Numerical Methods and Computation

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Lecture No # 4

Solution of Nonlinear Algebraic Equations (Continued)

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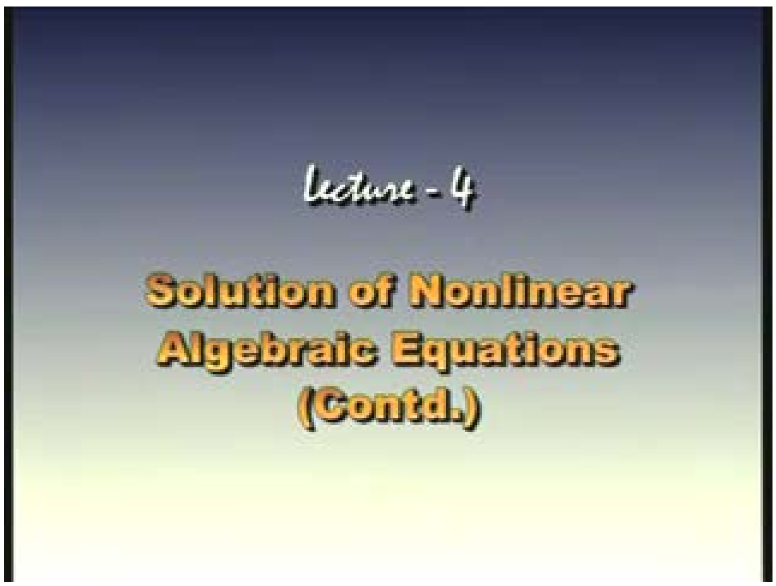

In our previous lecture we derived the Chebyshev method. The implementation of the method requires evaluation of three quantities. One is the function fx, the other is its derivative f prime x and its second derivative f double x. Further we also mentioned that the method is of third order, which would mean that if you are away from the exact root by say 0.1 then each attrition can produce an accuracy of ten to the power of minus three such that the method is sufficiently fast for any requirement. However in many practical problems it may not be possible for us to find out the second derivative or even the first derivative, therefore in such circumstances we need an alternative method where in the evaluation of the derivatives may not be possible. So in that direction we have what is known as the Muller's method.

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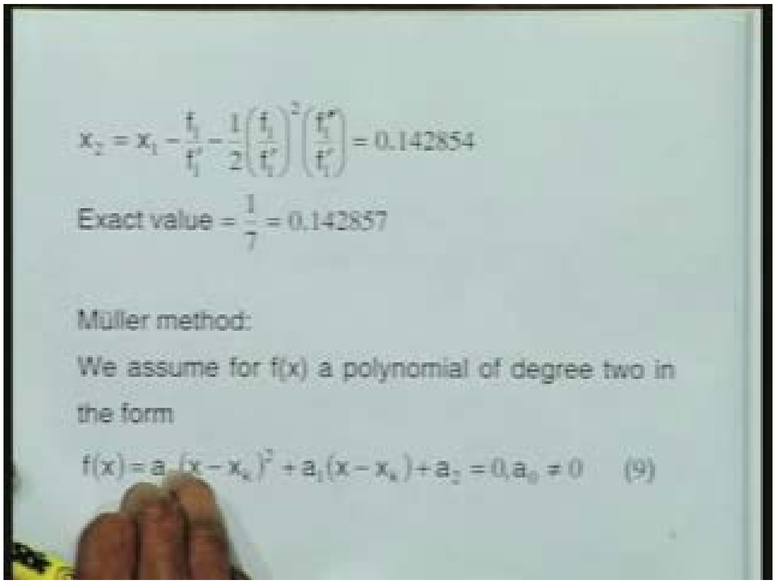

The Muller method also uses the concept of approximating the exact root by a quadratic equation. In the neighborhood of the root the curve is approximated by a quadratic equation. Now there are number of ways of writing a quadratic polynomial. One way that we used for deriving Chebyshev method was to write the polynomial a_0x square plus a_1x plus a_2 is equal to zero. However an alternative way of writing a quadratic polynomial is a_0x minus xk whole square plus a_1x minus xk plus a_2 is equal to zero, where we take a_0 not equal to zero because it is a quadratic polynomial. Now if I open it up and simplify what I would get is some a_0x square plus some b_1 into x plus b_2 is equal to zero. Therefore this is a suitable form for me in order that the method can be derived much more easily.

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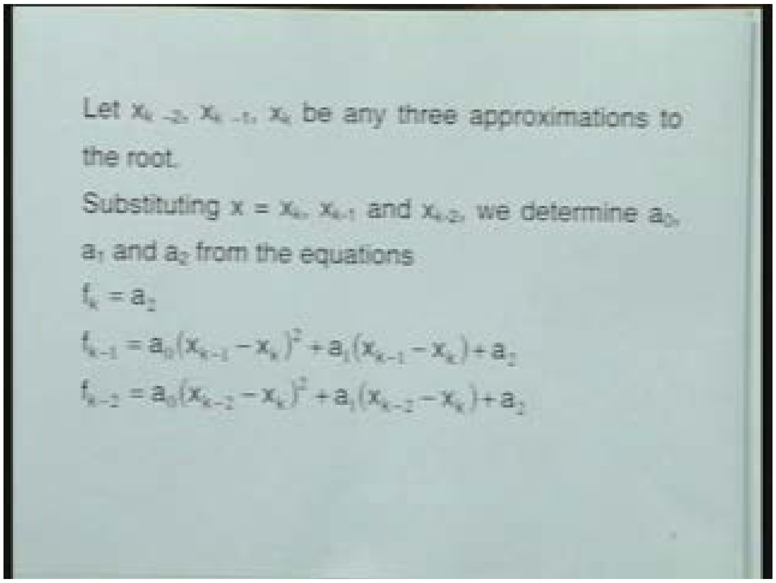

Now the application of the Muller's method requires three initial approximations. If you remember that the Newton Raphson required one initial approximation, Secant method required two initial approximations and the Chebyshev method required one initial approximation. But for implementing this Muller method we need three approximations to the exact root. These three roots are any three approximations near the root. The exact root need not lie between any one of these intervals. For example if the root is say 5, you could choose on one side of it; say 3, 3.5, 4, 4.5, 4.7 or arbitrarily one can choose the approximation. Once we know that the root lies between a certain intervals, any three points within this interval can be taken as the initial approximation to the root. Therefore I would take the three approximations to the root as xk, xk minus one, xk minus two; which means in order to start my implementation method I need x_0, x_1, x_2 to start and then find x_3, x_4, x_5 five from the method. Now since these are the three approximations, obviously they are three points on the curve i.e. xk f at xk, xk minus one, f xk minus one, xk minus two, f xk minus two; through three points we can always plot a quadratic curve, that is a parabola.

Now this parabola will intersect the axis and that point cuts the axis at two points, again out of which one root will be the exact and the other root will not be approximating the exact root. So what I would just substitute xk, xk minus one, xk minus two in this which I had written in the previous step. So you see the reason why we had written will be obvious now. When I substitute x is equal to xk in this, these two terms drop, so a_2 two gets evaluated automatically. So a_2 will be simply equal to fx. Now substitute xk; x is equal to xk minus one, then I would get a_0 xk minus one minus xk whole square, then a_1 into xk minus one minus xk plus a_2. Similarly I will substitute the approximation xk minus two here, so that I have fk minus two. This is xk minus two and this is xk minus two minus xk plus two is equal to zero. These are three equations and three unknowns; a_2, a_0 and a_1. Now since a_2 is determined I can take a_2 to the left hand side here, then I will have a system of two equations for determining a_0 and a_1.

