Numerical Methods and Computation

Prof. S.R.K. Iyengar

Department of Mathematics

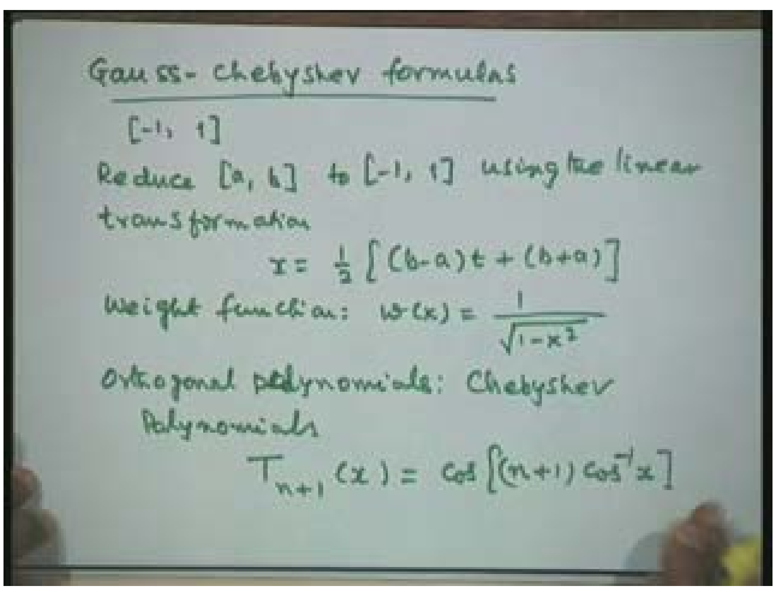
Indian Institute of Technology Delhi

Lecture No – 40

Numerical Differentiation and Integration (Continued)

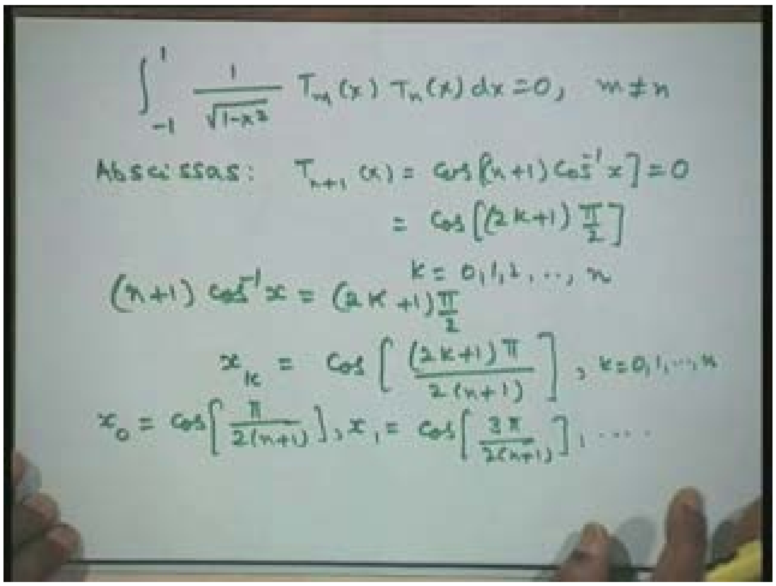
Now, in the previous lecture we have proved the general theorem on Gaussian quadrature rules, we have shown that with n plus 1 points we can derive a formula of order 2 n plus 1, in particular we have chosen the weight function as w(x) is equal to 1 taken the Legendre polynomials as the orthogonal polynomials and then constructed the Gauss Legendre quadrature rules. Let us now take the next set of formulas called the Gauss Chebyshev formulas; they are called the Gauss Chebyshev formulas.

(Refer Slide Time: 01:31)



Now here again the basic interval that we shall consider will be minus 1 to 1 therefore given an interval [a, b], we shall reduce it to minus 1 to 1 by using the same linear transformation, so if the given interval is [a, b], reduce [a, b] to minus 1 to 1 using the linear transformation, this linear transformation we have derived last time, we can use the same transformation. This transformation we have written it as x is equal to half of [(b minus a) into t plus (b plus a)], now t takes the values minus 1 to 1, when x takes the values a and b. Now the weight function for the case of the Gauss Chebyshev formulas is the following, weight function w(x) is 1 upon under root 1 minus x squared. The corresponding orthogonal polynomials, the orthogonal polynomials or the Chebyshev polynomials, Chebyshev polynomials $T_{n+1}(x)$ which are defined as cos of, we defined it as cos of [(n plus 1) cos inverse of x]. While discussing the approximation we have taken the properties of the Legendre polynomials and the Chebyshev polynomials also and the Chebyshev polynomial is defined as $T_{n+1}(x)$ is cos [(n plus 1) cos inverse x].

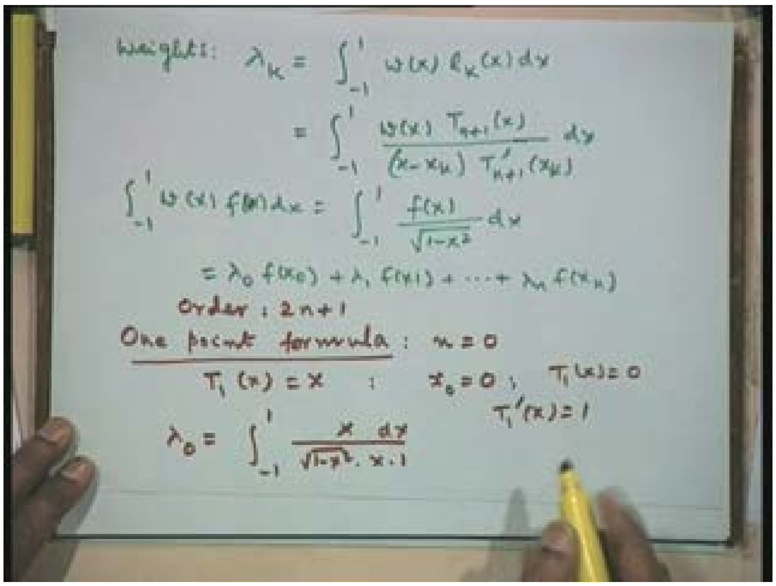
(Refer Slide Time: 03:47)



The orthogonal property of Chebyshev polynomials is that the Chebyshev polynomials are orthogonal with respect to the weight function 1 upon under root 1 minus x squared that is your w(x) $T_m(x)$ $T_n(x)$ dx is equal to 0 for m not equal to n, so that is a definition of the orthogonal polynomial, it is orthogonal with respect to the weight function w(x) over the interval minus 1 to 1, so this is the orthogonal property of the, or Chebyshev polynomials. Now when once we have fixed our weight function and the corresponding Chebyshev polynomials the remaining part is quiet straight forward in that we have to pick out our abscissas and weights and then write down the formula, so let us write down what will be the abscissas in this case. Abscissas are obtained by setting $T_{n+1}(x)$ is equal to 0, so I can put this cos of [(n plus 1) cos inverse x] is equal to 0. Now this is cos of certain quantity is equal to 0 so I can set it as cos of some [(2 k plus 1) into pi by 2], cosine of this quantity is 0 therefore this is equal to this, k is equal to 0, 1, 2, n, there are n plus 1 zeros, this is the polynomial of degree n plus 1, this has got n plus 1 zeros therefore we are setting k is equal to 0, 1, 2, 3 n thereby I can get the all the zeros from here.

