Numerical Methods and Computation

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Lecture No - 34

Numerical Differentiation and Integration

In our previous lectures we have defined the problems of interpolation and approximation and derived various methods for constructing the interpolating polynomials. We shall discuss now the application of interpolating polynomials in other areas; in particular we can apply them in the numerical differentiation and numerical integration areas.

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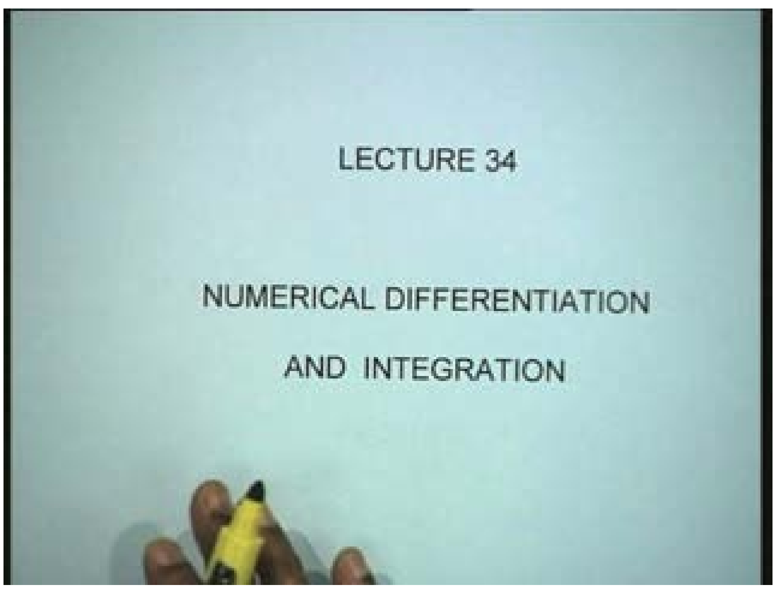

Let us now first of all discuss what we mean by numerical differentiation, so let us define numerical differentiation.

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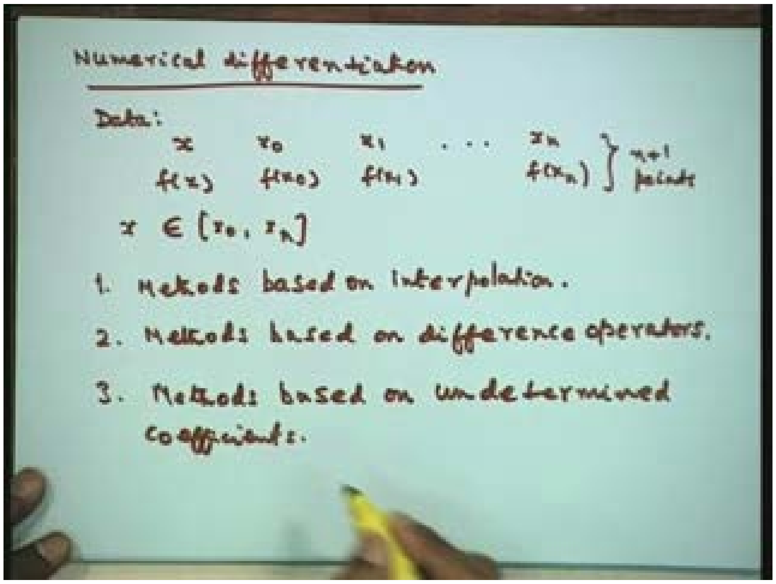

We are given a data of n plus 1 points which we were considering earlier so we have this data x f(x), x_0 f at x_0, x_1 f at x_1 and so on x_n f at x_n. So we have a data of n plus 1 points, we wish to find the derivative or higher order derivatives of the function f(x) which represent this particular data either at a nodal point that is x_0, x_2, x_n or at a non nodal points somewhere in between these data points, so for any point x contained in x_0 to x_n we would like to find out a derivative or higher order derivatives for the solution of this particular problem. There are 3 approaches for finding the numerical differentiation methods, the 3 approaches are as follows.

The first approach uses the interpolating polynomial that is we shall call this as methods based on interpolation, based on interpolation. In these methods what we do, given a data we construct the Lagrange interpolating polynomial or other forms of interpolating polynomial and then differentiate it required number of times to find the derivatives at the nodal or non nodal points, we can also find out the error estimate in the derivatives also. The second method is methods based on difference operators, methods based on difference operators. Now when the data is equispaced then we have defined the forward difference, backward difference, central difference, now when the data therefore is equispaced, we can use these difference operators to either construct the interpolating polynomial or directly obtain the value of the derivative at a required point, therefore this will be a simplification of what we have done in the first step that is here we are using the data which may be equispaced or which may not be equispaced, so this is general data that is given to us and the third one is methods based on undetermined coefficients, undetermined coefficients. Now in this case what we do is we write down a formula which we want, a particular type of formula depending on the application. We construct a formula which contains the ordinates that has given, the ordinates that is given to us, abscissa given to us and some parameters in that, these are arbitrary parameters.

We use the Taylor series expansion of this for all these terms, simplify them, compare the coefficients of the powers of h, write down the required number of equations as the number of parameters, solve them, if you can solve them the method which you have written exists otherwise it does not exists and the next non vanishing term would be the error in the particular formula. All these 3 methods give the similar type of methods and let us discuss each one of them and see how we can obtain the methods for numerical differentiation.

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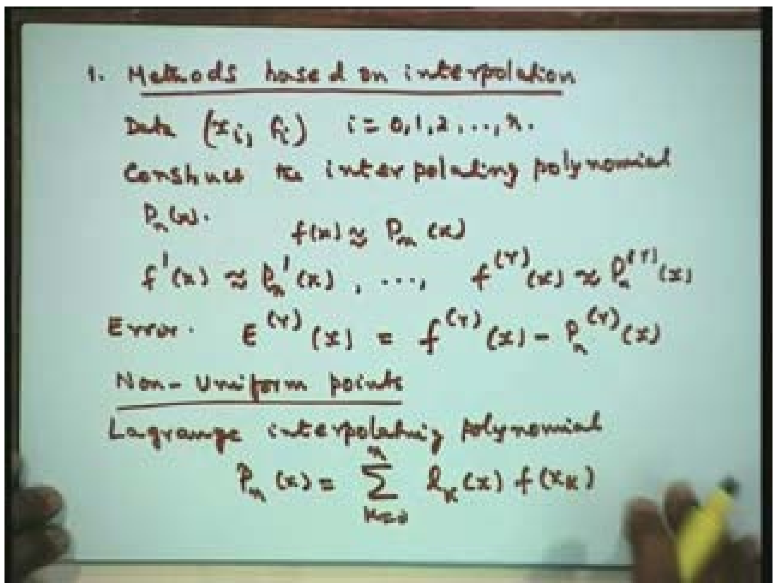

So the first one we shall take will be methods based on interpolation. Now we are given this data that is, is given to us (x_i, f_i), i is equal to 0, 1, 2, n what we shall say, construct the interpolating polynomial, construct the interpolating polynomial. Now it is our convenience which particular polynomial you are trying to construct, a Lagrange interpolating polynomial or a divide difference polynomial or any other polynomial, if it is equispaced we can go to Newton forward backward, so in general we are writing an interpolating polynomial $p_n(x)$, $p_n(x)$ that means f(x) is now approximated by $p_n(x)$. Now I can differentiate it and get f dash x is equal to approximately p_n dash of x, so in general I can differentiate it r times to get the $p_n^{(r)}$ (x). So I can differentiate it and get the value of the derivative either at the nodal point that is x_i or at non nodal points. Then we define the error also as E, again super fix will put r to define the order of the derivative that we are taking and that will be equal to $f^{(r)}$ (x) minus $p_n^{(r)}$ (x).

