Numerical Methods and Computation

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Lecture No - 32

Interpolation and Approximation (Continued)

Now in the previous lecture we have derived the least square approximation using the coordinate functions $phi_i(x)$ as polynomials or as orthogonal polynomials.

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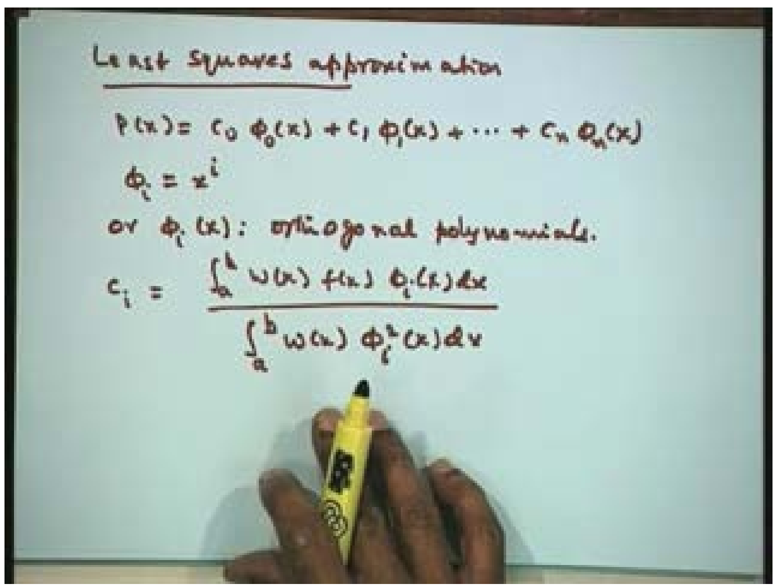

Now what we have written, let us just remember what we have done in the least square approximation, we have written the approximation p(x) is c_0 $phi_0(x)$, c_1 $phi_1(x)$ plus so on c_n $phi_n(x)$. We have chosen these $phi_i(x)$ as simply equal to x_i or we have chosen $phi_i(x)$ as orthogonal polynomials. Then we have derived the constants c_i such that the aggregate or the sum of squares of the errors are minimized and such approximation we called the best least square approximation, best approximation in the least square sense.

We have derived the constants c_i and we have also illustrated through an example and let us now see, illustrated to an example in the case when we choose them as orthogonal polynomials and now when, if $phi_i(x)$ are the orthogonal polynomials, we have derived the values of the constant c_i as integral a to b w(x) f(x) $phi_i(x)$ dx divided by integral a to b w(x) phi_i square x dx. Now when the data is given, this integrations have become summations and we have certain equations and we have illustrated examples when a data is given. Now let us illustrate the case when the function given is a continuous function and we want to write a least square approximation for that, so let us take it as an example.

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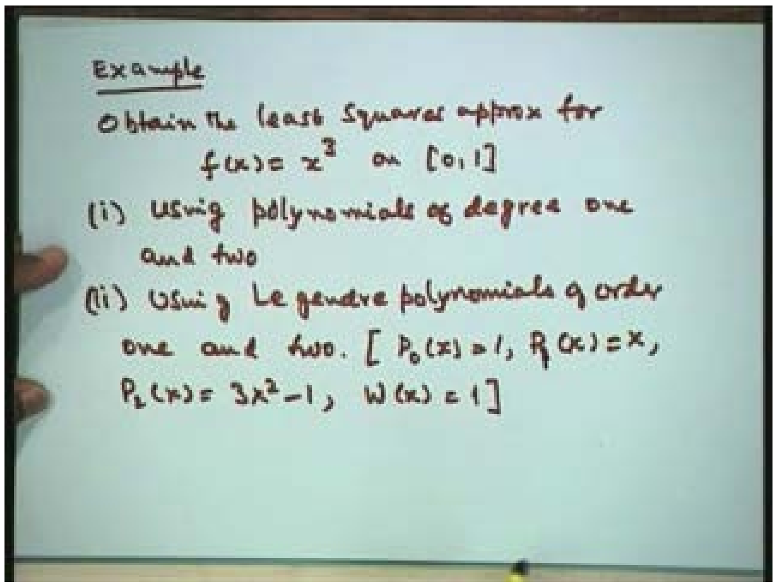

Obtain the least square approximation for, let us give the function as simply, a simple function let us take this on [0, 1], (i) using polynomials of degree one and two and secondly using Legendre polynomials of order one and two. Now we are formally going to define these polynomials later on, let us also give the data here that is required for us, $P_0(x)$ is equal to 1, $P_1(x)$ is equal to x, $P_2(x)$ is equal to 3 x square minus 1 and w(x) is equal to 1 for the case of the Legendre polynomials. Now here a continuous function is given to us, I would like to first of all approximate it by simply polynomials of degree one and two then I would also do the same thing by using the orthogonal polynomials that is your Legendre polynomials.

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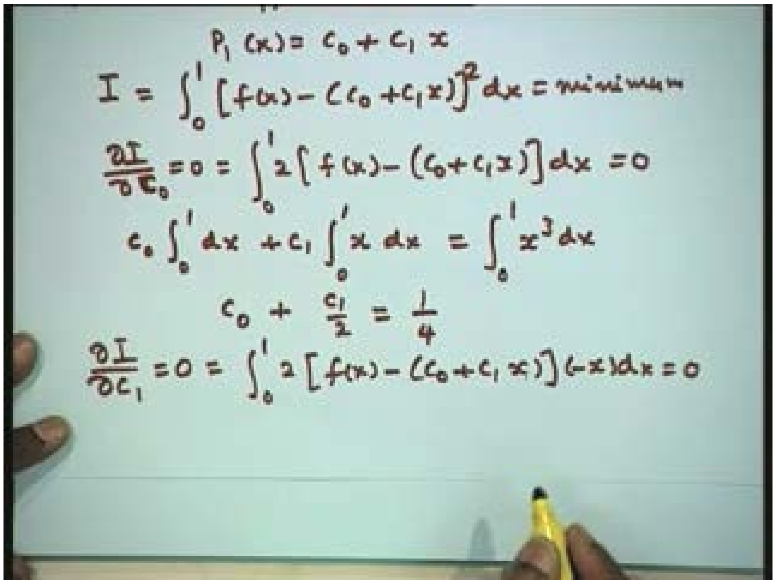

Let us first take the first part of it, now we need the linear approximation first therefore we are writing $P_1(x)$ is equal to c_0 plus c_1x and what we mean by least square is that we are saying that integral 0 to 1 [f(x) minus (c_0 plus c_1x) whole square] dx is equal to a minimum quantity. Now we get two equations from here by minimizing with respected to the parameter c_0 and c_1, so I will set delta I by delta c_0 is equal to 0. So let us follow the entire procedure even though we could have written the equation directly, let us just see how every step we proceed on. So I can now differentiate this 2 times [f(x) minus (c_0 plus c_1x)] dx is equal to 0, I have differentiated with respected to c_0 so I will have minus sign which I removed that minus sign, we have a minus sign coming, that I have cancelled it throughout. I can cancel 2 also from here, so what I have here is c_0 integral 0 to 1 dx plus c_1 integral of x dx and this is equal to right hand side 0 to 1, I would now substitute the value of f(x) given to us, f(x) is equal to x cubed here.

