Numerical Methods and Computation

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Lecture No - 36

Numerical Differentiation and Integration (Continued)

In our previous lecture we were discussing the effect of round off errors in numerical differentiation; we were illustrating the same through an example. Now let us just review our discussion of the example and study the effect of the round off errors. Now what we are considering where the computation of the second derivative in the, using the formula f double dash x_0 is approximately 1 upon h square [f of x_2 minus twice f of $x_1$plus f of x_0].

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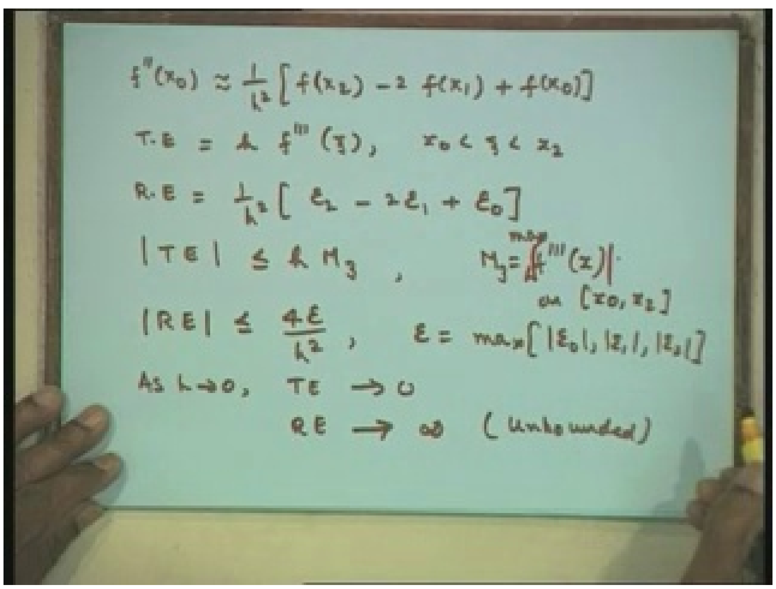

Now we are shown that the truncation error in this formula, it can be written as, truncation error is h times f triple dash of zhi, zhi lying between x_0 and x_2. we have also shown that the round off error is given by 1 upon h square of [epsilon$_2$ minus twice epsilon$_1$ plus epsilon$_0$] where epsilon$_1$, epsilon 1 and 2 are the round off errors in the $f(x_0)$, $f(x_1)$ and $f(x_2)$.

Then we have taken the magnitude of this and shown that the magnitude the truncation error is less or equal to h times M_3, where M_3 is the maximum magnitude of f triple dash x is equal to the maximum on the given interval x_0 to x_2. Then we have taken the round off error and written the bound of the round off error as less than or equal to, we have maximized these ones and retaken it as epsilon plus 2 epsilon plus epsilon that is 4 times epsilon upon h square, were epsilon is the maximum of the magnitudes of these 3 round off errors. Then I can write down, the round off error is bounded by this, then we made the observation that as h tends to 0 that is your convergence problem, then truncation error goes to 0 however because of the appearance of the h square is denominator round off error goes to infinity or goes unbounded. So as h tends to 0, truncation error goes to 0 and round off error goes to infinity or we can say it becomes unbounded, it becomes unbounded as h goes to very very small quantity. Now let us just have a look at these 2 values, let us try to draw a graph of the round off error and truncation error with respect to these step length h.

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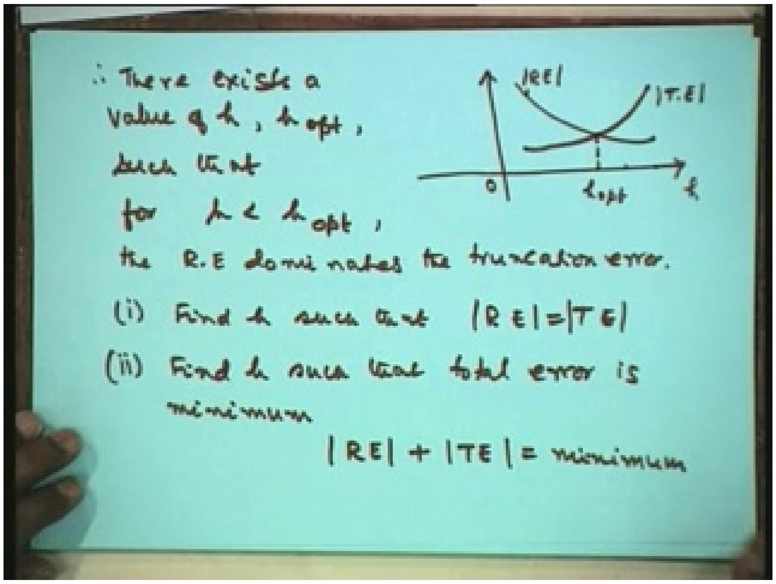

Let us take, draw the graph of this so I will take along this h and on this side I will take either the round off the error or the truncation error. Now if I take the truncation error, for a given h it has got a certain value, let us suppose I choose a value of h over here and I have a value truncation error here. As h goes to 0 that means h comes towards origin, this truncation error goes to 0 therefore the graph of the truncation error will be something like this, it is coming to 0 as h tends to 0. Now this is the graph of truncation error. Now if I draw the graph of the round off error for some value of h, it has got some value over here, some value over here, the round off error is, you know corresponding value of this h we take this as this, as h goes to 0 this round off error grows, so the round off error now grows and goes like this, so this is the magnitude of round off error.

Now this is the behavior of the round off error, h tends to 0 whereas truncation error goes to 0 and the round off error becomes unbounded as it goes to 0. Therefore this tells us that there can be a value of h such that for h smaller than that value say optimal value, the round off error dominates the truncation error. Therefore round off error dominates the truncation error and hence after few more values of h it is going to completely get unbounded that means we can say there exists a value of h, so let us call it as h optimum such that for h less than h optimum, for h less than h optimum the round off error dominates the truncation error and therefore this leads to our numerical instability. Now how do we find this h optimum, there are two ways of dealing with this problem. For example if you look at this graph, we find that there is a point at which these two graphs cut that is at the point where the truncation and round off error are equal. So I can call this value as an h optimal value, where the truncation error, the magnitude of the truncation error and round off error are equal.