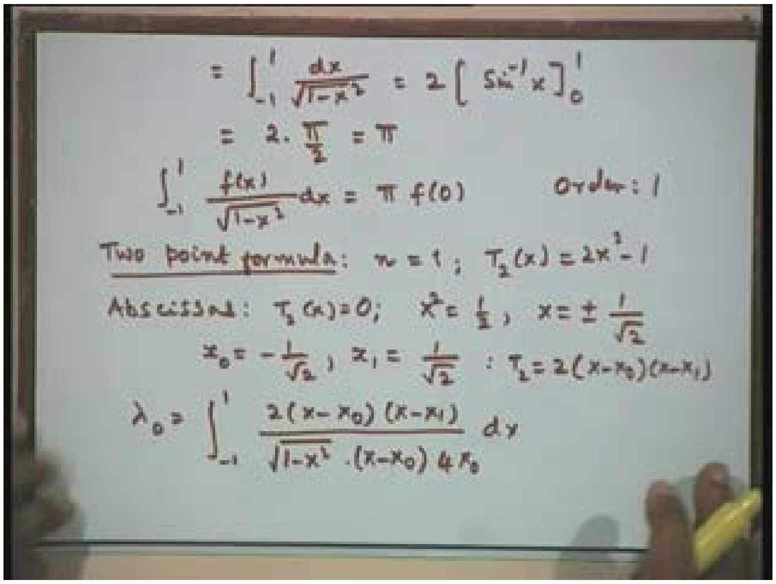
Now if I compare these 2 what I would get will be (n plus 1) cos inverse x is equal to (2 k plus 1) into pi by 2, I am comparing the arguments of the 2 functions. Then bring (n plus 1) to the right hand side and also cos inverse x to the side therefore I will have x, now I can put suffix k this is equal to cos of [(2 k plus 1) pi by 2 into (n plus 1)], this is [(2 k plus 1) into pi by 2 divided by (n plus 1)] and k of course, as we have written above 0, 1, 2, 3, n, these are the n plus 1 zeros of the Chebyshev polynomial of order n plus 1. Therefore in the general case this, when once n is fixed I can write down what are the zeros of this, I take k is equal to 0 so I will get pi by 2 into (n plus 1) as the first 0, I will get the next 0 as x_1 is equal to cos of, this is 3, [3 pi by 2 into (n plus 1)] and so on, so I can now get all the zeros of the Chebyshev polynomial as this one.

(Refer Slide Time: 06:43)



When once I get these abscissas, now I need the weights in the formulas, so let us write down what is our weights, weights are given by lambda_k and this we had shown earlier that this will be minus 1 to 1 of w(x) $l_k(x)$ dx, where $l_k(x)$ is the Lagrange fundamental polynomial built on this n plus 1 abscissas and an alternative form also we have written it, we can write in terms of these orthogonal polynomials so that will be w(x), we have earlier used as p_{n+1} but now it is $T_{n+1}(x)$ divided by (x minus x_k) into T_{n+1} prime x at x_k of dx, so these will be the weights of this formula. Now when once the weights and the abscissas are known then we can finally write down what is our formula, the formula is minus 1 to 1 f(x), w(x) f(x) dx that is minus 1 to 1 f(x) upon under root of 1 minus x square dx is a weight function that we have introduced and that is your lambda_0 f of x_0 plus lambda_1 $f(x_1)$ plus so on lambda_n f of x_n, so this will be the formulas that we will have. We know that the order of this formula is 2 n plus 1, order of the formula is 2 n plus 1. Now let us write down few formulas, 3 of them which are important and we shall use them in examples, let us write down 1 point formula. In the case of 1 point formula we have n is equal to 0. Therefore the corresponding polynomial will be T_1, T_1 of x is equal to x, T_1 of x is x and we need its 0 therefore the abscissa will be simply x_0 is equal to 0, by setting $T_1(x)$ is equal to 0 so we put $T_1(x)$ of x is equal to 0, I get x_0 is equal to 0. I need the derivative here for the weight so let us write down differentiate this T_1 dash of x is equal to 1 so it is derivative is only 1.Therefore lambda$_0$, let us put k is equal to 0 here, so I will get here minus 1 to 1, the T_1 is x in the numerator, I am writing T_1, w(x) is under root of 1 minus x squared, w(x) 1 minus x square, (x minus x_k) therefore I will have x minus 0, x_k is 0, derivative is 1 this is simply dx.

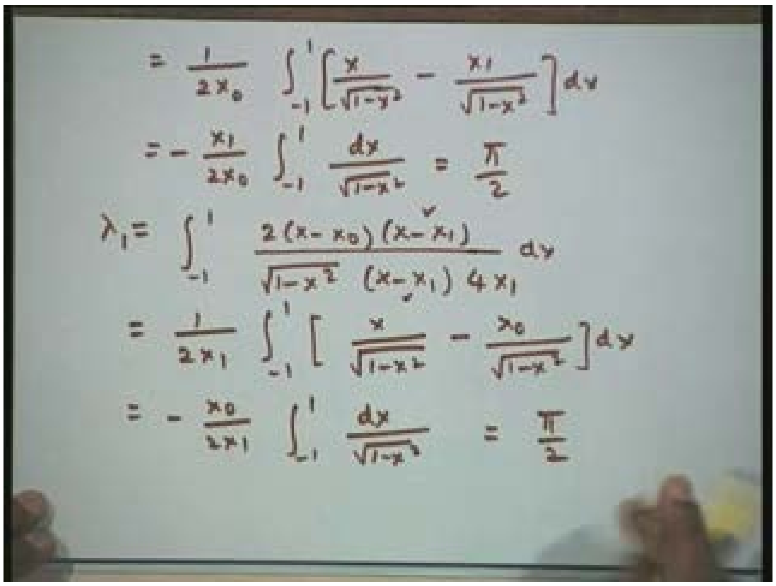
(Refer Slide Time: 00:10:14)



