Numerical Methods and Computation

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Lecture No - 28
Interpolation and Approximation (Continued)

In the previous lecture we have derived the Newton's divided difference interpolating polynomial, lets us just write down what is the formula that we have derived there.

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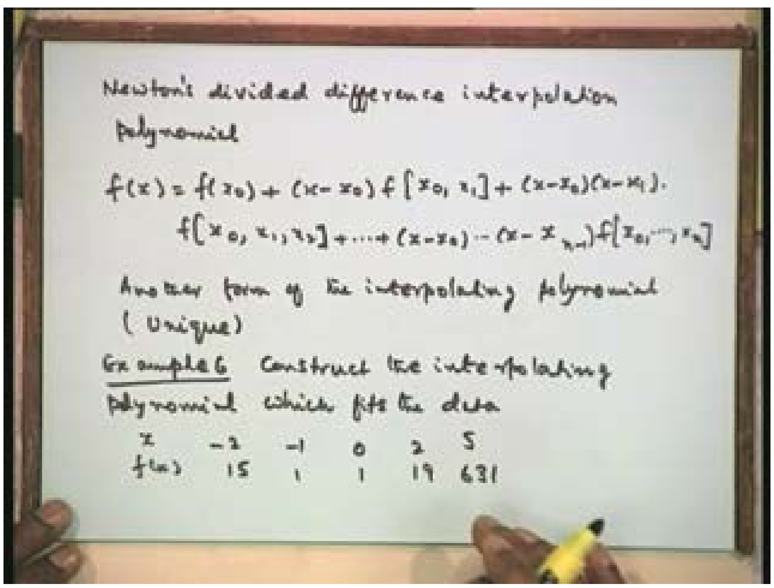

we called a formula as the Newton's divided difference interpolating polynomial or interpolation polynomial, we have written it as the polynomial f(x) is equal to f(x_0) plus (x minus x_0) into the first forward difference f [x_0, x_1] plus (x minus x_0) (x minus x_1) into the second forward difference f [x_0, x_1, x_2] plus so on, we have the products (x minus x_0) upto (x minus x_{n-1}) and a divided difference f[x_0, x_1, x_2, x_n]. To apply this interpolating polynomial what we need to construct is the divided difference table and from the divided difference table we can pick out the values of first divided difference, the second divided difference and so on the last divided difference, substitute it and that gives you, then we can simplify it and get the interpolating polynomial. If we want only the interpolation at a point, we need not simplify it, we can straight away substitute the value of x is equal to some given value zhi and we can just simplify this particular numbers to produce the value at the particular point. Now we have said that the interpolating polynomial is unique, therefore if you simplify this it will be same as the Lagrange interpolating polynomial because we cannot get two polynomials; it is only another form of the

Lagrange interpolating polynomial. So this is another form of the interpolating polynomial because that is a unique interpolating a polynomial, therefore the two forms that we have, whereas we found that the Lagrange interpolating is much more difficult because we need to find all the fundamental polynomials $l_i(x)$ which are polynomials of degree n. Let us first take an example on this to illustrate how we can apply this formula, now construct an interpolating polynomial, construct the interpolating polynomial, construct interpolating polynomial which fits the data x f(x) is equal to minus 2, 15, minus 1, 1, 0, 1, 2, 19, 5, 631. This example we have considered it in our last lecture wherein we, when we defined the Newton's divided differences, we constructed the divided difference table for this particular data.

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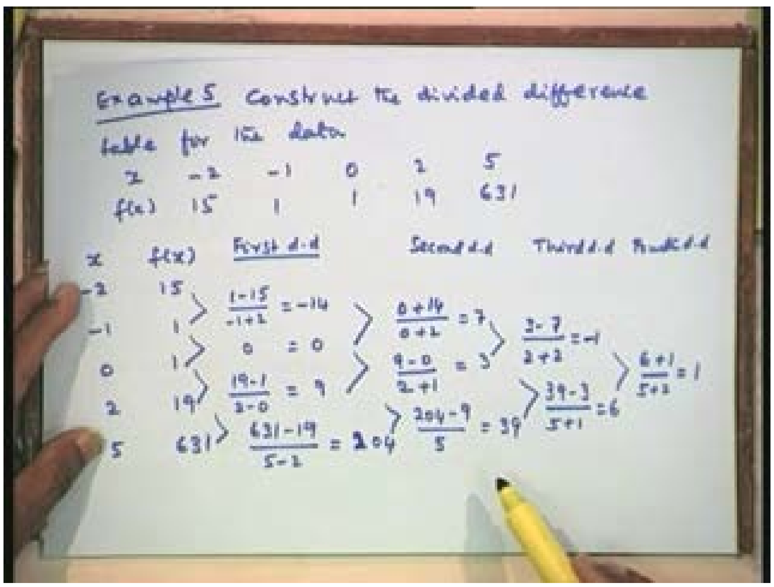

Now we would like to use that, we can just, I will just show what we have done last time, this was the example which we had constructed example five, wherein we have constructed the divided difference table, let us just copy these entries here, so that we can apply and get our interpolating polynomial. So I need to first of all construct the divided difference table for this data, so let us take this data.

