# Numerical Methods and Computation

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## Lecture No - 33

## Interpolation and Approximation (Continued)

Now in our previous lecture we have defined the uniform approximation, let us briefly review what we have done last time.

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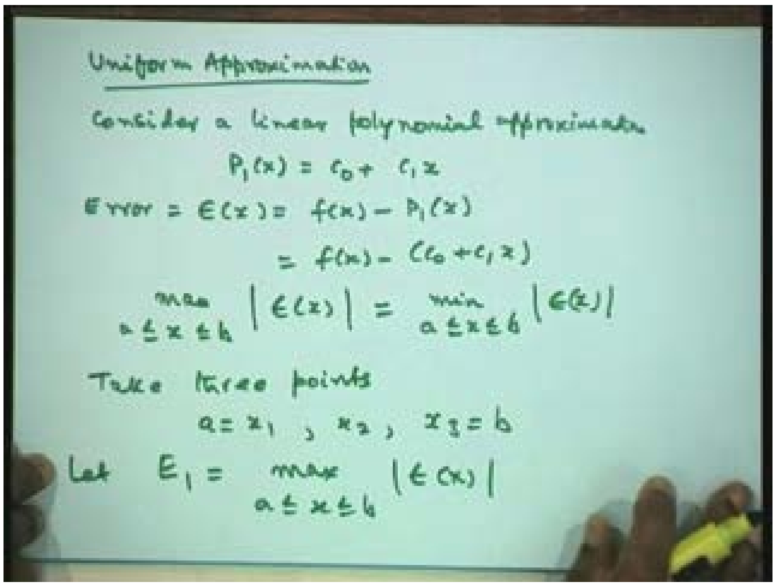


We consider the linear polynomial approximation, so let us write it as a uniform approximation and then to define this, we consider a linear polynomial approximation. We have taken this linear polynomial approximation as $P_1(x)$ is equal to some $c_0$ plus $c_1$ into x. Then we have defined the error, the error is defined as epsilon of x that is f(x) minus $P_1(x)$ that is f(x) minus ($c_0$ plus $c_1x$). Now the problem of uniform approximation therefore is, that the maximum of error should be equal to the minimum error in the interval, whatever in the interval are considered [a, b] that means maximum of the error in the interval a less than x less than or equal to b of this magnitude should be equal to minimum of a less than x less than or equal to b of error x. This would imply that all the errors are equal in magnitude because the maximum should be equal to the minimum, therefore the errors at all the points would be identically same and that is got magnitude of epsilon of x. Now for considering the linear polynomial approximation, we will take three points $x_1$, $x_2$, $x_3$, so take 3 points take three points. We will take the first point as $x_1$, the second point as $x_2$ and the third point as $x_3$, $x_3$ is equal to b. If we define the largest value of epsilon(x) in magnitude as $E_1$, let us define $E_1$ as maximum of a less than x less than or equal to b in magnitude of epsilon(x).

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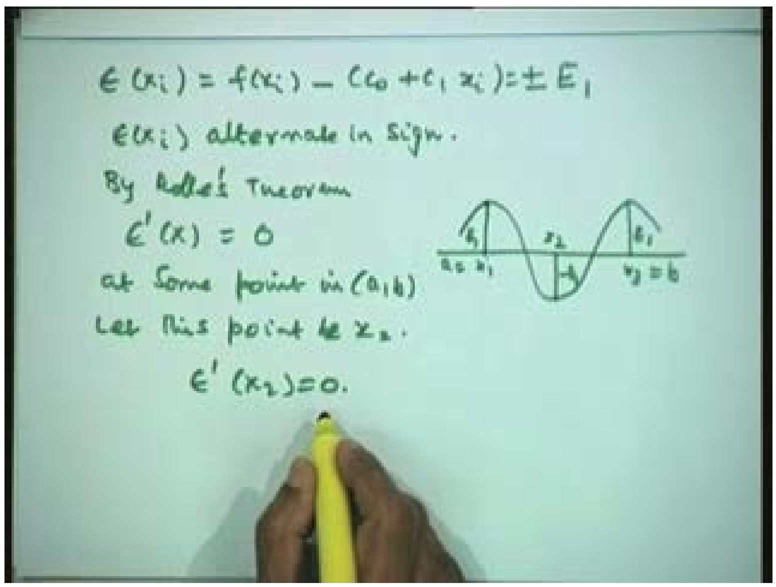


Then uniform approximation would mean that the error at these points epsilon ($x_i$) that is equal to f at $x_i$ minus $c_0$, I am substituting x is equal to $x_i$ in this equation so $c_1$ of $x_i$, they would be equal to either with a plus sign or with a negative sign of $E_1$, $E_1$ is the maximum magnitude therefore they would be having plus minus signs that means we require that epsilon($x_i$) alternate in sign. It may start with plus $E_1$ then it becomes minus $E_1$ and then plus $E_1$ or it may start with minus $E_1$ then it may go to plus $E_1$ and minus $E_1$. For example, in the simple case of a example of a graph of the error like this, looking like this, if I take this line as this, then I will take this as $x_1$, this as $x_3$ and this is my $E_1$ and this is minus $E_1$ and this is equal to $E_1$. Now I have not yet numbered this, I will just number it. Now since the error here of the graph is, this is $E_1$, this is the $E_1$ therefore I can apply the Rolle's theorem immediately on the error function, so we have by Rolle's theorem, by Rolle's theorem epsilon dash(x), derivative epsilon(x) is equal to 0 at some point in (a, b). This is $x_3$ is equal to b, a is equal to $x_1$ between some point and we take this point as $x_2$, so we will take this point as $x_2$, let this point be $x_2$. In other words we are setting epsilon dash ($x_2$) is equal to 0, so the $x_2$ satisfy this equation epsilon dash ($x_2$) is equal to 0.