So let us just simplify this, this is equal to minus 1 to 1 simply dx upon 1 minus x square, under root of, under root of this and this is simply integrally sin inverse x, so I can just 2 times I will write down 0 to 1 so I will write it as [sin inverse of x] 0 to 1 that gives me 2 into pi by 2, sin inverse 0 is 0 so I will get simply pi. Therefore the 1 point formula is minus 1 to 1 f(x) upon under root of 1 minus x square dx is pi into f of 0, at the abscissa is x_0 is 0 and we have lambda$_0$ is equal to pi so simply pi at x_0 which you can see that it is almost like a midpoint rule, at the middle point only it has been obtained and the order of this formula is 1 that is your 2 n plus 1 therefore n is 0 so I will have the order of the formula as 1. Therefore this will integrate polynomials of degree less than or equal to 1 exactly. So let us derive a 2 point formula, therefore I need to take n is equal to 1, therefore I need to T_2(x) which is we have derived it as 2 x squared minus 1, therefore I can immediately write down the abscissas from here by setting T_2(x) is equal to 0. Therefore I would get here x square is equal to half, therefore x is equal to plus minus 1 upon root 2, so let us choose our x_0 is minus 1 by root 2 and x_1 is 1 upon root 2, so we choose the 2 abscissas as x_0 is minus 1 upon root 2 and x_1 is minus. Now I need the expressions for lambda's so let us write down our lambda$_0$, now before doing that let us do one more thing T_2(x) as, let us take 2 common out and write this whatever is left out as (x minus x_0) into (x minus x_1) because easy then to cancel so I will write down T_2(x) as, 2 I have taken common and this is a factor of (x minus x_0) (x minus x_1).

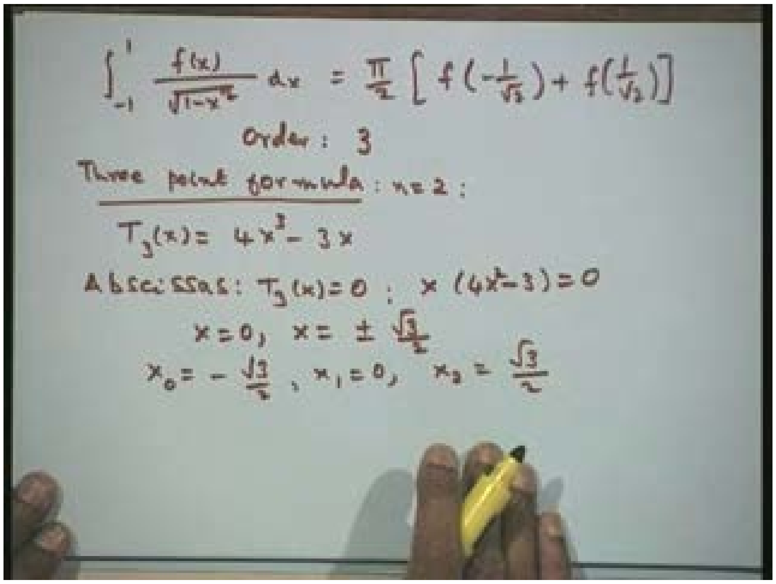
Therefore I will have here minus 1 to 1, this is T_2 in the numerator then the weight function 1 upon under root 1 minus x square this, into (x minus x_0), (x minus x_0) and T_2 prime is 4 times x therefore this will be 4 times at x_0 dx. Now this will, let us cancel of this (x minus x_0) x minus x this one and let us take x_0 is constant, so let us pull it out.

(Refer Slide Time: 13:44)



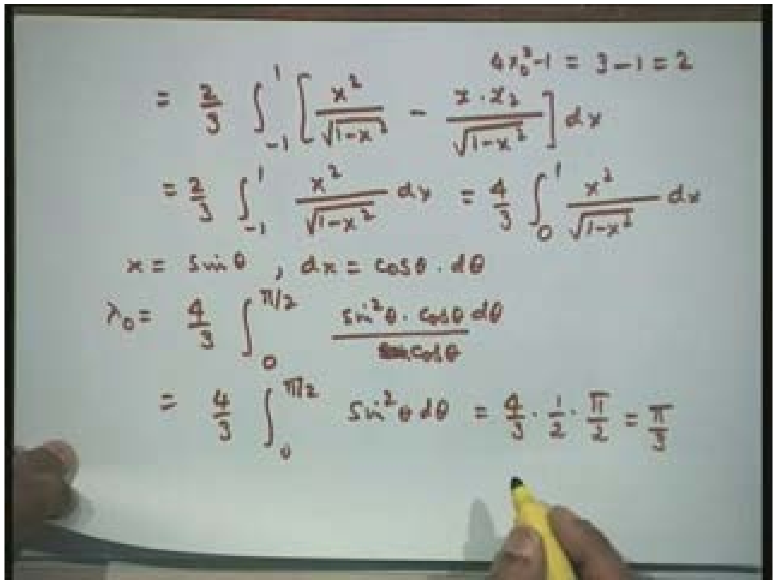
So it will be 1 upon 2 times x_0 and this is minus 1 to 1 x upon under root of 1 minus x square minus x_1 upon under root 1 minus x square. I just separated this (x minus x_1), x divided by under root of 1 minus x square minus x_1 upon under root of minus x square. This is an odd function therefore the value of this is 0 between minus 1 to 1 so this gives us a value and let us take the constant out, this is minus x_1 upon twice x_0, minus 1 to 1 dx upon under root of 1 minus x square. Now we have just now shown the value of minus 1 to 1 dx upon 1 minus x square is equal to pi so I can use this value as pi and here the ratio of x_1 and x_0 is minus 1 therefore ratio of x_1 upon x_1 is minus 1, so we will have a plus half into pi so I will have simply pi by 2. Now let us go to the next value lambda_1, lambda_1 gives us minus 1 to 1 2 times (x minus x_0) (x minus x_1) the weight function under root 1 minus x squared in the factor (x minus x_1) into T_2 prime is again 4, 4 times x at x_1 that is 4 times x_1 dx. Now let us again simplify this, we will take this x_1 outside so I will have 2 times x_1 minus 1 to 1 again x upon under root 1 minus x squared and this is, we have cancelled this (x minus x_1) and this (x minus x_1) what will left out is (x minus x_0) so I will have x_0 upon under root of 1 minus x squared dx. Now again this is an odd function therefore the value of this is 0 so again have this minus sign outside so x_0 minus 2 x_1, minus 1 to 1 dx upon under root of 1 minus x squared. Now this is same as this because ratio of (x_1 minus x_0) and (x_0 minus x_1) is minus 1, so I will again have this is pi by 2.