Now before we actually write down the numerical differentiation formulas a word of caution should be given for the numerical differentiation and what that is, there are in fact two points that are to be noted that is when a data is given to us, we really do not know what the function f(x) is actually representing in other words does the derivative of the function which represent that function exist at all or even if a first derivative exist, does higher order derivatives exist at all.

Therefore we really do not know whether the derivatives of a particular function that is representing data is existing and maybe we are trying to find out the derivative which does not exists, therefore in that case numerical instability would arise. And for example if you take y x is equal to root x, now let us write down the data artificially starting at x is equal to 0 to some x is equal to 5, now at x is equal to 0 its well defined so I have got a value 0 at 0 0 I can construct and interpolate polynomial. Suppose you are trying to find the derivative at x is equal to 0, now derivative dy by dx is 1 upon 2 root x, now at x is equal to 0 derivative does not exist therefore we will be landing into serious numerical instabilities if you are trying to construct or find out the derivative at a point where the derivative may not exist. Therefore numerical differentiation must be done with great care and the second observation that should be made is numerical differentiation is very sensitive to round off errors, which we are going to show and also see how badly the numerical differentiation will effect the solution.

If suppose you have taken a data of say 6 place accuracy data or a 4 place accuracy data, how many places can we expect in a numerical differentiation, now therefore the we were going to show that particular aspect of it. The numerical differentiation is very sensitive to round off errors so these are the two points that you must note for the numerical differentiation. Later on we discuss; when we discuss numerical integration the problem, it becomes just the opposite; it is always a stable process whereas a differentiation can be an unstable process. Now with this caution let us now construct the methods based on interpolation, so let us first start with non uniform points, non uniform points. Therefore we can straight away construct the Lagrange interpolating polynomial that fits this data so let us take the Lagrange interpolating polynomial and that is we are writing $p_n(x)$ is equal to summation of k is equal to 0 to n $l_k(x)$ f of x_k.

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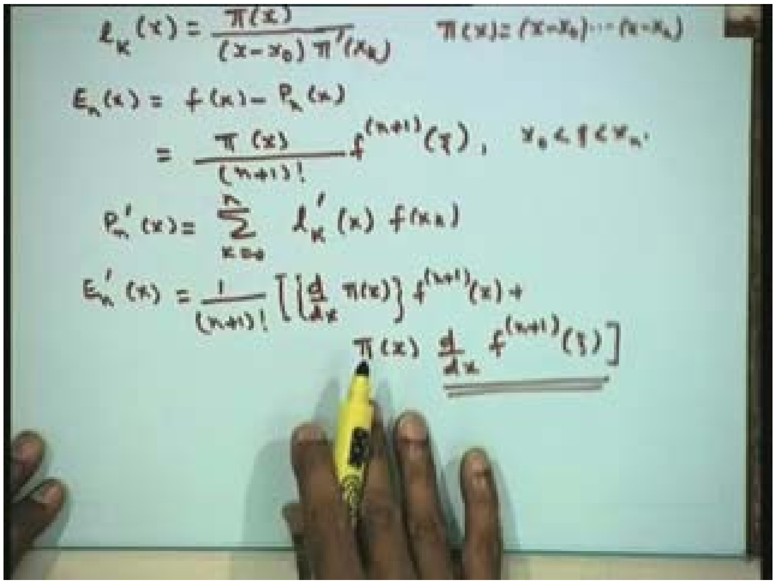

Where $l_k(x)$ are our Lagrange fundamental polynomials, let us write down what is our l_k of x, I shall now use a slightly different notation that I used earlier and I will tell you the reason. I will denote this w(x) that we were using earlier by pi(x), this is (x minus x_0) pi dash of x. Now our pi of x is the product of all these factors (x minus x_0) (x minus x_n). We shall reserve this w(x) that we were using earlier for the weight function that is going to come in the numerical integration. We have also proved that the error of interpolation is $E_n(x)$ is f(x) minus $p_n(x)$ and this is equal to the product of all this factors that is pi(x) divided by n plus 1 factorial $f^{(n+1)}$ of some zhi, where zhi lies between the first and the last points that is x_0 less than x_n.

Now what we have stated is that if I want the derivative now, take this particular polynomial 1 and then just differentiate it. So I can now differentiate it and write p_n prime of x is equal to summation k is equal to 0 to n, I have to differentiate l_k so differentiate it with respect to x $f(x_k)$. Then correspondingly I can look at what happens to the error also so I can differentiate this error also, so I will then have E_n dash of x is equal to 1 upon (n plus 1) factorial, now we should remember that zhi is a variable quantity whether it is a unknown quantity lying between x_0 to x_n. Now the value of zhi is going to change as we change your value of x therefore that will be taken as a variable. Therefore we shall have this as d upon dx of pi(x) into $f^{(n+1)}(x)$ plus the second term that is pi of x d upon dx of $f^{(n+1)}$ of (zhi). So I will take this as product of two functions and then differentiate it as a product, so derivative of the first one into this plus pi(x) into derivative of this. Now in the case when we are trying to find out the derivative at a norm nodal point that is not x_0, x_1, x_2, x_n then we do not know this particular quantity. Therefore we will not be able to give exactly what will be the error at a non-nodal point but at a nodal point it is easy for us because pi(x) in that case is 0 at a nodal point because there is a product of this factors, so if I take x is equal to anyone of the nodal points then pi(x) is equal to zero, therefore at the nodal points I can give the exact expression for this.