Now I integrated, this simply gives us x between the limit 0 to 1, I get 1 therefore c_0, x square by 2 within the limit 0 to 1 so I will get 1 by 2, this is x to the power of 4 by 4 within the limit 0 to 1 so I will simply get 1 by 4, so the first equation reads c_0 plus c_1 by 2 is equal to 1 by 4. Now let us take the second parameter that is delta by delta c_1 is equal to 0, so I would get 0 to 1, 2 times [f(x) minus (c_0 plus c_1x)] into, I differentiating with respect to c_1, I will get minus x over here, dx is equal to 0. Now again we retain the terms of c_0, c_1 on the left hand side and take f(x) to the right hand side.

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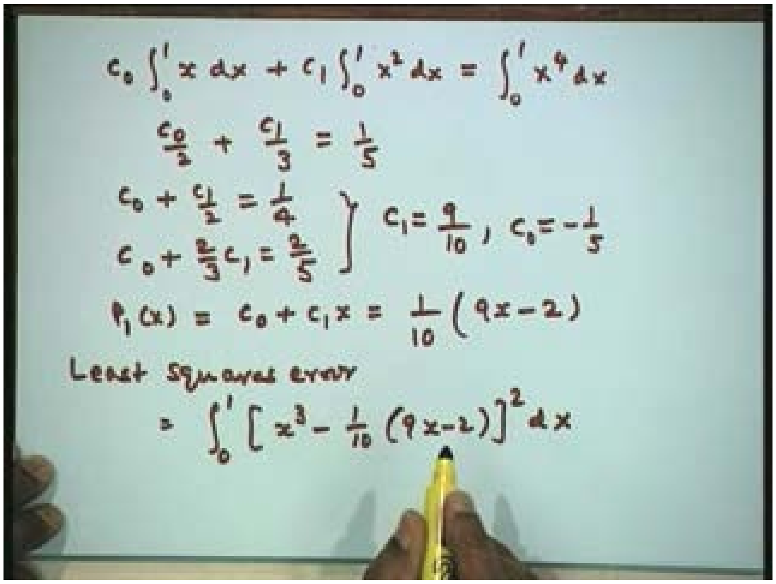

Therefore I would get c_0 integral 0 to 1 x dx plus c_1 integral 0 to 1 x square dx, on the right hand side 0 to 1 x cubed into x that is x to the power of 4 into dx. Integral of x gives us the x square by 2 that will give c_0 by 2, this gives x cubed by 3 that is c_1 by 3, x to the power of 5 by 5 so I will have 1 by 5. Therefore let us rewrite the equations here, c_0 plus c_1 by 2 is 1 by 4, let us take 2, multiply by 2 that is c_0 plus 2 by 3 c_1 is equal to 2 by 5. Now I can solve it easily, I can subtract and find c_1 and substitute it over here and what I get c_1 is 9 by 10, then using this I get c_0 is equal to minus 1 by 5, substitute it here and get this result. Therefore our polynomial approximation, least square approximation is equal to c_0 plus c_1x, so I can take 10 common out and write this as (9 x minus 2) that is 1 by 5, so I will have (9 x minus 2), this is the least square linear polynomial approximation.

Now I would like to know what is the total error that is committed here, so I can also find that, so I can call this as least squares error and that will be equal to integral of 0 to 1 f(x) minus (c_0 plus c_1x) that is 1 by 10 (9 x minus 2) whole square dx. This error was minimized and we will now, we are now finding out what is this minimum error that has been committed, so I can just integrate this and obtain the value of this and say that this is the least square error that has been committed in this particular approximation.

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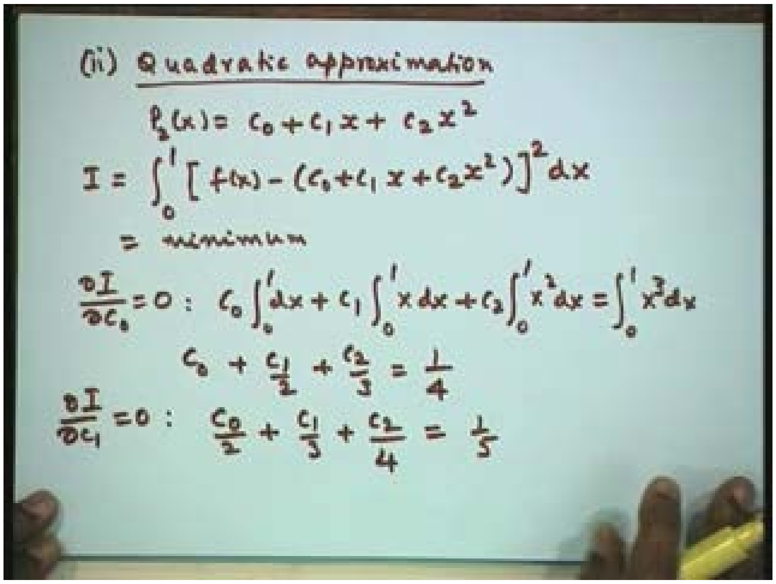

Now we have been asked to find the quadratic also, so let us do the quadratic approximation, quadratic approximation, now in this case we are taking the polynomial approximation as, we add one more term to the previous case that is your $c_2(x)$ square, c_0 plus c_1 x plus c_2 x square. Therefore the first step would be to define our least square approximation 0 to 1 [f(x) minus (c_0 plus c_1 x plus c_2 x square) whole square dx is equal to minimum. Now in this case what we have done is that we have added one more term here, therefore everywhere you will get an extra term, we in fact without writing any further step we can write from this case what will be the equations from here because there will be 1 more term here c_2 0 to 1 x square dx and there will be, here c_2 and there will be 1 more term one third equation also would be there.

Therefore we will get 3 equations which we can write immediately by setting delta by delta c_0 is equal to 0 and this gives us, let us write down straight away that will be c_0 integral 0 to 1 dx plus c_1 integral 0 to 1 x dx plus c_2 integral 0 to 1 x square dx and on the right side we have 0 to 1 x cubed dx. That is c_0 into 1, this is x, this is x square and this is cubed dx, therefore this is c_0, this is c_1 by 2, x cubed by 3 that is c_2 by 3, the right hand side is 1 by 4. Now if I said delta by delta c_1 is equal to 0, now I will have a product with x here, I will have x square here, I have x cubed here and x to the power of 4 here, therefore we would simply get c_0 by 2, c_1 by 3, c_2 by 4, this is x cubed therefore x 4, this is x^4 therefore I will have 1 by 5. The third equation gives us delta I by delta c_2 is equal to 0, now when I differentiate it, this with c_2 I am getting x square here in the product, so the leading term will be x square, x cubed, x to the power of 4, x to the power of 5, therefore this will be c_0 by 3 because of x square, x cubed therefore c_1 by 4, this is x to the power 4 that is c_2 by 5 and this is x to the power of 5 so I will have 1 by 6.