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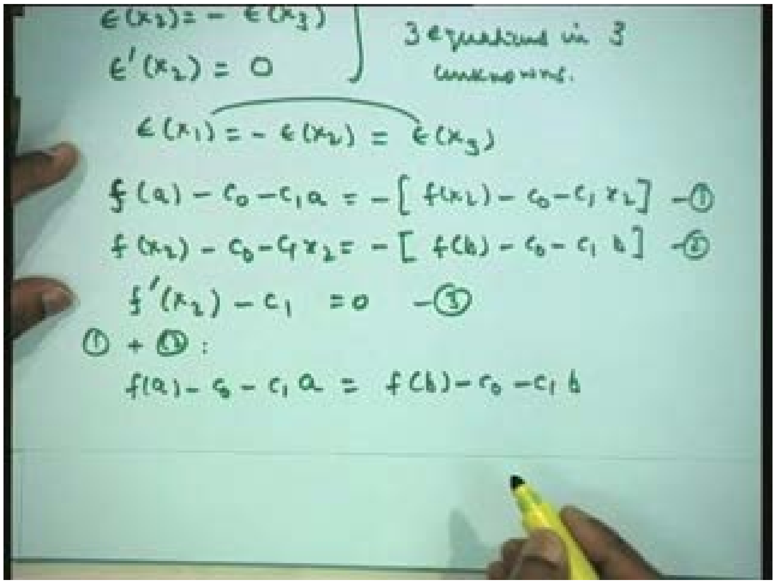


Now I have got 3 equations, let me repeat this equation here, so let us take i is equal to 1 and 2 from here so this first equation reads, put x i is equal to 1, $x_1$ is equal to minus epsilon($x_2$) they should alternate in sign. Error at the next point will be opposite sign of the error at the next point, of course you can view this as epsilon($x_1$) plus epsilon($x_2$) is 0, epsilon($x_2$) plus epsilon($x_3$) is equal to 0 and the third equation is derivative at $x_2$ is equal to 0. Now the quantities that we have to determine in our approximation are, $c_0$ is to be is determined, $c_1$ is to be determined, $x_2$ is unknown, so I need to determine $c_0$, $c_1$ and $x_2$, therefore I have got 3 equations in 3 unknowns therefore these are 3 equations in 3 unknowns.

Now we can solve this, however this is going to be non-linear equations, however there is a simple way of solving them because this equation is going to be simpler than this but if you look at this first two equations, interestingly it will be epsilon($x_1$) is equal to minus epsilon($x_2$), continue the next equation this is epsilon($x_3$). I just combine the 2 equations, now if I take these two as an equation out coming out of this for solving gives an interesting equation which simplifies and obtain the solution of one of the parameters immediately and that will give you substitution backwards and then how can obtain the solution of the problem. Now let us look at how we are going to do this in this problem, now our polynomial is the epsilon(x) is equal to this, I am substituting epsilon($x_1$) that is $x_1$ is equal to a, so that is f at a minus $c_0$ plus $c_1$ at a, so I will, let us take it f at a minus $c_0$ minus $c_1$ a that is this, on the right hand side I have epsilon at $x_2$ so this will be [f of $x_2$ minus $c_0$ minus $c_1$ $x_2$]. The second equation reads f at $x_2$ minus $c_0$ minus $c_1$ $x_2$, this is epsilon at $x_3$, $x_3$ is equal to b so I will set f at b minus $c_1$ minus $c_1$ of b. Now I differentiate epsilon, so if I differentiate I will get f dash at $x_2$, derivative of $c_0$ plus $c_1$ x so I will have simply $c_1$ is equal to 0.

Now I need to solve these three equations, let us use this equation that is I subtract these two equations, so let us number it as 1, 2 and 3. I take it to this side and then write it as add it, I can take addition of these two equations or as equating these two equations that will then give me f of a minus $c_0$ minus $c_1$ a is equal to f of b minus $c_0$ minus $c_1$ of b that is epsilon($x_1$) is equal to epsilon($x_2$). Now here I can see that $c_0$ cancels with $c_0$, so I can bring this $E_1$ b to this side and then solve for $c_1$.

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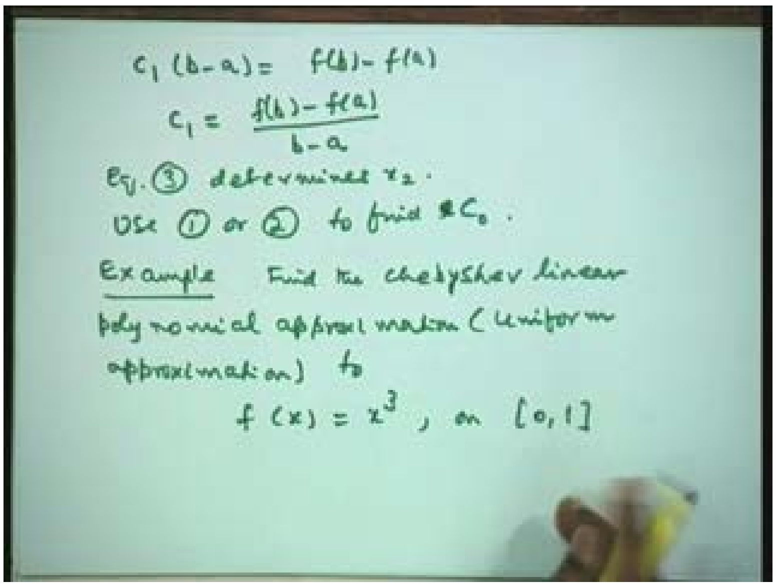


Therefore I can have $c_1$ of (b minus a) is f of b minus f of a, therefore solution of $c_1$ is obtained f of b minus f of a divided by b minus of a. Now when once I get $c_1$ from here, equation 3 gives me $x_2$, now equation 3 determines $x_2$. When once $c_1$ and $x_2$ are determined, I can use any one of equations 1 or 2 to find out $x_1$, so we use 1 or 2 to find to find $c_0$ and that completes the determination of the polynomial and we can also find out what is the maximum error, I can take anyone of the points and find out the error because the error is in magnitude is the same, so I can use the first point or the last point or any point to find out what will be the uniform error here in this particular case. Now let us take an example for this to see how we are going to use this, so we shall use the same example which we have done in the least square approximation so that we can have a comparison. So find the Chebyshev linear polynomial approximation which is same as the uniform approximation to the function, let us take the same function which we have done earlier, f(x) is equal to x cubed on [0, 1].

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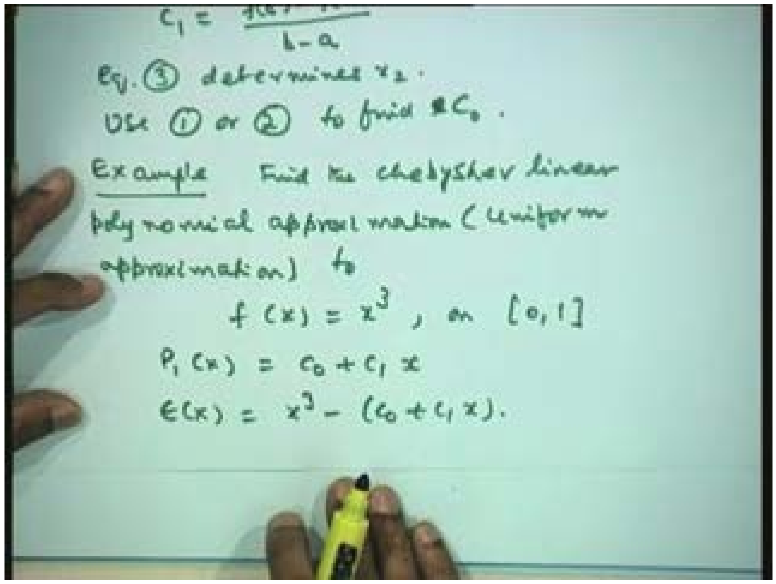


Now as we have done just now, let us take the required approximation $P_1(x)$ is equal to $c_0$ plus $c_1x$ and define our error as epsilon(x) is equal to x cubed minus $c_0$ plus $c_1$ x. Now I need to choose 3 points in the interval 0 to 1 including the end points.

