Numerical Methods and Computation

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Lecture No - 39

Numerical Differentiation and Integration (Continued)

In the previous lectures we have derived the Newton cotes integration rules, in particular the trapezoidal and the Simpson's rules, we have also derived the composite trapezoidal and Simpson's rules. The extra pressure schemes which we called as Romberg integration are very useful to obtain results which converge very fast. Now each column in the Romberg integration table is of at least of one order higher than the previous column but in the case of the trapezoidal and Simpson's rules these were two orders higher than the previous column. In the case of trapezoidal rule if the actual result of order of h square then the first extrapolation gives us an order of h to the power of 4 then the next extrapolation goes to order of h to the power of 6 and so on, similarly for the Simpson's rule. Therefore the Romberg integration rule is one of the most powerful rules to get the results which are, which converge very very fast.

In the case of the trapezoidal rule or the Simpson's rules we have taken the abscissas that is x_0, x_1, x_2, x as fixed therefore we were able use the Lagrange interpolation formula and then construct the Newton cotes formulas, however if you want to increasing the order of a method it is necessary that as for us to consider the abscissas also as unknowns and determine the unknowns so that the order of the formula can go up with using less number of points only. What we are saying is in the trapezoidal rule it is a two point rule if you started with and the two point rule has given a formula of precision 1 because it integrates exactly polynomials of degree 1, whereas in Simpson's rule we have used 3 points and it has given us a formula with precision 3 that is polynomials which the integration rule which integrates exactly polynomials of degree less than or equal to 3. However if I now allow the abscissa also to be unknowns are determine them it is possible for us to rise the order of the formula with less number of points, such formulas are called the Gauss integration rules, so let us now consider the Gauss integration rules.

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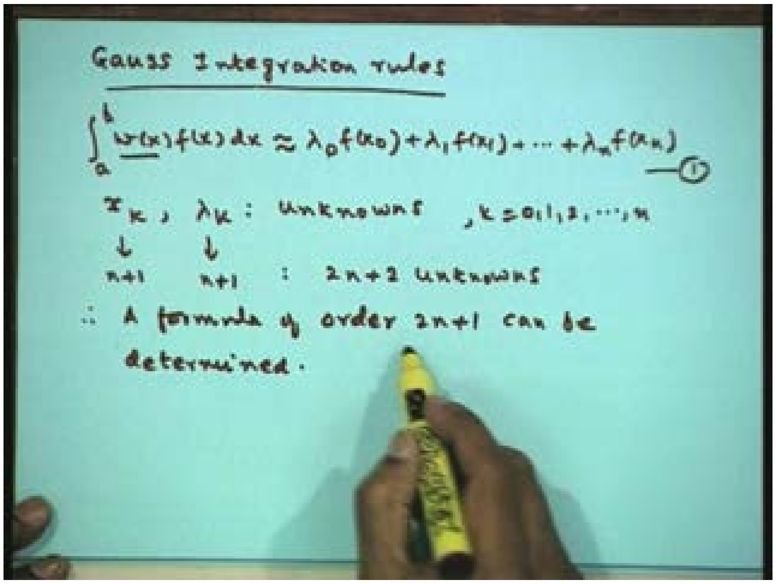

Therefore what we are looking for is a formula of the type, integral a to b weight function w(x) f(x) dx is approximately lambda$_0$ f of x_0 plus lambda$_1$ f of x_1 plus so on lambda$_n$ $f(x_n)$. Now we shall allow x_k's as well as f_k's unknowns, so we will take x_k and lambda$_k$ or unknowns to be determined and k going from 0, 1, 2 to n. Therefore here we have n plus 1 unknowns, here we have n plus 1 unknowns therefore we have total of 2 n plus 2 unknowns, we have 2 n plus 2 unknowns to be determined this particular formula. Hence a formula of order or precision 2 n plus 1 can be determined; therefore we can say a formula of order 2 n plus 1 can be determined. Now it is indeed true that a formula of such order really exists and we can derive that particular formula such that this n plus 1 point rule can become a formula of order 2 n plus 1.

Now the basic theorem for this is that if I choose this abscissas x_0, x_1, x_2, x_n as the zeros of an orthogonal polynomial, orthogonal with respect to the weight function w(x) over the interval a to b then it is possible for me to construct a formula of this type with precision 2 n plus 1. Depending on the weight function and the interval of integration we can construct the corresponding orthogonal polynomials, the most important orthogonal polynomials we know are the Legendre polynomials, Chebyshev polynomials, if you have finite interval minus 1 to 1 or if you have Laguerre polynomials in 0 to infinity, Hermite polynomials and minus infinity to infinity, so we have the orthogonal polynomials available to us, these orthogonal polynomials can be used to construct the integration rules of the order 2 n plus 1 over here. This result which I stated let us take it as a theorem; so let us call this as a theorem.

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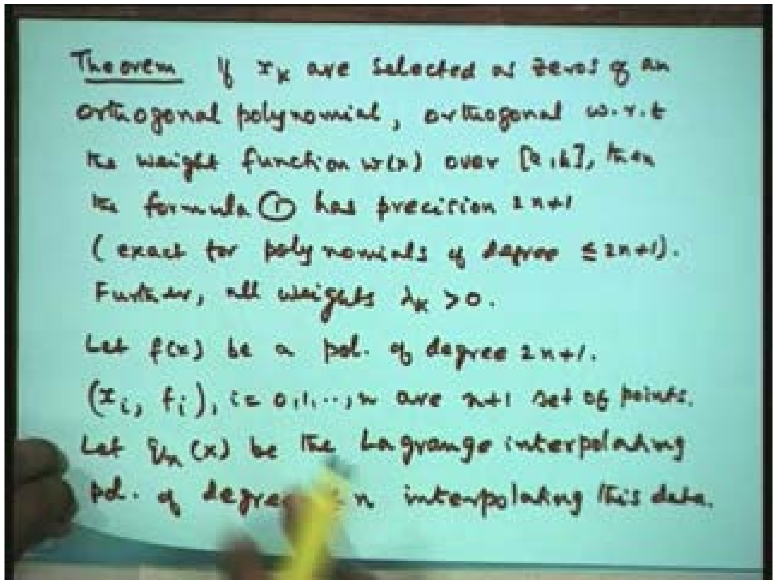

If x_k are selected as zeros of an orthogonal polynomial, of an orthogonal polynomial, orthogonal with respect to, I will write down w.r.t so that it is with respect to the weight function w(x), with respect to a weight function w(x) over the given interval [a, b] whatever the interval that is given to us then the formula 1 has precision 2 n plus 1 that means it is exact for polynomials of degree less than or equal to 2 n plus 1. It is going to integrate exactly polynomials of degree less than or equal to 1 and another important thing is further all weights are positive, all weights lambda$_k$ are greater than 0. Therefore what we are stating here is that in this particular formula if I choose these abscissas x_k's as the zeros of an orthogonal polynomial, orthogonal with respect to this weight function that is given to us, over the given interval [a, b] then it is possible to get the precision of 2 n plus 1.