Now this is a simple, solving the system, so we can solve this system and then substitute this values in $P_2(x)$ that is c_0 plus $c_1(x)$ c_2 square to get the quadratic approximation in the least square sense, we can similarly find out the least square error by substituting the values of c_0, c_1, c_2 in this expression and get what would be the minimum error that would be committed in the least square sense.

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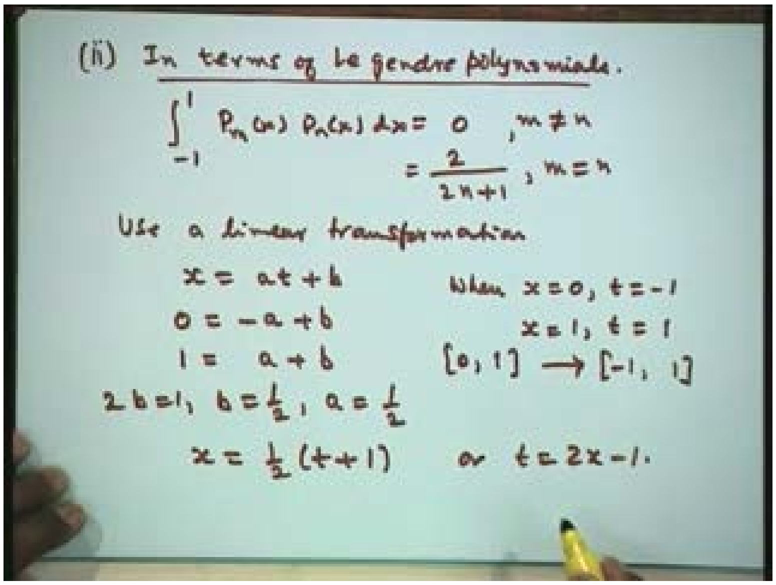

Now let us go to the second part that is in terms of the Legendre polynomials, in terms of Legendre polynomials. Now let us just remember that the orthogonal property of legendre polynomials are, minus 1 to 1 $P_m(x)$ $P_n(x)$ dx is equal to 0, for m not equal to n and 2 upon 2 n plus 1 for m is equal to n, so this is the orthogonal property of the legendre polynomials. Now one very important thing that we remember is that the Legendre polynomials are defined over minus 1 to 1 but the problem here was given in 0 to 1, therefore we must change given interval if it is [a, b] we reduce it to minus 1 to 1 by using a linear transformation, so I was first reduce this into minus 1 to 1 and then go and write down the least square approximation in terms of the Legendre polynomials.

We use a linear transformation; use a linear transformation, will write down some x is equal to at plus b, so let us write it as at plus b. Now what is it we want, we are given the interval 0 to 1 so, therefore when x is equal to 0, I need t is equal to 1, lower limit for t should be 1 when x is equal to 1 and when x is equal to 1, I want the upper limit to be 1, therefore 0 to 1 we are now bringing to minus 1 to 1. We shall do the same thing if arbitrary interval is given, some peak interval is given, we shall use the same transformation and then find out the values of a and b. So let us substitute it over here, therefore 0 is minus a plus b and x is equal to 1 that is equal to a plus b, so if I add I get 2 b is equal to 1 or b is equal to half therefore a is equal to half. Therefore the transformation is x is equal to half (t plus 1), if you want to look what is t, t is equal to (2 x minus 1).

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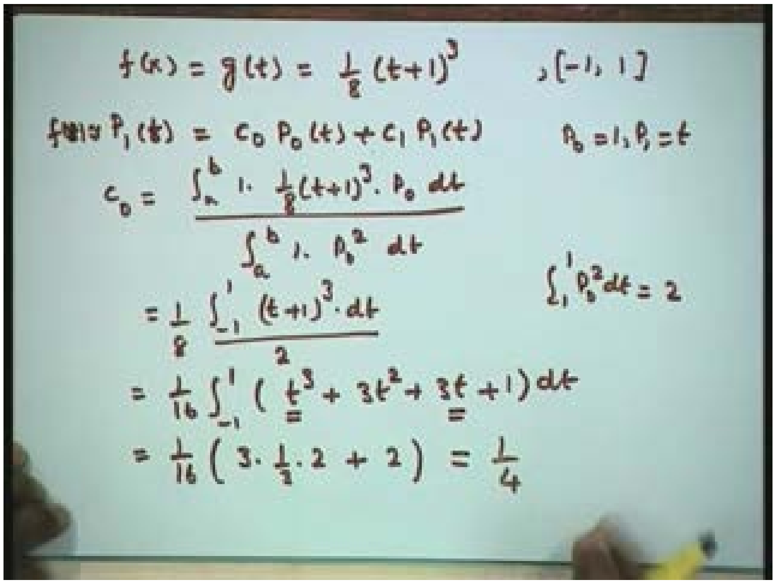

Therefore from here we can write down now what is our function given to us, so the function f(x) is equal to, let us call it as a function as g(t) is equal to x cubed that is 1 by 8 of (t plus 1) whole cubed and now the interval is in minus 1 to 1, the interval for t is minus 1 to 1. Now the least square approximation shall be written in term of Legendre polynomials in the variable t because we want the approximation over this interval therefore we write down the least square approximation P_1 of t is equal to c_0 P_0 of t plus c_1 P_1 of t, where we remember P_0 is equal to 1 and P_1 is equal to t. Therefore this is our approximation f(x) is, f of t is approximated or f(x) is equal to g(t) that is approximated by this one.

Now we have derived earlier the expression what will be in the case of the orthogonal polynomials, the values of the constants can be directly written down from here, so I can use this value by putting i is equal to 0 and so on and find out what will be the values of these parameters, so let us write down, substitute it here. So I will have here, c_0 will be equal to integral a to b weight function is 1, our variable is t now, sorry I will replace this minus 1 to 1 in the next step, so this is w(x) is 1, f(x) is equal to 1 upon 8 (t plus 1) whole cubed, 5 0 is P_0, P_0 dt divided by a to b 1 P_0 square dt. Let us now translate it correctly as minus 1 to 1, 1 upon 8 can be written outside, (t plus 1) whole cubed, p_0 is 1, now the denominator is minus 1 to 1 P_0 square dt and we would like to use the orthogonal property that we have just now described that minus 1 to 1 m n is equal to 0 therefore this is equal to 2, so this gives us p_0 square and this is equal 2, when m is equal to 0 we have 2 here.

Therefore what we have in the denominator is 2, therefore I just open it up and integrate this, integral minus 1 to 1, I can take this as (t cubed plus 3 t square plus t 3 plus 1) of dt. Now t cubed is an odd function therefore its contribution is 0, t is an odd function its contribution is 0 therefore only t square and 1 will contribute the solution. So this is 1 by 16 that is 3 into t cubed by 3 1 by 3 and t cubed within limits minus 1 to 1 that gives us 2, then integral of 1 that is t that gives us 1 plus 1 that is equal to 2 therefore this gives us 2 plus 2 4 that is equal to 1 by 4.