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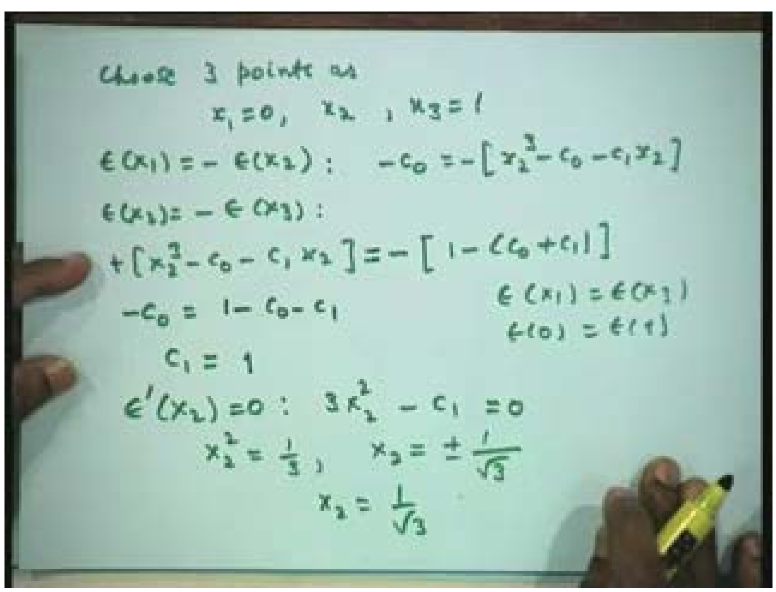

So I choose the 3 points as $x_0$, $x_1$ is equal to 0, $x_2$ is $x_2$, $x_3$ is equal to 1. Now the first equation reads epsilon at $x_1$ is equal to minus epsilon at $x_2$. Now let us substitute $x_1$ is equal to 0, so this gives us 0, this is 0 therefore I will have minus $c_0$ from here and that is equal to minus epsilon at $x_2$ that is [$x_2$ cubed minus $c_0$ minus $c_1$ $x_2$]. The second equation reads epsilon($x_2$) is minus epsilon($x_3$), so let us substitute [$x_2$ cubed minus $c_0$ minus $c_1$ $x_2$] and the right hand side, this is plus, will take this and minus epsilon($x_3$), now $x_3$ is equal to 1 so I will substitute x is equal to 1 in this that is [1 minus $c_0$ plus $c_1$]. Add these two first and then solve it and then write the third equation afterwards, so I add these two, so I will therefore have minus $c_0$, now I am adding these two so these two have canceled, so I am going to or we are looking at, writing the third equation as epsilon($x_1$) is equal to epsilon($x_3$) that is epsilon at 0 is equal to epsilon at 1. So that is what we mean by adding these two equations, so this is epsilon at 0 that gives you, epsilon at 0 gives us minus $c_0$ so that is what we have here and epsilon at 1 is equal to this quantity and therefore this gives you 1 minus $c_0$ minus $c_1$. $c_0$ cancels, I will get $c_1$ is equal to 1, so the value of $c_1$ is equal to 1.

Now I will write the third equation epsilon dash at $x_2$ is equal to 0. Now I differentiate my epsilon(x) that is 3 x square minus $c_1$, 3 x square minus $c_1$ and this should be equal to 0. $c_1$ is 1 so let us take it to this side and write $x_2$ is, $x_2$ square is 1 upon 3, $x_2$ is equal to plus minus 1 upon root 3 but we are in the interval 0 to 1 therefore I can throw away the value minus 1 upon root 3 therefore the required value of $x_2$ is 1 upon root 3. Now I can use any one of these equations to get my value of $c_0$ so let us take the first equation and write it.

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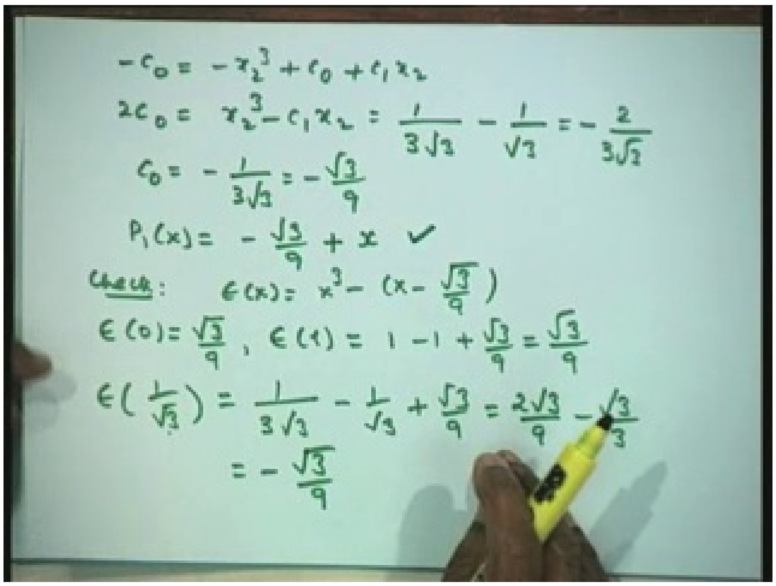

Let us just rewrite this first equation here, this is minus $c_0$ is minus $x_2$ cube plus $c_0$ plus $c_1$ $x_2$. So if i take it to this side, I will have here 2 $c_0$ and bring it to this side $x_2$ cubed minus $c_1$ $x_2$ and $x_2$ cubed let us take the cube of this that is 1 upon 3 root 3 minus 1 upon root 3, $x_2$, is $c_1$ is equal to 1. Therefore this is minus 2 upon 3 root 3. Therefore $c_0$ is equal to minus 1 upon 3 root 3 or we write this as root 3 by 9. Therefore $P_1(x)$ the required polynomial is minus root 3 by 9 and $c_1$ is equal to 1 that is equal to x. Now let us just see whether what we have started with and what is uniform approximation is really satisfied by this, just let us have it as a check, let us call it as a check. So our error is x cubed minus (x minus root 3 by 9) $c_0$ plus $c_1$ x we are substituted, let us get the error at 0, error at 0 is root 3 by 9 substitute x is 0.