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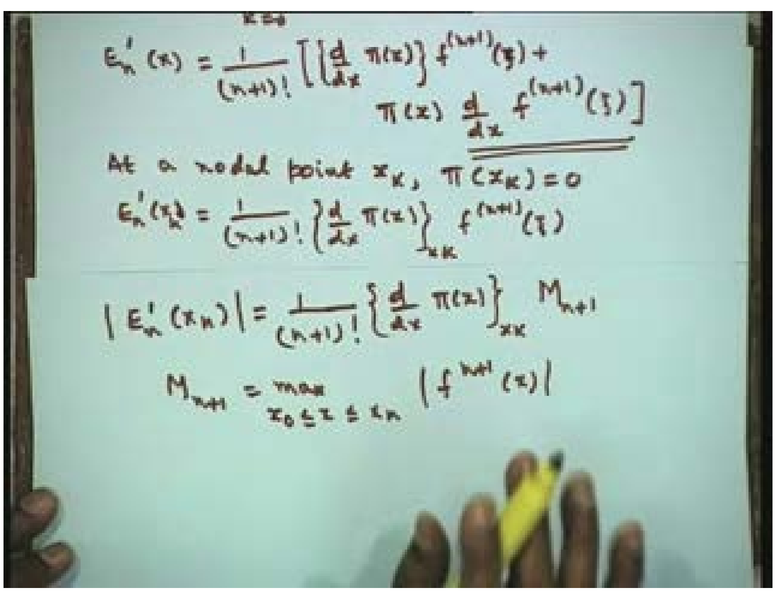

So let us write down at a nodal point, at a nodal point say x_k, pi of x_k is equal to 0. Therefore our derivative at this point x_k is 1 upon (n plus 1) factorial d upon dx pi(x) evaluated at x_k $f^{(n+1)}$, I think this is zhi, this is zhi, therefore I will be able to write down the bound of this immediately, I can take the magnitude from both sides and write down what will be the bound of this error from this.

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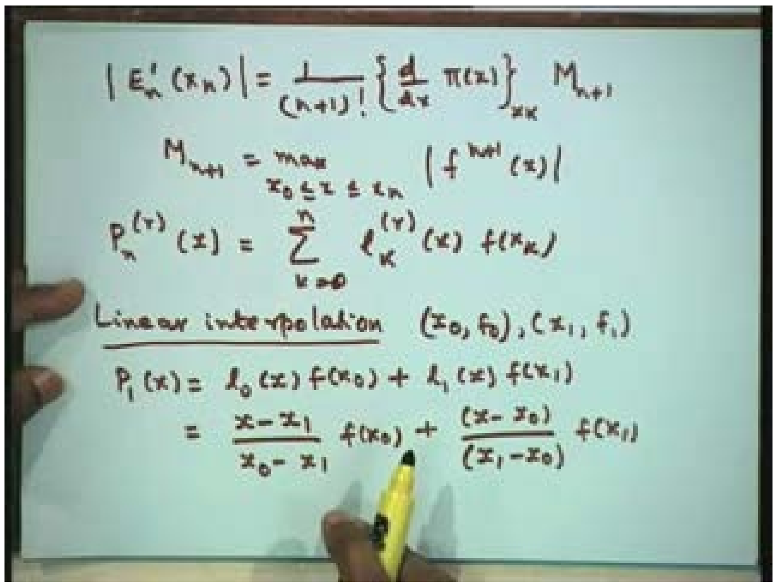

That will be 1 upon (n plus 1), now this is a number because this has been differentiated and evaluated at x_k also, so this is only a number so this is d upon dx of pi(x) evaluated x_k and this we will write it as M_{n+1}, M_{n+1} is the maximum of this, where M_{n+1} is the maximum in the interval x_0, x_n magnitude of f^{n+1} of x. So I can write down the expression for the derivative of the error at a nodal point much more easily. As I said if it is not a nodal point then there will be contribution from the second term also and we do not know how this particular quantity can be bounded. Now let us take a particular case from here, let us just look at some of the cases; let us start with linear interpolation. Now if I want the higher order derivatives, I just have to go ahead and then differentiate it further, so I can differentiate it r times and I can have here differentiate it r times. So if I want the, for example in the general case if I want the r^{th} derivative, I can just differentiate the expression that we have written earlier that is l_k, differentiate it r times of x $f(x_k)$ and correspondingly I can get the expression for the derivative also the second derivative the third derivative, again you will see that the contribution, now I will have to differentiate this further, so when I differentiate it further there will be more number of terms that will be available and at a nodal point we are happy because we can able to get the bond, at a non nodal points also again it will be difficult for us to write down what will be exactly the error.

Now let us start with linear interpolation, so we have got only 2 points which we are considering (x_0, f_0) and (x_1, f_1), these are the only 2 points. And the Lagrange interpolating polynomial is $p_1(x)$ is equal to $l_0(x)$ f of x_0 plus $l_1(x)$ f of x_1. There are only 2 points, so let us write down our interpolating polynomial (x minus x_1) by (x_0 minus x_1) $f(x_0)$, this is (x minus x_0) by (x_1 minus x_0) f of x_1. So this is the linear interpolation, Lagrange interpolating polynomial which we have open it up for $l_0(x)$ and $l_1(x)$ retained it in the suitable form.

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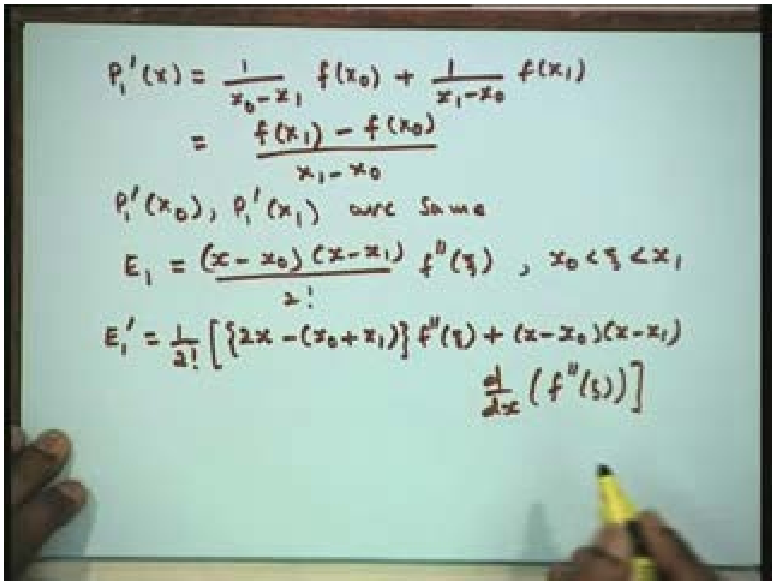

Now let us differentiate it, so if I differentiate it p_1 prime of x is equal to, derivative of this 1 upon (x_0 minus x_1) $f(x_0)$, derivative of this is 1 upon (x_1 minus x_0) f of x_1. Which I can write it as, I can take (x_1 minus x_0) common so I can write down f of x_1 first and this with an opposite sign x_0 divided by (x_1 minus x_0). Which of course is a constant because we started with a linear polynomial, polynomial degree 1, so we have differentiated it therefore you will get a constant. Now what that this imply, it implies that the slope is the same at both the points, if you evaluate it at x_0 it will be the same, if you evaluate at x_1 it is the same, so that means p_1 dash at x_0, p_1 dash at x_1 are same, slopes at both these points are the same and it is given by this particular value. When that is to be expected as I mentioned because we are taken a linear interpolating polynomial, polynomial degree 1 and differentiate it therefore we are getting constant, it is constant at all points x_0, x_1 and also at any point in between also, slope will be a constant across x_0 to x_1.