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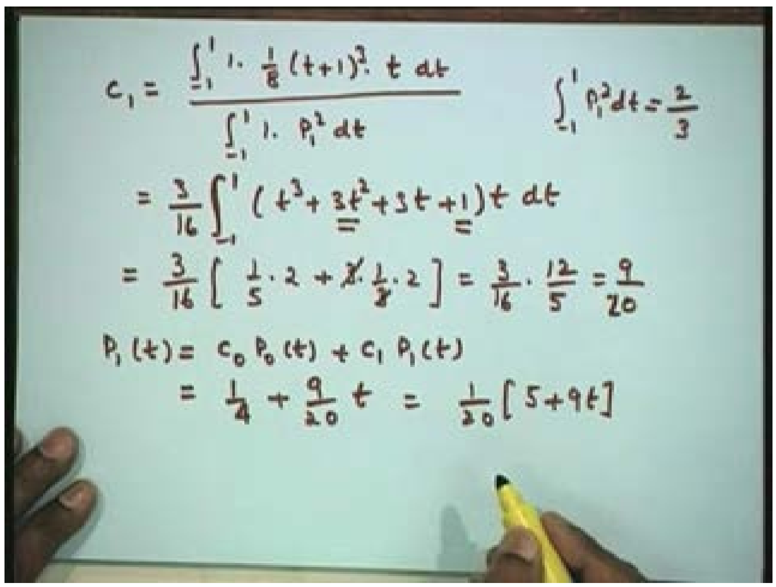

Now let us write down what is c_1, c_1 is integral of minus 1 to 1, 1, 1 by 8 (t plus 1) whole cubed into p_1, p_1 of t is t dt, in the denominator minus 1 to 1, 1 into p_1 square, p_1 square of dt. Now minus 1 to 1 p_1 square dt that is 2 upon 2 n plus 1 therefore 2 upon 2 plus 1 that is equal to 3 therefore the denominator the value is 2 by 3. Therefore this we can write it as 3 by 2 and this is 8 that is 16, 3 by 16 and let us open this up that is your (t cubed plus t 3 square plus 3 t plus 1) into t dt.

Now we are multiplying by t therefore this is odd function t cubed and this is t, this is an odd function therefore the contribution of these two will be 0, so will have contribution only from these two, this is t to the power of 4 therefore 1 by 5 into 2, this is 3 t square that is 3 into 1 by 3 into 2, so I can cancel of this, this is 10 plus 2 12 therefore 3 by 16 into 12 by 5 that is equal to, I can cancel of 4 from here so 9 by 10, 9 by 20 therefore we obtain the value of c_1 also. So let us substitute and get down what is our approximation, this is equal to c_0 p_0 of t plus c_1 p_1 of t, c_0 is 1 by 4 plus c_1 is 9 by 20 into t or simply 1 by 20 into [5 plus 9 t].

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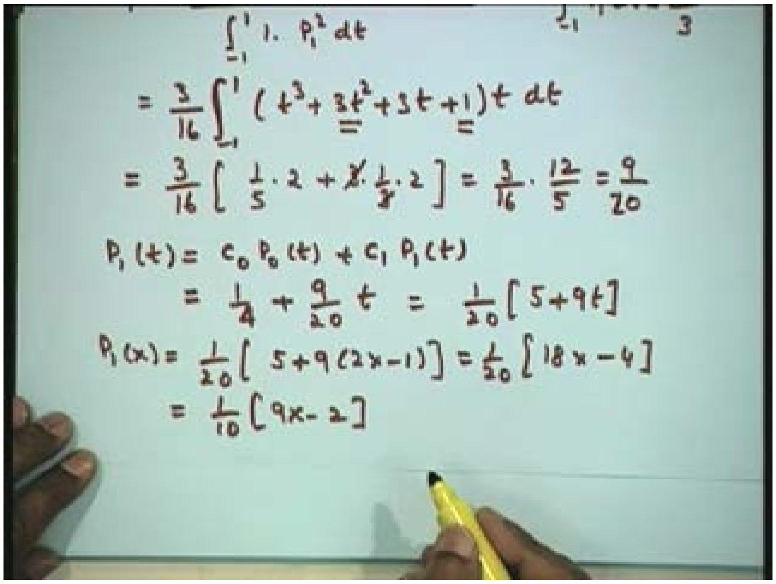

Now I can go back and then find out what is in terms of x also, so I can just substitute what is now t, value of t we have written as (2 x minus 1) is the value of t, so I can substitute the value of that, so [5 plus 9 into (2 x minus 1)] so this is equal to 1 by 20 that is 18 x, this is minus 9 so I will have minus 4, so I will have here 1 by 10 [9 x minus 2]. Therefore the linear approximation in terms of the orthogonal polynomials turned out to be the same as what we have done in the, what we have done in the other case, when we have done it directly using the polynomial approximation of order one, so we have got the same result over here by using the orthogonal polynomials also. Now let us do the quadratic also, so I would now write down the quadratic approximation.

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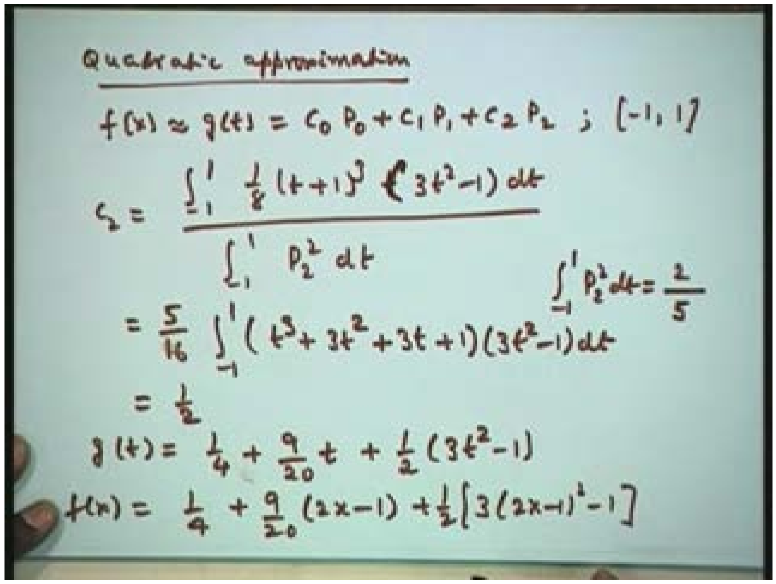

Therefore I will now write down f(x) is approximately that is g(t) that is equal to c_0 p_0 plus c_1 p_1 plus c_2 p_2, they are all in terms of t and t lying between minus 1 to 1. Now as in the case of the polynomial approximation, what we have added is, only c_2 p_2 we have added, therefore in a sense what we are doing is a correction to the linear approximation that we have performed, therefore the values of c_0 and c_1 is going to be the same because when once you use this approximation and then write down this formula, that c_0 c_1 formula, it is going to be the same, therefore we have, we need to find only c_2 to get the quadratic approximation as an extension of the linear approximation, so I need only to find c_2 to get our result.