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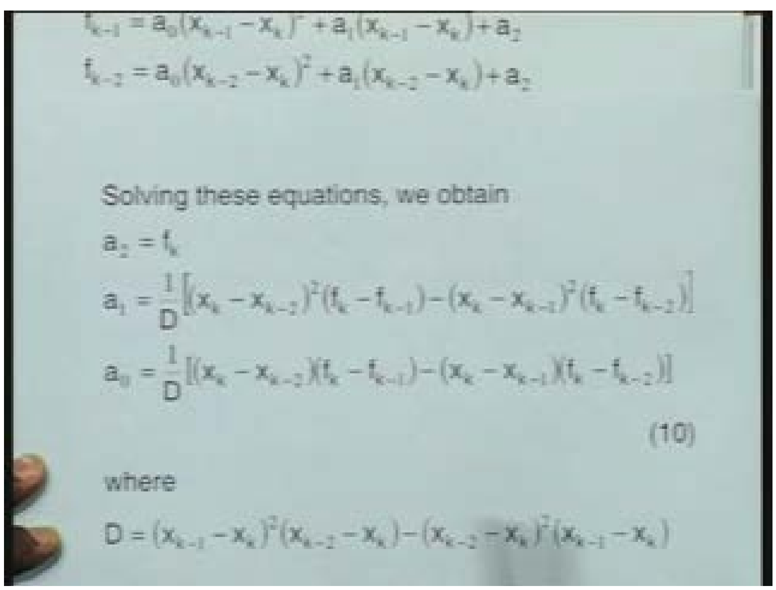

So I would solve these equations. Now a_2 is determined from this. So a_2 is equal to fxk. I have taken this a_2 to the left hand side, then I am finding first of all D as the determinant of the coefficient matrix. So I evaluate the determinant coefficient matrix i.e. D that is xk minus one. If I take the determinant of the coefficient, what I would have here is xk minus one into xk whole square into xk minus one minus xk minus the product of these two theorems i.e. xk minus two minus xk whole square into xk minus one minus k. So this is the determinant of the coefficient matrix for a_0, determinant a_0a_1. Therefore I can find out a_1 by using the Cramer's rule. I have evaluated the numerator and similarly I evaluated for a_0 - this is the numerator that comes from there. It was evaluated simply by using the Cramer's rule for two by two systems of equations. Therefore I am able to determine the constants a_0 and a_1 and a_2 very easily by assuming the polynomial form in the special form that we have taken there.

If I had taken it as a_0x square plus a_1x plus a_2 , I will have to solve a system of three by three equations and the system of three by three equations would give me the same solution but in a different format wherein I will have to combine the number of terms to get this particular format . But we will see this format later on. We shall be using such an approximation for a polynomial degree to find out some other numerical methods also. Therefore once I determine a_0, a_1 and a_2 term here, I would go back and substitute in this polynomial.

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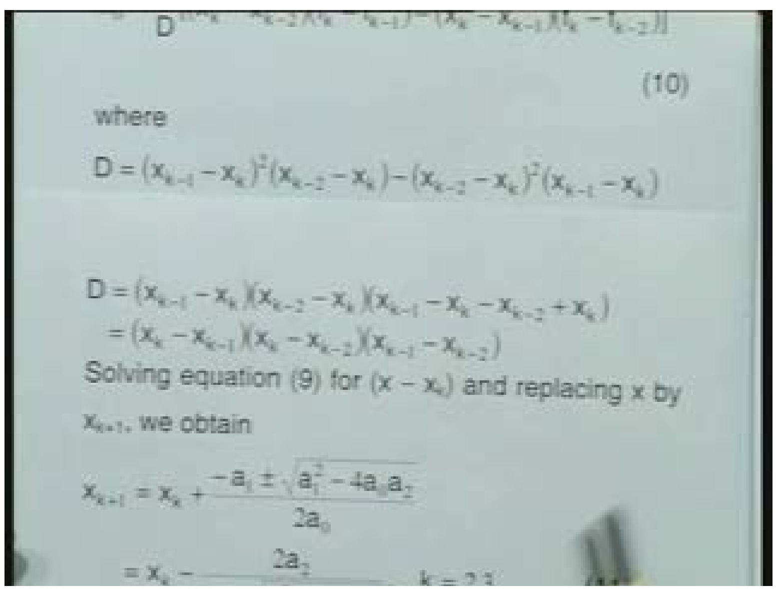

Before let us first of all simplify this D that we had written over here because this can be simplified further. So if I now simplify this, you can see that xk minus one minus xk is a common factor; xk minus two minus xk is a common factor. So I can take out both these common factors xk minus one minus xk and xk minus two minus xk and what is left out is the xk minus one minus xk and minus xk minus two plus xk. So xk cancels. I have a factor of this particular form.

Now you can see that the denominator of this can easily be written down once we know the three approximations. You start with any approximation either xk that means from xk or the other end. So we have the three points, xk minus two xk minus one and xk. The starting one is xk minus two and the last one is the xk. We can start with any one of them. So what I do is subtract xk from the remaining two previous approximations which are xk minus xk minus one, xk minus xk minus two. Then I move to the next point in the order xk minus one minus xk minus two, so this will be the denominator. So you can look at it in the reverse way, you can start from the other end.

You start from xk minus two. Let us take a minus sign throughout. So xk minus two minus xk minus one then xk minus two minus xk, that is over. So I have to move to the next point which is xk minus one minus xk with a negative sign. So if I move from that sign I will have a minus sign, so I can do from either side but it is always good that we start with the current approximation that is available like previous one that is xk. So that we do not have to take care of any sign, so that is simply in the product in the denominator. Now since we have solved for a_1, a_0, a_2 we can know go back and substitute in this. So I substitute it for a_0, a_1 and a_2 here and once I substitute it over here I can write down the next approximation. So the next approximation is f of xk plus one. By substituting this is equal to zero, I am putting xk plus one here, xk plus one here. This is a quadratic equation for xk plus one minus xk, so I solve by the ordinary method which is xk minus one xk plus one minus xk is equal to minus one plus minus under root a_1 square four times a_0 a_2 divided by two times a_1. So I have just solved this equation for xk plus one minus xk that is I have taken xk to the right hand side. So it is trivial, so finding the root of this quadratic equation and both the roots we are taking over here. This particular form can be simplified by rationalizing this.

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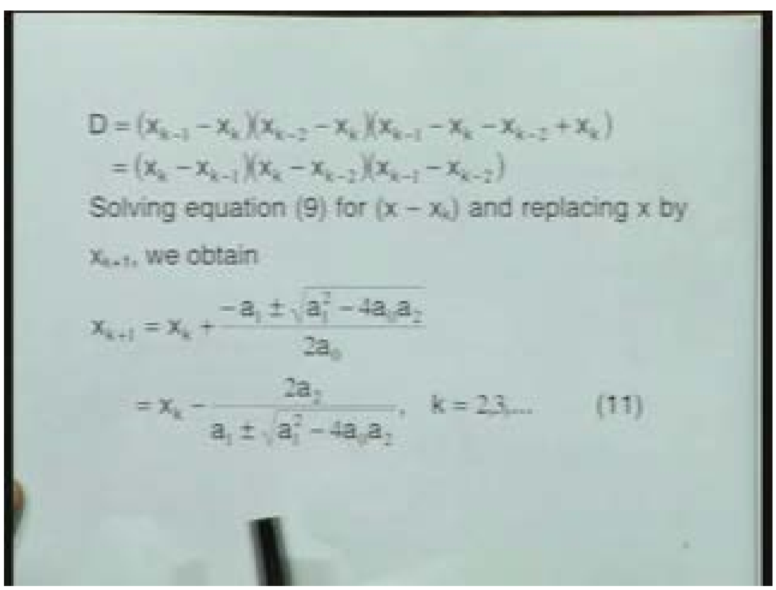

I multiply the numerator and denominator by a_1 plus minus under root of a_1 square minus four a_0, a_0. When I multiplied it you could see a_1 square cancels with a_1 square and I have a plus minus four a_0 a_2; out of this four a_0 there is two a_0 in the denominator, it cancels. So what is left out is simply two times a_2 two and in the denominator I have a_1 plus minus under root of a_1 square minus four a_0 a_2. So the minus sign has been taken care of by writing this. Now there is a reason for writing this rationalization in this particular form.

We have started with initial approximations xk minus two, xk minus one and xk. Since the solution is a continuous solution the next root that we are going to get or next approximation we are going to get should be closer to xk. It cannot jump far away from the xk. Therefore what we do here is in order to choose the correct value to be added to xk, I would see that the change that comes here is small; because there are two values here. I will choose that particular value which gives me the smaller of two in magnitude.