Let us start the proof of this with assuming that f(x) in our integral is a polynomial of degree 2 n plus 1 and then we shall show I can get the values of lambda$_k$'s and x_k's uniquely from there, then that means we obtained the required result. So let us say let f(x) be a polynomial of degree 2 n plus 1. Now x_k's and f x_k's are unknown, so let us take this as our data points i is equal to 0, 1, n are n plus 1 set of points. So I am taking the abscissa and the corresponding ordinates and then I will take this as n plus 1 set of points. Now this is x_i's or random here, arbitrary, they not of equal length therefore what I will do is I would construct a Lagrange interpolating polynomial passing through this, this is n plus 1 set of points therefore I can get a polynomial degree less than or equal to n passing through this one. So let us say let $q_n(x)$ be the Lagrange interpolating polynomial, be Lagrange interpolating polynomial, interpolating of degree less than is equal to n interpolating this data. We have taken these points, abscissas, corresponding weights, then we called them as n plus 1 set of points and then using this n plus 1 set of points we are constructing the Lagrange interpolating polynomial $q_n(x)$ of degree less than or equal to n, so that means $q_n(x)$ is of the form,

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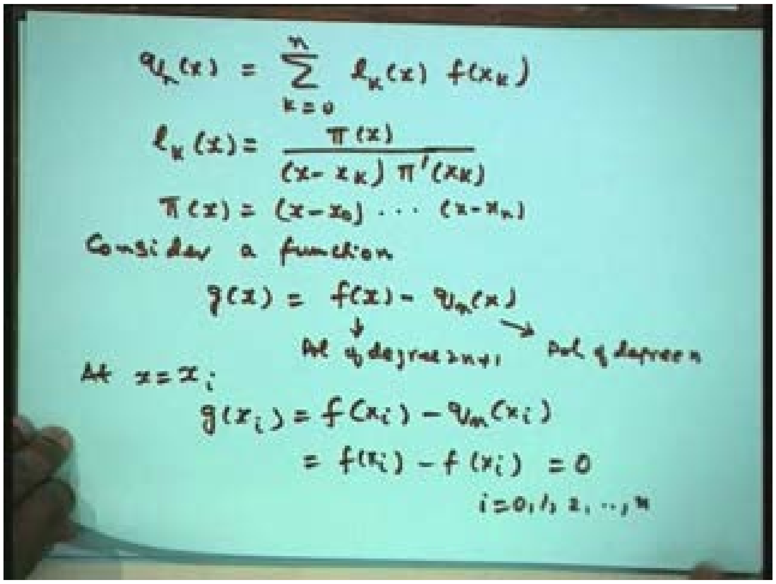

Therefore $q_n(x)$ will be of the form our Lagrange interpolating polynomial, it will be summation of k is equal to 0 to n $l_k(x)$ $f(x_k)$, where $l_k(x)$ are the Lagrange fundamental polynomials which we had written it as pi(x) upon (x minus x_k) pi dash of x_k and pi of x is the product of all the factors (x minus x_0) (x minus x_1) x_n, this is the factor. This is the Lagrange interpolating polynomial interpolating at the n plus 1 data points given to us and these are the Lagrange fundamental polynomials that we are using it. Now what we do is, I would construct a new function so I will consider a function; let us call it as g(x) this is equal to f(x) minus $q_n(x)$. Now we know that f(x) is, we started with a polynomial of degree 2 n plus 1, this is a polynomial of degree 2 n plus 1, now this is a polynomial of degree n, this is a polynomial of degree n. What happens at x is equal to x_i, I will have g at x_i is equal to f at x_i minus q_n at x_i. Now $q_n(x)$ is a Lagrange interpolating polynomial interpolating at the data and hence it is exactly equal to $f(x_i)$ therefore this is equal to f of x_i minus f of x_i that is equal to 0 for all i. This is obvious because $q_n(x)$ is the Lagrange interpolating polynomial interpolating the given data and therefore this is equal to f at x_i and this is equal to 0.

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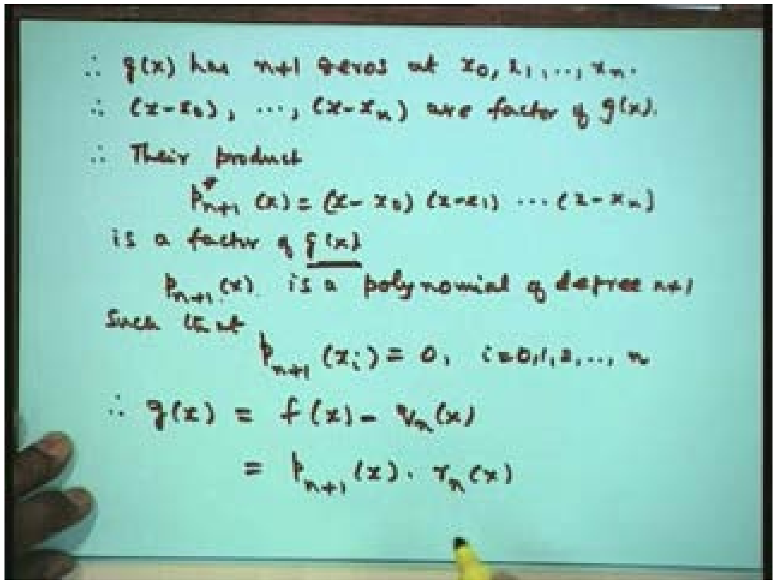

Therefore this implies that our function g(x) has n plus 1 zeros at x_0, x_1, x_2, x_n because g at x is equal to 0 at these points, therefore this implies that g(x) has n plus 1 zeros at x_0, x_1, x_n. What does that mean, these are n plus 1 zeros means (x minus x_0) (x minus x_1) (x minus x_n) can be factors therefore (x minus x_0), so on, (x minus x_n) are factors of g(x). If that is so their product is a factor of g(x), obviously the product is a, therefore the product, their product, let us call this as $p_{n+1}(x)$, let us put here star, p_n star, (x minus x_1) (x minus x_n) is a factor of g(x). I will now choose a polynomial of degree n plus 1, polynomial of degree n plus 1 such that, is a polynomial of degree n plus 1 such that $p_{n+1}(x_i)$ is equal to 0, i is equal to 0, 1, 2 so on n.

Both will be the same if the leading coefficient of $p_n(x)$ is 1, here if I open it up, expand it, the leading coefficient is x to the power of n with leading coefficient is 1, p_{n+1} I am allowing leading coefficient to be something arbitrary, for example if you have a quadratic you may have 3 x square minus 2 x plus 1, so the leading coefficient is 3 so therefore in p_2 I am allowing the leading coefficient to be a factor which is not equal to 1. Whereas if I take this simplest factors and multiply it out the leading coefficient will be 1, so either one of them can be taken because p_{n+1} excise automatically 0 therefore this has got the same factors. If that is so we are said that this product is a factor of g(x) therefore I can write down g(x) as equal to, which is we started with f(x) minus $q_n(x)$, this is the definition of this, this can now be written as $p_{n+1}(x)$. Now g(x) is a polynomial of degree 2 n plus 1, I have taken out n plus 1 polynomial, degree polynomial therefore what is left out will be polynomial of degree n therefore it will be some $r_n(x)$.