So we can say find h such that magnitude of round off error is equal to magnitude of truncation error. Alternately you can look at in a different direction, you can ask find h such that the total error is minimum that means some of round off error and the truncation error is minimum, so we can also use this criteria say find h such that total error is minimum that means we are talking of magnitude of round off error plus magnitude of truncation error is minimum. Now this will completely tell us why we are talking of possibility of numerical instability in the numerical differentiation methods. Now let us use this first criteria for the problem that we are just now discussing.

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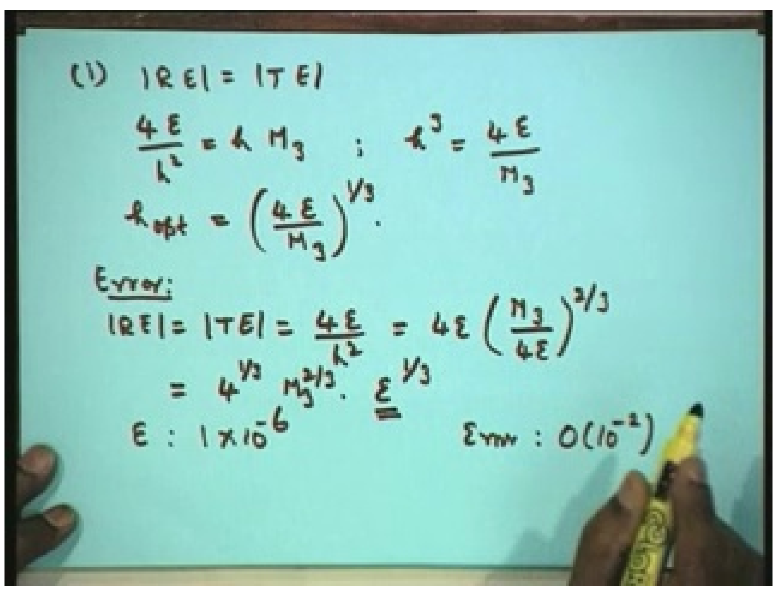

So let us take, go back to this to find this. Now magnitude of round off error is equal to magnitude of truncation error. Therefore I need to have 4 epsilon by h square is equal to h into

M_3, so that is 4 epsilon upon h square is equal to h into M_3. Now I can multiply it out and write this h cubed is 4 epsilon upon M_3; therefore I can find the value of h from here therefore h optimal value is equal to 4 epsilon upon M_3 to the power of 1 by 3. Now let us see what is the error here, the error is magnitude of round off error is equal to magnitude of truncation error, we can use anyone of them that is your 4 epsilon upon h square suppose I use this, the left hand side that will be equal to 4 epsilon h square, 1 upon h square so I can invert this and take to the power of square so I can invert this as M_3 upon 4 epsilon and write this as 2 by 3. So this is 4, this is divide by 4 to the power 2 by 3 so I will have 4 to the power of 1 by 3, I have M_3 to the power of 2 by 3 and this is epsilon and epsilon 2 by 3 so I will have here epsilon to the power of 1 by 3.

Now you can see the effect of the round off error, if in a particular problem you have said that your maximum round off error is 1 into 10 to the power of minus 6 that is your results the accurate to 6 decimal places in the data then I find that the error is of the order of epsilon to the power of 1 by 3. So the error is going to be of the order of epsilon to the power of 1 by 3 is of the order of 10 to the power of minus 2, cube root of 10 to the power minus 6 so we will have 10 to the power of minus 2. Now you see how much is round off error is eaten away your decimal places, there were 6 places accuracy initially and we find that the derivative, second derivative is now we can guarantee only 2 decimal places that is 10 to the power of minus 2 is beyond that derivative error, therefore you can guarantee. Therefore 4 decimal places have really being destroyed because of the round off error, so this is where the numerical instability can arise in numerical differentiation. Now if I, I leave this is as an example for you, let us look at the second part.

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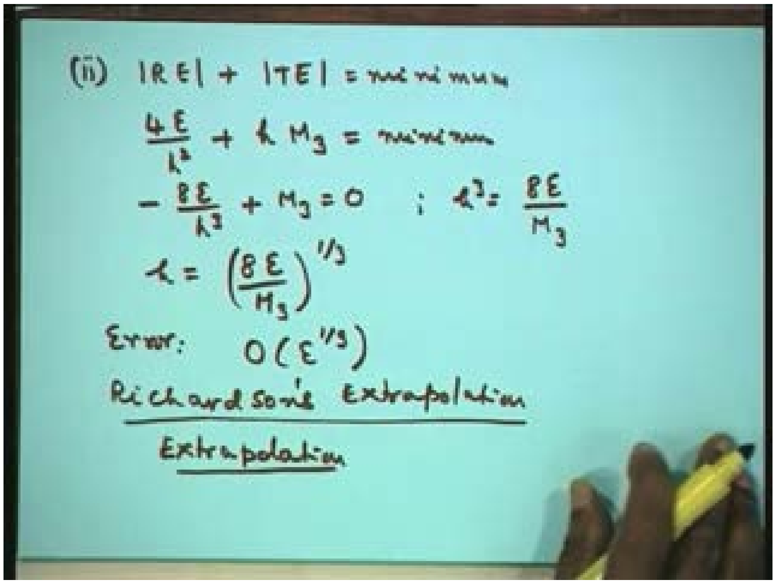

The second way of looking it is, magnitude of round off error plus magnitude of truncation error is minimum therefore I can add these 2 quantities and make this minimize it, so I will have 4 times epsilon upon h square plus h times M_3 is minimum, is minimum. So I can differentiate with respect to h and then find out the minimum value so this will give us, minus 8 epsilon upon h cubed plus M_3 is equal to 0. Therefore I can find out h, M h cubed from here, I go to this side and write this as 8 epsilon upon M_3, so that h is now 8 epsilon upon M_3 to the power of 1 by 3. Now this is the, now we can what is the total error, now this is h times M_3 so this is simply epsilon to the power of 1 by 3, this is also epsilon to the power of 1 by 3, just summed up and show it and again I have this as epsilon 1 by 3 except that the constant that is multiplying this factor is different from what we have the previous example. Therefore even though we make the sum of these errors is minimum even then we are still not improving the result substantially except that this figure that we are now having that is constant that is multiplying may be smaller in one case and bigger in the other case. Therefore this is how the loss of the significant digits can obtain in the numerical differentiation.