(Refer Slide Time: 16:40)



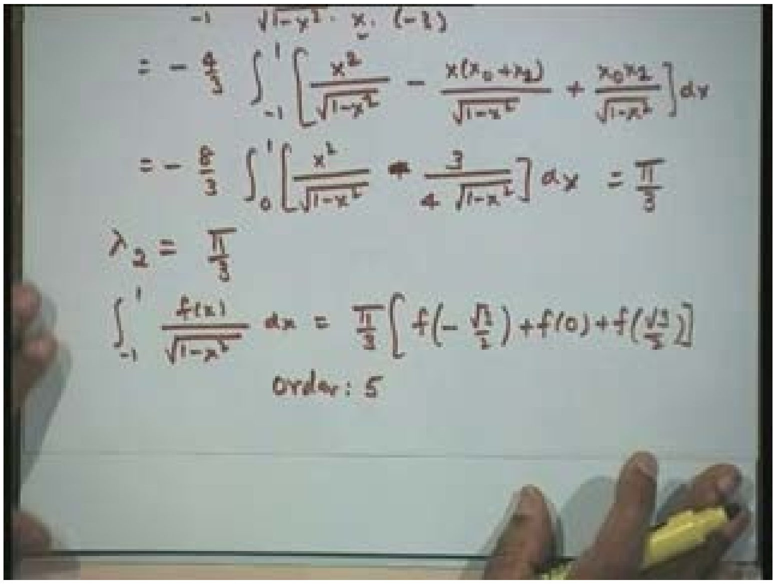
Therefore we are in a position to write down the formula, the formula is minus 1 to 1 f(x) upon under root 1 minus x square dx is pi by 2 into [f of (minus 1 upon root 2) plus f of (1 upon root 2)] and the order of the formula is 2 n plus 1 that is equal to 3. The order of the formula is 3 so it is going to integrate exactly polynomials of degree less than or equal to 3, again we can compare with the Simpson's rule wherein we have 3 points and they are equispaced points and here we have relaxed that condition therefore we are able to get a higher order formula. Now let us derive one more formula before we take an example, let me take the 3 point formula, therefore I need to take n is equal to 2 and the polynomial is T_3 of x which is defined as 4 x cubed minus 3 x, n is equal to 2 therefore the Chebyshev polynomial of order 3 that is $T_3(x)$ is 4 x cubed minus 3 x. Now I can again find the abscissas from here, I set $T_3(x)$ is equal to 0 so I would get here x into (4 x squared minus 3) is equal to 0 so I have the value is 0, x is equal to plus minus root 3 by 2. Therefore we shall choose the points as x_0 is minus root 3 by 2, x_1 is 0, x_2 is equal to root 3 by 2 and $T_3(x)$ we shall write this as, I will take common factor 4 out and then write the remaining as (x minus x_0) (x minus x_1) into (x minus x_2). Let me illustrate the finding out one of the lambdas, the remaining I will give the values of them. Now let us write down lambda$_0$ that is minus 1 to 1 T_3 that is 4 into (x minus x_0) (x minus x_1) that is 0 let us written it as 0, (x minus x_2) under root of 1 minus x squared into (x minus x_0) into, now I need the derivative of T_3 so let us write down T_3 dash x that is 12 times x squared minus 3, this is 12 x square minus 3 so that I can write it as 3 into 4 x square minus 1, so I will this as, 4 times x_0 square minus 1. Now we will cancel of (x minus x_0) (x minus x_0) from here.

(Refer Slide Time: 20:08)



Let us compute what is this value, this is 4 times x_0 square minus 1, x_0 is minus root 3 by 2 so that is 3 by 4 that is 3 by 4 into 4 that is 3 minus 1 that is 2, so this value of 4 x_0 square minus 1 is equal to 2. Now there is a 2 here and 4 let us take it out so I will have here is 2 by 3, I have taken 2 by 3 and what is left out is minus 1 to 1, now let us multiply it out x squared upon under root 1 minus x square, I am combining these 2 x square upon this minus x into x_2 upon under root 1 minus x square. I have taken this as 2 factors x into x x square and this is minus x into x_2, now you can see that this is an odd function now, therefore the value of the second integral will be 0 so only the first integral is going to contribute that is equal to 2 by 3 minus 1 to 1 x squared upon under root 1 minus x squared. It is an even function so I can write this as 4 upon 3 0 to 1 x squared under root 1 minus x square dx. So let us use your standard substitution x is equal to sin theta, then we have dx is equal to cos theta d theta, so that this integral simplifies to 4 upon 3, theta is equal to 0, sin 0 is 0, x is 1, theta is equal to pi by 2, x square that is sin square theta, dx is cos theta d theta, this is 1 minus cos square is sin square under root of this that is sin of, cos of theta, 1 minus sin square we have cos square under root of this cos theta. Therefore this is simply equal to 4 upon 3, 0 to pi by 2 sin squared theta d theta. Now let us use a reduction formula, this is equal to 1 by 2 into pi by 2, 0 to pi by 2 sin square theta 1 by 2 into pi by 2 therefore this is equal to pi by 3. Now let us write down lambda_1 also because it is slightly different from this coefficient, so let us just write down what is our lambda_1.

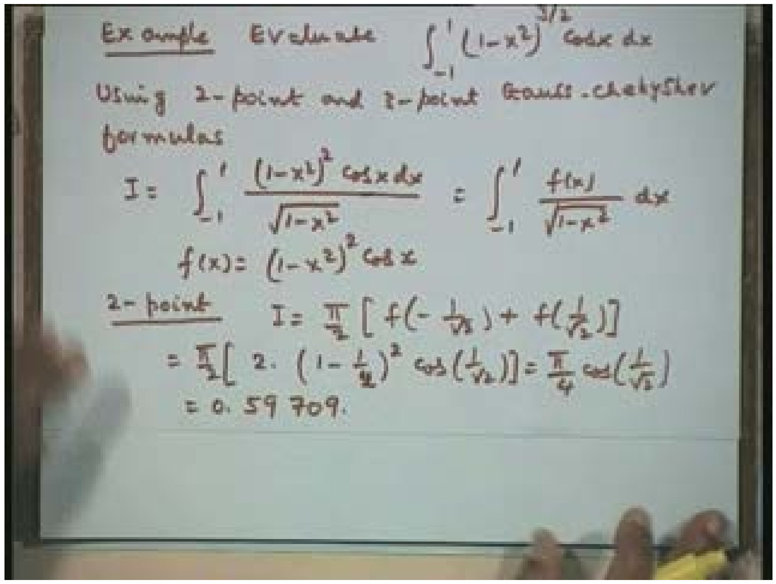
(Refer Slide Time: 23:10)



Lambda$_1$ is minus 1 to 1 4 times (x minus x_0) into x (x minus x_1) under root of 1 minus x square, we have a x here, then 4 x, the T_3 dash is 3 times 4 x square minus 1 at x is equal to 0 so I will have a minus 3 coming from here, so I will have minus 3 here and dx. Now this, therefore this gives us minus 4 by 3, now x x cancels here so I will have minus 1 and this product I will open it up as x squared upon under root 1 minus x squared that is the first term, then I am multiplying it out therefore we will have here x into (x_0 plus x_1), I am the writing the product of these 2 divided by under root of 1 minus x square, then the product of these 2 is plus x_0 x_1 by under root 1 minus x square dx, so we will have 3 terms in this case. Now this is 0, this is an odd function therefore the middle integral gives contribution is 0 so you will have contribution only from these 2 integrals and this is even, this even so I can now write this as minus 8 by 3 0 to 1 x squared upon under root 1 minus x squared plus, now we have found out the values of x_0 and x_1, x_0 into x_1, x_2 we have, using it as x_2, the middle one x_1 was 0 so x_0 plus x_2, x_0 into x_2 is the product of these two that is minus 3 by 4 so we will have this as minus 3 by 4, so let us make this as minus 3 by 4 into under root of 1 minus x square dx.