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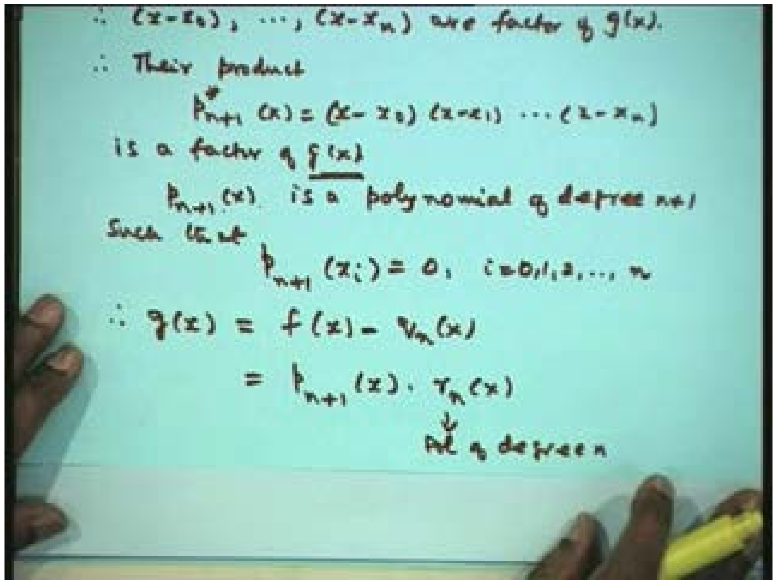

It will be some $r_n(x)$ where $r_n(x)$ is a polynomial of degree n, this will be polynomial of degree n so that the total degree is 2 n plus 1 which is the degree of the polynomial g(x). Now this is the basis of our theorem, let us number it 2. Let us multiply both side by the weight function and integrate over [a, b], so multiply by the weight function w(x) and integrate.

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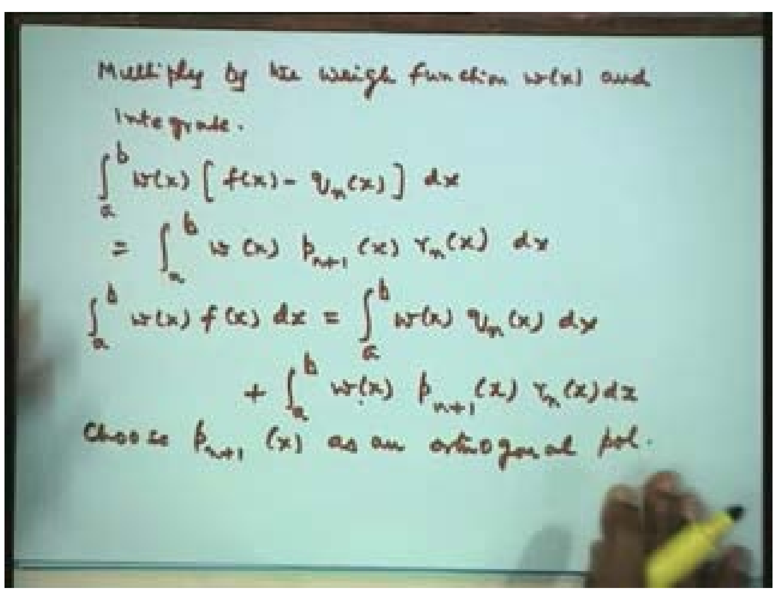

Now that would give us, I am just multiplying both sides by w(x) and integrating therefore I will have integral a to b w(x) [f(x) minus $q_n(x)$] dx and that is equal to integral a to b w(x) p_{n+1} of x $r_n(x)$ dx. Now we shall open up the left hand side and simplify, take this to the right hand side so I will write this as integral a to b w(x) f(x) dx is the integral a to b w(x) $q_n(x)$ dx plus this second term integral a to b w(x) $p_{n+1}(x)$ $r_n(x)$ dx. So I have just separated out this into 2 integrals, sum of 2 integrals, taken one integral to the right hand side that is w(x) $q_n(x)$ to the right hand side and added to this one. Now choose this $p_n(x)$ that we have taken here, $p_{n+1}(x)$ this or this, choose this as an orthogonal polynomial, orthogonal with respect to weight function w(x) over the interval [a, b]. If I choose $p_n(x)$ as an orthogonal polynomial what would happen? We know if you have an orthogonal polynomial some q, $q_i(x)$ then interval a to b w(x) $q_i(x)$ $q_j(x)$ dx is equal to 0 for i not equal to j, therefore if I once choose this as an orthogonal polynomial then this is the polynomial degree of n, this is the polynomial degree n plus 1 hence by the property of the orthogonal polynomials the value of this integral is 0. Therefore now choose, choose $p_{n+1}(x)$ as an orthogonal polynomial, as an orthogonal polynomial, of course of degree n plus 1,

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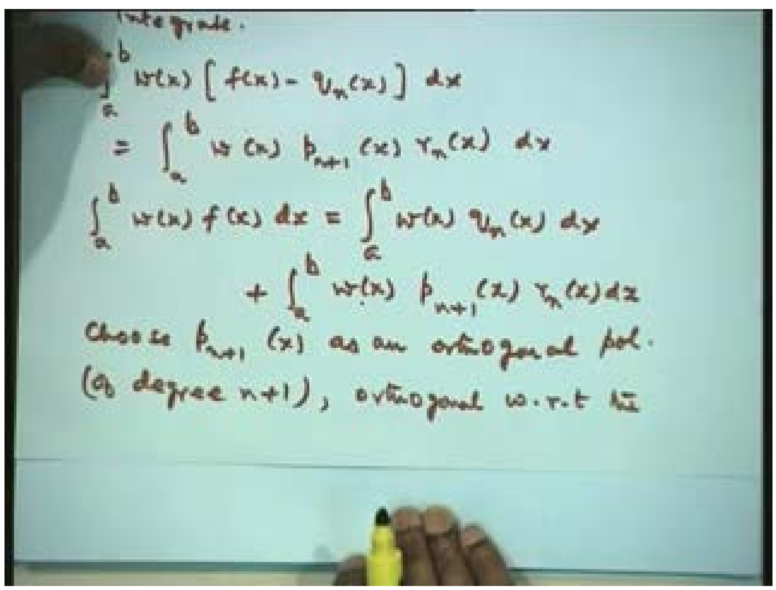

Of degree n plus 1, as an orthogonal polynomial, orthogonal with respect to the weight function, with respect to the weight function w(x) over [a, b].