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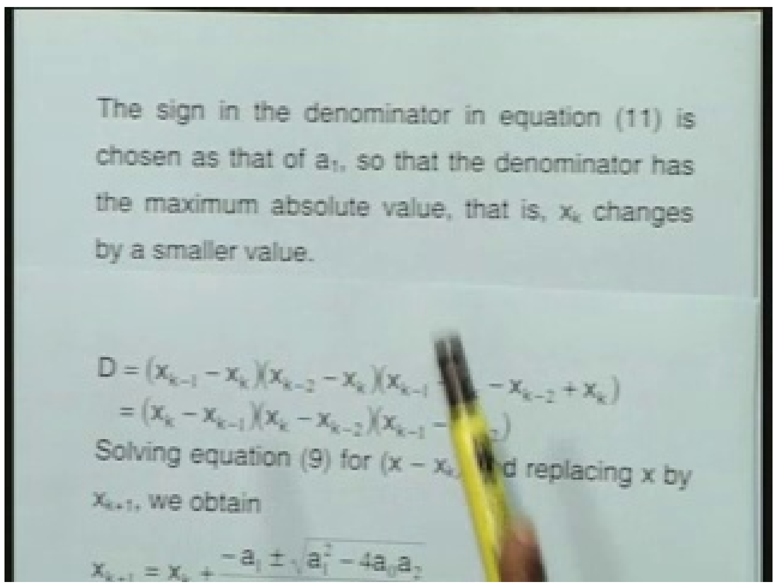

So what I would do is the sign in the denominator in equation eleven is chosen as that of a_1, so that the denominator has a maximum absolute value, that is, xk changes by smaller value. Let me explain it again. We are taking the square root; therefore this is a positive square root. Now this quantity will be small if the denominator is large in magnitude. Therefore depending on the sign of a_1, if a_1 is positive I will take positive sign here, then the denominator is the largest value. If a_1 is negative, I will choose the negative value here, so that in magnitude it is large and in this quantity it will be smaller in magnitude.

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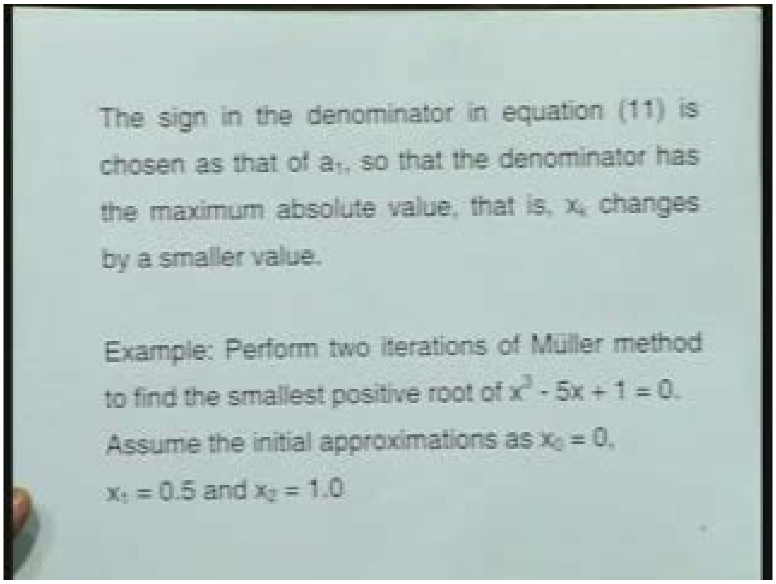

We discussed that the sign in the denominator in this equation is chosen as that of a_1, so that the denominator has the maximum absolute value, that is, xk changes by the smallest possible value. This will automatically determine throwing away of the other root because the fact that we had rationalized it and brought in this particular form, we are automatically throwing away the second root by saying the other root is going to give a bigger change to xk. Therefore that is how we have solved the problem.

Now let us discus what we have to do for the next iteration, what is the computational cost per iteration. We have the approximations xk minus one, xk minus two, xk minus one and xk. We have computed xk plus one. Now we go to the next step where I need evaluation of fxk plus one. I do not need anything else here because a_0, a_1 and a_2 do not contain any other thing. Therefore the cost of evaluation is simply one function evaluation i.e. f of xk plus one, so the cost of evaluation is only one function vanishing. However we shall show later on that the method is is not quadratic or cubic. It drops down a super linear at about 1.8 which is superior to the secant method but it will only be 1.8, as the rate of convergence. We state here that what we have derived here is not superior to Chebyshev method or superior to the Newton Raphson method.

Now let us take this example. Perform two iterations of the Muller method to find the smallest positive root of x cubed minus five x plus one is zero and here in the problem the initial approximations are given to us as 0.5 and 1. We already know and we have shown that the roots lies between 0 and 1 so we have just taken three points randomly 0, 0.5 and 1. If it so happens, depending on the root if you are able to take a different approximation, you may reduce one or two iterations and if it is not, it will take one or two more iterations to adjust and converge from one side.

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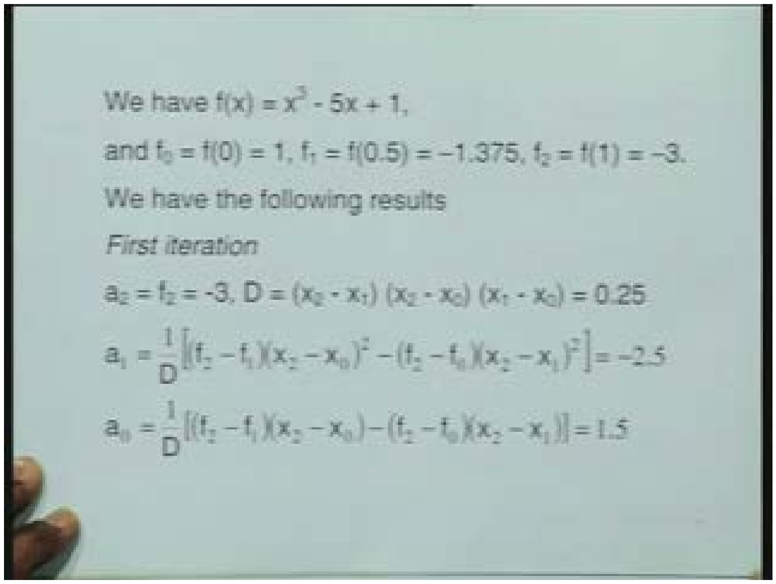

Now let us solve this problem. Now here we have fx is equal to x cubed minus five x plus one. Now I evaluate f. I have taken the three approximations, which are f at zero i.e. f_0, f at point five i.e. f_1, f at one is f_2. So these three values are 1, minus 1.375 and minus 3. Now we will substitute it and then get first iteration. Now if you remember that a_2 is computed as f_2 i.e. minus 3, then we said the denominator is starting with x_2, x_2 minus x_1, x_2 minus x_0, x_1 minus x_0. So I am starting from x_2 and then taking all the subtractions from all the remaining approximations and this I can compute easily and it comes out as 0.25. Once D the denominator is obtained then the numerator also forms really a particular way of writing it. You can see the way in which the sequence goes in the numerator also. This is f_2 minus f_1, x_2 minus x_0 whole square, f_2 minus f_0, x_2 minus x_1 whole square. So this f_2, x_2, f_2, x_2, f_1 opposite value, f_0 opposite value, this is how a particular sequence can also be remembered in writing even a_1 one and a_2. So I can substitute this f_2 and f_1 here; x_2 and x_0 here; f_2 f_0 here; x_2 x_1 here, on simplifying this, I get minus 2.5.Similarly this a_0 is f_2 minus f_0 x_2 minus x_0 and there is no squares except that both these quantities are same expect there are no squares here and this quantity is 1.5. So I have now computed a_0, a_1 and a_2. Once I compute a_0, a_1 and a_2 I would then try to evaluate the quantity.

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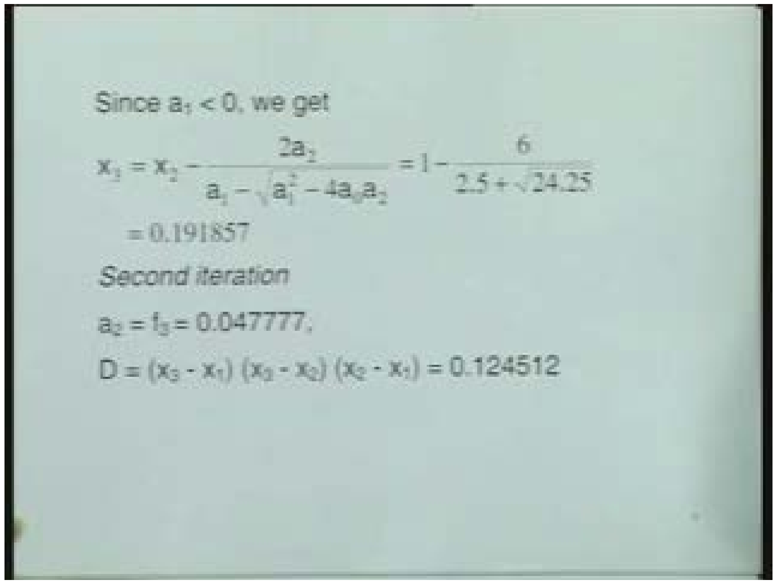

Now a_1 is less than zero. So you can see that a_1 is minus 2.5. Therefore a_1 is negative. Therefore in the denominator I will take the negative sign here out of this plus minus. So since a_1 is negative, I take x_3 as x_2 minus two times a_2 two; this is a_1 with a negative sign minus under root a_1 square minus four $a_0a_1a_2$ and this is a simple computation. I can substitute the values of a_0, a_1 and a_2 and get 0.191857. Now this gives us the first step of Muller iteration i.e. first iteration of the Muller method. Now I repeat the same thing and I go find out what is my a_2; a_2 is now f of x_3. The current estimate is x_3; so f of three. I compute the value of f_3 and then D. Now look at this x_3. I could have written this first – x_3 minus x_2, x_3 minus x_1 into x_2 minus x_1. So now we need three approximations, we have not come to x_3, so we drop the original x_0. So we have x_1, x_2 and x_3. So I take x_3 minus x_2, x_3 minus x_1 i.e. x_3 minus x_2, x_3 minus x_1. Now I move further and get x_2 minus x_1, so that will be the denominator and this product is simply equal to this.