Now given any formula for us, the procedure for doing this is just write down what is the truncation error, just write down the round off error and then apply this particular rule to find out what should be the step length or h optimum such that the round off error is controlled. Now, if this is the difficult in numerical differentiation what should be the alternative to get better results. If I can take an higher order formula by including more number of points that means instead of, in this case we had f_0, f_1, f_2, instead of having f_0, f_1, f_2, I can include few more points f_3, f_4, f_5, so that I can have the formula for first derivate, of second derivative which will be higher order than this because I am using this, but then if once I introduced more number of ordinates here automatically I am introducing the round off error by few more. So instead of 4 epsilon it could become 6 epsilon; so the factor that we are multiplying can go up, go higher in this particular case.

Second observation is, suppose that in a practical problem you really do not know what is it, what is the function that is governing the data, we did not know what is its derivative, first derivative, second derivative and so on. So in practice what we normally do in any problem is that you solve the problem with a particular step length h, we are not sure that the result is good or not so we repeat the same computation with another step length say smaller than that. If you are started with say point 5, you may integrate again, solve it; use it by again point 2 5 or so. If the magnitude of the difference between the 2 results is the less than tolerance, we accept that the result is good. Now if this difference is greater than the tolerance which you have, what do we do? We have to go further down one more step length and then see how the result is. Now in this procedure we have already spent lot of computing time in the evaluating the first result, evaluating the second result, now we are computing third result. Now the question is can we use these results that you have already computed and which are of no use for us because they are not accurate enough, can we use these results that have been obtained from the computer to manipulate a result which will give you better result than what we are obtained from these 2 results.

The answer is yes it is possible for us to manipulate those results which we think they are useless but they can be manipulated to get a result which will behave as if it has been obtained by a higher order formula and not by present formula that we are using and that is called Richardson's extrapolation. So let us discuss what is Richardson's extrapolation; this is some time simply called extrapolation, we may not use the word Richardson we may simply call it as extrapolation. Let us generalize this Richardson extrapolation whether we can use it for the first derivative, second derivative or any derivative in fact we are going to use this in numerical integration also the same procedure extrapolation. What we have here is, we suppose let g, let us just call g, suppose g is the quantity being determined; being determined by a method of; let us choose a method, by a method of order p, say with a step length h.

Now we can identify this quantity being determined as first derivative, second derivative or any derivative so we can identify it as first derivative, second derivative or any one of them. Therefore your, what we have is g(h) that means g obtained with step length h is a g exact, e for exact, let us denote g_e for exact that is the exact quantity g that is being determined. I add to this a truncation error that we have method of order p therefore c h to the power of p plus order of h to the power of p plus 1. I add it truncation error to the exact solution or to the numerical solution to get this as the expression for the exact solution. Now we repeat this computation with one more step length, repeat with step length, repeat the computation with step length, let us take q times h and we take q smaller than 1 so that we are now decreasing the step length. Now therefore this equation 1 will now read as g(q h) is equal to g exact that remains as it is, c q times h to the power of p plus order of q times h to the power of p plus 1.

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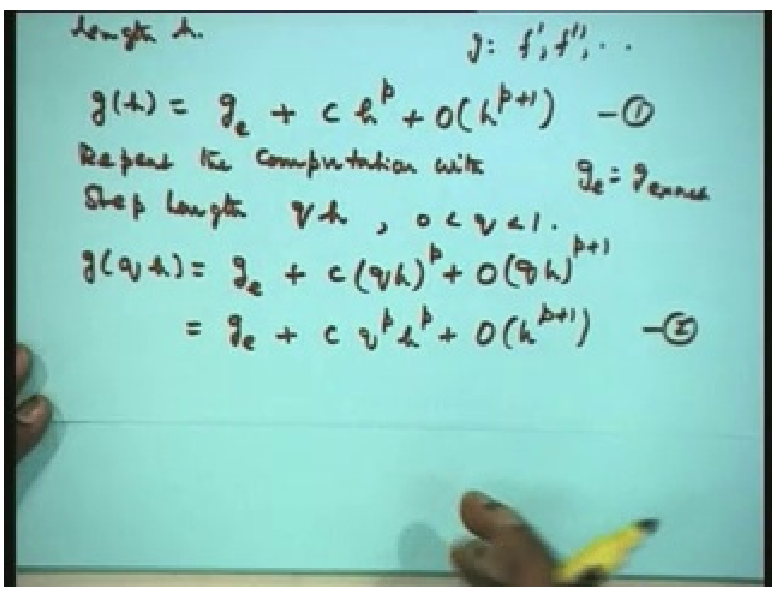

So let us repeat this, g exact c q to the power of p h to the power of p, order of h to the power of p plus 1. This q to the power of p plus 1 is a number which can be absorbed in this order so we absorb it and write it as simply order of h to the power of p plus 1. Therefore we now have two computed result, one result is g of h that has been computed another result g of q of h has been computed.

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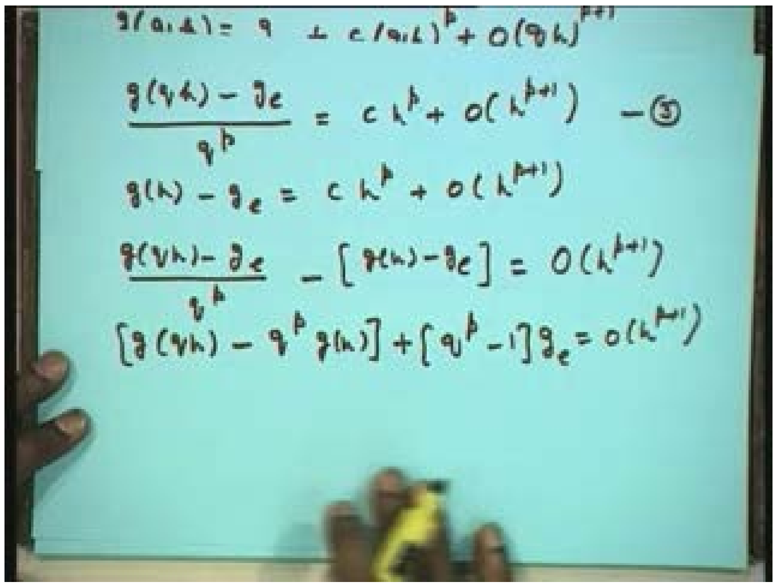