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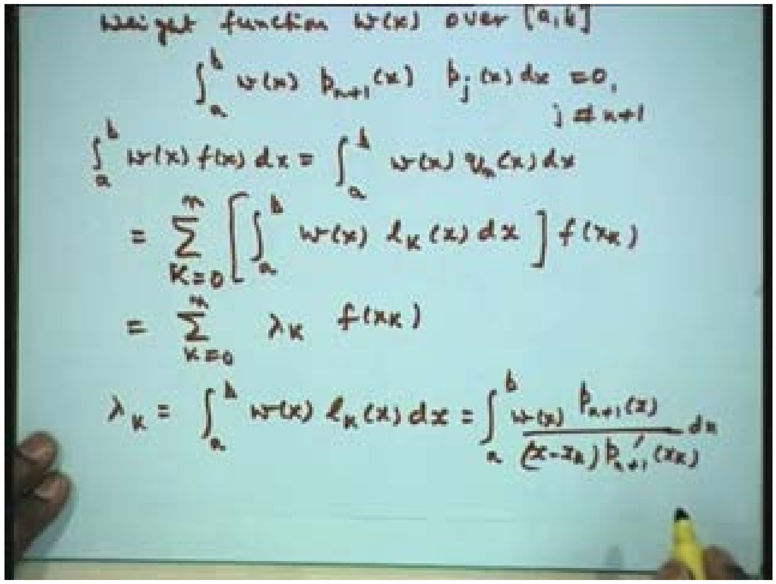

That means integral a to b w(x) $p_{n+1}(x)$, any polynomial $p_j(x)$ dx is 0 j not equal to n plus 1, if j is not equal to n plus 1 then this is going to be 0 by the definition of an orthogonal polynomial. Now in our integral we have the $r_n(x)$ as a polynomial degree n therefore this integral will be 0 therefore I will just have this expression as my result, therefore I conclude from here, integral a to b w(x) fx dx is equal to a to b w(x) $q_n(x)$ dx. Now we have achieved what we want, we have thrown away the second term and we have got this but $q_n(x)$ is known to us already, it is the Lagrange interpolating polynomial so I can substitute the value of $q_n(x)$. So if I substitute it I will just get simply i we have written, we have written k is equal to 0 to n [integral a to b w(x) $l_k(x)$ dx] into f of x_k. I substituted the value of $q_n(x)$ that is your $l_k(x)$ $f(x_k)$ integrate over a to b, $f(x_k)$ is independent of integral so I have written it out of this one. And this I will write it as lambda$_k$ $f(x_k)$, which is our required formula where your lambda$_k$ is integral a to b w(x) $l_k(x)$ dx.

This formula looks identically same as we have done in the Newton codes formula except that here the abscissas there to be chosen are the zeros of an orthogonal polynomial, there we have in trapezoidal Simpson's rule we have fixed the abscissas already but here when once you choose them to be the zeros of an orthogonal polynomial the order of the formula goes up. Now either I can use this $l_k(x)$ as the fundamental polynomial or I can write $l_k(x)$ back in terms of $p_{n+1}(x)$ that we have chosen, so I could as well I have written this as a to b w(x) and I can write $l_k(x)$ as polynomial $p_{n+1}(x)$ divided by (x minus x_k) p_{n+1} prime at x_k dx. Now what I would, what we had really written here is that $l_k(x)$ are given by this, this numerator pi(x) is nothing but (x minus x_0) (x minus x_n) so in place of this we are now using that $p_{n+1}(x)$ which we had derived it here as p_{n+1} star is this product or if this as a leading coefficient, we allow the leading coefficient to be there and that leading coefficient shall cancel with the leading coefficient it is there. If there was a 3 leading coefficient there is a 3 here also so that 3 cancels of and this ratio would be identically same as the $l_k(x)$. So either I can actually use $l_k(x)$ as determined by this or I can take whatever the orthogonal because pn+1 is orthogonal polynomial, so I can insert the orthogonal polynomial itself here in place of writing $l_k(x)$, integrate this and get my lambda$_k$'s. Now this, since we are able to uniquely determine lambda$_k$'s this shows that effects is indeed a polynomial of degree 2 n plus 1 and hence the formula is of precession 2 n plus 1.

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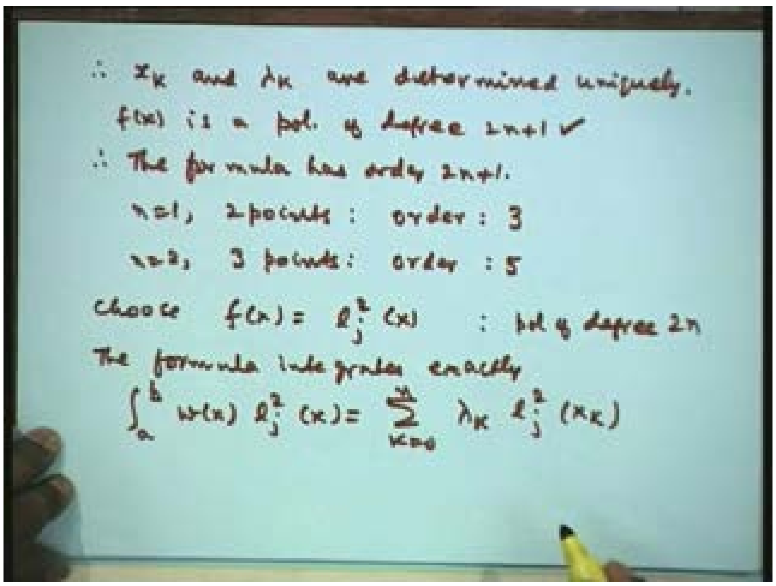

Therefore we can say x_k and the lambda$_k$ are determined uniquely. And therefore f(x) is a polynomial of degree 2 n plus 1 therefore the formula has order 2 n plus 1, therefore the formula has order 2 n plus 1. We have to show that lambda$_k$'s are positive but before that let us see what this implies, let us suppose I take 2 point formula, say n is equal to 1, 2 points, this corresponds to the trapezoidal rule in the fixed abscissa case and what will be the order here, the order is going to be 3 n is equal to 1 therefore 2 n plus 1 that is order 3 that means the corresponding to the trapezoidal rule we have there, we have here which is order 3 there it is order 1. If I take n is equal to 2 that is 3 points, this corresponds to the Simpson's rule in the case when the abscissas are fixed or pre chosen but here we have the order as 5, 2 n plus 1 order is 5, so you can see the order is 5 here Simpson's rule is only of order 3. So the order of the formula shoots up like anything when you allow these abscissas to be not equispaced but they will be arbitrarily spaced but they would be zeros of an orthogonal polynomial. Now we shall show that lambda$_k$'s are positive, now since it is integrating exactly polynomials of degree 2 n plus 1 it will also integrate any polynomials which is less than that degree therefore let us choose f(x) as simply l_j square x. $l_j(x)$ is a Lagrange fundamental polynomial which is of degree n, hence f(x) is the polynomial of degree 2 n. Therefore this formula should integrate this exactly, the formula integrates, the formula integrates exactly that means integral a to b w(x) l_j square x is summation k is equal to 0 to n lambda$_k$ l_j squared (x_k).