Now again I compute my a_1; a_1 is one upon D, f_3 minus f_2. Now you see here x_3 minus x_1 whole square, f_3 minus f_1, x_3 minus x_2 whole square and a_0 is f_3 minus f_2 x_3 minus x_1 f_3 minus f_1 x_3 minus x_2 and I get this. Now these are the two computed values for a_1 and a_0 which we have written from the original expression that we had here earlier. Again I find a_1 is negative. Therefore I will have to take in denominator the negative sign, so that the change in xk, the current value x_3 is smaller.

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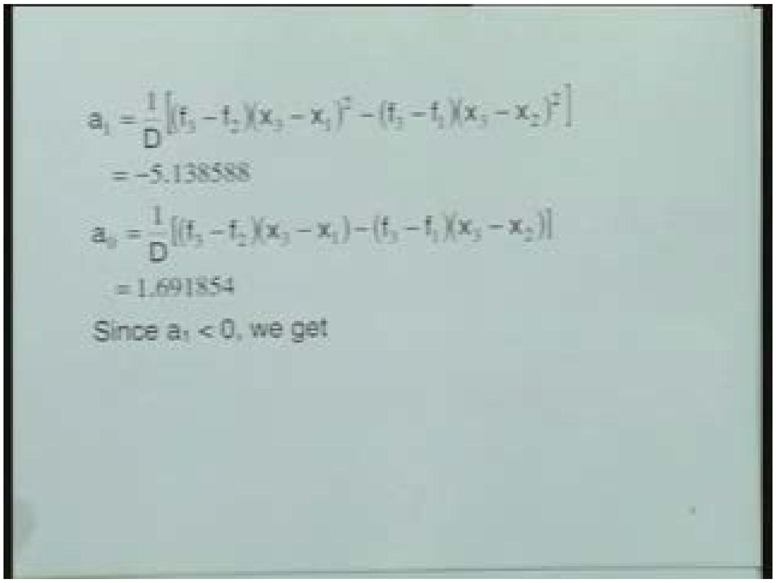

So I would therefore write x_4 as x_3 minus twice a_2, a_1 minus under root of a_1 square minus four, a_0a_2. Now I have just written the values of x_3, a_2 here, a_1 with a negative sign which has been taken care of as a plus sign here and this has become plus under root of this and evaluation of this reduces to this value. Now this completes the evaluation of two iterations, we have done x_3 and this is the value that we have got as x_4.

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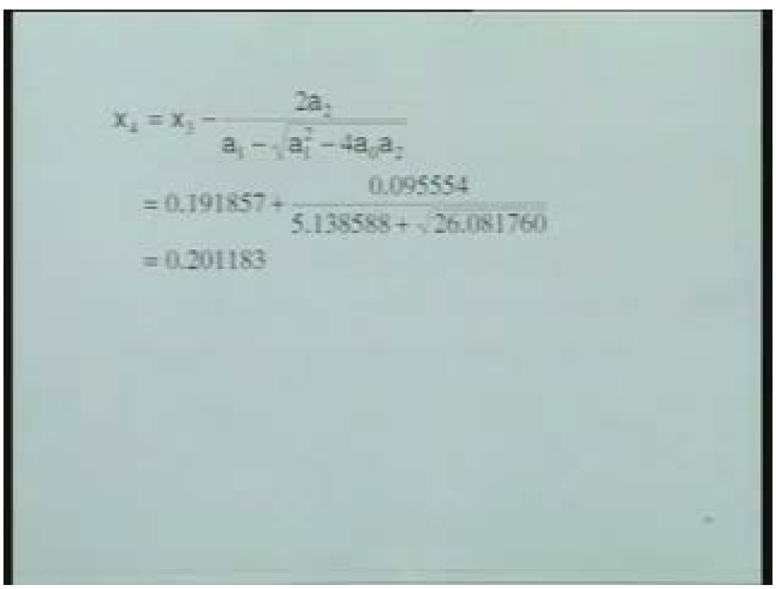

Now as I mentioned earlier the reason why we have derived Muller's method is because, in practical applications it may not be possible for us to find out the second derivative as well as the first derivative. Therefore we opted for a method which may not be as fast as the Chebyshev method but reasonably fast whose order of accuracy is 1.8 or rate of convergence is 1.8.

Now it may be possible that we must give an alternative that it may be possible for us to find the first derivative but not the second derivative. For that we have already Newton Raphson which uses first derivative. But if it is possible for me to get first derivative at some other point also it may be possible for me to get the same order as the Chebyshev method which is a third order method, that means it will work much superior to this. Therefore what we would like to do now is to start with the Chebyshev step but the way we have manipulated Chebyshev method; the Chebyshev method was obtained starting with the Taylor series expansion up to the second order, we drop the third order terms and then from there we manipulate it to get the Chebyshev method. However we would like to show now that if I manipulate it in a different way, I can avoid the evaluation of second derivative and I will evaluate one more first derivative which means I will have two first derivative evaluations and one function evaluation, at the same time achieve the third order accuracy which is order of the Chebyshev method.

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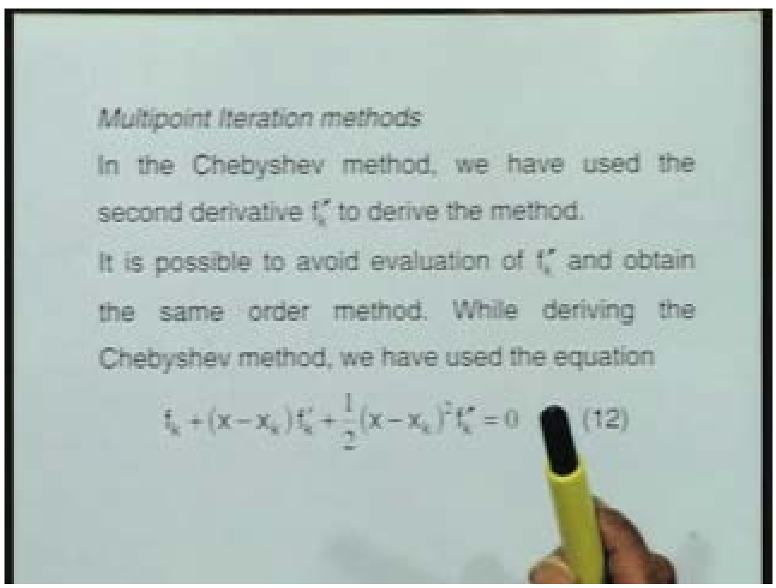

Such methods are called multipoint iteration methods. They are simple manipulations. Why it is called multipoint is trivial; we are evaluating f prime at two points, so it is a multipoint. More than one point it is being evaluated. So in a Chebyshev method we have used the second derivative. Now to avoid the evaluation of f double prime and obtain the same order method we shall use the following method. So we will start from where we have started our Chebyshev method. In the Chebyshev method we have written the Taylor expansion. We took the first three terms, the order of h cube term or order of x minus xk whole cube term is dropped. So from this we have manipulated the Chebyshev method. We are going to give multipoint iteration methods and both of them start from the same step.