Now let us write down the equation 2 as, let us bring this is g_e to the left hand side that is g(q h) minus g_e and divide by q to the power of p that will be c h to the power of p plus order of h to the power of p plus . I bring this to the left hand side and divide by q to the power of p this one. Now let us write down the first equation also again, g of h minus g of e is c h to the power p plus order of h to the power of p plus 1. Now if I subtract these 2 results then c h to the power of p gets eliminated, so therefore I have got g(q h) minus g_e by q to the power of p minus [g(h) minus g_e] is equal to order of h to the power of p plus 1. Now c h to the power of p has been cancelled. Now let us simplify this, this is g(q h) minus, I will first write this q to the power of p into g of h, I multiplied this and combined this, then this is plus [q to the power of p into g of e minus 1] of g_e. Now if I divide throughout by q to the power of p minus 1, I would have the expression g(q h) minus q to the power of h g of h divide by q to the power of p minus 1 plus g of e is order of h to the power of p plus 1.

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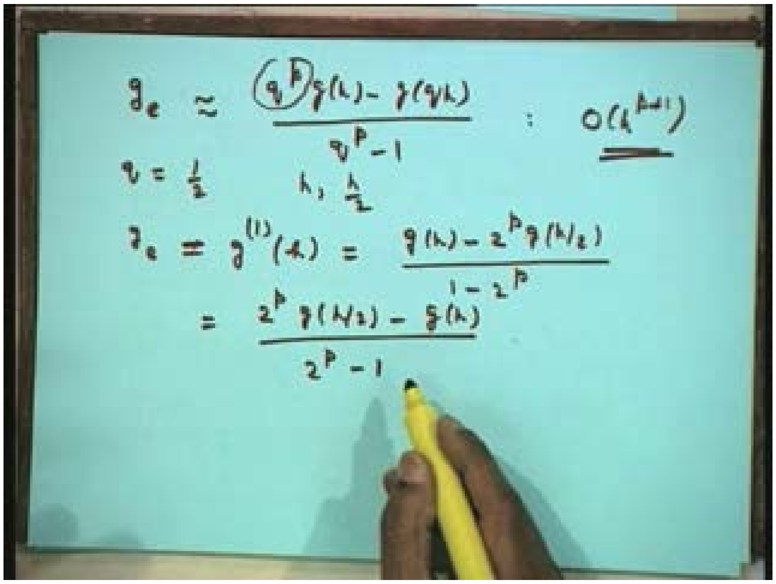

Now if I take this to the right hand side and approximate it, this as the new value g of e is a value that is being computed that is the exact value that is been, so I can approximate by this quantity, I take it to the right hand side so I will make this as plus sign q to the power of p g(h) minus g of (qh) divided by q to the power of p minus 1 and the error is now order of p plus 1. Now you see the interesting thing g of h is the computed result, g of (qh) is also computed result, q is the number that we have already know, q is the number that we have reduced the step length that is q into h so q is also known quantity so all the quantity here is completely known. Now I have got the new approximation for the g exact as this particular quantity and what is the order of the method? The order of the method is p plus 1 therefore I have manipulated 2 computed results to get a result which is 1 order higher than the previous one, so it behaves as if you have obtained the higher order method. Now if you can go back into the formula that we have, if you have used the 3 point formula and if I want a higher order, I need to add one more point or two more points to get the higher order formula that means it is as if you are using more number of points and getting better results.

For example if you have decreased by a factor of 2 that means q is equal to half that means we are using step length as h and h by 2 then this will read as g of e and let us equate it as $g^{(1)}$ of h. So we would call this as the approximation as super fix 1 and this, the initial once will take at g 0 and this is equal to g of h minus 2 to the power of p g of h by 2 by 1 minus 2 to the power of p. This is 1 upon 2 to the power of p, I am multiplied it and 1 upon 2 to the power of p, I have cancelled of 2 to the power of p. This we can write as 2 to the power of p (h by 2) minus g(h) upon 2 to the power of p minus 1. Now you can see the coefficients, if you look at the coefficients in the numerator q to the power of p and minus 1, denominator is q to the power of p minus 1. The denominator is just the sum of the coefficients in the numerator in this formula; this is called the Richardson extrapolation formula. Now if I repeat the next computation with one more step length, I can again do the same manipulation with respect to 2 successive values and get a method as if it is obtained by the next higher order. Now let us write down what it would look like if I write this in the sequence that I will call this as extrapolation table, Extrapolation table.

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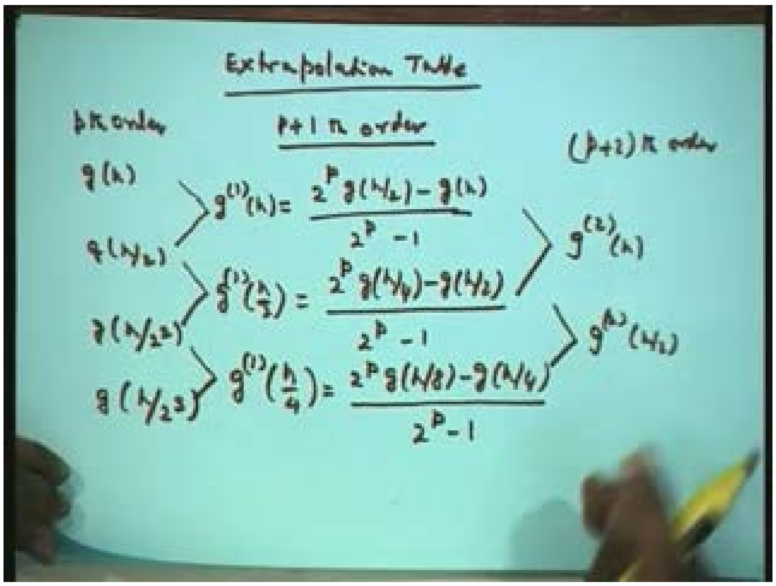

So let us start with a p^{th} order formula that we started with and we have computed g of h, we have computed g of h by 2, let us say we have computed g of h upon 2 square and let us say g of h upon 2 cubed, let us suppose that we have the computations, 4 computations we have done. Now if I manipulate these 2 computations which I am denoting as $g^{(1)}$ of h that is 2 to the power of p, I am taking the 1 upon 2, 1 upon 2 square therefore this will be 2 to the power of p g of (h by 2) minus 1, minus 1 into g of h by 2 to the power of p minus 1. Now this will be as if this has been obtained from a p plus 1^{th} order formula, it would behave as if it has been obtained from p plus 1^{th} order formula. I can manipulate these 2 successive once and call this as $g^{(1)}$ h by 2, the coefficients do not change, 2 to the power of p g of (h by 4) minus g of (h by 2), these are the 2 successive values that we are using divided by 2 to the power of p minus 1.