Now let us look at the error, now error E_1 is (x minus x_0) (x minus x_1) divide by factorial 2 f double dash of zhi, x_0 less than zhi less than x_1. Now let us differentiate it, so I will have E_1 dash is equal to 1 upon factorial 2, whatever we explained here let us repeat this one so I am differentiating this as a product, now this is x square minus x_0 plus x_1 into x plus x_0 x_1. So let us differentiate it, 2 x minus (x_0 plus x_1) that is the derivative of this, into f double dash of zhi plus (x minus x_0) (x minus x_1) d upon dx of f double dash of zhi. Now what we were mentioning earlier is that this quantity is a quantity which is completely unknown for us, I would be able to find out the error much more easily at the nodal point that means the value of the error in the derivative at the point x_0 that will be 1 upon factorial 2, I substitute x is equal to x_0 here so I will get here (x_0 minus x_1) f double dash of zhi, at x_0 this is equal to 0, so I am able to write this particular quantity. So that I can write down the bound for this as a derivative E_1 dash x_0 magnitude is less than x_0 minus x_1 in magnitude divide by 2 M_2, where M_2 is our usual notation that is the maximum of f double dash of x in the interval x_0 to x_1.

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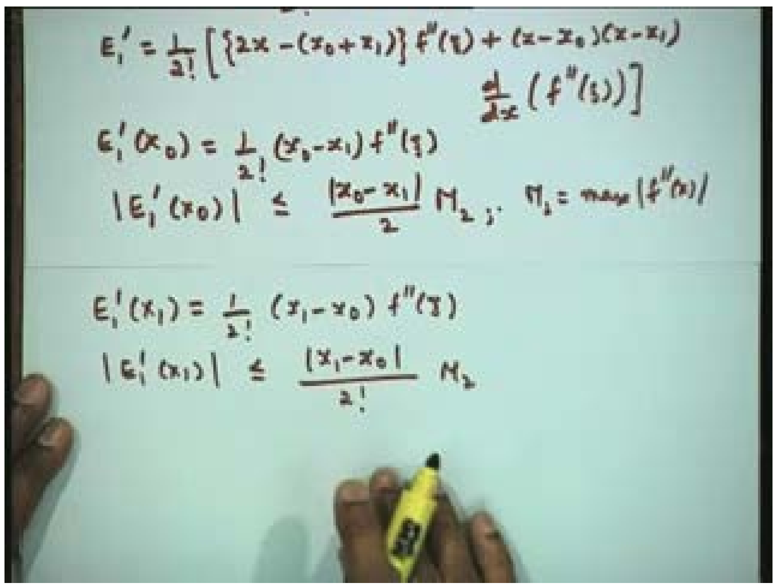

Now I can find out what is the error at the other point x_1. Now if I substitute x is equal to x_1 here, I will get {2 x minus (x_0 minus x_1)} so (x_1 minus x_0) so I will have 1 upon factorial 2 (x_1 minus x_0) f double dash of zhi. Again the second term vanishes at x is equal to x_1.Now this error is same as this error except it is of opposite sign, so this is here (x_0 minus x_1), this is (x_1 minus x_0). Therefore the slope of error is of opposite sign but it is magnitude is the same, so the magnitude of this is, magnitude of this is less than or equal to x_1 minus x_0. We need not write magnitude because x_1 is greater than x_0, we can actually write as x_1 minus x_0 into M_2, therefore it is possible for us to find out what is exactly the error.

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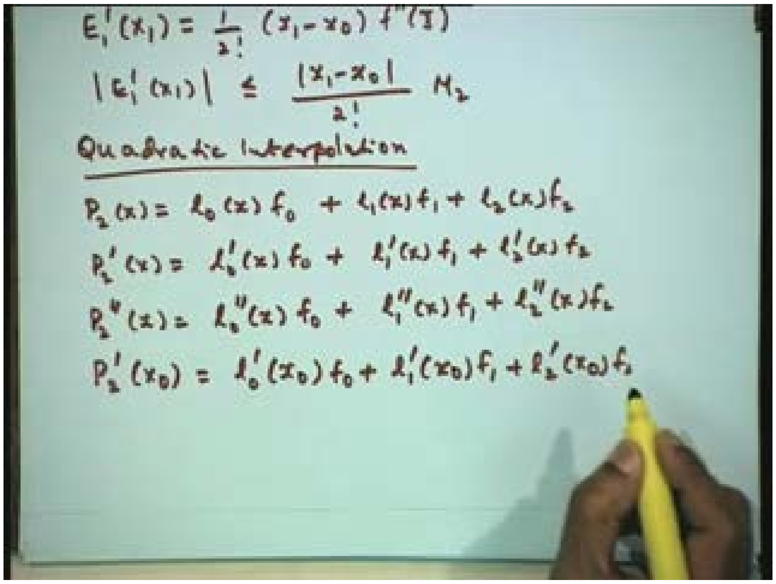

Now I can go ahead and then get any derivative, second derivative, third derivative, only thing is I will have to increase the order of interpolating polynomial to second order, third order, fourth order so that I can find the first derivative, second derivative, third derivative and so on. Let us just restrict up to quadratic interpolation, so let us take what is quadratic interpolation. So we need to write the Lagrange polynomial $l_0(x)$ f_0, $l_1(x)$ f_1 plus $l_2(x)$ f_2. Now if I want the derivative, I will just write down P_2 prime x derivative of $l_0(x)$ f_0 plus derivative of $l_1(x)$ f_1 plus derivative of $l_2(x)$ into f_2. Now this is a quadratic polynomial so I can differentiate it once more and get the second derivative also, so I can also write down what will be the second derivative in this case that is l_0 second derivative f_0, l_1 second derivative into f_1 plus l_2 second derivative into f_2. Now what I do is, I write down all this polynomials l_0, l_1, l_2 differentiate it number of times that are required, substitute it and then evaluate whatever the point that we want. For example let us derive one particular formula, the others you can, it is trivial, you can get it, so let us try to get what will be the value of P_2 prime at x_0, so let us try to get what is the value of this at one of the nodal points. This will be I substitute x is equal to x_0 therefore l_0 prime x_0 f_0, l_1 prime at x_0 f_1, l_2 prime x_0 f_2.