So I can now write down my c_2 from here as minus 1 to 1, 1 by 8 (t plus 1) whole cubed, now I have the, p_2 is equal to, now we have written here, the expression for p_2 is given in our problem, p_2 is equal to 3 x square minus 1, variable is t now, this is 3 t square minus 1. So I will have to use the value of p_2 as, or I will put a bracket here, (3 t square minus 1) dt divided by minus 1 to 1 p_2 square dt. Now the denominator value, we know the value of this, minus 1 to 1 p_2 square dt is 2 upon 2 n plus 1, n is 2 that is 4 plus 1 that is 5, 2 by 5. Therefore this will be simplify 5 by 16, this is 1 by 8 outside, so I will have minus 1 to 1, I can open it up as (t cubed plus 3 t square plus 3 t plus 1) multiply by (3 t square minus 1) of dt. I will leave this for you to; the value of this comes out to be half. Therefore the approximation g(t) is equal to c_0 that we already obtained as 1 by 4, c_1 is 9 by 20 into t plus half (3 t square minus 1). If I want to convert it back to x, yes I will substitute t is equal to (2 x minus 1) and write down the approximation in terms of x as 1 by 4 plus 9 by 20 into (2 x minus 1), (2 x minus 1) plus half [3 into (2 x minus 1) whole square minus 1].

Therefore we can see the advantage of using the orthogonal polynomials that we are able to obtain the this paramitive values c_0, c_1, c_2, c_i directly and then it is a matter of simple integration to perform and get the values and the advantage that we mentioned was that there is no ill conditionness if we use this particular format, whereas if we use the polynomial approximation and if you go to higher order approximation say 5, 6, 7, 8 like that then there is a chance that these polynomial, these can become ill condition system of equation, in fact the coefficient matrix that we have seen here, if you look at this, it is called Helbert matrix actually, if you see this coefficients 1, 1 by 2, 1 by 3, 1 by 4, 1 by 5, 1 by 2, 1 by 3, 1 by 4, 1 by 5, 1 by 3, 1 by 4, so you go 1 by 4, 1 by 5, 1 by 6 and that matrix is for very high value, this matrix is an ill condition matrix.

Therefore that is a reason why it becomes a ill condition system of equations and we do not use very high order approximations taking as polynomials but in practice, normally in the engineering applications one probably do not go beyond even a cubic approximation, when given a data of points that is given, you may not like to go behind a cubic polynomial to be fitted, in that case you are safe, you can use the polynomial approximation but if you use the orthogonal polynomials, we can do both the Legendre as well as the Chebyshev polynomials. You can still go higher order, higher order in this case and you can have better a better approximation in the sense the minimized error will be smaller and smaller as we increase the order of the approximation. Now the least square approximation was obtained by using the Euclidean norm, now if I use the other norm that is the maximum norm, uniform norm, then I get different type of approximation and that approximation we call it as a uniform approximation,

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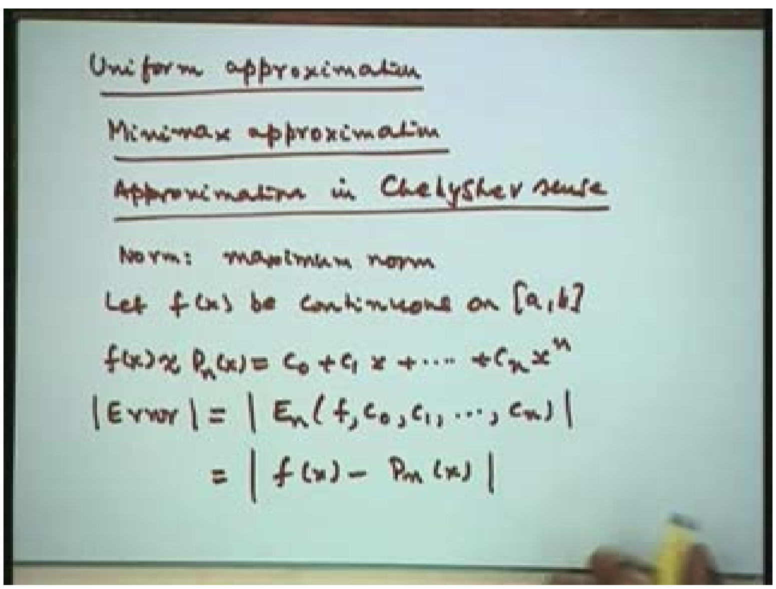

you call this as uniform approximation and this approximation is also called, it has another name, it is called the minimax approximation, it is also called the mini minimax approximation and it is also sometimes called approximation in the Chebyshev sense, why it is so? Will just see it, it is also called approximation in Chebyshev sense. Now here we are using the, norm that we are using is the maximum norm, we are using the maximum norm here, now let us define what is this uniform approximation.

Now we have a continuous function given to us, so let us take let f(x) be a continuous function, be continuous on [a, b] and let us write down our approximation f(x) is equal to $p_n(x)$ is equal to c_0 plus c_1x, c_n x to the power of n. Now let us write down what is the error, the magnitude of the error will be equal to E suffix n to denote the degree of the polynomial approximation and that is equal to f(x) minus p_n of x. Let us review what is the least square approximation; we applied the principle that the aggregate of the sum of squares of errors must be minimum. In uniform approximation we shall require that the maximum of the error, we are talking of all local maxima which may occur at a number of points in the interval [a, b] and the minimum of the error again we are talking of the local minima which may also occur at a number of points in the interval [a, b] must be equal in magnitude. Further this is very important; we shall also require that the maximum and minimum errors which are of equal magnitude must occur alternatingly, we call this approximation also as the minimax polynomial approximation.

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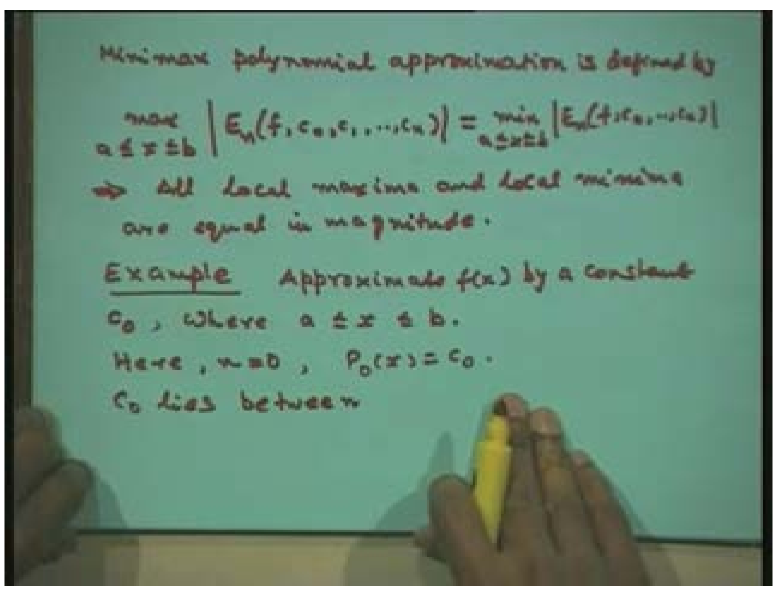

Therefore the minimax polynomial approximation is defined by maximum of magnitude E_n which is a function of f (c_0, c_1, c_2, c_n) x lying in the interval a to b is equal to minimum of E_n f(c_0, c_1, c_2, c_n) again x lying in the interval a and b.