If I manipulate these two, again I will have g of 1, I will now call this as (h upon 4), this is 2 to the power of p g of (h upon 8) minus g(h upon 4) divided by 2 to the power of p minus 1. Now these, all these 3 results are of p plus 1^{th} order, the last one being the best value because this is h, smaller h, smaller h, so this last value is the best in this column and this last value is the best in this column. Now since the method is of p plus 1^{th} order that is h to the power of p plus 1, I can now similarly manipulate exactly the same way that we have done but now the coefficients are going to be 2 to the power of p plus 1 and minus 1 because the method is of order p plus 1^{th} order. So now if I reduce h by h by 2, h to the power of p plus 1 divided by 2 to the power of p plus 1, so we have to eliminate that particular coefficient so I have to multiply by 2 to the power of p plus 1. So I can now have here $g^{(2)}$ of h, we will write down the value in the next page so $g^{(2)}$ of h by 2, so this behaves like p plus 2^{th} order formula.

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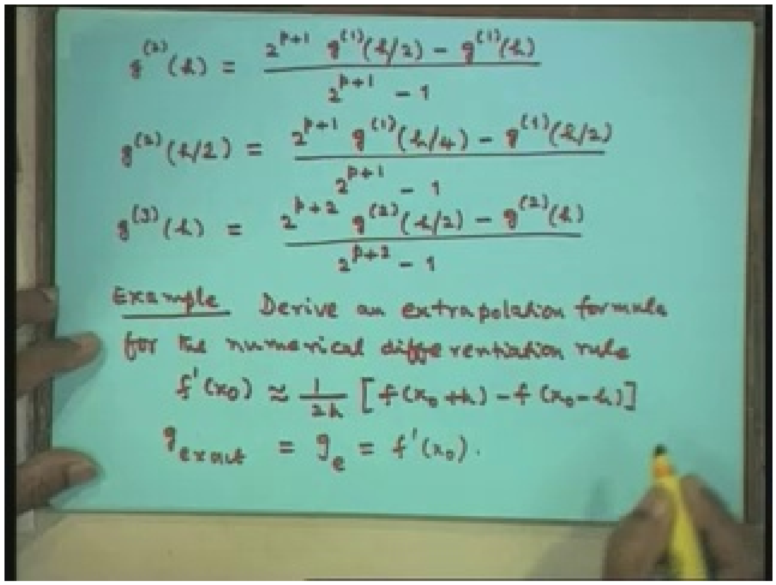

Where that $g^{(2)}$ h is equal to 2 to the power of p plus 1 $g^{(1)}$ that we are now computing this, $g^{(1)}$ of (h by 2) minus $g^{(1)}$ of h that is this, this and this divided by 2 to the power of (p plus 1) minus 1. And similarly $g^{(2)}$ (h by 2) is 2 to the power of (p plus 1) $g^{(1)}$ (h by 4) minus this value, the previous value $g^{(1)}$ h by 2 by 2 to the power of (p plus 1) minus 1. Now depending on the number of computation that we have done we can complete the table until we get 1 value. Now I can use these two and then get, I will write it here $g^{(3)}$ of h and which will behave like a (p plus $3)^{rd}$ order formula. Where, now we are eliminating the method of order (p plus 2) that is order of h to the power of p plus 2 therefore the next coefficients will be 2 to the power of p plus 2 minus 1. So I would therefore have the next computation $g^{(3)}$ of h is 2 to the power of (p plus 2) $g^{(1)}$ $g^{(2)}$(h by 2) minus $g^{(2)}$ of h that is we are now manipulating these two values, divided by 2 to the power of (p plus 2) minus 1.

Let us now take an example for application, example, derive an extrapolation formula for the numerical differentiation rule f prime at x_0 is approximately equal to 1 upon 2 h [f(x_0 plus h) minus f(x_0 minus h)]. Now in the notation that we have used earlier, let f dash at x_0 be denoted by g exact that means we are writing g exact and let us call it as g of e is equal to f prime of x_0. Now let us find the truncation error of this rule.

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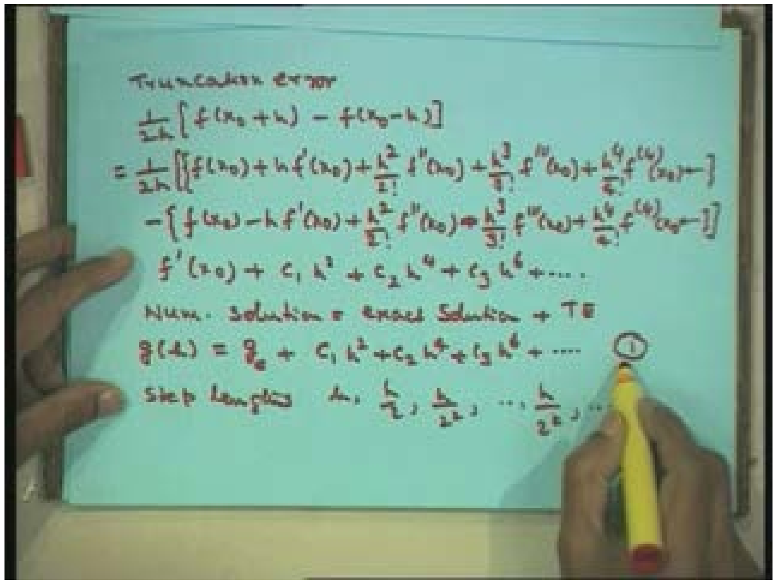

Now we write 1 upon 2 h [f of (x_0 plus h) minus f of (x_0 minus h)] we expand f(x_0 plus h) and f(x_0 minus h) in Taylor series, simplify and then collect the terms. Hence we have 1 upon 2 h [f of (x_0) plus h f dash of (x_0) plus h square by factorial 2 f double dash at (x_0) plus h cubed factorial 3 f triple prime at (x_0), h^4 by factorial 4 $f^{(4)}$ of (x_0)] minus [f of (x_0) minus h f prime at (x_0) plus h square by factorial 2 f double prime at (x_0) plus h cubed by factorial 3, minus h cubed factorial 3 f triple dash at (x_0) plus h^4 by factorial 4 $f^{(4)}(x_0)$ and so on]. Now we can see f of x_0 cancels with f of x_0, second derivative terms cancel, 4^{th} derivative term cancel and the first derivative term add up, third derivative term adds up to give us 2 times of these terms. Now we cancel 2 times h then we would produce here f prime at (x_0) plus some c_1 h square plus c_2 h to the power of 4 plus c_3 h to the power of 6 and so on.