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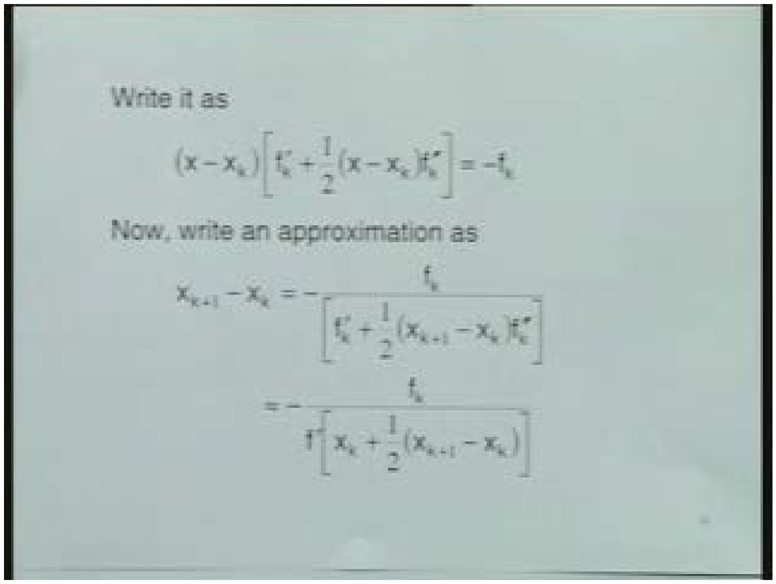

Now we write this step like this. What I do is I take fk to the right hand side, retain these two terms on the left hand side, take x minus xk as common factor. So what is left out is x minus xk f prime of k plus half x minus xk f double dash k is minus fk. So it is simply taking this right hand side and taking the common factor. Then I would write from here the next approximation. I will take this factor to the denominator here and write down my next approximation. Therefore I will write xk plus one minus xk minus fk divided by this entire factor f prime k plus half of xk plus one minus xk f double dash k. Now what I would do is the manipulation which we talked of will be done here. I will write this as the first two terms of the Taylor expansion of this. Lets look at this quantity here, f prime of xk half xk plus one minus xk. The Taylor expansion of this is, f prime at xk plus this factor into next derivative i.e. second derivative. So this is half xk plus one xk. Let us even think this as h; if you think this as h, this is simply h into f double dash of xk.
So that means these two terms get absorbed in the first two terms of the Taylor expansion of f prime of xk. Therefore I have now avoided taking the evaluation of f dash but I have now brought it in this form, so that f dash is being evaluated at one more value besides f dash at k. Therefore the the justification of calling a multipoint iteration that, f prime is now being evaluated at a new value one more value besides the value at f xk.

Now as we have done in the Chebyshev method we had xk plus one on the right hand side, xk plus one minus xk was approximated by the Newton Raphson method and the method retained the order of three. We shall do the same thing here, xk plus one minus xk and we shall approximate it by the Newton Raphson method i.e. minus f of k divided by f prime of k.

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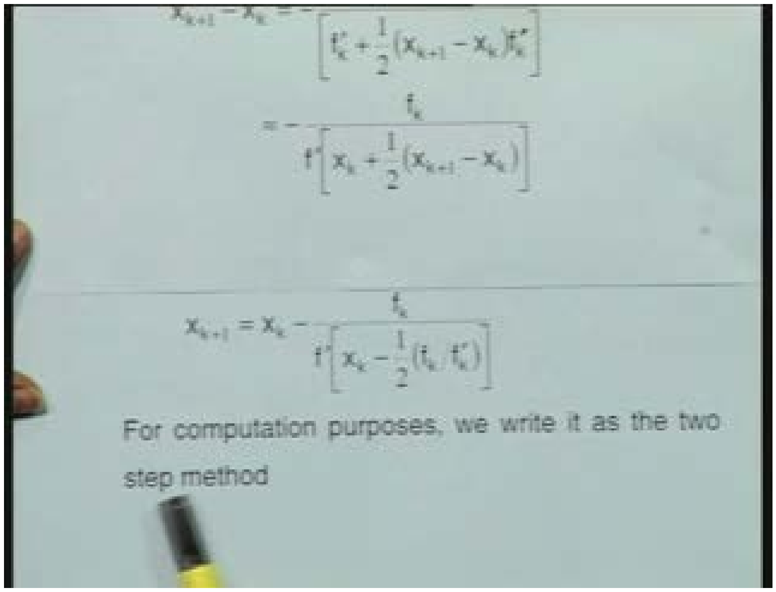

So we will now approximate this quantity xk plus one minus xk by fk minus f prime k. So we simply are approximating this by Newton Raphson method so this is being replaced by minus fk minus f dash k. This method is of the same order as the Chebyshev method. The order does not fall down. The order is three. Now this is a multipoint iteration method, but this can be written in a nice fashion what we call as two step method. That means we shall write this method in two steps. One is evaluation of this particular argument of f dash k and then evaluation of this.
So we would therefore write this method as follows.

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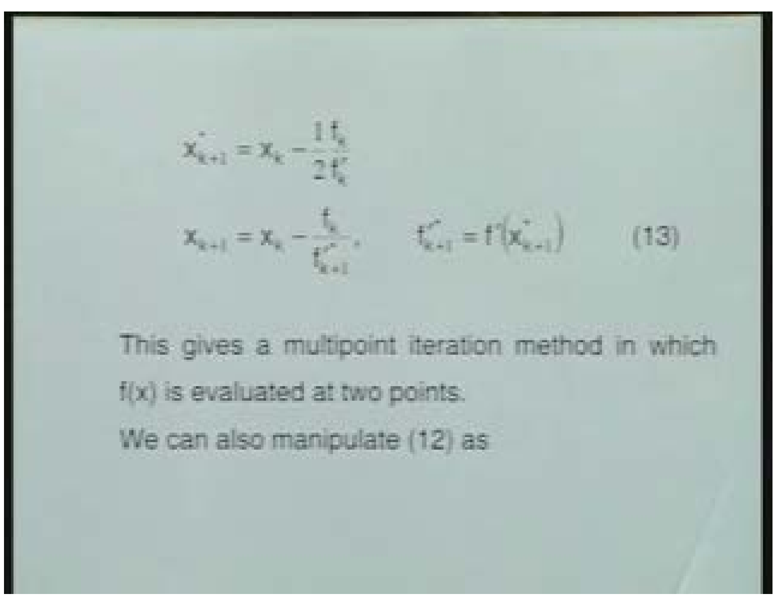

So as you can see, I am now denoting this quantity that we have here as xk plus one star as some intermediate value, xk minus half fk of dash k and then this method will be writen as xk plus one is xk minus fk, evaluation of f prime at this, so that this f prime is being evaluated as f prime at x star k plus one this one. So this is almost looks like Newton-Raphson method except the other half here. If there was no half here it would have been a Newton-Raphson method. So this particular quatity is being used here and we compute xk plus one star, then compute f prime at this particular argument and then we substitute it here and get the multipoint iteration method. Therefore this is called the multipoint iteration method, as one way of writing it.

Now the starting point for manipulation of this was this step. So we have manipulated this and brought this one step. If I manipulate it in a slightly diferent way I can get a differnt multipoint iteration method. Now let us see how we manipulate this particular thing here.

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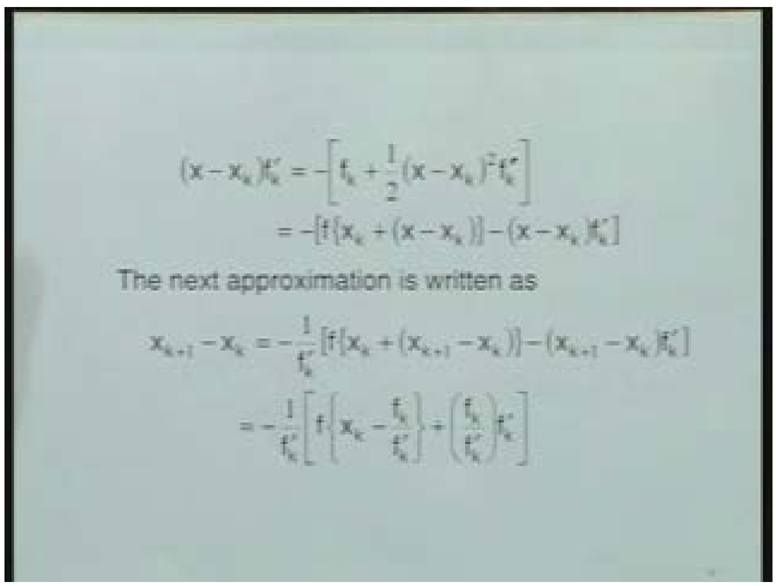

Now I retain the middle factor here and take these two to the right hand side, so xk plus one minus xk f prime k is retained on the left hand side. I take fk half fk minus one to the whole square to the right hand side. Then the manipulation that I want to do is on the right hand side. So I would now try to manipulate this particular thing to avoid my second derivative. Let us forget for this moment, let us take this f of xk plus x minus xk; xk and xk cancelach other. So if I open it, it gives you f at xk plus x minus xk f prime of xk which I haveve subtracted. So x minus xk f dash k cancels with x minus xk f prime xk and the next term is half x minus xk whole square f double dash k and that is this term. So to get this what we have done is we have added and subtracted x minus xk f dash k and the these three terms have become the first three terms of theTaylor expansion of f xk plus x minus xk.