Now let us go to the last point epsilon at 1 that is 1 minus 1 plus root 3 by 9 that is equal to root 3 by 9. Then the $x_2$ is 1 upon root 3, $x_2$ is 1 upon root 3, so this gives me 1 upon 3 root 3 minus 1 upon root 3 plus root 3 by 9. So this is root 3 by 9, this is root 3 by 9, so I can write this as 2 root 3 by 9 minus root 3 by 3, so this gives you minus root 3 by 9. Now you can see that the error epsilon(0) is plus root 3 by 9, epsilon at 1 by root 3 is minus root 3 by 9 and epsilon at 1 is root 3 by 9, the errors of equal magnitude and they are alternating inside, so the maximum error is equal to minimum error and we have got this as the best approximation in the uniform sense that is the Chebyshev polynomial approximation. Now would like to generalize the result for any polynomial, so let us generalize this result and we call this result as Chebyshev equi-oscillation theorem.

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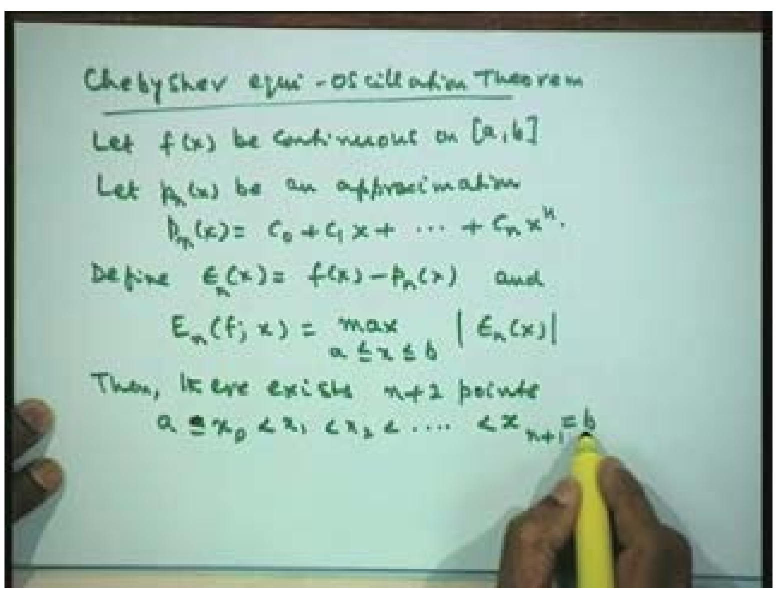


So we call this as Chebyshev equi-oscillation theorem, this is the basis of the uniform approximation for a general polynomial. Now we are given a continuous function so let us start with a continuous function, let f(x) be continuous on a given interval [a, b]. Then we have an approximation, so let $p_n(x)$ be an approximation, let $p_n(x)$ be an approximation that means we are writing $p_n(x)$ is a polynomial $c_0$ plus $c_1$ x so on $c_n x$ to the power of n. Then we define the error, define epsilon(x) is equal to, $\text{epsilon}_n(x)$ we can now put, is f(x) minus $p_n(x)$ and the maximum value will denote by capital A, so $\text{epsilon}_n$ x (f, x) is equal to maximum of this error in the given interval this. These are all the definitions that we have used earlier, now we state equi-oscillation theorem out of this, we will say then there exist n plus 2 points, there exist n plus 2 points, will take a is equal to $x_0$ less than $x_1$ less than $x_2$ $x_n$ plus 1 is equal to b. We are considering $x_0$ to $x_n$ plus 1, n plus 2 points, the first point is a, the last point is equal to b.

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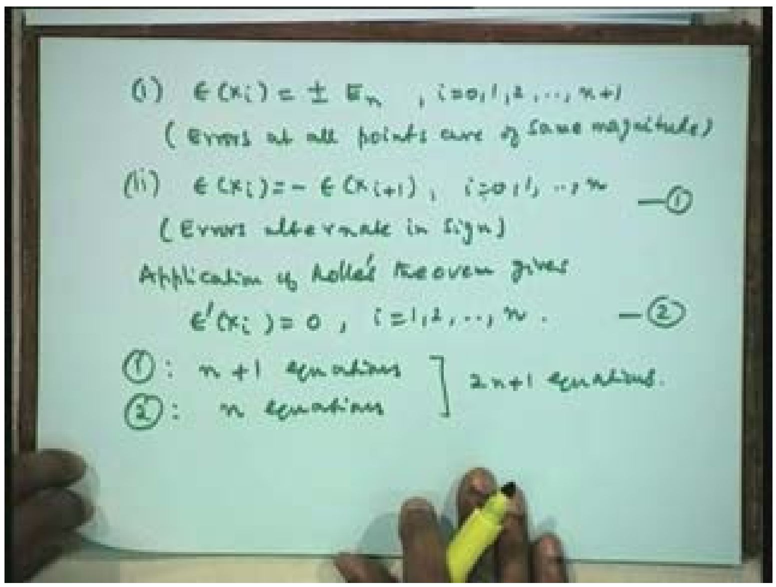


Now there exist n point such that (1) error at $x_i$ is plus minus $E_n$ that is your, $E_n$ is the maximum magnitude of the error therefore error at this will have either plus or minus sign of $E_n$ and this is for i is equal to 0, 1, 2, n plus 1 that means this actually is telling us that errors at all points are of same magnitude, so this is stating that errors at all points are of same magnitude. Secondly error at $x_i$ is equal to minus error at $x_{i+1}$ starting from i is equal to 0, 1, so on n. Epsilon at $x_0$ is equal to minus $\text{epsilon}_1$ and lastly error at $x_n$ is equal to minus error at $x_{n+1}$ and this states that the errors alternate in sign. Now this second part would therefore give us, the application of Rolle's Theorem can be applied using this, so we can now state that this implies application of Rolle's Theorem gives that derivative at $x_i$ is equal to 0, i is equal to 1, 2, 3, n. There are total of n plus 2 points and we have got errors at all these points are same in magnitude and hence the Rolle's theorem can be applied and we will have n intermediate points at which the errors will be equal to 0. Now we can say that 1 gives us n plus 1 equations, these are n plus 1 equations and this gives us n equations, therefore n gives us, 1 gives us n plus 1 equations and 2 gives us n equations, so that we have a total of 2 n plus 1 equations.