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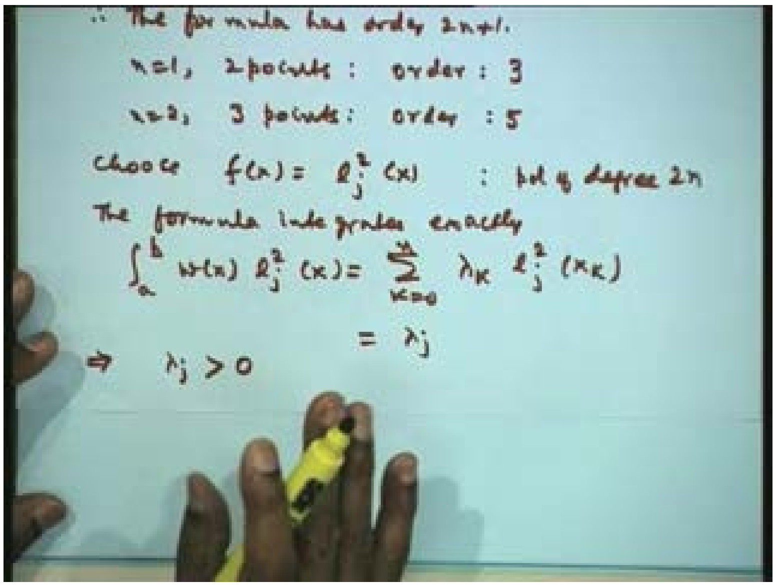

But by definition l_j square x_k all zeros expect j is equal to k, therefore this is equal to simply lambda$_j$ all of them would vanish except one where j is equal to k and that is equal to 1 therefore this entire sum is simply equal to lambda$_j$. Now w(x) is a weight function positive quantity, $l_j(x)$ square is a positive quantity, b is greater than a therefore the value of the integral is always greater than 0 therefore this implies that lambda$_j$ is strictly greater than 0. So that all the weights are positive and the, what is the advantage? The advantage is that there will be no errors due to cancellations because if the weights, some of the weights are negative then there will be cancellation then there will be growth of errors. We shall now derive some Gaussian integration rules using the known orthogonal polynomials. We shall derive 1 point, 2 point and 3 point integration rules and then generalize them.

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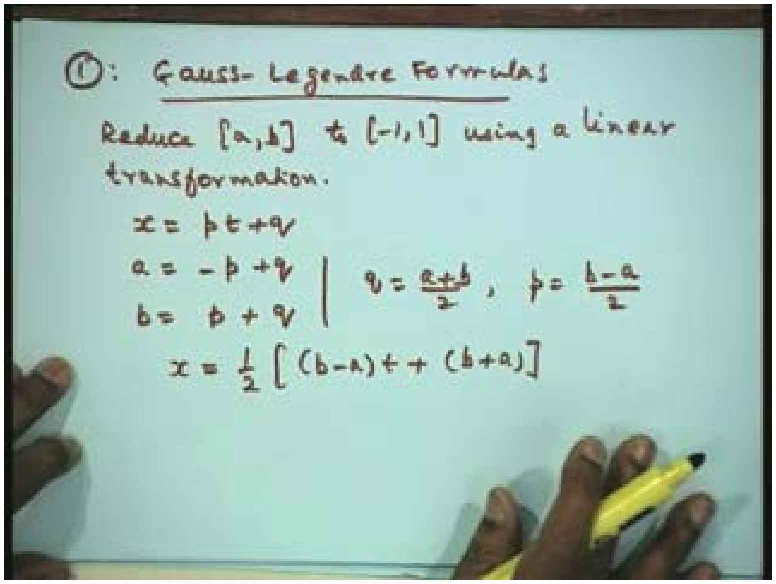

Let us first consider the derivation of gauss Legendre formulas, as the Gauss Legendre formulas; these are called the Gauss Legendre formulas. The Legendre polynomial is defined on minus 1 to 1 therefore given an interval [a, b] reduce it to minus 1 to 1 by using a linear transformation, so reduce [a, b] to minus 1 to 1 using a linear transformation, using a linear transformation. That is something like we are putting x is equal to some pt plus q, you are putting x is equal to pt plus q then when x is equal to a, t is minus 1 so that is p plus q. When x is equal to b t should be 1 that is p plus q, I can add these two, I find the value of q therefore q is equal to a plus b by 2 and I can subtract these two, b minus a by 2 is p, b minus a by 2 is equal to p. Therefore the required linear transformation which we can always remember it easily as half of (b minus a) into t plus (b plus a), so this is the transformation, linear transformation which will reduce [a, b] to the interval minus 1 to 1.

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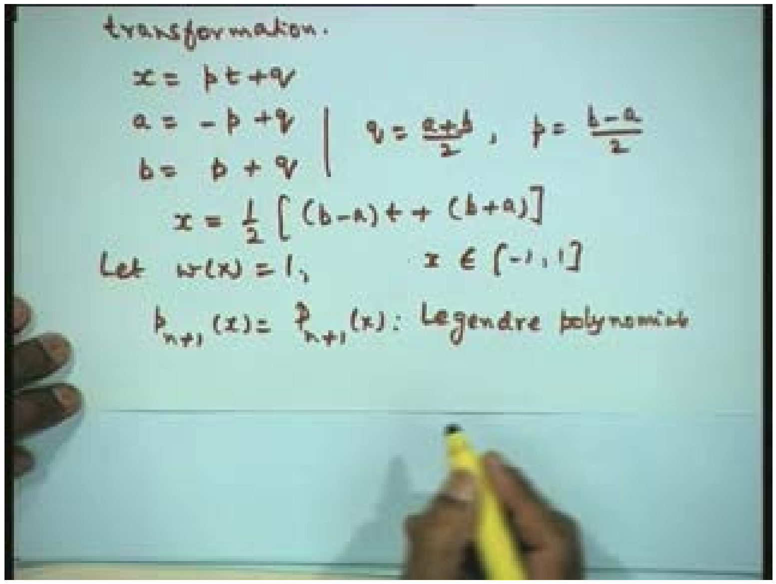

Further we will take weight function as 1. Now x is lying between, now we are taking the dummy variable as x, now we replaced it by t and replace t by the dummy variable x, now, so will talk in terms of x only, so x is lying between minus 1 to 1. The orthogonal polynomials which are orthogonal over minus 1 to 1 with weight function 1 are the Legendre polynomials therefore the p_{n+1} which we are talking of in the rule is the Legendre polynomials $p_{n+1}(x)$, these are the Legendre polynomials, these are the Legendre polynomials.