Now numerical solution is equal to exact solution plus truncation error or exact solution is numerical solution plus truncation error. So let us write down the numerical solution is equal to exact solution plus truncation error therefore if I denote g of h by the numerical solution, I can write this as g of exact plus these error terms c_1 h square plus c_2 h to the power of 4 plus c_3 h to the power of 6. Now we need to take the step lengths starting with step length h and then decrease it further, for our discussion we shall take the remaining step lengths as h by 2, h by 2 square, h by 2 cubed and so on. Therefore we use this step lengths as h, h by 2, h by 2 square and so on, h upon 2^k and so on. Now I can set h is equal to h by 2, h is h by 4 in this equation and then write down what would be the expressions in that case.

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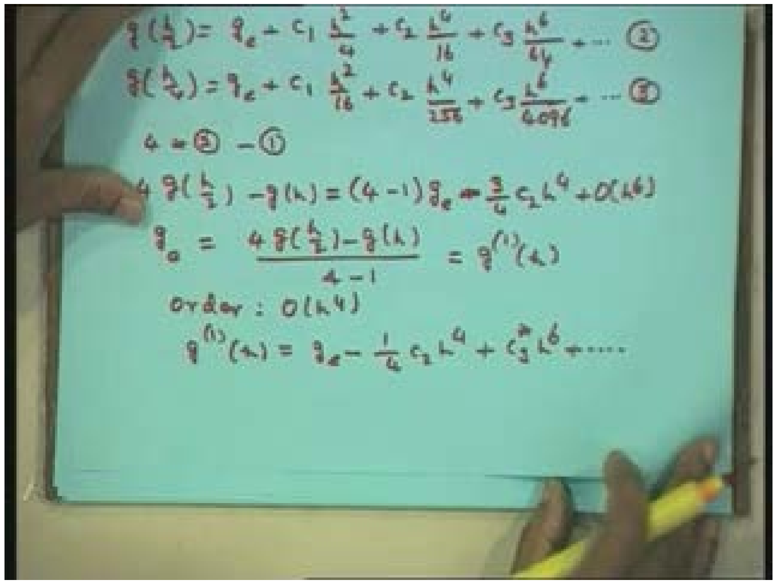

Then we get here, g of (h by 2) is equal to g of e plus c_1 h square by 2 square that is 4, plus c_2 h to the power of 4 by 2 to the power of 4 16, plus c_3 h to the power of 6 by 64 and so on. g of (h by 4) is g of e plus c_1 upon 4 square that gives you h square by 16, then I have 4 to the power of 4 h to the power of 4 by 256, plus c_3 h to the power of 6 by 4 to the power of 6 that is 4096. Now let us stop here and then write down, derive our extrapolation formula. Now I would use equations 1 and 2, I will use equation 1 and 2 and eliminate c_1 h square from this, now we can see if I multiply this equation by 4 and subtract this then c_1 h square cancels. Therefore I would write down 4 into equation 2 minus equation 1, then I get 4 g (h by 2) minus g of h is equal to (4 minus 1) I will retain it in this particular form, yes, plus this has cancelled, now I multiplied this by 4 therefore this is 1 by 4 minus 1 so I will get minus 3 by 4, so I will have 3 by 4 c_2 h to the power of 4 plus order of h to the power of 6, so let us write for the moment order of h to the power of 6.

Now similarly now I will write down what is my g of e, g of e is 4 times g (h by 2) minus g of (h) divided by 4 minus 1. So I find the value of g by just dividing by 4 minus 1 and this I will denote it by $g^{(1)}$ of h. Now here h square term has been removed therefore this is result would behave like n order of h to the power of 4 result therefore this value is of the order of h to the power of 4. Hence I can write $g^{(1)}$ h is equal to g_e minus, I divided by 3 therefore I will have here 1 by 4 c_2 h to the power of 4 plus some c_3 star h to the power of 6 plus so on. Now I would eliminate, now I would eliminate similarly c_1 h square from, I would eliminate c_1 h square from equations 2 and 3 therefore again you can see, I multiply this by 4 then I subtract this equation, I would then again cancel this.

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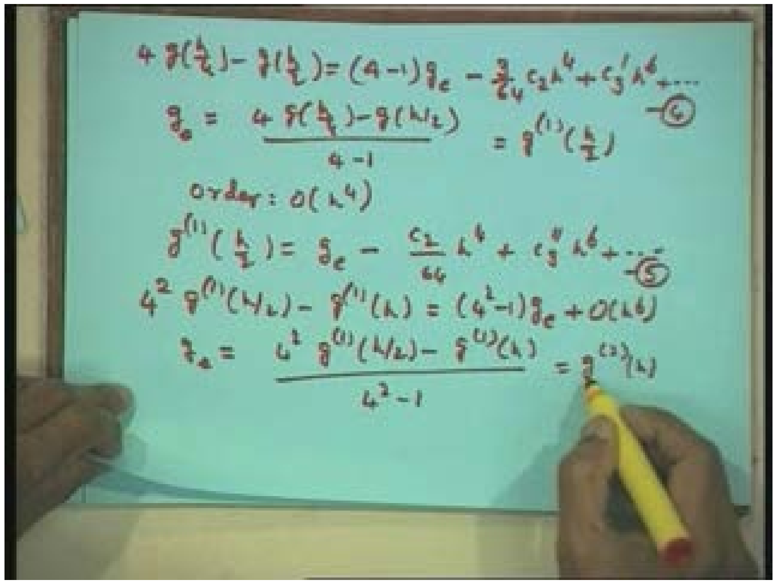

Therefore I have 4 g(h by 4) minus g(h by 2) is 4 minus 1 of g of e, now this has cancelled, this is 1 upon 64 minus 1 upon 16 therefore I would then have the required term as minus 3 by 64 c_2 h to the power of 4 plus some c_3 prime h^6 and so on. Now I can determine g_e from again, g_e is equal to 4 into g(h by 4) minus g(h by 2) divided by 4 minus 1 and this I will denote it as $g^{(1)}$ of (h by 2). Now again c, c_1 h square has been remove from here therefore this result again behaves like a 4^{th} order results therefore this is also of order of h to the power of 4 and $g^{(1)}$ (h by 2) is g of e minus, I divided by 3 therefore I will have c_2 by 64 h to the power of 4 plus some c_3 double prime h to the power of 6 and so on. Let us number this 4, number this as 5.