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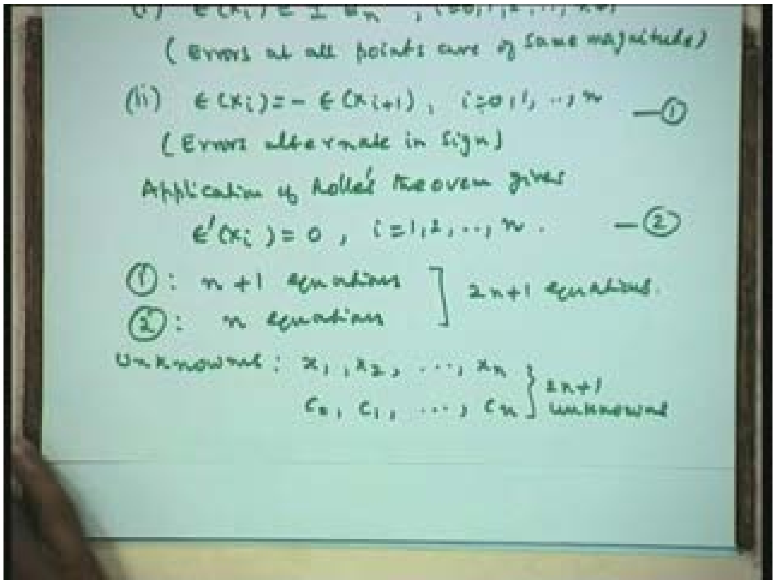


Now what are the unknowns that we have here, the unknowns that we have are $x_1$, $x_2$, the first point and the last point are known to us therefore $x_1$, $x_2$, $x_n$ are the points in the middle and then we have the parameters in the polynomial $c_0$, $c_1$, $c_2$, $c_n$ and these are 2 n plus 1 unknowns. Therefore we have 2 n plus 1 equations in 2 n plus 1 unknowns and these are of course non-linear equations and we have way of as we have shown earlier, there is a way of solving them, we can solve this and then get the uniform approximation and finally we can determine, find using any point $x_0$ or the first or the last point, we can finally determine what is the maximum value of this and then take that as our uniform error. Now let us take a an example for a quadratic polynomial.

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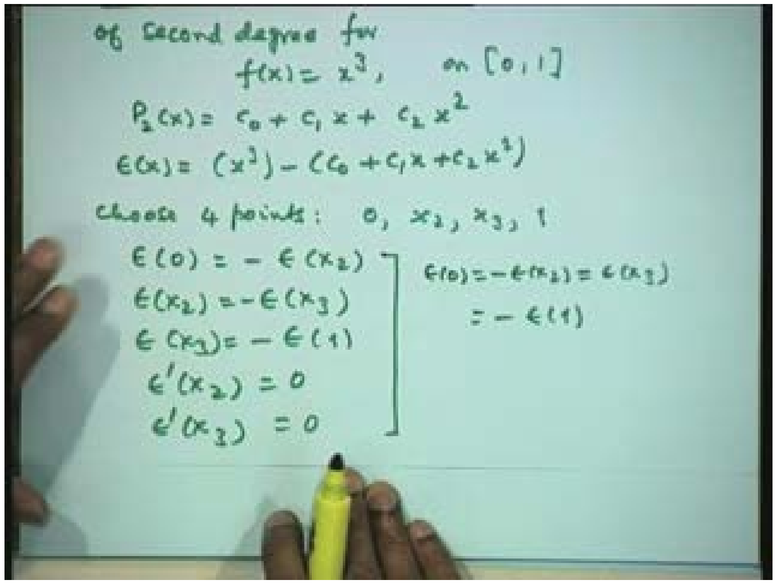


Let us do the same example for the quadratic approximation so let us take it as, obtain the best uniform approximation of second degree for f(x) is equal to, I will take the same example x cubed on [0, 1]. Now I need to consider a polynomial of degree 2 therefore I will consider $P_2(x)$ as my polynomial ($c_0$ plus $c_1$ x plus $c_2$ x square) and error is equal to (x cubed) minus ($c_0$ plus $c_1$ x plus $c_2$ x square). Now given a polynomial, the number of points that should be chosen is given over here, so we need to choose a total of n plus 2 points given a polynomial, so given a polynomial of degree 2 so I need to have n plus 2 that is 4 points. Therefore we say choose 4 points, the first and last will take as this 0 then will take $x_2$, will take $x_3$ and $x_4$; $x_0$, $x_1$, $x_2$, $x_3$, $x_4$.

Now let us first write down all the equations which we should get here that is epsilon (0) is equal to minus epsilon at $x_2$, epsilon at $x_2$ is equal to minus epsilon at $x_3$, epsilon at $x_3$ is minus epsilon at $x_1$ then we have the derivatives epsilon dash at $x_2$ is equal to 0, epsilon dash at $x_3$ is equal to 0. Now while solving this system as I mentioned earlier, we shall first solve the which one that combines the first and the last because then there is no parameter, so if you look at that one, what that equation would be epsilon (0) is minus epsilon($x_2$) so substitute a minus sign here that is plus epsilon($x_3$) and plus epsilon ($x_3$) is minus epsilon (1), minus epsilon (1). So that particular equation which you are looking for which simplifies the first equation is, we start with epsilon(0) is minus epsilon(1) and that will give me, it will have only one parameter therefore I can solve that particular equation immediately and then substitute backwards in this equations to get the values of the other parameters.

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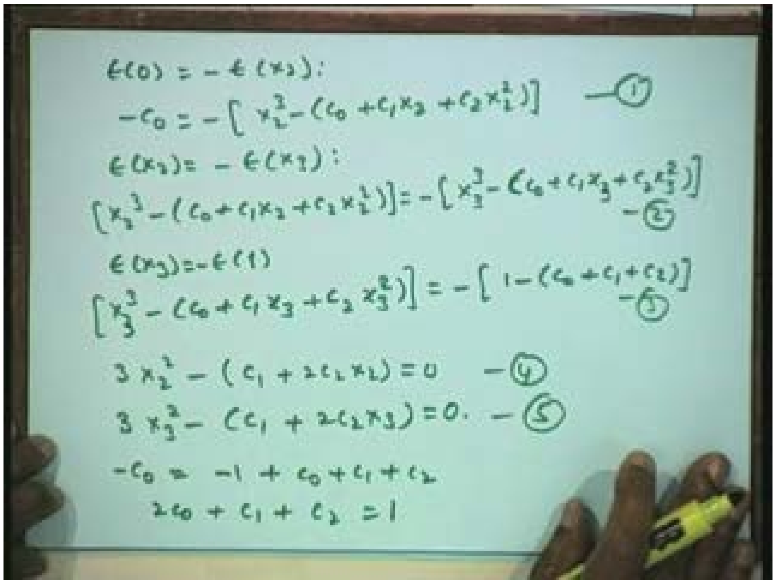