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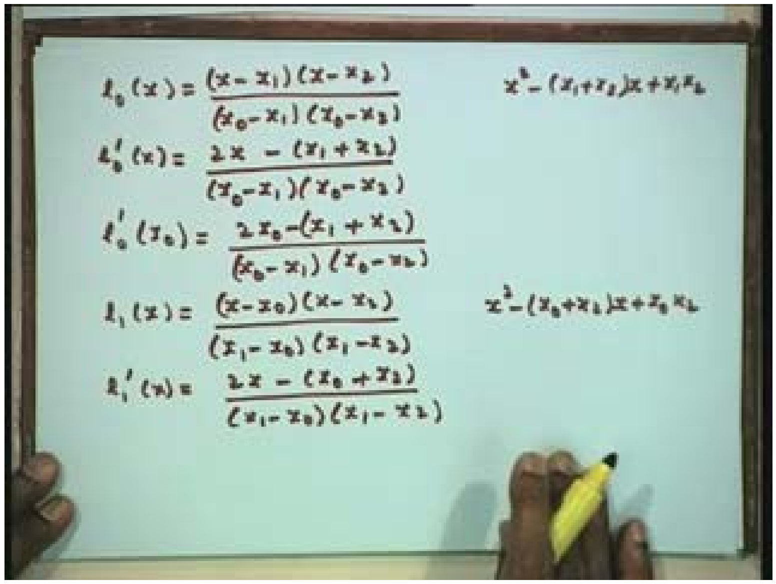

Now let us write down the expressions for $l_0(x)$, the expression for $l_0(x)$ is equal to x minus, except x_0 that is x_1, (x minus x_2) by (x_0 minus x_1) (x_0 minus x_2). Let us write down what is the expression for the numerator, the numerator is x square minus (x_1 plus x_2) into x plus x_1 x_2 so that is our numerator. Therefore I will get the derivative of this as; I differentiate this, 2 x minus (x_1 plus x_2) divided by the same quantity (x_0 minus x_1) (x_0 minus x_2). Now I can substitute x_0 here and then evaluate what is our, the first term that is required for us that is your l_0 dash x_0. So I can substitute here l_0 dash x_0 that is equal to 2 times x_0 minus x_1 plus x_2 divided by the same quantity (x_0 minus x_1) (x_0 minus x_2).

If I want the derivative at some other point x_1, x_2, yes I would now use that particular quantity here and if it is not a nodal point then I will take that particular value of x is equal to some eta so it will be 2 eta minus x_1 minus x_2 divided by this quantity. Now let us write down $l_1(x)$, $l_1(x)$ is (x minus x_0) (x minus x_2) divided by (x_1 minus x_0) (x_1 minus x_2). Now the numerator is, let us multiply it out x square minus (x_0 plus x_2) x plus x_0 x_2, this is our numerator. Now let us differentiate this, l_1 prime of x is equal to 2 x minus (x_0 plus x_2) divided by (x_1 minus x_0) into (x_1 minus x_2).

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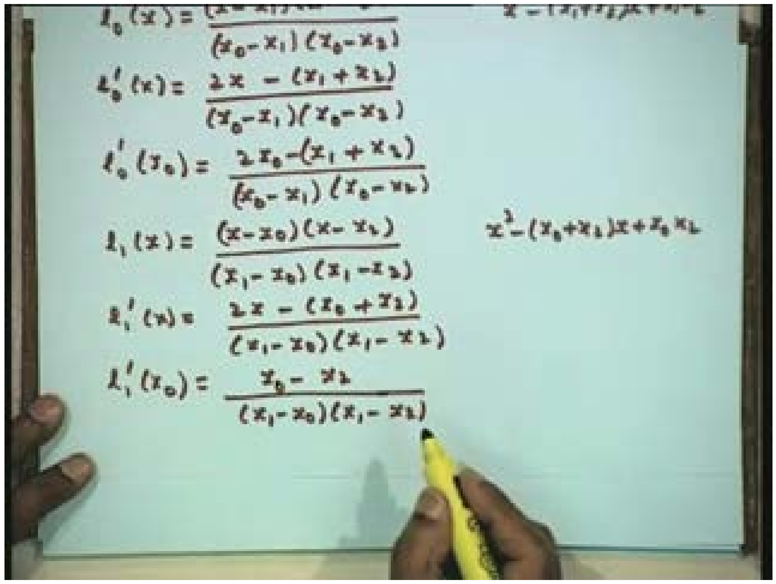

Again it can be evaluated at any nodal or non-nodal point. Now I need this derivative at x_0 so I substitute it, now here this is 2 x_0 minus x_0 so I will have only x_0 here, minus x_2 divided by (x_1 minus x_0) into (x_1 minus x_2).

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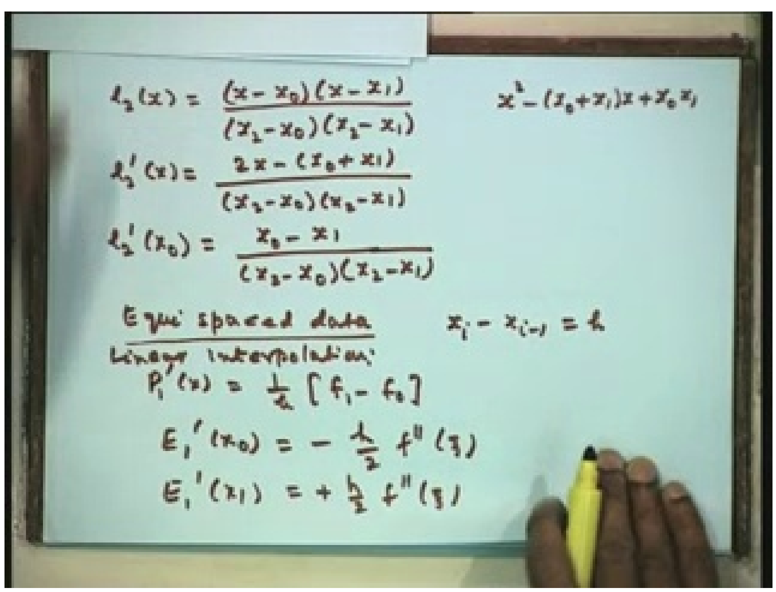

Now similarly let us get the third quantity that is l_2 of x, l_2 of x will be (x minus x_0) into (x minus x_1) divided by (x_2 minus x_0) (x_2 minus x_1). Now let us write the numerator, the numerator is x square (x_0 plus x_1) x plus x_0 x_1. Therefore I can find the derivative l_2 dash x is, derivative of this that is 2 x minus (x_0 plus x_1) and the denominator is (x_2 minus x_0) (x_2 minus x_1). Therefore again I can evaluate this at any nodal point or non-nodal point, I get 2 x_0 again minus x_0 that is your (x_0 minus x_1) by (x_2 minus x_0) (x_2 minus x_1). Now all the quantities have been evaluated so we go back and substitute it here, the values of, the values of l_0 dash x_0, l_1 dash x_0, l_2 dash x_0, f_0, f_1 and f_2 are given to us as the ordinates in the data, substitute it and we valuate it and find the value of that particular expression. So we can get the value at any nodal point or non-nodal point straight away using our Lagrange interpolation.