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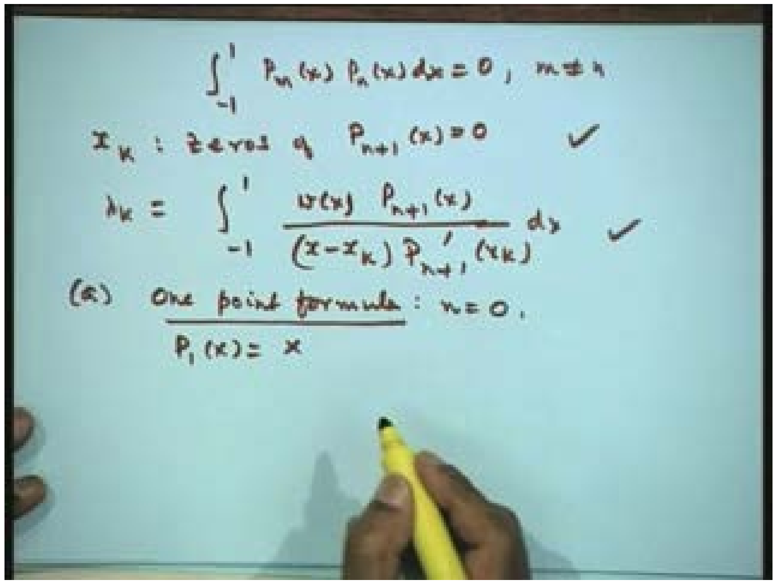

Now we earlier discussed the orthogonal property of the Legendre polynomials, we have shown it as minus 1 to 1 some $p_m(x)$ $p_n(x)$ dx is equal to 0, for m not equal to n. Now let us recognize our values of x_k and $lambda_k$, x_k's are zeros of $p_{n+1}(x)$, the zeros of $p_{n+1}(x)$ that means if you want a particular formula, fix up your n, find out what is the Legendre polynomial, find its zeros and those are will be the abscissas in our formula and $lambda_k$ would be equal to, we have derived that $lambda_k$ is equal to, $lambda_k$ is equal to w(x) $p_{n+1}(x)$ upon this, so I could write this as integral minus 1 to 1 w(x) that is 1, w(x) that is 1 $p_{n+1}(x)$ this is the Legendre polynomial divided by (x minus x_k) p_{n+1} prime at x_k dx, with w(x) is equal to 1. Now that is all we need, we know what will be x_k's, what will be the weights of this and once this is know I can substitute in the formula then these formulas are called the Gauss Legendre formulas. Now let us derive some particular formulas, let us write down one point formula. Now one point formula means n is equal to 0 and I have to consider p_{n+1} that is p_1 so I must consider $p_1(x)$ which is equal to x.

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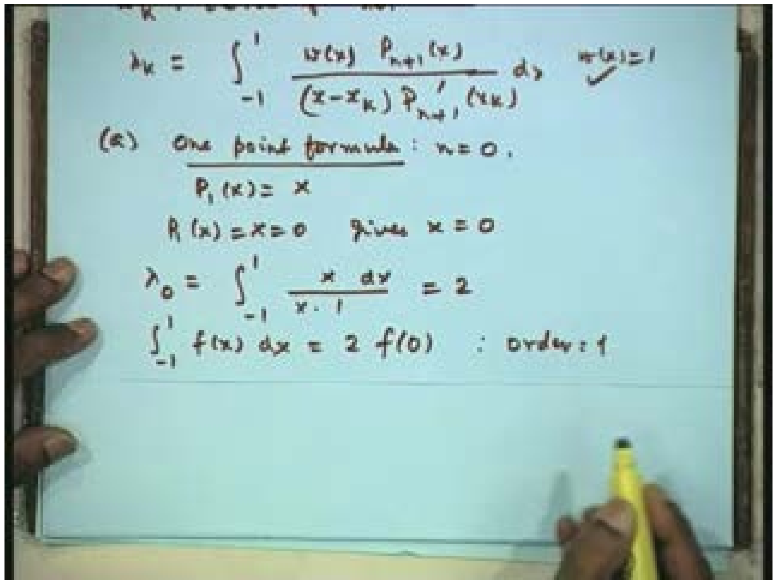

The abscissa will be obtained by setting $p_1(x)$ is equal to 0, so $p_1(x)$ is equal to 0 gives x is equal to 0. Of course I should, so there is no confusion w(x) is 1. Now let us substitute what is our lambda_0, lambda_0 will be minus 1 to 1 numerator $p_1(x)$ that is x, (x minus x_0) that is x, derivative of p_n p_1 derivative of p_n is 1 that is equal to 1 dx. This is p_{n+1} that is p_1, this p_1 prime this (x minus x_0) that is simply x that is simply integral dx that is equal to 2. Therefore we have derived the formula minus 1 to 1 f(x) dx is equal to 2 times f of 0, lambda_0 is 2 x is equal to 0 therefore 2 times f of 0 and what is the order of the formula that is 2 n plus 1 that is 2 into 0 plus 1 therefore order is 1, so this is the one point formula.

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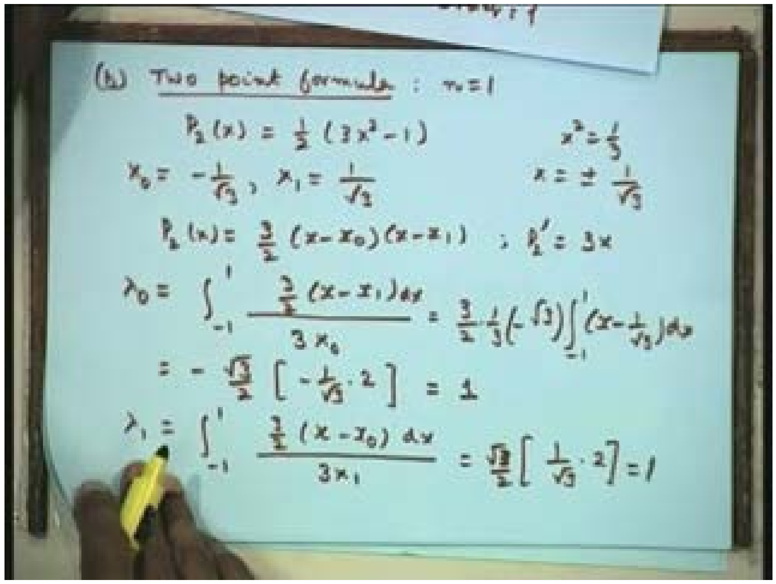

Let us take a two point formula, so then I will have n is equal to 1. Therefore I have to choose my $p_2(x)$, the polynomial will be $p_2(x)$, p_{n+1} that is your $p_2(x)$, $p_2(x)$ is equal to half (3 x square minus 1). Now the roots of this is, if you set it equal to 0 you will get x squared is equal to 1 upon 3 therefore x is equal to plus minus 1 upon root 3. Therefore the abscissas are x_0 is minus 1 upon root 3, x_1 is 1 upon root 3. In terms of x_0, x_1 let us write down what is our $p_2(x)$, now this is what were stating when we are proving the theorem that this one may have a coefficient in the leading power of x, here we have a coefficient of 3 by 2 here and it is not one so what we will have here is 3 by 2 (x minus x_0) into (x minus x_1). So if I take out factor from here, the roots are these x_0 and x_1 so this is same as 3 by 2 (x minus x_0) into (x minus x_1). Now with this I would like to find out what are our weights, so let us find out what is the weight lambda_0, minus 1 to 1 this is $p_2(x)$ so this is 3 by 2 (x minus x_1).