What does this imply, it implies that all local maxima and local minima are equal in magnitude, all local maxima and local minima are equal in magnitude. Now let us first consider a very simple example to illustrate the uniform approximation, so let us call it as an example. Approximate f(x) by a constant c_0, where a is less than or equal to x less than or equal to b, here n is equal to 0, now we are talking of n is equal to 0 therefore we simply have p_n, $p_0(x)$ is equal to c_0, $p_0(x)$ is equal to c_0. Now since we are approximating f(x) by the constant c_0, c_0 should lie between the maximum and minimum values of f(x).

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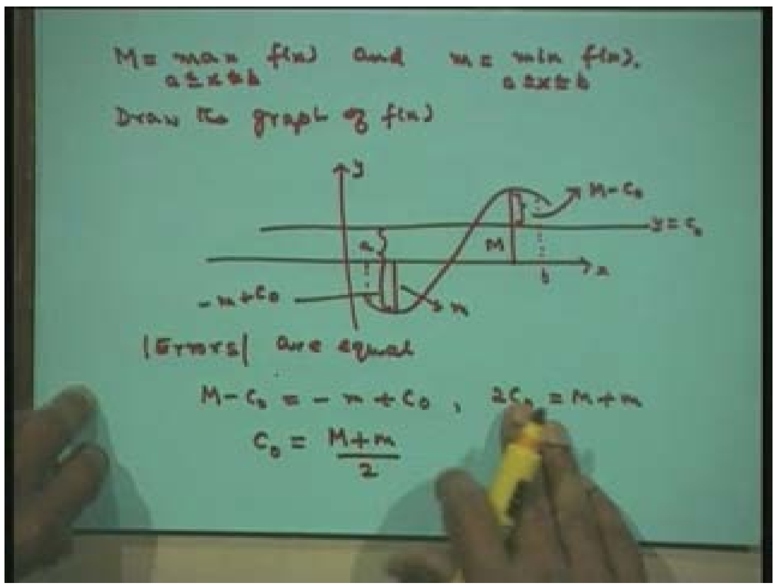

Therefore c_0 lies between, let us call it as capital M, the maximum value of f(x) in the interval [a, b] and the minimum value by small m of f(x). Therefore the value of c_0 that we are talking of should lie between these two values. Now to understand it, let us first draw the graph of the function f(x), the function f(x) that is given to us; let us draw its graph, now let us say draw the graph of f(x), let us take this point as a and let us take this as point b. We are approximating by y is equal to c_0, the maximum is this, this is capital M, the minimum is this, this is our small m and the error is now M minus c_0, notice that the graph is below the x axis therefore m is a negative quantity and therefore the error would be, this whole thing is the error and therefore this would be minus m plus c_0. Now we are requiring that errors in magnitude are equal which would imply that M minus c_0 and minus small m plus c_0 they must be equal. Now both are positive quantities, so I will not write the magnitude, M minus c_0 is equal to minus m plus c_0 or 2 c_0 is equal to M plus m, therefore c_0 is equal to M plus m by 2.

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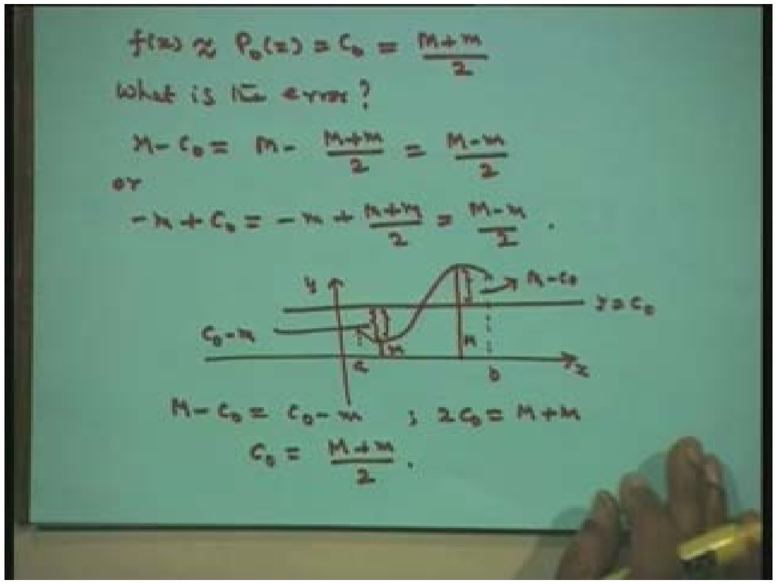

Therefore our uniform approximation is, f(x) is approximately equal to p_0(x) is equal to c_0 is equal to M plus m by 2. Now if this is the approximation to our function f(x) in the interval [a, b] let us try to find out the error, what is the error we have committed here. Now the error would be, this is the error, this is the error, we have made them equal therefore I can find out the error either from this or from this, so let us see what will be in both the cases. So I can now write this as M minus c_0 is M minus M plus m by 2 that is M minus m by 2 or I can also look at this quantity that is minus m plus c_0 that is minus m plus M by 2 that is equal to M minus m by 2. Obviously both these errors have to be equal because that is what the condition that we have put here, errors are equal therefore it is sufficient for us to use anyone of these points to find out what is the error that we have committed in this particular approximation.

Now it is not necessary that the graph of the function should be like this, the graph could be above the x axis also, let us see what would happen in that case; let us draw the graph. This is the point our a, this is the point b, this is our M, this is our small m and this is the approximation that we are talking of y is equal to c_0. Now the error would be M minus c_0 over here and the error will be this quantity, the error will be this quantity and this quantity will be equal to c_0 minus m, this is our c_0 minus m, this will be the error c_0 minus m. Now the magnitude of the errors are equal therefore I will again get M minus c_0 is equal to c_0 by m therefore I will get 2 c_0 is M plus m, thereby giving us the same value for c_0 that is M plus m by 2. Therefore this result would be the same whether we would take the graph cutting the x axis or it does not cut the axis that means this result is independent of the function that we are now choosing. Now let us consider a linear approximation.