Now both these integrals have been just been evaluated earlier so we can, you can simplify this and I will give the value of this also as equal comes out to be pi by 3. We have evaluated the second integral while finding lambda$_0$ and the first integral was also evaluated earlier, lambda$_1$ comes out to be pi by 3 and now show that lambda$_2$ is also equal to pi by 3, lambda$_2$ is also equal to pi by 3. Therefore we will have the 3 point formula as f(x) upon under root 1 minus x square dx is pi divided by 3 f of minus root 3 by 2 plus f of 0 plus f of root 3 by 2 and the order of this formula is 5, order of this formula is 5.

(Refer Slide Time: 27:01)

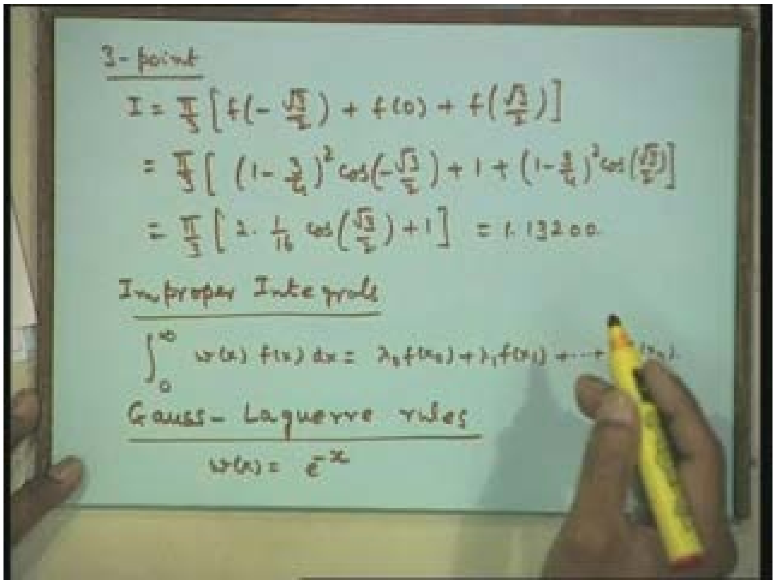


Interestingly it can be shown theoretically that in the general case all the lambdas are equal as it has happened here and indeed in the general case lambda$_0$, lambda$_1$ they are all equal to equal and they are all equal to pi upon n plus 1. In the case of 1 point formula n is equal to 0 we get the weight as pi, in the case of 2 point formula n is equal to 1 we get the weight as pi by 2, in the case of 3 point formula n is equal to 2 we get the weight as pi by 3 but in the general case we can show by integrating the Chebyshev polynomial, we can show that all the weights are equal and they are equal to pi by n plus 1. Now let us take an example, so let us say evaluate minus 1 to 1 (1 minus x square) to the power of 3 by 2 cos x dx using 2 point and 3 point Gauss Chebyshev formulas. Now before actual working out of the problems two things, important thing that we should notice whether the interval was given as minus 1 to 1 if it is not we use the linear transformation first reduces to minus 1 to 1.

Secondly we given the particular formula whether the suitable weight function is available or not, if the weight function is not available we provide the weight function that means multiply and divide by the weight function to produce the weight function for the formula to be used. In this example you can see that the weight function is not available for us that is under root of 1 minus x square, so we divide by under root minus x square multiply by under root 1 minus x square. So we shall first of all write this as minus 1 to 1, I have to divide by under root of 1 minus x square which is the weight function I need so I multiply by this, so would have this numerator (1 minus x square) whole square cos x dx. So I now provided the weight functions in the formula so that we can apply the Gauss Chebyshev rule and the limits of integration is minus 1 to 1 so we can apply the formula directly. Therefore I can recognize now my function f(x) as, this is of the, this is of the form f(x) upon under root of 1 minus x square, this is the Gauss Chebyshev form, therefore f(x) will be (1 minus x squared) whole squared into cos x, (1 minus square) whole square into cos x.

When once you recognize this, the application is really trivial so let us write down our 2 point formula. The 2 point formula gives us I is equal to pi by 2 [f of (minus 1 upon root 2) plus f of (1 upon root 2)]. So let us substitute it pi by 2, so we just substitute this in the, you can see that it is we have all even function completely so for both the values going to be same, so we can do multiply 2 times straight and this is 1 minus x square, 1 minus 1 by 4, 1 by 2 whole squared cos of 1 upon root 2. So this gives you 1 by 4, so this is pi by 4, this is simply pi by 4 cos of 1 upon root 2. I can evaluate this value and write this as point 5 9 7 0 9.

(Refer Slide Time: 31:17)