Now let us first of all write down what these values are, so you will have here, let us write down again epsilon (0) is minus epsilon($x_2$). Now this gives us, I am substituting in this particular equation therefore I will have simply minus $c_0$ that is the left hand side and the right hand side is [$x_2$ cubed ($c_0$ plus $c_1$ $x_2$ $c_2$ $x_2$ square)] that is our first equation, so let us again number it as 1, 2, 3 here. Now the second equation is epsilon($x_2$) is minus epsilon($x_3$) so that is your [$x_2$ cubed minus ($c_0$, $c_1$ $x_2$, $c_2$ $x_2$ square)] this is at $x_3$, will have [$x_3$ cubed minus ($c_0$, $c_1$ $x_2$, $c_1$ $x_3$, $c_2$ $x_3$ square)] and the third equation reads epsilon($x_3$) is epsilon(1) that is [$x_2$ cubed ($c_0$ plus $c_1$ $x_3$, $c_2$ $x_3$ squared)] is minus of, substitute x is equal to 1, I will get here [1 minus ($c_0$ plus $c_1$ plus $c_2$)].

Now let us write down the last two equations that we have here, that is the derivatives at $x_2$ and derivative at $x_3$ is equal to 0. So if I now differentiate, if I now differentiate this epsilon(x) here, I would get here 3 $x_2$ square minus ($c_1$ plus twice $c_2$ $x_2$) is equal to 0, I differentiated this, ($c_1$ plus twice $c_2$ x) is equal to 0 and similarly 3 $x_3$ square minus ($c_1$ plus twice $c_2$ $x_3$) is equal to 0. So I have got this equation 2, this is 3, 4 and 5. I would like to use this particular equation which gives me a solution immediately, epsilon (0) is equal to minus epsilon (1), so if I do that, that is epsilon(0) is minus $c_0$, epsilon(0) is minus 0 and minus epsilon(1) is here, so I can lift it as it is, so this will be equal to minus 1 plus $c_0$ plus $c_1$ plus $c_2$. Right, this gives me 2 $c_0$ plus $c_1$, plus $c_1$ plus $c_2$ is equal to 1. Now I think I will leave this is an exercise for you to complete the problem, I will give the solution of this problem; you can verify the values of $c_0$, $c_1$, $c_2$, so on.

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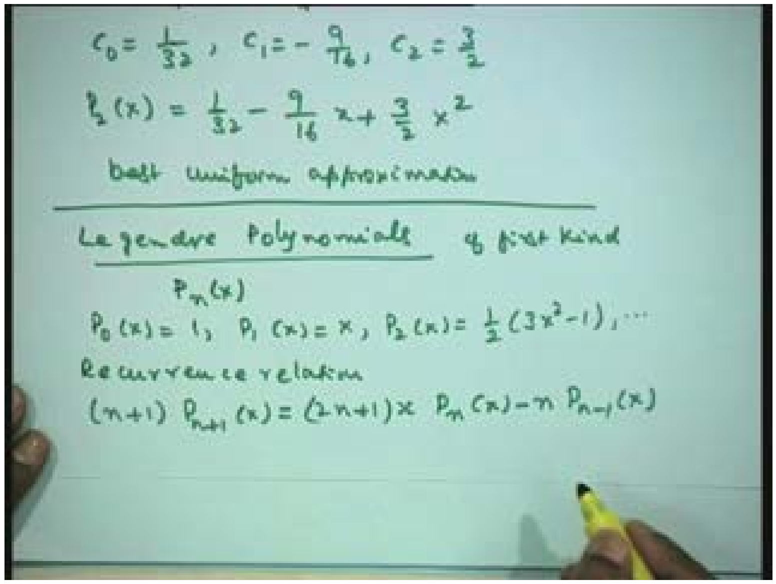


So I would give the values of $x_2$ is 1 by 4, $x_3$ is equal to 3 by 4, $c_0$ is 1 by 32, $c_1$ is minus 9 by 16 and $c_2$ is 3 by 2. Therefore our best uniform approximation is 1 upon 32 minus 9 by 16 x plus 3 by 2 x square, so this is the best uniform approximation and we can actually find out what the errors, what the error is, we can find out the error that will be equal at any of these points particularly with respected to, at the first point we can find out what will be the maximum error and that will give you the maximum error in that in this equation.

Now we have earlier talked about least square approximation and in the least square approximations we wanted to use the orthogonal polynomials which will give the values of the coefficients immediately as a ratio of two integrals, if it is a continuous function or the ratio of sums of the some w $(x_i)$ into the orthogonal polynomials and would now would like to define the two polynomials which can be used in least square approximation as well as we shall be using in the next topic that is a numerical integration also. So let us define these polynomials, orthogonal polynomials which you must have done in your earlier course also but let us just briefly revise what are those polynomials and what are the properties of those polynomials which makes them very powerful. Of all of them, the Chebyshev polynomial is supposed to be the best and will see why Chebyshev polynomial is the best. So let us just briefly describe these polynomials, the Legendre polynomials, now we are really talking of Legendre polynomials of first kind because the polynomials, Legendre polynomials that we talk of normally is only the polynomials of the first kind that is the polynomials $P_n(x)$. The first polynomial is 1, $P_1$ of x is equal to x, $P_2(x)$ is equal to half 3 x square minus 1 and so on. I can generate all of them through a recurrence relation, if once I know two of them it is sufficient for me, I can now write down the recurrence relation and get any Legendre polynomial I want from here.

So the recurrence relation for this is this, (n plus 1) $P_{n+1}(x)$ is equal to (2 n plus 1) x $P_n(x)$ minus n $P_{n-1}(x)$. Therefore when once I know $P_0$, $P_1$, I can now generate from here by taking n is equal to 1, I can get my $P_2$, I can get my $P_3$ all of them, so I can generate all the polynomials, whichever polynomial I want using this recurrence relation. Now these Legendre polynomials are obtained as the solutions of a differential equation, they all belong to the problem called the strum levelly problem in which the co efficients can be taken differently and the solutions of this differential equations of the strum levelly problem gives rise to this orthogonal polynomials that is a one class of problems which produces a large number of orthogonal polynomials and these polynomials are defined in the interval, let us write it here itself, they are defined in the interval minus 1 to 1, they are defined over the interval minus 1 to 1.