However all these derivations gets simplified if you are using the uniform spaced data that is your equispaced data, so let us see what is the simplification for equispaced data. Therefore the step length x_i minus x_{i-1}, let us take it as h. Now in this case we can use the formula Lagrange interpolation or since it is equispaced we can use the Newton's forward backward difference formulas which we will take it in a moment but let us now first concentrate on how we can simply what we have done already earlier. So let us start with what we have written here, earlier the formula for the first derivative, the formula for the first derivative using the Lagrange interpolation is this. Now denominator is (x_1 minus x_0) that is simply equal to h therefore I can straight away write down, if I take P_1 dash x that is we are discussing the linear interpolation then P_1 dash x is equal to 1 upon h of [f_1 minus f_0], this is f_1, this is f_0 and this is only step length h over here. And also let us write down what is our E_1, so we have the expression for E_1 dash at x_0 that is equal to half (x_0 minus x_1) this is minus h, (x_0 minus x_1) is minus h. Therefore the error at x_0 will be equal to minus h upon 2 f double dash of zhi. Whereas the error at x_1 is (x_1 minus x_0) by factorial 2, so will have here h upon 2, so this is the error at x_1 is plus h by 2 f double dash of zhi.

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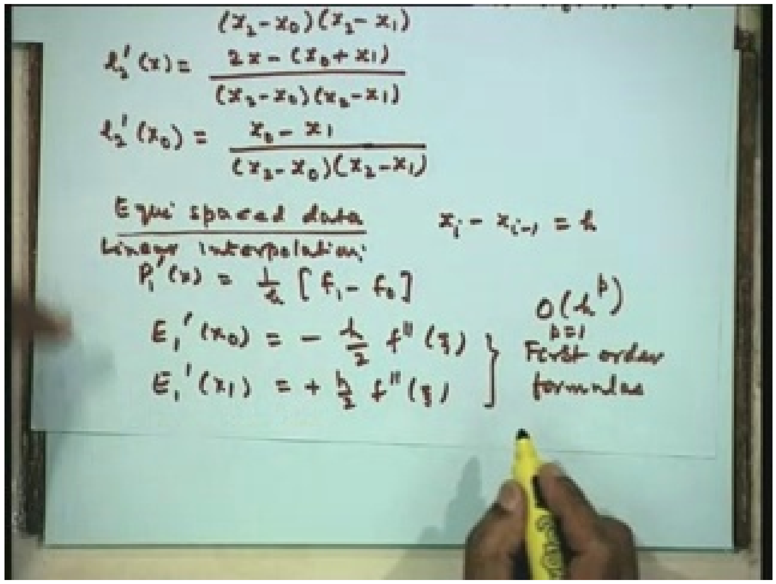

Now therefore the error is of opposite sign, they are of same magnitude and we shall call these as the order of the formula its error is h to the power of p, p is equal to 1. Therefore these are all the first order formulas, so these are called the first order formulas. That is we are talking of order of h to the power p as the error and here in this case p is equal to 1 therefore we are calling them as a first order formulas.

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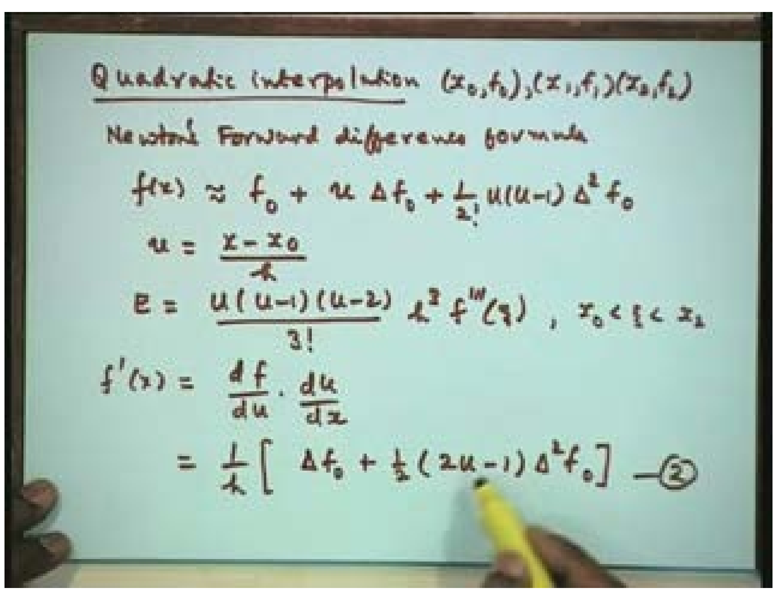

Now let us take quadratic interpolation, now for deriving the quadratic interpolation I will not use the Lagrange because we have the much more simpler formula Newton's forward backward difference formula, let us use one of them for this so let us take the 3 points here is (x_0, f_0), (x_1, f_1) and (x_2, f_2), let us take these 3 as the points. Since these are only 3 points, only second order forward difference or second order backward difference exists. So therefore our Newton's formula will be, let us take the Newton's forward difference formula, Newton's forward difference formula that is f(x) is equal to approximately f of 0, I will use that we have derived formula in terms of u so I will write down that formula, u into this, 1 upon factorial 2 u into (u minus 1) delta square f_0. If you remember we have defined u is equal to x minus x_0 divided by h and the error expression for this is the next coefficient, binomial coefficient that is u (u minus 1) (u minus 2) divide by factorial 3 h cubed f triple dashed of zhi. zhi lying between the first and the last points x_0 less than zhi less than x_2.

Now we mentioned earlier that for computational purposes this formula would be useful because u is a small number and this binomial coefficients would reducing very fast as we go along therefore if you write down the forward differences and write the polynomial in terms of this, these later terms are going to be very very small numbers, it is as if you are adding some correction terms to this previous terms and then getting a final value. Now let us write down what is our derivative, I want f dash x, the variable is u therefore I will write this as df upon du into du by dx, df upon du into du by dx. Now du by dx let us first write it, du by dx is 1 upon h, u is equal to x minus x_0 by h, du by dx is 1 over h, so this is 1 over h. Now we differentiate this with respected to u therefore differentiate it, I will get delta f_0, this is u square minus u, so differentiate it (2 u minus 1) delta square f_0, so let us number this as 2. Therefore we have the expression for the derivative, very simple; we have for first forward difference, second forward difference. If I want at any nodal point, I will take the nodal point, if it is a non nodal point then u has got a different value, I can substitute that value and find the derivative at any nodal or non nodal point from here. Now let us write down what will be the error in this case, before write the error let us write down the, let us just look at how this formula looks like at a nodal point.