Now I would eliminate c_2 h^4 from these 2 equations, I would now eliminate c_2 h to the power of 4 from this equation and this equation. Now you can see if I multiply this by 16 and subtract from this then c_2 h^4 cancels from here therefore I will have 16 that is 4 square $g^{(1)}$ (h by 2) minus $g^{(1)}$ of h will be equal to (4 square minus 1) of $g_{e,}$ this and this, plus order of h to the power of 6. Now I solve for g_e from here therefore I will get here g_e is equal to 4 square $g^{(1)}$ (h by 2) minus $g^{(1)}$ of h divided by 4 square minus 1. This we can write it as $g^{(2)}$ of h. Now the previous 2 values have given as the first 2 elements of the second column of the extrapolation table, this gives me the first element of the third column of the extrapolation table.

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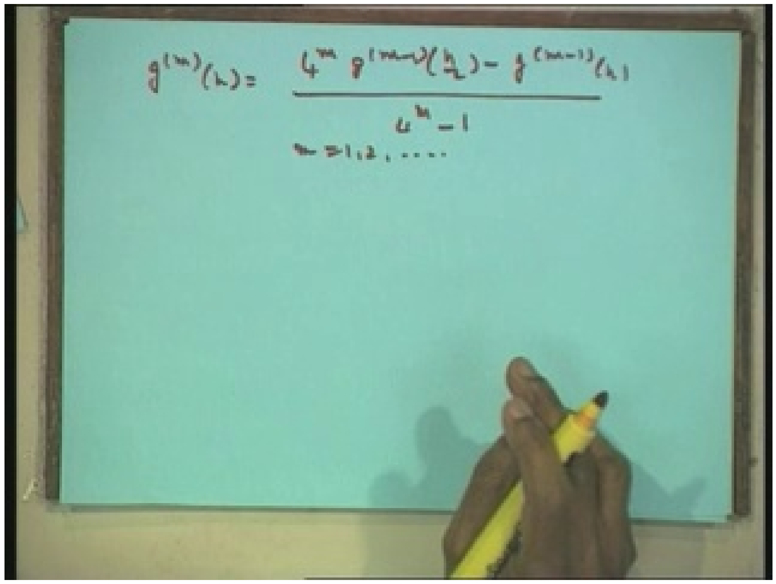

Now I can continue this procedure and then arrive at the formula that $g^{(m)}(h)$, I am now writing the column of this $g^{(m)}(h)$ is equal to 4 to the power of m $g^{(m-1)}$ (h by 2) minus $g^{(m-1)}$ of h by 4 to the power of m minus 1, m is equal to 1, 2, 3 and so on.

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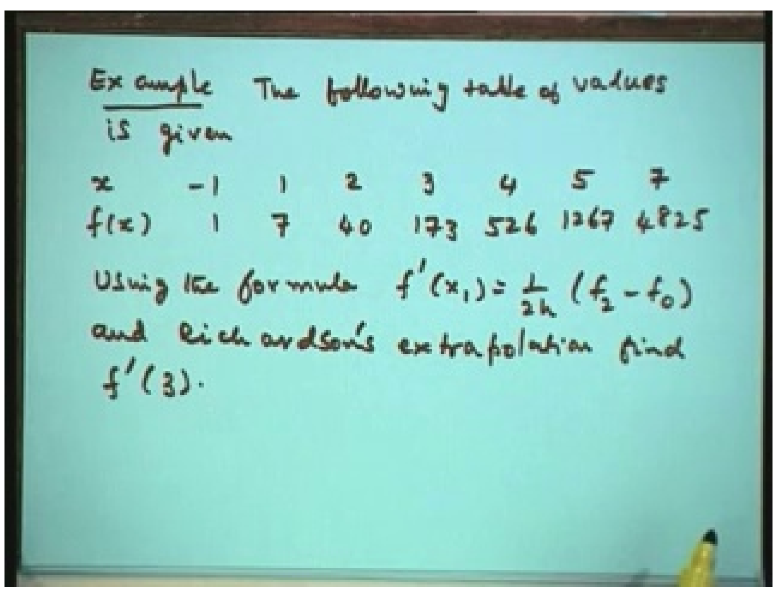

Now let us take an example on this, the following table values are given, the following table values are given x, f(x) minus 1, 1, 1, 7, 2, 40, 3, 173, 4, 526, 5, 1267, 7, 4825. Using the formula f dash of (x_1) is 1 upon 2 h (f_2 minus f_0) and Richardson extrapolation, find f dash of 3; find f dash of 3. We are giving 7 values in the table and we are using the central difference approximation and the Richardson's extrapolation to find the derivative at the point 3. Now this is the example which we have just now discussed that is why I have written it but however the first step for you would be to first Taylor expand this, while write down what is the error in this formula.