Now I have saved one step because there is denominator is (x minus x_0) here, I have cancelled (x minus x_0), I have not written that and let us write down one more step here, what is our derivative of this, p_2 dash is equal to 3 by 2 into 2 x that is equal to 3 x, p_2 dash is equal to 3 x. Therefore this is p_2 dash at x_k that is p_2 dash at x_k that is 3 times x_0, we are talking of lambda_0 so I will have this as x_0 and dx of course, therefore this is 3 by 2 into 1 by 3, x_0 is the constant and x_0 is minus 1 by 3 so let us take it out, minus root 3 this, x_0 I have taken back, integral of minus 1 to 1 x, x_1 is 1 by root 3 so x upon 1 by root 3 dx. So I have here minus root 3 by 2, I cancelled through this one x, x dx is an odd function therefore it is 0 so the contribution will come only from the second factor and that is minus 1 upon root 3 into integral of dx that is x that gives you 2, therefore this is equal to simply 1. Now let us write down lambda_1, lambda_1 is minus 1 to 1, again I will have numerator as 3 by 2, this time I will have (x minus x_0) dx, (x minus x_1) has cancelled, denominator will be p_2 dash at x_k that is p_2 dash at x_k that is 3 x_1, 3 x_1 dx. Now this is, let us cancel of 3 and this 1 upon root 3 so therefore root 3 by 2 and again integral x dx is 0 it is an odd function. x_0 is minus 1 by 3 so this is plus 1 by root 3 into 2, minus x_0, x_0 is minus 1 by 3 so I will have plus 1 by 3 integral of dx is 2 so this is also equal to 1.

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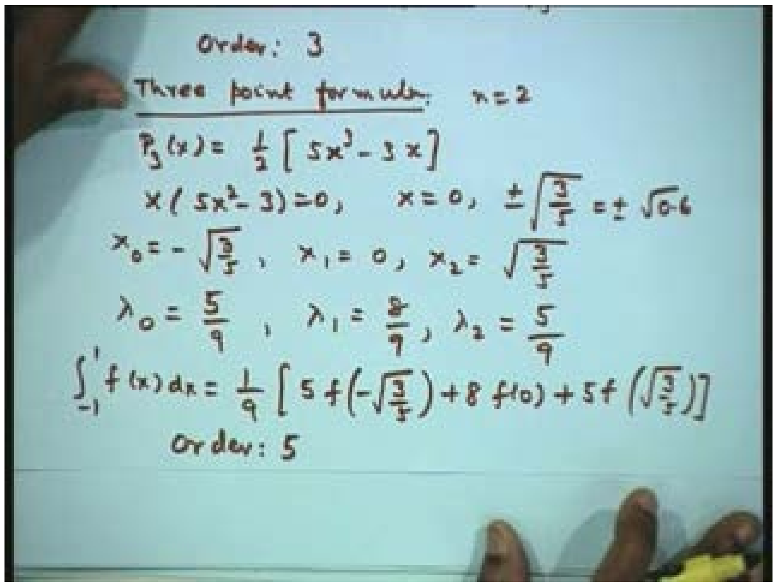

Therefore the 2 point formula, Gauss Legendre formula is minus 1 to 1 f(x) dx both the weights are equal and that is 1 therefore we have f of minus 1 by root 3 plus f of 1 by root 3 and what is the order of the formula is 2 n plus 1 that is 3, so the order of the formula is 2 n plus 3. Now you can compare with the trapezoidal rule which uses 2 points, which are fixed points and now we have got a formula of order 3. Now I can quote the three point formula also because we often use the first 3 formulas, three point formula that is n is equal to 2. Therefore I must choose the orthogonal polynomial, Legendre polynomial p_3, so I will take p_3 of x that is half [5 x cubed minus 3 x], [5 x cubed minus 3 x]. Therefore if I set it equal to 0, I will get here is x into (5 x square minus 3) is equal to 0, therefore x is equal to 0 and plus minus 3 by 5 under root of this, that is under root of point 6, so I can use this as under root of point 6. Therefore the 3 abscissas are x_0, I will first take the negative value under root of 3 by 5, x_1 as 0, x_2 as plus under root of 3 by 5.

Now I will leave this as simple exercise for you, just as we are substituted lambda we can find out the value, I will give the values, just find it out, lambda_0 comes out to be 5 by 9, lambda_1 is equal to 8 by 9 and lambda_2 is equal to 5 by 9. So that the formula, the three point formula is given by f(x) dx is 1 upon 9 [5 times f (minus under root of 3 by 5) 8 times f of 0 plus 5 times f of under root of 3 by 5] and the precision of this formula or the order of this formula is 2 n plus 1, n is 2 therefore order is 5. Now we can compare this with the equispaced Simpson's rule wherein the order was 3, with 3 points we have got a formula of order 3, now here we have got a formula of order 5. now let us illustrate this through an example, let us take an example.

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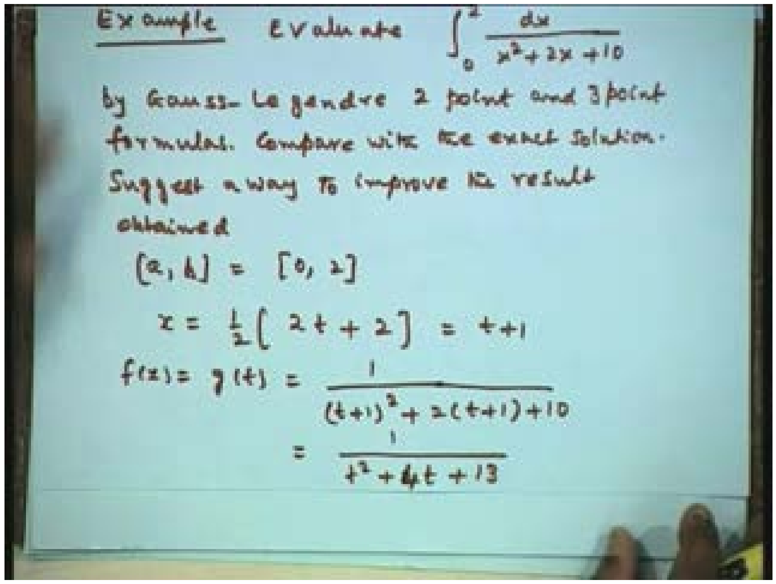

Evaluate integral 0 to 2 dx upon x square plus 2 x plus 10 by Gauss Legendre 2 point and 3 point formulas. Now compare with the exact solution. Suggest a way to improve the result obtained. Now to apply the formula we notice that the interval given to us is 0 to 2 that is a and b so we must first of all reduce it to minus 1 to 1, so we have been given the interval as [a, b] is equal to [0, 2]. Now this is always a linear transformation therefore I can straight away write down what is my transformation here which reduces to minus 1 to 1, so I can immediately write down that x will be equal to half of b minus a into t plus b plus a that is equal to t plus 1. So this is the transformation that is required, you can verify when x is 0 t is minus 1, when x is 2 t is equal to 1 so we are reduced it to the required formula. Therefore our f(x), let us call it as now g(t) that is equal to 1 upon x square that is (t plus 1) square 2 times x plus 10, so I am just substituting in this t plus 1 whole square 2 times (t plus 1) plus 10 that is equal to 1 upon t square plus 2 t plus 2 t that is 4 t, 1 plus 2 plus 10 that is 13, therefore our function g(t) is this.