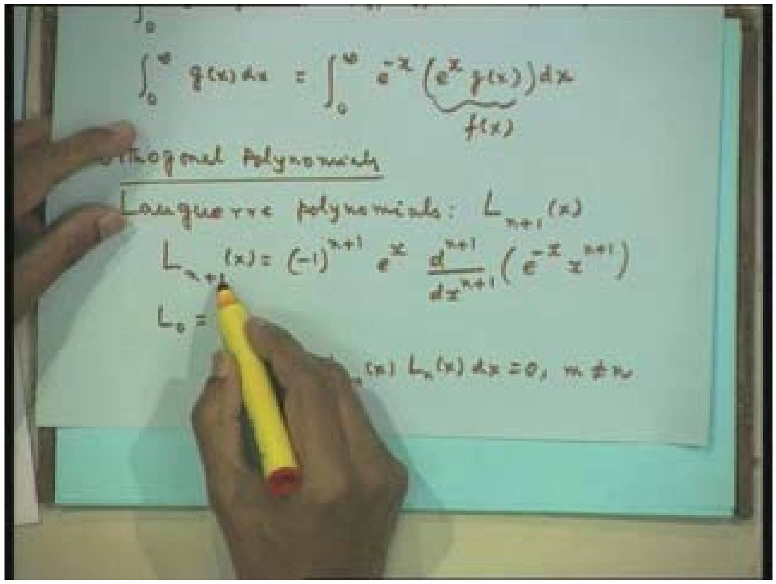


Let us now use the 3 point formula, so I can now write this as I is equal to pi by 3 [f of (minus root 3 by 2) plus f of (0) plus f of (root 3 by 2)]. Therefore we can evaluate them pi by 3, we substitute minus root 3 by 2 here, so we will have (1 minus 3 by 4) whole square cos of (minus 3 by 2), f of 0 is 1, cos 0 is 1 that is 1 and this is (1 minus 3 by 4) whole squared cos of (root 3 by 2). So these two we can combine them and write this as pi by 3 [2 times 1 by 16 cos of (root 3 by 2) plus 1] and the value of this is 1 point 1 3 2. We can see that evaluation of the integral is not simple, with 2 point formula we got the value as 0 point 5 9 7 and with 3 point formula we got the value as 1 point 1 3 2 because this order 5 and in fact we are not sure whether this result is still correct, we still have to take a 4 point or 5 point formula to see the difference between them and find out what is the real result here because you can see that even this simple problem turned out to be not that is simple, it turned out to be quite difficult that the convergence is not immediately seen, so if I am able to show that the difference between these two values is a particular tolerance then we can stop at that particular value. However the most important thing here is the tables of all the Gaussian formulas are available up to 16 decimal places that means the weights and abscissas are available for ours in the standard tables for 16 decimal places.

Therefore even if you do in double precision the entire computation you would not lease any accuracy because we have shown that the, in all the integration rules the round of error is not having great effect, it the at the most multiply by b minus a that the length of the interval that we have taken, therefore if you retain 16 decimal places for both weights and abscissas we can expect at least 12, 13, 14 places to be accurate in our result. Therefore you can get the results of any accuracy, you can go to the any higher order formulas as I said; it is available for any order in the tables. Now let us go to the other type of formula which is our improper integrals; which are also very important and that is your 0 to infinity, minus infinity to infinity problems, so let us look into those problems.

We shall first consider the formula of the type 0 to infinity w(x) f(x) dx is lambda_0 $f(x_0)$, lambda_1 $f(x_1)$, lambda_n $f(x_n)$. The first class of the formulas that we consider is where the weight function is w(x) is exponential of minus x; the reason is that the orthogonal polynomials are available for me which I have the weight function exponential of minus x and orthogonal over 0 to infinity and these are the solutions of the Laguerre differential equation, they are called the Laguerre orthogonal polynomials. As I mentioned earlier these orthogonal polynomials are the solutions of the differential equations that belong to a sturn levelly family, which are variable coefficient differential equations. For suitable expressions for the coefficients of y double prime, y prime and y we get Legendre's equation, chebyshev differential equation, Laguerre differential equation and Hermite differential equation. Now since these are variable coefficient differential equations, the solutions obtained in a series form by the frobenius method. The solution of these differential equations lead to orthogonal polynomials, the solution of Legendre's differential equation gives Legendre polynomials, the solution of Chebyshev differential equations gives Chebyshev polynomials, the solution of the Laguerre differential equation gives Laguerre polynomials and the solution of hermite differential equations gives hermite polynomials, so based on this what we really require is called the Gauss Laguerre, Gauss Laguerre rules or formulas, Gauss Laguerre rules. The weight function in this case is exponential of minus x, exponential of minus x.

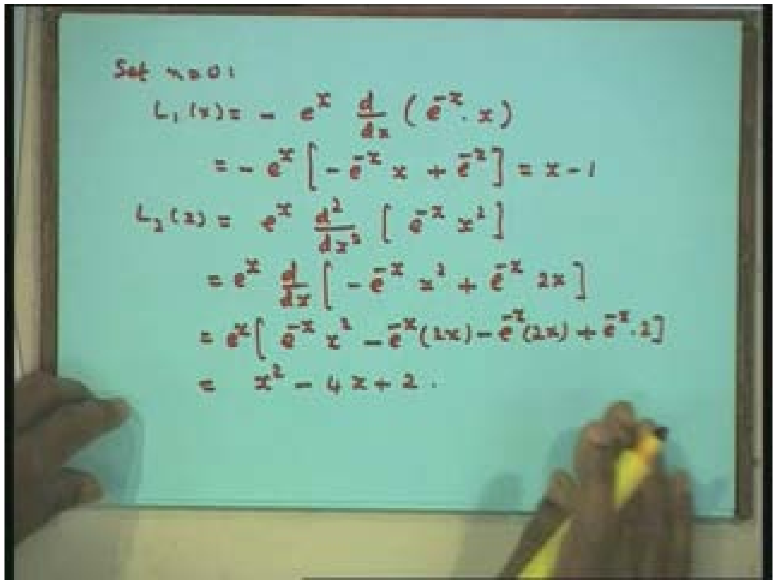
(Refer Slide Time: 37:03)



That means what we are looking for is an integration rule for 0 to infinity e to the power of minus x f(x) dx is equal to lambda_0 $f(x_0)$ plus so on lambda_n $f(x_n)$. As in the case of the Chebyshev polynomials if this weight functions was not available for us in a given problem, we shall supply the weight function that is multiply by exponential of minus x, divided by exponential of minus x that means if suppose you are given only a problems as g(x) dx, g(x) dx then what I would write it here is, I will write this as e to the power of minus x into e to the power of x g(x) dx, this will be my f(x), this will be my f(x). So even if the weight function is not available for us we shall provide the weight function and then get this one.

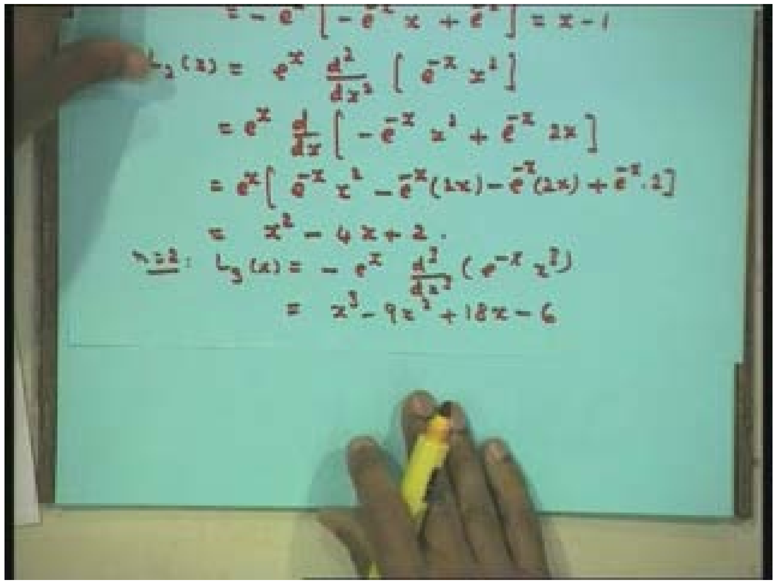
Now I need the set of orthogonal polynomials for this, orthogonal polynomials are called the Laguerre polynomials; they are called the Laguerre polynomials denoted by L_{n+1} of x. The definition of Lauguerre polynomial it can also be put in closed form as we are able to do Chebyshev polynomial, so we define it as this, minus 1 to the power of n plus 1 exponential of x d^{n+1} upon this, e to the power minus x to the power of n plus 1. It is analogous to the Rodriguez formula for Legendre polynomials with L_0 is equal to 1, where L_0 defined as 1. Therefore the orthogonal property of this should be 0 to infinity exponential of minus x $L_m(x)$ $L_n(x)$ dx is 0, so this is the orthogonal property of the Laguerre polynomials. Let us now find few of the Laguerre polynomials, now set n is equal to 0 in this equation then we get L_1 here, minus 1 to the power of 1 therefore a negative sign, n is 0 therefore is a first derivative, n is 0 I will have here x here.