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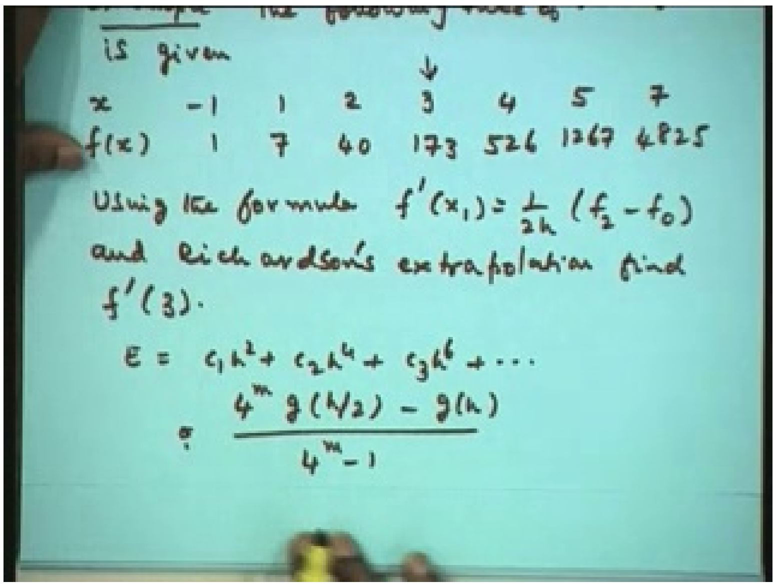

So we know that the error in this formula is of the order of c_1 h square, c_2 h to the power of 4 plus c_3 h to the power of 6 and so on, it is a central difference formula which is symmetric about x_1, so we have this. So the table that we have given earlier that we shall use this therefore our formula would be that 4 to the power of m g of (h by 2) minus g of h divided by 4 of m, this is our formula that we shall use in the table. Now we have been asked to use the Richardson extrapolation and therefore we must be able to compute the derivative using a particular step length, let us see what are the possible step lengths that we have here so we are you to use f of x_0 plus h, f of x_1 plus h, x_1 minus h, this is our point. Now you can see 7 is here which is at a distance of 4, now if there is a value available for us at a distance 4 behind, I can use the formula, this is 4 ahead and this is 4 behind.

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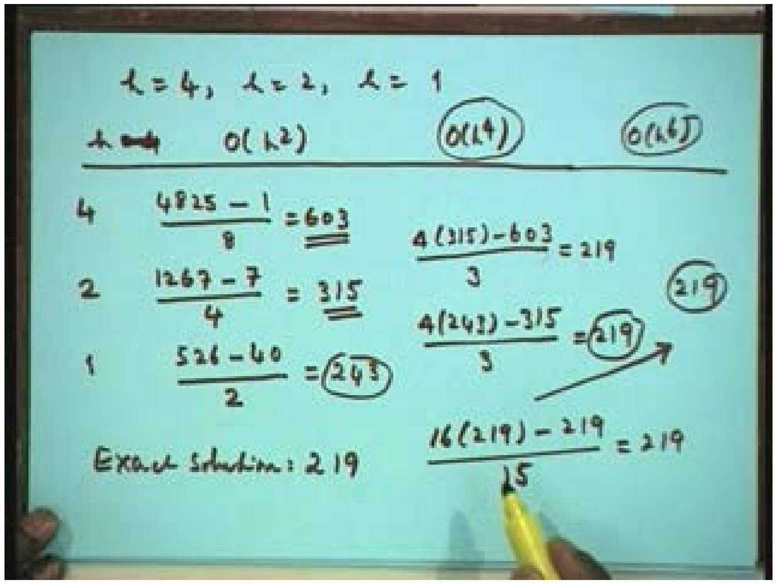

So I can use these 2 values with a step length of h is equal to 4, I can use h is equal to 4 and get an approximation for this. Now with 3, 5 is also given which is the distance of 2 and at a distance of 2 behind also is given to us, so I can also compute with h is equal to 2. Now again 3, 4 is given at a distance of 1 ahead and I have a distance 1 behind also is given to us that means I can also compute with h is equal to 1. Therefore I have got 3 possible values of h that can be computed h is 4, h is equal 2 and h is equal to 1 and each one is reduced by factor of 2, so again it is reduces to the factor of 2, so that I can use this particular formula that we have used. This is only for the case when you are reducing by factor of 2. Now let us write down the solution, so let us take h is equal to 4 and the formula that is given is the starting computation is order of h square formula. The leading term is order of h square that is a second order method, h is equal to 4. Now what is the value 1, f of 2 is equal to 4825 minus 1, this is f of x_2 minus f at x_0 divided by 2 times h, h is equal to 4 that is equal to, by 8, therefore 603, 603.

Now let us use with step length 2, with step length 2 f of x_2 will be 1267, so this will be 1267 and the value f_0 will be 2 steps behind that is 7, minus7, divided by 2 times h, h is now 2 that is divided by 4, 1260 divided by 4 that is 315. Now we have step length 1, with step length 1 the value at the next point is 526 minus 40 divided by 2 times h that is 2, that is 486 by 2 that is 243. Now I would like to manipulate this column, now of all these the best value in this column will be this because this is the best value of h. Now the next order formula will be order of h to the power of 4 because the next order in this is h to the power of 4 so it will behave like the 4th order formula and what is the value, this is 4 times (315) minus (603), 4 times this value minus this value divided by 4 minus 1 that is equal to 3 this gives us 219. And now I have 4 times (243) minus 315 divided by 3, this is your, this is your basic numerical method that we have used.

Now we are starting, yes this is 4, this is m is equal to 1, you have got 4 times g of (h by 2) minus g of (h) divided by 4 to the power of 1 minus 1 that is 3 and this also equal to 3. This is also equal to 219, it turns out that it is also equal to 219. Now let us write down complete this table, order of h to the power of 6, the next term in the series is h to the power of 6. Now let us, we will write down the value here, let us write it here this will be 4 square that is 16 times (219), 4 square that is 16 times this, minus 219 divided by 16 minus 1 that is equal to 15 and this of course is again 219, 16 minus 1 by 2 times so this is again 219. Now in this column this is the best value, in this column this is the best value. Two successive approximations are become identical that means this is your exact solution because 219, 219, two successive approximations identical the difference between them is 0, therefore error is here is 0, therefore what we have got here is the exact solution otherwise our best solutions will be going up in this direction.

The last value in this column is best, last value in this column is the best; the last value in this column is the best and so on. So we have the best approximation going in this direction. In fact in most of the problems Richardson extrapolation really works wonders because the values which were so much away from the exact solution, this is 603, this is exact solution is 219, we are able to manipulate these results which are so bad to get the results which are very very accurate and this is, we are now getting the values as if they have been obtained by order of h to the power of 4, order of h to the power of 6 and so on. By just getting 3 values, I am getting the value as if it has been obtained by order of h to the power of 6. Now if I want to write down the formula h to the power of 6 using this I need at least 9 points are so, therefore round off error will be more. Sorry, therefore the most important tool in getting better results is Richardson's extrapolation for numerical differentiation.