Now if you now look at this one the this is now being evaluated at a new point, xk plus x minus xk. Now once I am able to write this, I can immediately write down the next approximation replacing x by xk plus one. So this is xk plus one minus xk. I bring f prime k to the right hand side, so minus of one upon f prime k f xk plus xk minus one minus xk and this x also is replced by xk plus one. As we have done in the Chebyshev method as well as the previous multipoint iteration method, I would replace this xk plus one minus xk by the Newton Raphson method again. So this will this part will be read as fk minus divided by f prime k and this will read as fk by f prime k.

So let us simplify this one step further. Now if I simplify this I take a xk to the right hand side then you look at the second term first; f prime k cancels with the f prime k, there is fk here f prime k here. So the second term is fk minus f prime k and third term is minus of f xk minus f fk f prime k divided by f prime k. So this is another way of manipulating the starting approximation of the Chebsyshev method to arive at this particular new method. Now here again we have used the Newton Raphson method. Now I would like to write this also as a two step method.

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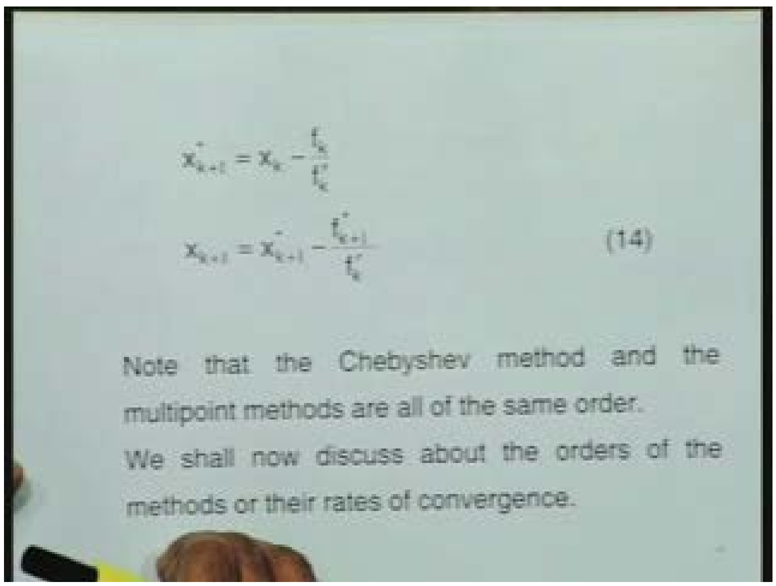

Now what I would do here is I would first of all call this a new approximation. This is x star k plus one is xk minus fk by f dash k. Then I would write this xk plus one is x star k plus one minus f star k plus one divided by f dash k. A very interesting thing here is the frist step is nothing but Newton Raphson method. So this is indeed a modification of the Newton Raphson method to arive at the next higher order method. We know this will be a second order value and now I am computing a third order value. Here you can see that we are using two evaluations of f. We are using f star k plus one. Therefore I am now using here the evaluation of f two times; whereas in the previous multipoint iteration we had two evaluation of f prime. So even though both of them are having the same number of evaluations (three evaluations) but the choice is with us whether the evaluation of f is easy for me or evaluation of f prime is easy for me. If the evaluation of f is easy for me, I would chose this because what I am using is f star at k plus one.
If evaluation of f prime is easy I will use the other multipoint iteration method. Therefore this is exactly a modification Newton Raphson method to get a new method.

Now we have all along been saying that the order of Newton Raphson method is two order; order of Secant method is 1.6; order of Chebyshev method is 3. How do you get about it? The orders or rate of convergence of all these methods is very simple and very straight forward. Every thing depends on Taylor series. We define the error at any particular step, that means the curent numerical solution minus exact solution is the error. Based on that I will substitute the expresions of that in any formula that I have got. I would bring any quantity that is there in the denominator. I will open up by Taylor series first then, whatever quantity is there in the denominator I will take it up and put it to the power of minus one, open it up as a binomial series, multiply everything and then cancel off. Whatever is left out will be a an expression conecting the error i.e. the curent step which is k plus one with the error at k, k minus one and k minus two depending on the method that we are using.

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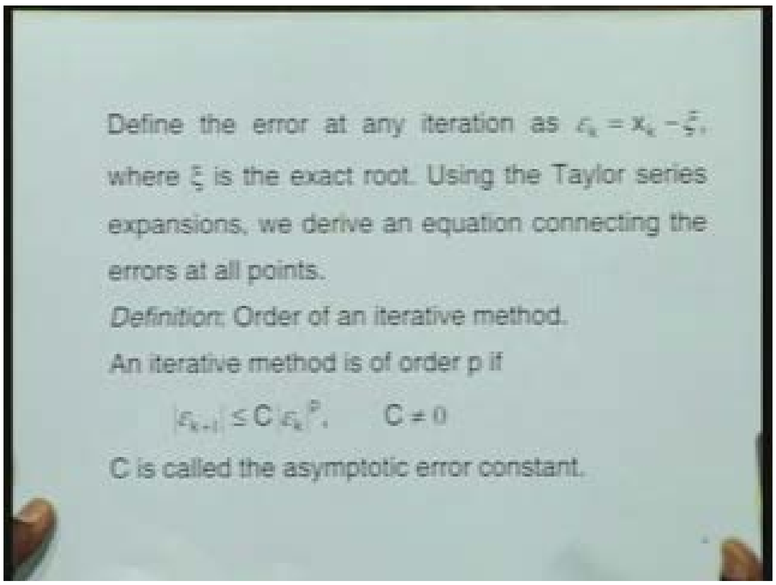

So to define this., let us first define the error as - xk is a curent approximation. the approximation minus xi as epsilon k. Similarly epsilon k plus one will be defined as xk plus one minus xi, epsilon at k minus one is equal to xk minus one minus xi and so it depends upon the values that xks are using. If you are using Newton Raphson we have only one value xk; if you are using Secant method we have to use xk and xk minus one; if you are using Muller method we have to use xk, xk minus one and xk minus two.

As I mentioned how the order of the method is defined by connecting how the error at the present step is connected to the error at the previous step. For example, the present step is you are computing xk plus one; how is the error at k plus 1^{th} step is connected to error at the k^{th} step? That is how epsilon k plus one is connected to epsilon k. Now in the Muller method or the secant method have more than one point like xk, xk plus one, we shall see that using this particular definition we can define the epsilon k minus one automatically back in terms of xk. So finally we arrive at an expression only connecting the the errors at two steps.

Now we define the order of an iterative method. The order of an iterative method is, the iterative method is of order p if error at the curent step in magnitude is bounded by some constant C epsilon k to the power of p. This p is called the order of the iterative method. It is also called the rate of convergence of the method.We will be showing that for Newton Raphson method, that p terms have to be two which means we will call the rate of convergence as two or order of convergence is two. Therefore epsilon k plus one is less than equal to C epsilon k whole square.

Similarly for Secant method we will show that this p is 1.612. Therefore we call it as super linear convergence. Now this C that is multiplying is called the asymptotic error constant. Of course we expect C is bounded quantity. When you write down the Taylor expansion you will get the value of c also. Depending on the value of C we can make few comments as to when the method may possibly fail; if C also grows and it also goes to infinity then, even though you are multiplying by a very small quantity or by a very large quantity the method can still diverge or it may converge very slowly.