(Refer Slide Time: 40:02)



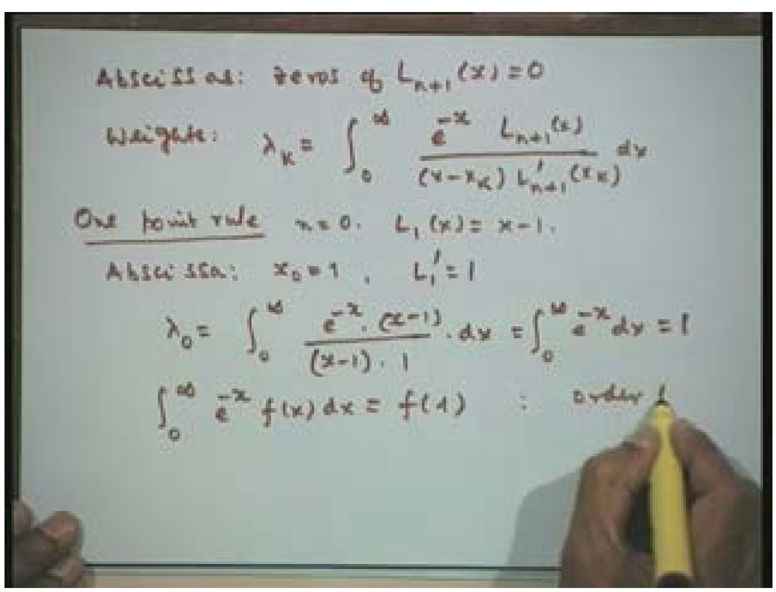
Therefore I will say set n is equal to 0 then I will have $L_1(x)$ is equal to minus exponential of x d by dx exponential of minus x into x. Now I can differentiate it and get minus e to the of power x, it is a product of two functions so I can differentiate it, I will get minus e to the power of minus x into x plus e to the power of minus x that will be equal to x minus 1, I have absorbed the minus sign inside, this gives me the first Laguerre polynomial. Now set n is equal to 1 in this x equation to get the Laguerre polynomial $L_2(x)$, therefore when I set n is equal to 1, I get here minus 1 whole square is a positive sign, this the second derivative and I will have here x square therefore $L_2(x)$ will be equal to exponential of x d square upon dx square e to the power of minus x into x square. Now again I will differentiate it is a product therefore I will get here e to the power of x d upon dx [minus e to the power of minus x squared plus e to the power of minus x into 2 x]. Again I differentiate it therefore I will get here e to the power of x, now minus minus becomes plus, e to the power of x x squared then I have minus 2 x into 2 x, this is again minus e to the power of minus x into 2 x, I differentiate it, I will get 2 from here so e to the power of minus x into 2. Now let us simplify it, e to the power of minus x cancels with e to the power of x, this is x square, this is minus 2 x, this minus 2 x therefore that gives us minus 4 x and this gives us 2 therefore I will have this as x square minus 4 x plus 2, therefore the Laguerre polynomial $L_2(x)$ square is given by x square minus 4 x plus 2.

(Refer Slide Time: 42:30)



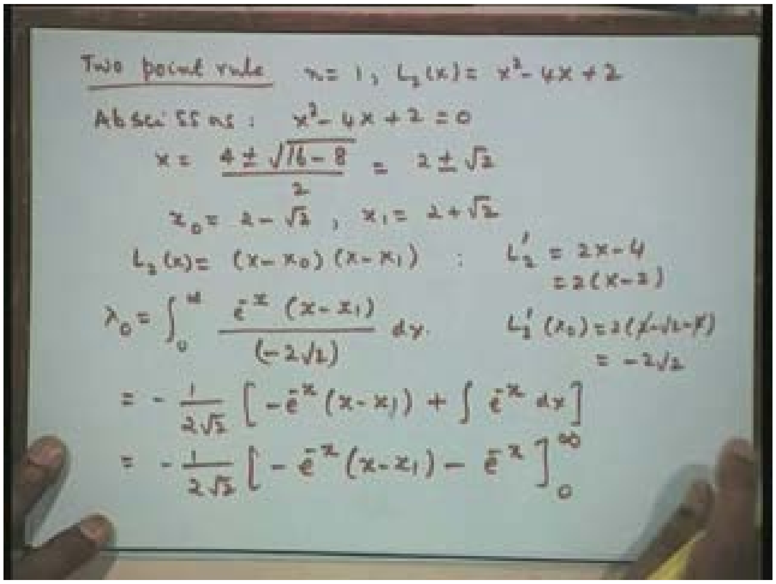
Now set n is equal to 2, I will get the Laguerre polynomial $L_3(x)$ that will give me minus e to the power of x d cubed upon dx cubed (e to the power of minus x into x cubed). Now I will give, leave this derivation for you, this value comes out to be x cubed minus 9 x squared plus 18 x minus 6. Therefore in the integration rule we substitute the zeros of the Laguerre polynomials and then write down the required formula,

(Refer Slide Time: 43:14)



Therefore the abscissas in the integration rule, abscissas in the integration rule are given by the zeros of L_{n+1} x dx, L plus 1 x is equal to 0 and the weights are given by lambda_k is 0 to infinity, the weight function e to the power of minus x $L_{n+1}(x)$ divided by (x minus x_k) L_{n+1} prime of x dx. The abscissas are zeros of $L_{n+1}(x)$, we have derived the expressions for $L_1(x)$, $L_2(x)$ and $L_3(x)$ from which their zeros can be obtained, the weights are given by the corresponding lambda_k. Let us get 2 rules and then take an example, let us take 1 point rule, so we will have n is equal to 0 and $L_1(x)$ is equal to x minus 1. Therefore there is only one abscissa, the abscissas is x_0 is equal to 1. Now we need the derivative here so let us differentiate that is 1. Therefore lambda_0 0 to infinity e to the power of minus x, this is x, minus 1, (x minus x_k) that is (x minus 1) into derivative is 1, derivative is 1 and this is (x minus 1), L_{n+1} is (x minus 1), the leading coefficient is only one here. So this is integral e to the power of minus x dx and minus e to the power of minus is 0 to infinity so it will give you simply 1, so the value of this is 1. Therefore our 1 point rule gives us 0 to infinity e to the power of minus x f(x) dx is equal to simply f of 1, this is order 1, it is simply evaluated f at 1.