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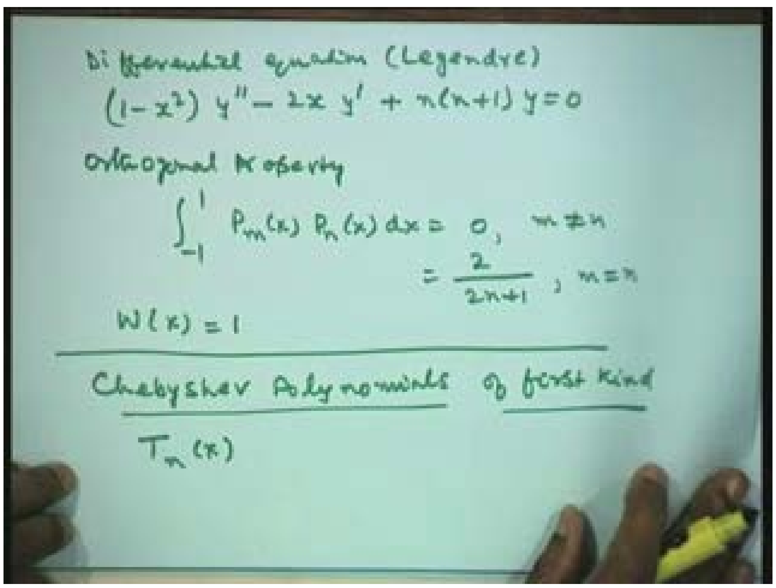


Now the differential equation governing this is this, (1 minus x square) y double dashed minus 2 x y dashed plus n into (n plus 1) of y is equal to 0, this is also called the Legendre's differential equation; this is called the Legendre differential equation. I can write down the series solution for this differential equation, you have done the series solution of differential equations, so I write the series solutions of the differential equations and one of the solutions would give me the orthogonal polynomial that is Legendre polynomials of first kind with a suitable coefficient that has been prefix to it and that coefficient is determined with some property that $P_n$ of 1 is equal to 1. You can see that $P_n$ of 1 is 1 always, so using that one can determined what will be the coefficient that is there. The orthogonal property is the one that we have been used earlier, the orthogonal property is with the weight function 1 here, minus 1 to 1 $P_m(x)$ $P_n(x)$ dx is equal to 0, for m not equal to n, 2 upon 2 m plus 1 for m is equal to n and in this case the weight function w(x) is equal to 1.

Now this is one of the set of polynomials that we shall use, the Legendre polynomials of second kind are not used because that they are not of great use, it is only the Legendre polynomials of first kind are of use. Now let us define the Chebyshev polynomials, here again we shall be considering only the Chebyshev polynomials of first kind, these polynomials are denoted by $T_n$ of x.

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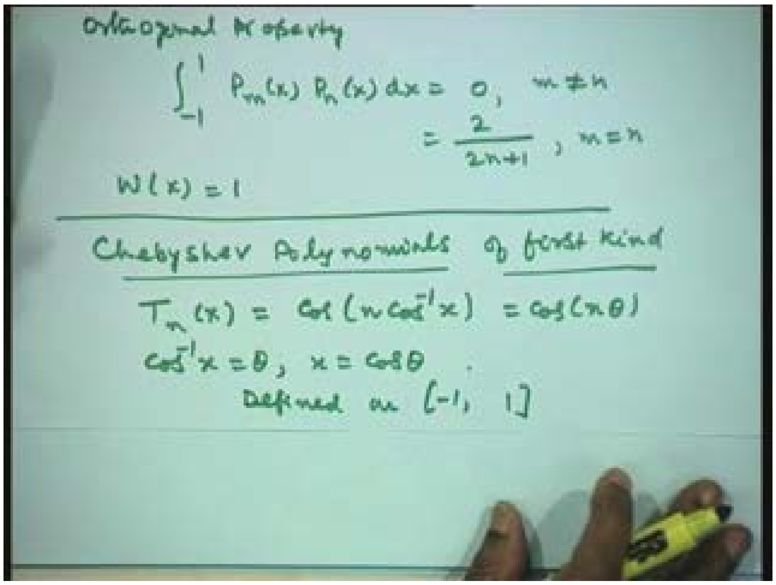


Interestingly even though it is a polynomial, it can be written in a nice close form in the trigonometry function form as cos (n cos inverse of x) or simply cos of (n theta), cos n theta that means we have set cos inverse of x is equal to theta or x is equal to cos of theta. So if I set cos inverse x is theta, I would get x is equal to cos theta and therefore they are all defined on minus 1 to 1, defined on minus 1 to 1, the same as in the Legendre polynomial.

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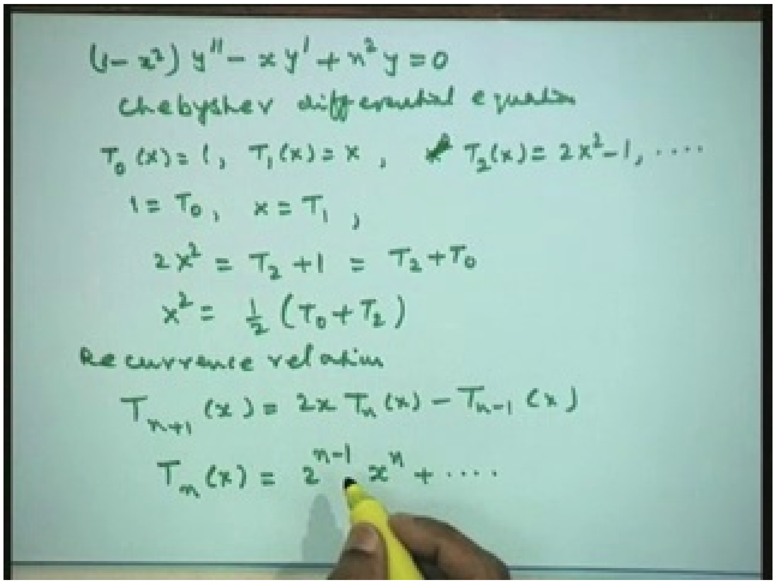


The differential equation that governs this is (1 minus x square) y double dashed minus x y dashed plus n square y is equal to 0, this is the Chebyshev differential equation. Now I can set n is equal to 0, 1, 2 here and then get the values of the Chebyshev polynomials of first kind again $T_0$ of x is 1, $T_1$ of x is equal to x, the first 2 polynomials are the same $T_2(x)$ is equal to 2 x square minus 1. In both the Legendre polynomials as well as Chebyshev polynomials, we can write x in terms of, in terms of polynomials also, so I can write down reverse relation also that is 1 can be written as $T_0$, x can be written as $T_1$ then I can solve for x square from here, 2 x square is equal to $T_2$ plus 1 but 1 is $T_0$ so I can write this as $T_2$ plus $T_0$ and therefore I can write down x square as half of ($T_0$ plus $T_2$). Therefore it is possible for me to write down, express the polynomials in terms of the Chebyshev polynomials starting with that degree that is $T_2$ and lower ones and here since it is even so it will contain only ($T_2$ plus $T_0$). If it is an odd power, it will contain your odd Chebyshev polynomials, degree Chebyshev polynomial that is $T_1$, $T_3$, x cubed will be in terms of $T_3$ and $T_1$.