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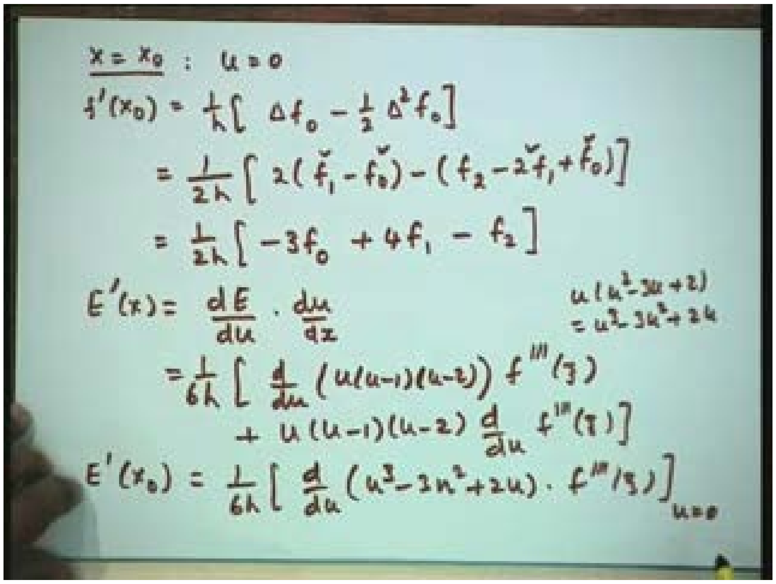

Let us take x is equal to x_0, how does it look like, then derivative at x_0 will be, now when x is equal to x_0 the value of u is 0, when x is equal to x_0 value of u is equal to 0, so we will have u is equal to 0. Therefore this is 1 upon h, I am substituting u is equal to 0 in this, so this is delta f_0 that is minus half delta square f_0. Now let us simplify this that is equal to, let us take 2 outside so 1 upon 2 h 2 times delta f_0 is (f_1 minus f_0) delta square f_0 is (f_2 minus 2 f_1 plus f_0), this is the second forward difference. This is equal to 1 upon 2, there is minus 2 f_0 here and a minus f_0 here, I have minus 3times f_0, I have plus 2 f_1 here and a plus 2 f_1 here so I have 4 times f_1 then I have minus f_2, so this is the formula that I have for the derivative at x_0. Now notice the, I mean whether you are writing it correctly or not the sum of the coefficients have to be 0, this is 4 3 minus 3 minus 1, sum of the coefficients is going to be 0.

Now let us write down what is the error expression for this, now I would take this error expression and differentiate it. So I can differentiate E prime of x is equal to again dE upon du into du by dx. Now du by dx is again equal to 1 upon h so I will retain this over here and then differentiate it, we have a factorial through, so let us put our 6 also outside. Then I will have here d upon du of the derivative of this, u (u minus 1) (u minus 2) into f triple dash zhi plus u into (u minus 1) (u minus 2) d upon du of f triple dash of zhi. Now this is the general expression for all the cases, for all the derivatives. Now let us get what is the value at x_0, so I will have derivative at x_0, error derivative at x_0 is 1 upon 6 h, now at x_0 u is equal to 0 so this is going to be 0, second expression is going to be 0. So let us differentiate this, this is your, this is your product u into (u square minus 3 u plus u) this is your plus 2. So that is your u cubed minus 3 u square plus 2 u. I differentiate it and then put u is equal to 0, so if I differentiate it I will get 3 u square minus 6 u plus 2. So let us write down one or may be one more term we can write down, this is equal to d upon du (u minus 2 u square plus 2 u) into f triple dash of zhi, evaluated at u is equal to 0.

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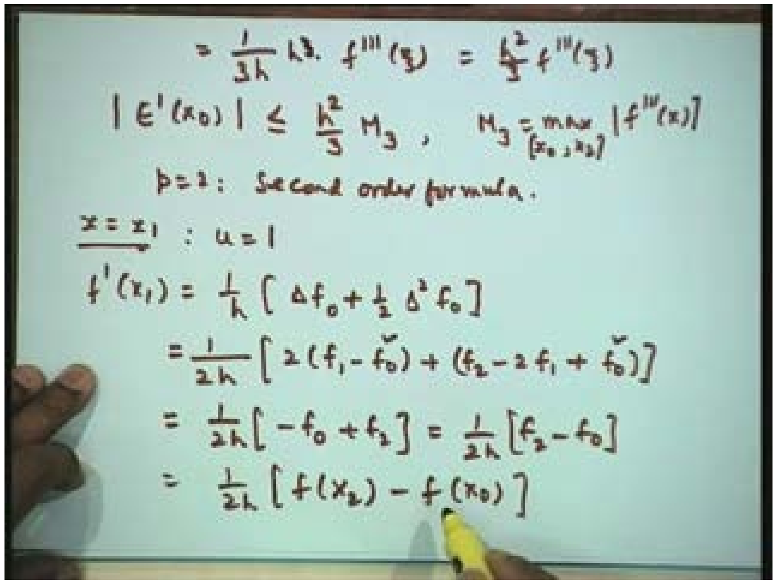

Therefore I get here only 2, 2 by 6 that is 3 1 upon 3 h. Where is our, I should put h cubed also here, we have a h cubed also here, so I write it outside h cubed, write it h cubed and h cubed into f triple dashed of zhi. Therefore this is equal to h squared by 3 f double dashed of zhi. Therefore the magnitude less than or equal to h square by 3 M_3, where we have written M_3 as maximum of f triple dash of x in the given interval x_0 to x_2. Now you can see error is of the order of h to the power of p, p is equal to 2 therefore we have p is equal to 2, therefore this is a second order formula that we have obtained, this is second order formula. Now it is interesting to see how the same formula would behave at different points, so let us take this formula at x is equal to x_1, let us take now at x is equal to x_1 that is the middle point, there are 3 points we have taken (x_0, f_0), (x_1, f_1), (x_2, f_2), so the middle point at x_1, the value of u is x_1 minus x_0 divided by h, x_1 minus x_0 is h therefore h upon h is 1. Therefore in this case u is equal to 1, at x is equal to x_1 u is equal to 1.