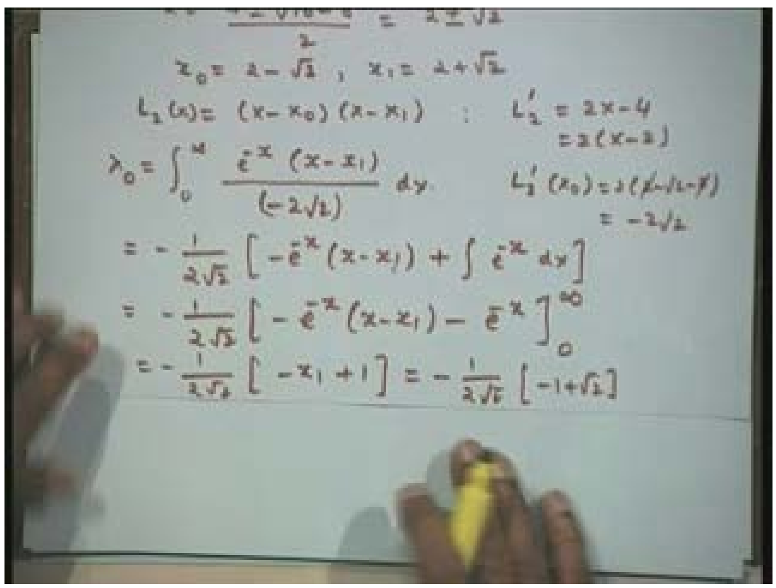
(Refer Slide Time: 45:49)



Now let take the 2 point rule, n is equal to 1 so we will have $L_2(x)$ is equal to x square minus 4 x plus 2 therefore let us put $L_2(x)$ 0 and get our abscissas. So will get abscissas x square minus 4 x plus 2 is equal to 0, so we will have x is equal to 4 plus minus b squared minus 4 ac by 2 so that gives us 2 plus minus, this is 8 2 root 2 so we will have 2 plus minus root 2. Therefore the two abscissas are 2 minus root 2 and x_1 is 2 plus root 2 and $L_2(x)$ again leading coefficient is 1, simply I can write it as (x minus x_0) into (x minus x_1), (x minus x_0) into (x minus x_1) and we need its derivative L_2 dash is 2 x minus 4, 2 x minus 4 or 2 into x minus 2, I can have it like this.

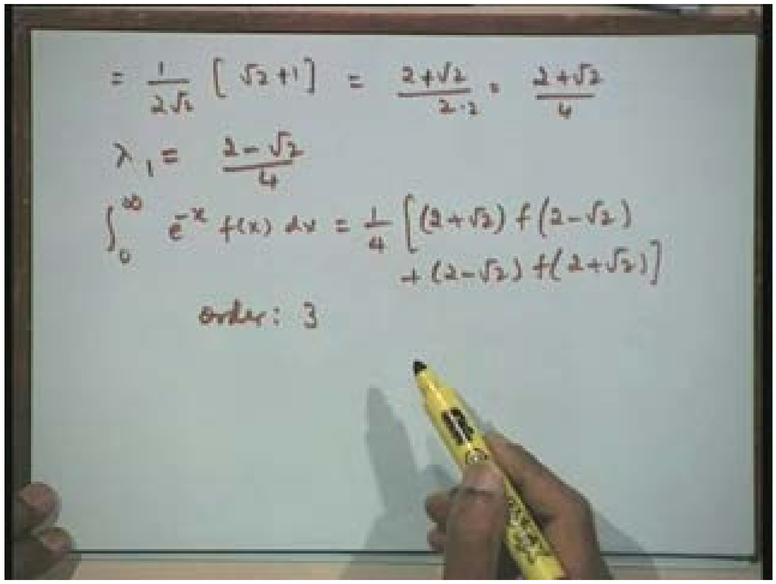
Therefore let us find out lambda$_0$, 0 to infinity exponential of minus x, let us reduce one step (x minus x_0) comes in the denominator so let us cancel of that, so (x minus x_1) so I have canceled of (x minus x_0) and what is left out is your L_2 dash at x_k so L_2 dashed at x_0, x_0 is 2 minus root 2 so I will have here is 2 minus root 2 minus 2 so that will be simply minus 2 root 2, 2 2 cancels off and I will have minus 2 root 2, so I will have here minus 2 root 2 of dx. Therefore this gives me minus 1 upon 2 root 2, now let us integrate it as a product so will have this as integral of this as (x minus x_1), (x minus x_1) minus integral of, there is a minus minus plus so I will have integral of e to the power of minus x dx, we will put 0 to infinity in a moment, I just integrated by parts so e to the power of minus x with a negative sign into this, plus e to the power of minus x derivative of this as 1. So this is minus 1 upon 2 root 2 minus [e to the power of minus x (x minus x_1) minus e to the power of minus x] 0 to infinity.

(Refer Slide Time: 49:00)



At infinity both of them are zeros and at the lower limit will have the, will have here 1 upon 2 root 2, this is the plus sign so the minus x_1 I will have and this is plus 1 at the lower limit. So x_1 is we have it here 2 plus root 2 and this is a plus 1 so what we have here is minus 1 upon 2 root 2, this is [minus 1 plus root 2].

(Refer Slide Time: 49:39)



Let us write down one more step 1 upon 2 root 2 into [root 2 minus 1], I have taken minus sign inside and let us remove this root 2 from the denominator so I will multiply that, that is 2 minus root 2 by 2 into 2. Minus x_1 plus 1 so, yes minus, yes minus 1 minus root 2, minus 1, you are right, minus 1 minus root 2, correct, this is minus 1 minus root 2 that is [root 2 plus 1], 2 plus root 2 that is 2 plus root 2 by 4. Now I will leave the second one to you to do this, this is comes out to be 2 minus root 2 by 4. Therefore our 2 point rule is 0 to infinity e to the power of minus x f(x) dx is equal to 1 upon 4, I will take this 1 upon 4 outside, the lambda_0 is 2 plus root 2, I have written 4 outside and the abscissa corresponding to this is the first abscissa 2 minus root 2 and the second one is 2 minus root 2 and the abscissa is 2 plus root 2, 4 I have written outside. So this formula, 2 point formula is very easy to remember that the, this particular argument that we have goes as the argument of this and this argument comes as this so these numbers are same, these numbers are same and this is of order 3, order 3. As I mentioned all the weights and abscissas for this Gauss Laguerre formula also are available in tables up to 16 decimal places and whenever the weight function is not available, we shall provide the weight function and then construct the solution of problem. Okay will stop with this.