Now again I can generate all of them through the recurrence relation, the recurrence relation for the Chebyshev polynomials are $T_{n+1}$ of x is 2 x $T_n(x)$ minus $T_{n-1}(x)$. Now I can expand this cos and cos inverse x to determine what is the leading coefficient for me in $T_n(x)$ because that is something very important and useful for us. The $T_n(x)$ is equal to, I can show that this is equal to $2^{n-1}$ x to the power of n plus lower degree terms, so that the leading coefficient of $T_n(x)$ is 2 to the power of n minus 1. Now I can define from here, what is the monic Chebyshev polynomial that is a polynomial with leading coefficient as 1, so if I define T, divide $T_n(x)$ by $2^{n-1}$ then I will have a monic Chebyshev polynomial.

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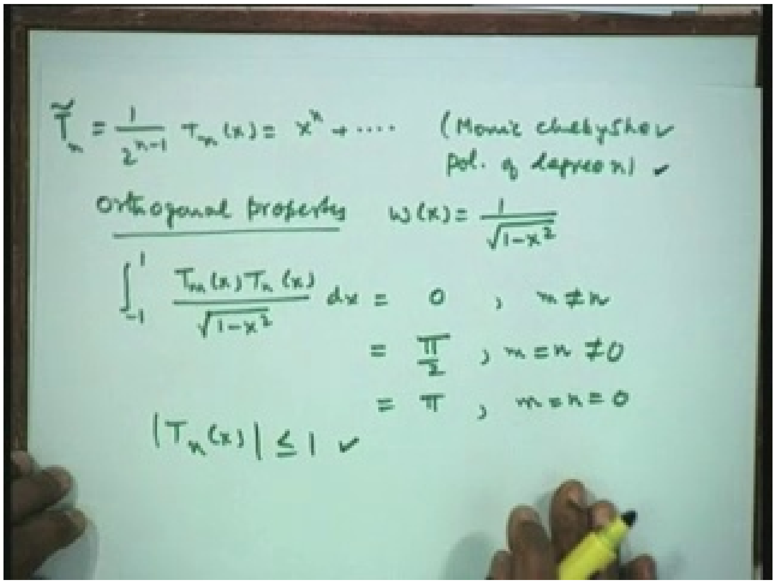


So I can write this as 1 upon 2 to the power of n minus 1 $T_n(x)$ is equal to x to the power of n plus so on, so this defines a monic Chebyshev polynomial of degree n. Now before I mention where I am going to use it in a moment, let us just also write down what is the orthogonal property, orthogonal property. The weight function for Chebyshev polynomials is 1 upon under root 1 minus x square, so the weight function for Chebyshev polynomial is not a constant but it is this particular form. Therefore integral minus 1 to 1 w(x) into $T_m(x)$ $T_n(x)$, so will now substitute for that $T_m(x)$ $T_n(x)$ into w(x) that is 1 minus x square under root dx this is equal to 0, for m not equal to n. If I put m is equal to n, I will get pi by 2, m is equal to n not equal to 0 but this will be equal to pi if it is equal to 0.

Now the most important thing here is that magnitude of $T_n(x)$, now what is the definition of $T_n(x)$, it is a cosine function, it is a cosine function therefore its magnitude is going to be less than 1 always, therefore magnitude of $T_n(x)$ is always less than or equal to 1. Now this is something very useful for us that the magnitude of any Chebyshev polynomial is less than or equal to 1. Now as I was saying the, I wanted to use this particular property, what is the, what is so great about this one, so if I, you can just denote this by tilde, T tilde x, tilde $T_n$.

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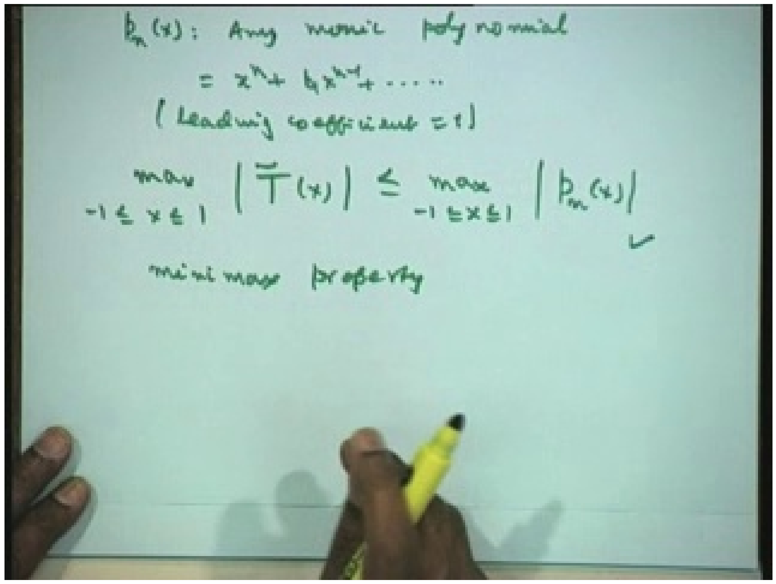


Now I consider, now any polynomial with leading coefficient 1, $p_n(x)$ is any monic polynomial that means leading coefficient is 1 that is of the form x to the power of n plus some $b_1$ x to the power of n minus 1 and so on, therefore the leading coefficient is equal to 1. Now if I consider the monic Chebyshev polynomial and arbitrary monic polynomial on the interval minus 1 to 1 and consider its magnitude, then the monic Chebyshev polynomial is the smallest in magnitude of all the polynomials on minus 1 to 1, there is no polynomial whose magnitude will be less than this particular Chebyshev polynomial. So the property that we want to show is, the maximum over minus x to 1 of T this, that is your monic Chebyshev polynomial is less than or equal to minus 1 to 1 this of $p_n(x)$. This is the property which makes Chebyshev polynomials as a rarest of the orthogonal polynomials because you cannot find any polynomial whose magnitude will be less than the polynomial, therefore in all the approximations we would prefer to use Chebyshev polynomials because its error is going to be smallest and this property is called the minimax property of Chebyshev polynomials. That is why you will find that when we are applying in any particular problem even though Legendre polynomials can also be used, Chebyshev polynomials can also be used, Chebyshev polynomials are preferred because of this particular property. Okay would stop at this.