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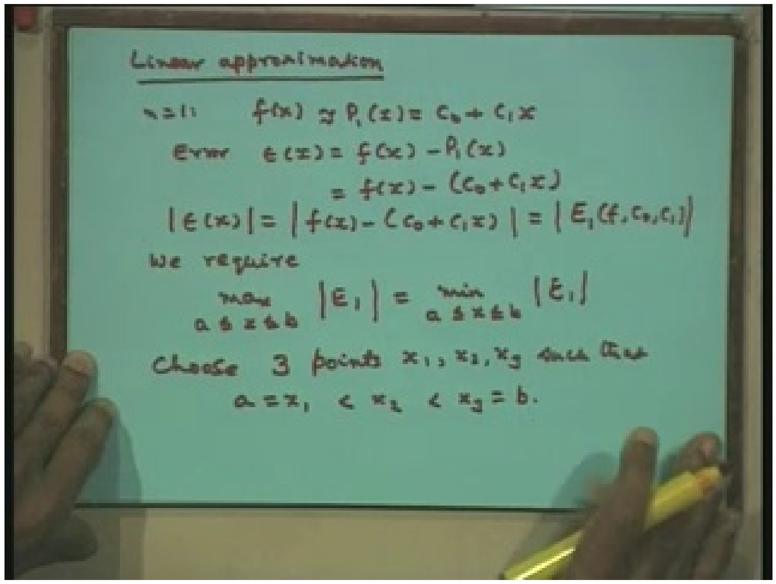

Now in this case n is equal to 1 therefore I want to approximate f(x) by $p_1(x)$ is equal to c_0 plus c_1 x and error is epsilon x is f(x) minus $p_1(x)$ that is equal to f(x) minus (c_0 plus c_1x). Therefore the magnitude of the error is equal to magnitude of f(x) minus (c_0 plus c_1 x) is magnitude of $E_1(f, c_0, c_1)$ this is the function of f and the two parameters that we have here c_0 and c_1. Now what is that we shall require, we shall require that the maximum of E_1 should be equal to minimum of E_1, therefore we shall say we require maximum x lying between a and b of magnitude of E_1 is equal to minimum a less than or equal to x less than or equal to b of E_1. Furthermore you remember that we mentioned that uniform approximation we shall require that the error should also alternate in sign, to do that we would choose 3 points, let us call them x_1, x_2, x_3 such that a is equal to x_1 less than x_2 less than x_3 is equal to b that means out of the 3 points the first point is the starting point of the interval a, the last point is the end point of the interval b and we need to determine x_2 also, x_2 will be an unknown.

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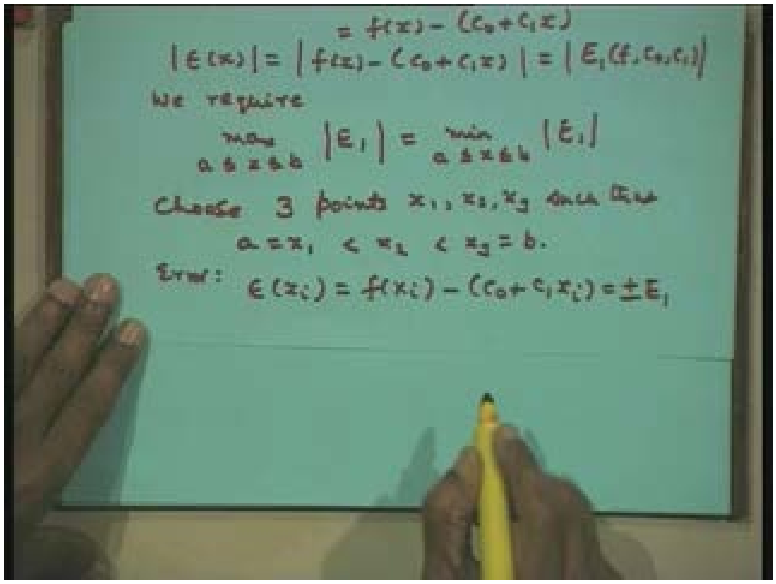

Now error at these points will be epsilon at x_i is equal to f at x_i minus (c_0 plus c_1 of x_i). Now this error will be, either it will be plus E_1 or it will be minus E_1 depending on graph that we have there. Now in order to have a understanding of what we see by plus E_1 or minus E_1, let us draw the graph of the error function that is we are talking the graph of epsilon x is equal to f(x) minus p_1 x, let us draw its graph.

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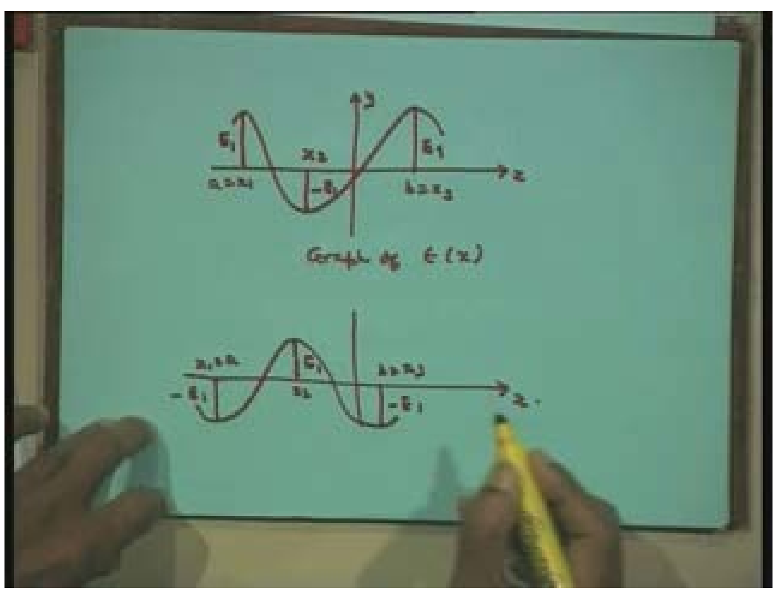

So we are talking of the graph of epsilon x, we take this point as a that is equal to x_1, we take this point as b that is your x_3, now I will choose this point as x_2. Now we are saying that this value is E_1, this value is minus E_1, this value is E_1 therefore the error is equal in magnitude but the errors alternate in sign, this is E_1, minus E_1 and this is plus E_1. However it is not necessary that the graph should be like this, we can also take graph like this, I can have this point as a, this will be my x_2 and this will be my b. In this case it will be minus E_1; this will be E_1; this will be minus E_1, but it has to alternate in sign that is all, for example if the error was say of magnitude point 5 then it could be either point 5, minus point 5, point 5 or it could be minus point 5, plus point 5 and minus point 5.

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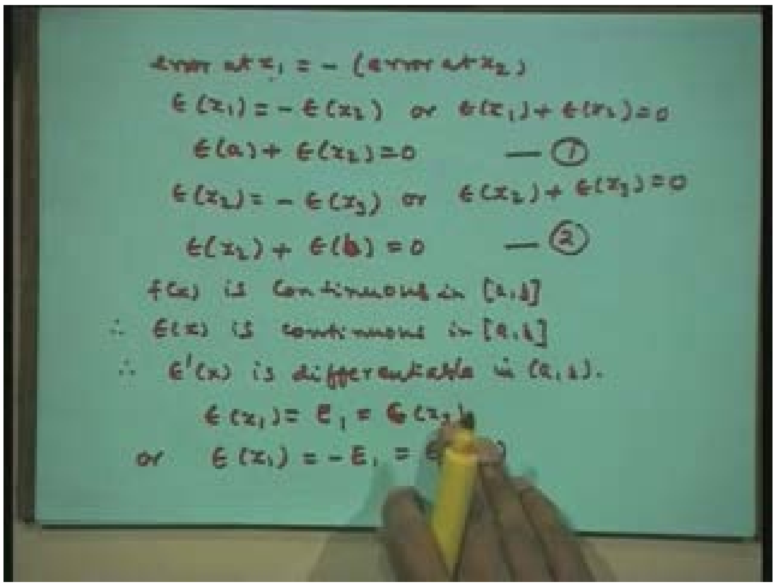

