitute the initial values x_0 is one, y_0 is equal to two, then it is a simple matter of computation. So this is one divided by two times. I have computed what is x square minus y square; set the value of x as one; two x is equal to two; minus 2_a minus 0.4; minus y is minus 0.2. Now this is the evaluation of f at xk f at xk, yk; that means I have evaluated f (x, y) and g(x, y) from this and written these values as minus 0.08 and minus 0.03. Now just multiply it out and divide it out by this quantity and produce this as 0.0354, 0.0229 and we have x_1 is equal to x_0 plus h_1, y_1 is equal to 0.2 which is your initial approximation given to us plus 0.0229 that is 0.2229. This is the solution of this particular problem. The initial approximation was given to us as 1 and 0.2 and now we have got the new solution as 1.20354 and this.

Now it is possible for us to check the error and satisfy the given equation, that means we can just put these values in the equation and see whether f is sufficiently close to zero or g is sufficiently close to zero. The values are f at x_1y_1 is equal to x_1 square y_1 square minus 0.00174 and g (x_1y_1) is this. So when we started it initially the values of f and k were here; this is your f at xk, yk; g at xk, yk. The value of minus 0.08 is now reduced to 0.00174 and 0.03 is reduced to 0.00079. So that means we are talking of how the equation is satisfied. So we are now testing whether the equation has been satisfied sufficiently. Now we are able to get the accuracy faster; probably you need just another three or four iteration to get an accuracy of four or five decimals places. Therefore it is possible for us to solve a system of nonlinear algebraic equation also by simple extension of the Newton's method.

Now earlier we made a comment that of all the methods that are available the Newton Raphson method is the most commonly used even today for any scientific and engineering applications.

The reason we have given was, the reasonable order of the method and the computational cost. Now if you look at these nonlinear systems, you will find that the extension of secant and other methods will be much more complicated and much more difficult and therefore the Newton Raphson method is very valuable for the system of nonlinear equations also.

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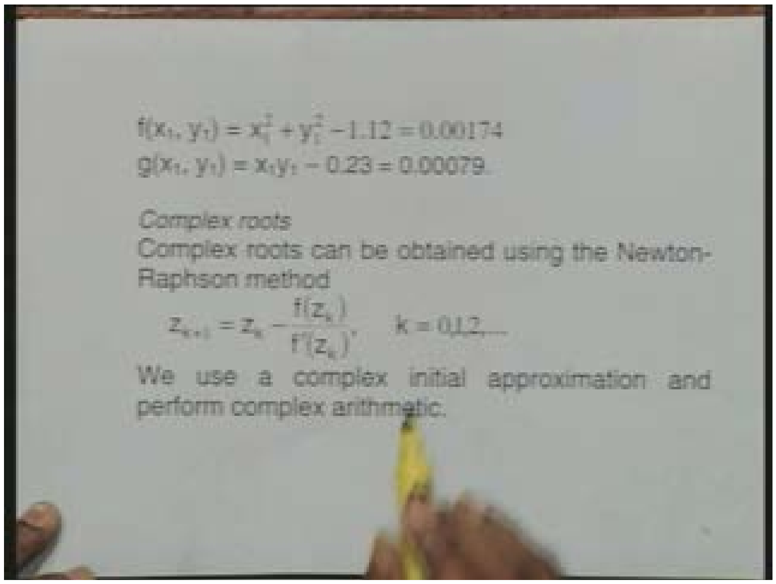

We all along talked of the real roots; a simple root and a multiple root. Now how do you get the complex root; say you want to solve x square plus one is zero, so there is no real roots; they are all complex roots, how do you get the complex roots? All the methods that we have done, all of them work; except that you have to do complex arithmetic. You have to start with the complex initial approximation and do the complex arithmetic. Every method that has been done so for can be used. Therefore that is what we have. For example, I can write down the Newton Raphson method as zk plus one is equal to zk minus f of zk by f prime at zk; k is equal to 0, 1, 2 and so on. Now the only thing is that, a real approximation cannot lead to the complex. So I need to have the complex initial approximation and I do the complex arithmetic. So you can define in advance the computation as complex and then go start with initial approximation and perform it; and the order of convergence retains as the same, whether it is a real root or complex root there is no difference. Therefore we do not need any special method for obtaining the complex root. However we can derive an alternative method for the solution. We shall illustrate this method through an example.

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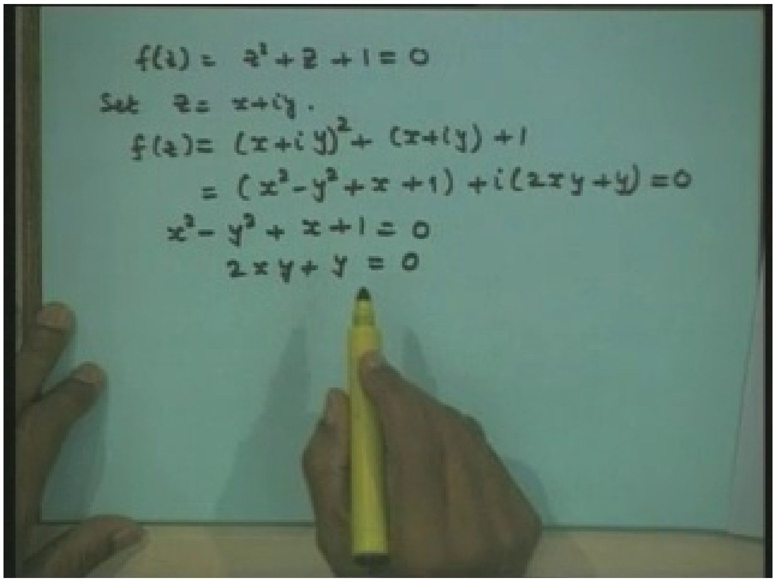

Let fz is equal to z square plus z plus one is equal to zero be the given equation. Set z is equal to x plus iy. Then we get f of z is equal to x plus iy whole square plus x plus iy plus one is equal to x square minus y square plus x plus one plus i into two xy plus y is equal to zero. Now setting the real part zero and imaginary part to zero, we get the two equations; x square minus y square plus x plus one is equal to zero two xy plus y is equal to zero. This gives a system of two nonlinear equations in two variables x and y. We solve this system by the Newton's method that we have derived earlier. However in this case the computation may take more time. We shall choose this procedure if we cannot perform the complex arithmetic and using complex initial approximation. Therefore the general technique in this particular case would be that substitute z is equal to x plus iy in fz is equal to zero; separate the real and imaginary parts, put them equal to zero, solve them as a system of two simultaneous nonlinear equations in two variables of x and y. That gives the alternate method of the Newton's method that we have discussed earlier. We shall stop it for today.