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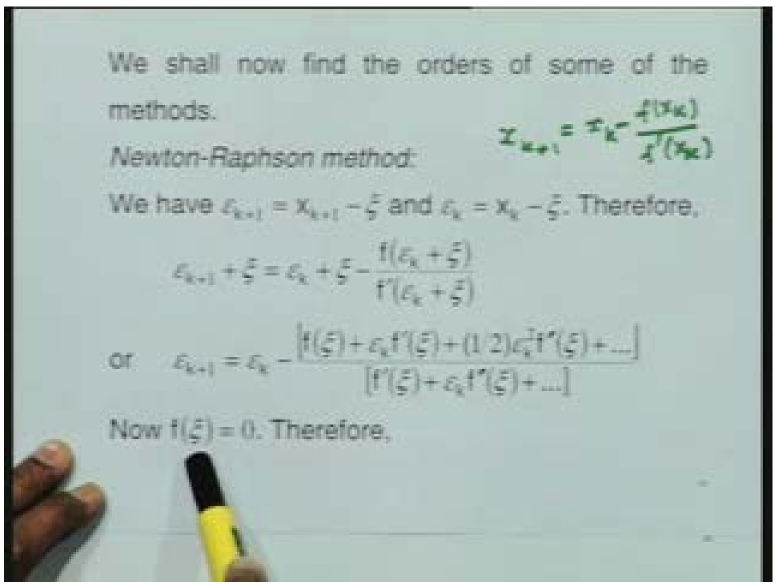

The simplest of all is the Newton Raphson method. So let us take the Newton Raphson method. So the curent error is epsilon k plus one is xk plus one minus xi. The previous error is epsilon k xk minus xi. Now Newton Raphson method is xk plus one. If you remember the method that we had written is xk plus one is equal to xk minus f of xk by f prime of xk. So this is the Newton Raphson method. Therefore I am substituting for xk plus one here i.e. epsilon k plus one plus xi; xk is epsilon k plus xi minus f of x; xk is again epsilon k plus xi f prime epsilon k plus xi. Now we Taylor expand all these quantities or functions that are there. The denominator shall be taken to the numerator and then expanded. So let us open this up.

Now here you can see xi and xi cancels over here and so that epsilon k plus one is eqaual to epsilon k minus, (when I open it and Taylor expand it) so f of xi plus epsilon k f prime of xi plus half epsilon k square f double dash xi plus; so on the denominator will be f prime at xi plus epsilon k f double dash xi. We started with that, xi is the exact root of our function fx, therefore f of xi is zero.We shall see later on that we are talking of here is only a simple root not a multiple root. If it is a multiple root then we know that f prime and f double prime can also be zero. So the other terms can also be zero. So let us assume for the moment we are not talking of a multiple root, we are talking of only a simple root so that f of xi is zero, f prime xi is not equal to zero. So I can set f of xi is equal to zero over here.

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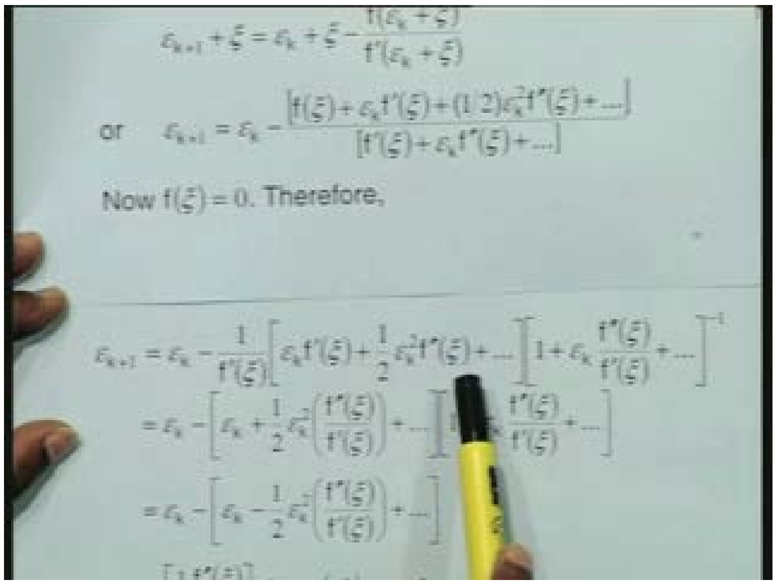

Now I said f of xi is equal to zero here. If I want to take this to the numerator and expand the binomial form of one plus some capital x to the power of minus one, I will always take it as one plus f to the power of minus one or one minus f to the power of minus one form. Therefore I would take this particular quantity i.e. the first term always out. There is a constant f dash at xi, so I would take f dash of xi constant over here and you can see this term here, by taking f dash a as common, I will have here one plus epsilon k f double dash xi by f dash xi to the power of minus one. So this quantity that I have here, I have taken it to the numerator and written as f dash xi, the numerator f of xi is zero. So I have written the two remaining terms epsilon k f prime of xi plus one by two epsilon k square f double dash xi.

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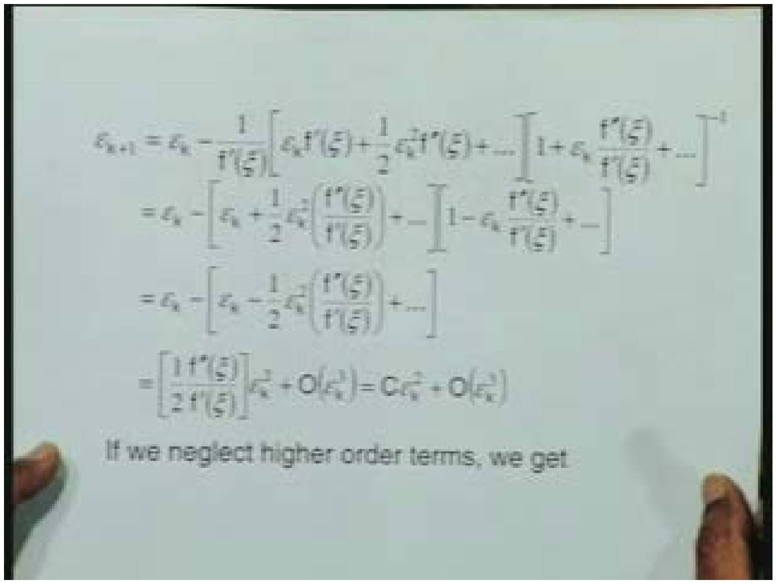

Therefore as I said the concept is to take the quantity that we have in the denominator to the numerator, put it to the power of minus one. Now let us assume this whole quantity as some capital X. It is easier to write it. So it is something like one plus capital X to the power of minus one. If I open it up, I will get one minus x plus x square and so on. So here I am not writing more terms because we do not need it. So it will be simply one minus capital X which is one minus epsilon k f double dash f dash plus so on. We do not require them therefore I have not written them. We have taken f dash xi inside, absorbed it into this bracket, so that f dash xi cancels with f dash xi and the second term is half epsilon k square f double dash xi by f double dash xi. So we have simplified by taking f dash xi inside in this. Now I just have to multiply these two. Now if I multiply you can see that epsilon k is epsilon k; for epsilon k square there are two terms, one into this is epsilon square, and epsilon k into epsilon k is epsilon k square. So therefore this will contribute; this will contribute; and this will contribute plus half epsilon square; this will contribute minus epsilon k square. So if I simplify I would get minus half epsilon square, whole square; f double dash minus xi plus one. Now once you multiply the whole thing I cannot simplify the whole right hand side; epsilon k cancels and I am left out with half f double dash xi by f dash xi epsilon k square. Now the other terms, I will write it in order of epsilon k cubed or I will call this term as C epsilon k square plus order of epsilon k cubed. Therefore I am able to derive that the error at the current step which xk plus one step is equal to C into epsilon k square plus order of epsilon k cube. So I now neglect this higher order terms.

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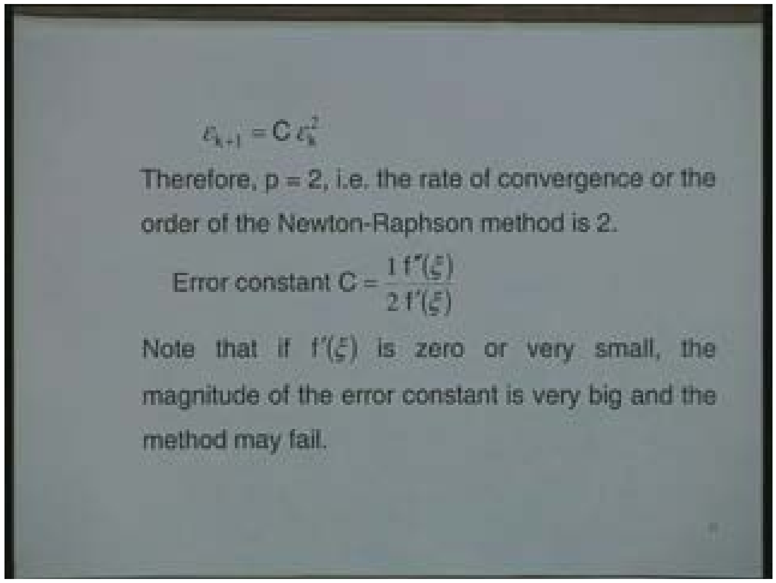

Now we are justified in neglecting because we are saying epsilon k is a very small quantity; so order of epsilon k cubed is much smaller than order of epsilon k square. Therefore if you neglect the higher order terms, what I get here is epsilon k plus one is equal to C epsilon k square, where this constant C is half f double dash by f dash xi. Therefore we have proved that p is equal to two and that is the rate of convergence or the order of the Newton Raphson method is two.