Now with this observation let us now write down the requirement in the mathematical terms, what do you mean by this mathematically, now what we are writing here is that error at x_1 is equal to minus error at x_2. The error at this point and the error of this point, they of opposite signs therefore I will set error at x_1 is equal to minus error at x_2 that means epsilon(x_1) is equal to minus epsilon(x_2) or I can take it to this side also and write epsilon(x_1) plus epsilon(x_2) is equal to 0. Now since x_1 is equal to a, I can also write this as epsilon(a) plus epsilon(x_2) is equal to 0, let us number it as 1. Now if you go to the next points, error at x_2 and error at x_3, they of opposite signs therefore I can again write error at x_2 is minus error at x_3 or error at x_2 plus error at x_3 is equal to 0. x_3 is the point b therefore I can substitute for b, therefore I will have epsilon(x_2) plus epsilon(b) is equal to 0, let us number it as 2.

Now we know that the given function f(x) is continuous, so f(x) is continuous. Therefore our error which we have defined here as f(x) minus p_1(x), p_1(x) is a polynomial therefore p_1 is continuous, therefore epsilon(x) is also continuous, therefore epsilon(x) is continuous. Furthermore we have taken that f is differentiable also, therefore epsilon also is differentiable therefore we have epsilon(x) is differentiable. We are talking of our interval [a, b] so I can include that, furthermore we also have that epsilon at x_1 is equal to E_1 is equal to epsilon at x_3. Now what we are talking of is this particular graph epsilon at x_1 is E_1, epsilon at x_3 is E_1 or if you having a graph like this, they are of negative sign, therefore or we will have epsilon at x_1 is equal to minus E_1 is equal to epsilon at x_3. Therefore the errors at these 2 points are the same therefore I have a function epsilon(x) which is continuous in the closed interval [a, b], differentiable in the open interval [a, b] and epsilon at x_1 is equal to epsilon at x_3 therefore epsilon(x) satisfies the requirements of the Rolle's theorem.

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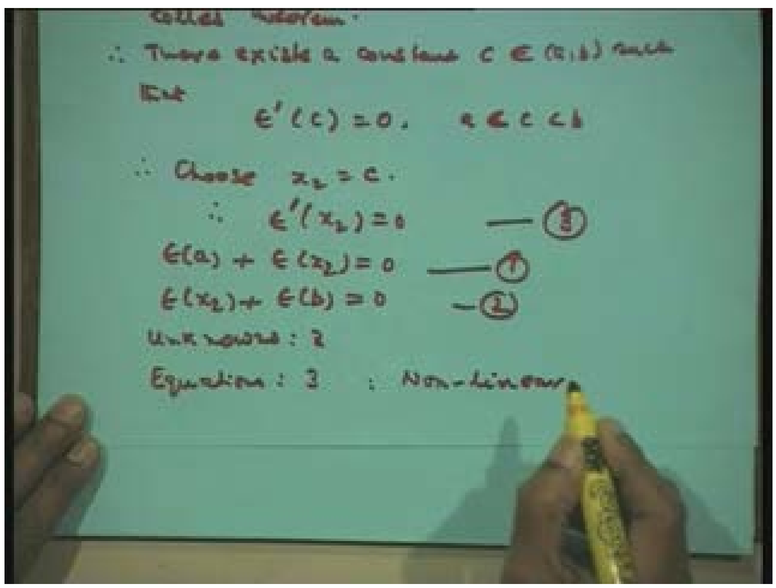

Therefore epsilon(x) satisfies the requirements of Rolle's theorem. Therefore there exists a constant c contained in the interval (a, b) such that epsilon dash(c) is equal to 0. Now it is, at this particular stage we will bring in this point x_2 that we are talking of, now since Rolle's theorem states that there exists a point c such that epsilon dash(c) is equal to 0, we shall choose this point c as our x_2 therefore we shall say choose x_2 is equal to c, which would imply epsilon dash at x_2 is equal to 0, I will take this as my third equation. Now we have earlier 2 equations and this is our third equation and we would like to solve these equations. Now we have the equation 1 as epsilon(a) plus epsilon(x_2) is 0, let us write them all at the same place, epsilon(a) plus epsilon(x_2) is equal to 0 then epsilon(x_2) plus epsilon(b) is equal to 0 and the third equation is here.

Now let us see what are the numbers of unknowns that we have in our problem, the unknowns that we have in our problem are, we have to determine c_0 and c_1 and we have also determine x_2 therefore we have got three unknowns in our problem and what is the number of equations that we have here, we have 3 equations. Therefore we have equations, 3 equations; therefore I can solve the system of 3 equations in 3 unknowns c_0, c_1, and x_2. However these 3 equations are nonlinear equation, these are non linear equations, however if I solve them in a systematic way, will say what is that systematic way, we will just see it later, then it is very easy for us to solve even this nonlinear equations.

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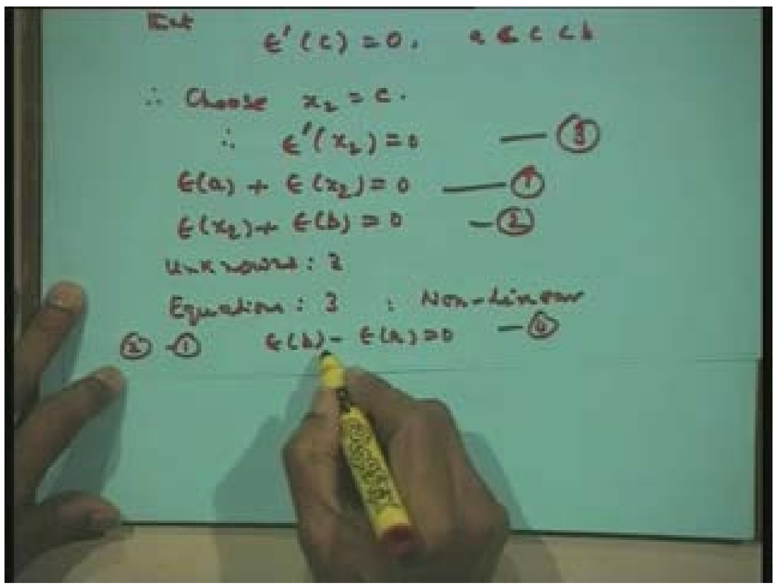

For example, one equation that would really help us is, that if I subtract 2 and 1 that is 2 minus 1 then I get here epsilon(b) minus epsilon(a) is equal to 0, so I get epsilon(b) minus epsilon(a) is equal to 0 and this does not contain x_2 therefore the work will be reduced by using such an equation. Now how we shall use it, we will see later on, in our next lecture will do a problem on it and see how we are going to apply this particular thing, will close it for today.