Now let us substitute what will be the value of f prime, so let us put it this way, therefore f prime at x_1 is 1 upon h delta f_0, now at u is equal to 1 this gives you plus 1 so I will have plus half delta square f_0 and let us again open this up, 1 upon 2 h [2 times (f_1 minus f_0) plus (f_2 minus 2 f_1 plus f_0)]. This is 1 upon 2 h, this is minus 2 f_0 plus f_0 so minus f_0 and this is 2 f_1 minus 2 f_1 it cancels so I will have here f_2 or simply I will rewrite it in, first f_2 and this f_0. Therefore the formula is simple, it is the symmetrically placed about x_1 that is, this is f at x_2, this is x_1, this is f at x_2 minus $f(x_1)$ that means we are really looking at a central difference formula, corresponding to a central difference formula, this is f at x_0 f at x_1. The step length is h therefore x_2 is 1 step h ahead, x_0 is 1 step behind therefore this is symmetrically placed approximation.

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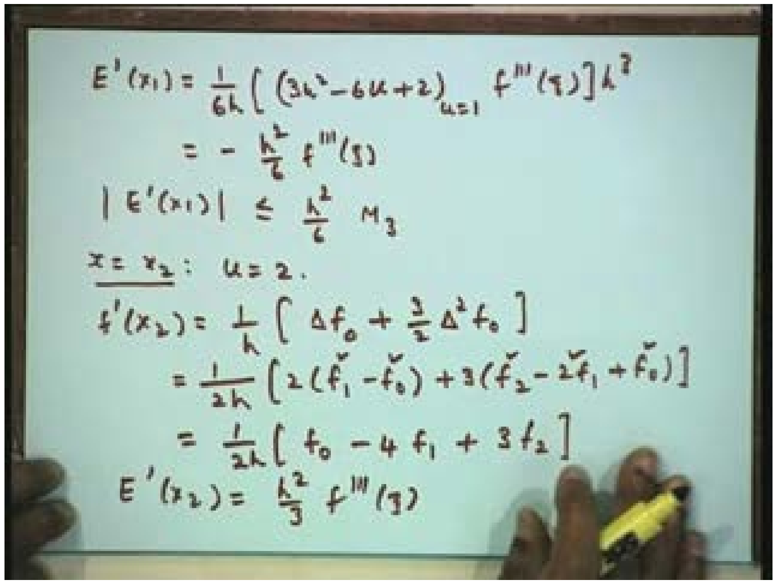

Now let us see how the behavior of the error would be at this particular point and for writing the error expression I just need to substitute here u is equal to 1 because the same expression we have got at u is equal to 1 therefore this will be 1 upon 6 h into, we can substitute it over here, this is equal to derivative of this is (3 u square minus 6 u plus 2) evaluated at u is equal to 1 f triple dash of zhi into h cubed and this gives us minus 1, this is at u is equal to 1, this 5 minus 6 that is minus 1 so I will have minus h square by 6 f triple dash of zhi. So error is minus h square by 6 f triple dash zhi so that the bound is less than or equal to h square by 6 M_3. Now you can see that the error in the previous case was h square by 3 and now we have h square by 6 which is reduced by factor of 2 and that should be excepted, this formula turned out to be a central difference approximation that is we have got symmetrically placed points so that if you tailor expand this, the odd derivatives would cancel. And similarly if I take the third formula x is equal to x_2 then when x is equal to x_2, if I go back and substitute x_2 here, x_2 minus x_0 is 2 times h divide by h u is equal to 2, so I will then have u is equal to 2 in this case. Now then I can substitute it straight away in this.

So let us substitute u is equal to 2 in this so I will have here f prime at x_2 is 1 upon h delta f_0 plus, this is 4 minus 1 that is 3 by 2 delta square f_0. Let us simplify this 1 upon 2 h [twice (f_1 minus f_0) plus 3 times (f_2 minus 2 f_1 plus f_0)]. That is 1 upon 2 h, this is minus 2 f_0 and plus 3 f_0 so I will have f_0, this is plus 2 f_1 this is minus 6 f_1 so I will have minus 4 f_1 and plus 3 times f_2 plus, 3 times f_2. Now interestingly if you look at the formula at the 2 end points, we are taking quadratic only interpolation x_0, x_1, x_2, at x_0 this is of the sign minus 3, minus 1 and 4 and if you go to the other point, the roles of this is reversed with an opposite sign, so this becomes f_0 with a negative sign through out and the roles of the first and last ordinates gets interchanged, this is f_0 and f_2. So these are, this is how it comes the first and last points the interpolation like this. Now I leave this as an exercise to you, you can just find out derivative at x_2 is again equal to h square by 3 f double dash of zhi. It is the same as at the first point it would be. Now if you want the second derivative, second derivative would be possible because we are doing the quadratic interpolation.

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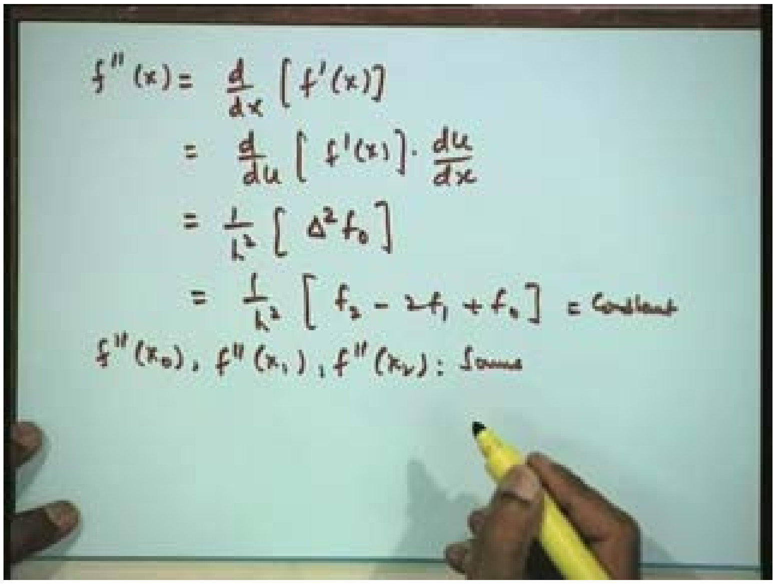

So I can find out my second derivative also f double dash of x that will be by definition d upon dx of f dash x. Now I need to use this to get the next derivative, so this will be d upon du of the expression that we have obtained for f dash x into du by dx. Now du by dx is 1 upon h, there is already a 1 upon h here so I will have 1 upon h square. I am differentiating with the respect to u therefore I will get 1 so simply I will get delta square f_0. Therefore only the second forward difference divide by h square, in fact earlier when we have gave the relationship between the derivatives and the forward differences this one we have got it there also, so this is the expression for f double dash that is simply equal to 1 upon h square of [f_2 minus 2 f_1 plus f_0]. Obviously this is a constant because you have taken a quadratic polynomial, differentiate it 2 times and therefore you will get a constant. Therefore second derivative at all the 3 points in fact at every point in the interval are the same, they are all the same. Okay we will stop at this today. We will take an example for this next time.