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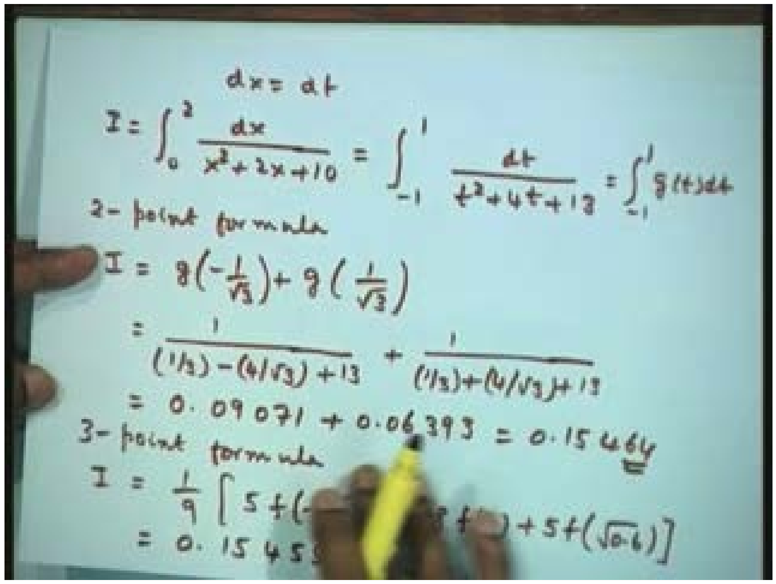

Further in our integration dx is equal to dt, dx is equal to dt. Therefore our integral, let us call it as I 0 to 2 dx upon x square plus 2 x plus 10 is equal to minus 1 to 1 dx is equal to dt and we have just derived this is t square plus 4 t plus 13. Now we shall apply our 2 point 3 point formulas on this integral minus 1 to 1 f(t) dt or some g(t) dt therefore what we, I think we call this as g(t) so let us call this as minus 1 to 1 g(t) dt. Now let us use the 2 point formula, therefore value of the integral I is equal to g of minus 1 by root 3 plus g of 1 upon root 3. So I just substitute the values and then evaluate it so this will be simply 1 upon 3 minus 4 upon root 3 plus 13 and this is 1 upon 3 plus 4 upon root 3 plus 13. Now this I will use for the next value therefore I will give you these values, these values are 0 9 0 7 1 plus 0 point 0 6 3 9 3, 0 point 1 5 4 6 4. So with the 3 point formula, I will be equal to 1 upon 9, 5 times f of minus under root of root 6, I will now use 3 by 5 is point 6, 8 times f of 0, 5 times f of root of point 6. This is 0 point 1 5 4 5 5. Now let us try to find the exact solution of the problem.

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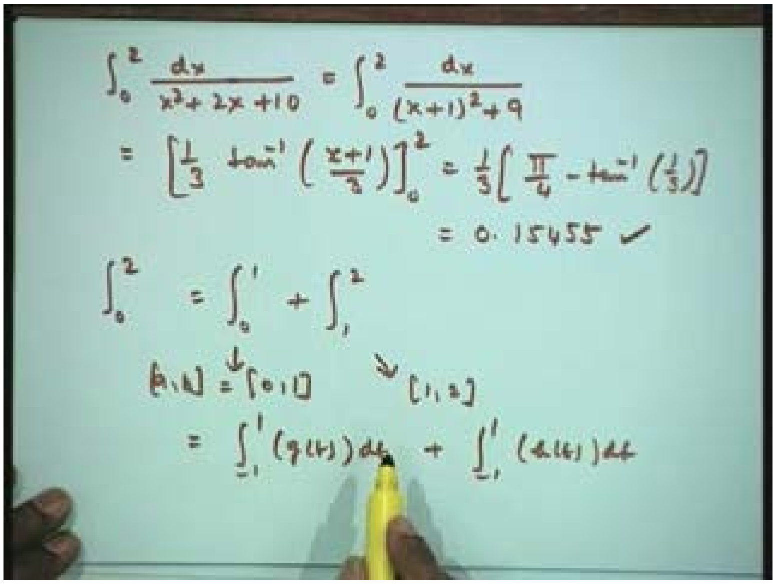

Our integral is 0 to 2 dx upon x square plus 2 x plus 10 this I can very easily write it as dx upon (x plus 1) whole squared plus 9. Therefore this is 1 upon 3 tan inverse of x plus 1 by 3 within the limits 0 to 2 that is 1 by 3, tan inverse of 1 that is pi by 4 minus tan inverse of, x is 0 1 by 3 and if you compute this, this is 1 5 4 5 5. 3 point formula is exactly matching with 5 decimal places and the 2 point formula is matching for the first 3 decimal places, therefore the order of 3 point formula was 5 therefore it is integrated this very very well. Now the second question that we asked was that suggest a way to improve the result obtained, now we are not thinking of an extrapolation or anything, the accuracy of the formula would increase if I am able to break this into number of integrals just as we have done the composite trapezoidal rule in which to improve the accuracy the given integral is broken into number of integrals and an each integral we have applied the trapezoidal or Simpson's rule, same thing can be done here. Therefore what we would like to suggest, I will write down 0 to 2 for example as, I will write it as 0 to 1 plus 1 to 2, I will write it as 0 to 2 as 0 to 1 and 1 to 2, now I will reduce this to minus 1 to 1, this also to minus 1 to 1, this is like almost like a composite Gauss Legendre formula it is like this.

Now for this the [a, b], for this is 0 to 1 and for this the [a, b] is 1 to 2. Now I can write down the transformation for this to make it minus 1 to 1, so I can bring this to minus 1 to 1 of some g(t) dt and this I can bring it to minus 1 to 1 some h(t) dt, by using the suitable transformation. Then I can apply the 2 point formula on this, apply the 2 point formula on this and they are the 3 point formula, 3 point formula, the corresponding result will be definitely be superior to what we have obtained here in this particular previous case as these two results, it will be much more superior than this and of course we can carry on to number more decimal places and this. It is something analogical to what we have derived the composite trapezoidal or Simpson's rule and this is the way of attaining the higher accuracy in the Gauss Legendre formulas, even though we can exactly give the error, find out the error and give the error, however computationally if you want to have better results over an integral which is the very large, the length of the interval is very large you can cut that into number of parts like this and each one we make the transformation to bring it back to minus 1 to 1 and apply the Gauss Legendre formulas. Okay we will stop this.