Now when we say the Newton Raphson method is of order two what we mean is; if we are sufficiently close to the root, let us say 0.5, it means if C is sufficiently small and that C is multiplying with every epsilon; for example it is multiplying with epsilon zero square, it is multiplying with epsilon one square epsilon one square. So C is bounded quantity and it is a finite quantity. Now at each stage you are 0.5 away from the exact solution, in the next stage you will be away from 0.5 whole square i.e. 0.25. We are multiplying each one by C, the same factor. In the next step we are away from 0.25 whole square, so 0.0625.

So now see how fast the convergence will come. So you will have the accuracy coming so fast, therefore this is reasonably a good method for any practical application. So this is how initially it will take few iterations to close up. Suppose as I said the root was 1.5 and you started at 3. There is no meaning between the errors, which here is 1.5. It is quite far away. So it will take few iterations to reduce the error sufficiently small, only then you will be able to see the convergence is very fast because you have already crossed the stage wherein the epsilon k is sufficiently small. Therefore the accuracy is achieved much faster.

As I said earlier if I look at this error constant I may be able to say when the iteration may become little bit slower or when the errors initially could be big. If you can see that f dash xi is close to zero which means dy by dx slope is almost equal to zero; that means the graph is cutting the x axis almost horizontally. In that case f dash xi is going to be very small. Therefore in that case the error constant will be small but still bounded but that is going to multiply each time with this factor.

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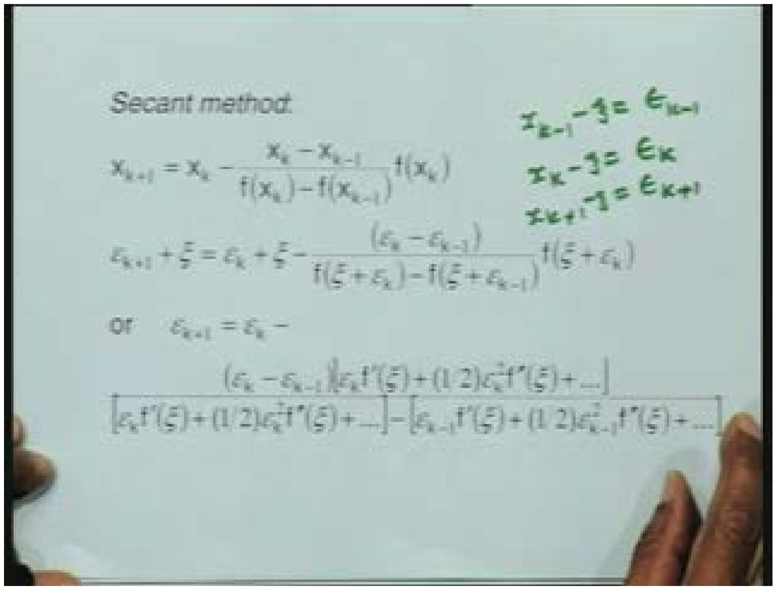

Now the secant method that we have written is xk plus one. There are two forms of secant method that we have written. One is for computation purposes and the other is for the error analysis. We shall use it - xk plus one is equal to xk minus xk minus xk minus one f of xk minus f of xk minus one fk. This is the method that we have written so that error analysis can be made. The procedure is identically same. We substitute the xk by epsilon k plus xi, and then xk minus one epsilon k minus one is written as in the previous step. We will write it again here; xk minus one minus xi is equal to epsilon k minus one and we have xk minus xi is equal to epsilon k. So I now substitute this and xk plus one minus xi is epsilon k plus one. Therefore I substitute xk plus one is epsilon k plus one plus xi; xk is epsilon k plus xi; in the numerator xk minus xk minus one. So I am subtracting these two, so xi cancels off and I will be left out with only the epsilon k minus epsilon k minus one. This is f of xi plus epsilon k and the second terms is minus f of xi epsilon plus k plus one and this factor is f of xi plus epsilon k.

Now the reason why I have chosen is, xi here automatically cancels; one of them easily cancels here, this xi and xi cancels over here so that I can write this as epsilon k plus one epsilon k. I have retained the numerator as it is. Now I open up all these three terms by Taylor series; f of xi plus epsilon k, epsilon k f dash xi, half epsilon k square f double dash xi plus one, in the denominator I have already used f of xi is zero so we need not write f of xi again. So we have taken f of xi zero. The next term is epsilon k f dash xi half epsilon square f double dash xi. Similarly f of xi is zero epsilon k minus one, first derivative at xi, half epsilon k minus one square f double dash f xi. I have to first simplify the denominator. Write it as whatever leading factor I take out and write it as some one plus capital X. So I would like to bring it in that particular form.

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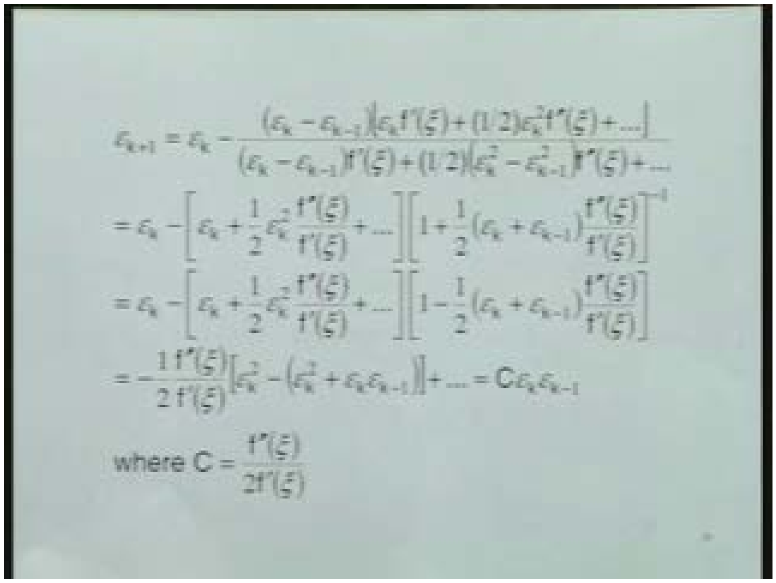

So as you can see the denominator is f dash xi f dash xi. So I can take epsilon k epsilon k minus one as a common factor here. Second term is half epsilon k square minus epsilon k minus one square. So these two are subtracted and this is f double dash xi. Now as I said in order to take it up I will take this particular factor out. If I take this factor out, epsilon k epsilon k minus one is common; so it cancels off. If I take this factor out, epsilon k epsilon k minus one cancels off. Now let us first look at the denominator. The denominator will be one plus half. This is epsilon k square minus epsilon k minus one square, factorize it, it becomes epsilon k minus epsilon k minus one into epsilon k plus epsilon k plus one. Epsilon k minus epsilon k minus one cancels because we are dividing by this. So what is left out is epsilon k plus epsilon k minus one. This is f double dash minus f dash xi, this to the power of minus one. In the numerator epsilon k minus epsilon k minus one was cancelled. We take f prime xi inside. So I will have here epsilon k half epsilon k square f double dash divide by f dash xi. Then as we have done in the previous case we will expand it by binomial expansion. This is one minus capital X. I should have written plus this this; this is one minus epsilon k epsilon k minus one f double dash by this. Now multiply it out. If I multiply it out, I would get here epsilon k into epsilon k and epsilon k cancels off and I will take the common factor of f double dash by f dash xi. If I take that I will have minus sign outside, epsilon k square here and I will have here epsilon k into this factor which means epsilon k into this is epsilon k square epsilon k into epsilon k minus one. So the product of these two is this and this. Now if I simplify this I would get here C epsilon k epsilon k minus one, where this constant is f double dash by twice f dash xi. This constant is same as in the Newton Raphson method but it has got a different form.

We will see now how we are going to manipulate this from this to get the order of this particular method; because when we have defined the order it is to link only epsilon k plus one and epsilon k but here it has linked epsilon k plus one with epsilon k and k minus one. So we will have to eliminate epsilon k minus one from here to get that particular step and we have the solution.