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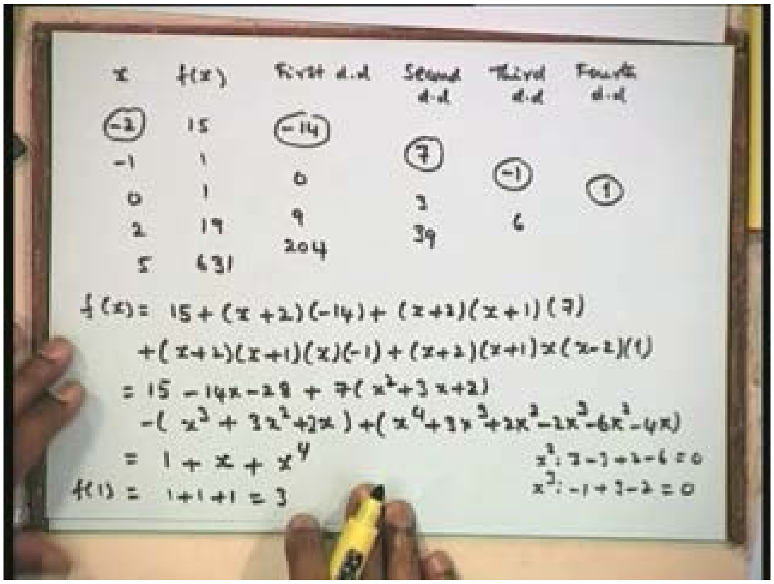

So that is x f(x) then we have first divided difference, then we have second divided difference, the third divided difference and the fourth divided difference. So the data points are minus 2, 15, minus 1, 1, 0, 1, 2, 19, 5, 631. I would give the entries from our the last lecture values, so I would just write down what are the values which we have obtained over here minus 14, 0, 9, 204. Then the second divided difference was 7, 3, 39. Third divided difference was minus 1 and 6 and fourth divided difference was 1.

Now I would write down our interpolating polynomial f(x) is equal to, here we note that this is our x_0, this is our first divided difference with respect to x_0, this is the second divided difference, third divided difference and fourth divided difference, so these are the four divided difference we shall be using with respect to x_0. This will be equal to f at x_0 that is 15 plus (x minus x_0) that is (x plus 2) into the first divided difference that is minus 14, that is minus 14, then we have (x minus x_0) into (x minus x_1) that is (x plus 1) into the second divided difference 7 plus (x plus 2) into (x plus 1) into (x minus 0) so I simply have x and the third divided difference is 1 and we have (x plus 2) (x plus 1) into x into (x minus 2) into 1 that is the fourth divided difference. Now I can simplify this result, let us write this, simplify this, minus 15, minus 14x, minus 28 and this is 7 into (x square plus 3 x plus 1) then we have a this product, I just have to multiply this by x that is (x cubed plus 3 x square plus x) and we have to multiply this by (x minus 2) only and the result is x to the power of 4 plus 3 x cubed, I am just writing it without simplifying it, so that we can use this values straight away. I just multiplied this by (x minus 2), so there is, you have here x to the power 4 minus plus 3x 4 plus plus 2 x square and so on, so multiplied by this, by this. Now we can just collect the coefficients, this is 15, this is minus 28, this plus 7, there is no constant term here.

x square plus 3 x plus 2, isn’t it, yes, this is equal to 2x here, correct that is right, x cubed plus 3 x cubed plus 3 x square plus minus 2 x cube, this is okay. So this is 15 minus 28 plus 14 that is 29 by 28 that is 1. This is minus 14 x plus 21 x and there is minus 2 x here and there is minus 4 x also, so I would get here plus x. Then x square is 7 here, minus 3 here and you have here plus 2 and minus 4, right, this is, I am writing the coefficient of x square that is equal to 7 here and this is minus 3 here, this is plus 2 here, minus 6 here, so this becomes 0, so the coefficient of x square is 0. So let us take x cubed coefficient this is minus 1 here, plus 3 here, minus 2 here, so this is equal to 0. So x 4 is only 1, so I have here this. So the simplification of this gives us the polynomial as (1 plus x plus x to the power of 4) as the required interpolated polynomial which fits the given data. Now if I want to interpolate at a particular value, let us say some point 1, let us say f of 1, if I want to find f of 1 either we are substituting it over here and getting it as equal to 3 or alternatively if we do not want to use this interpolating polynomial and if you just been asked to find what is the value at x is equal to 1, I could straight away substitute in this expression x is equal to and just multiply the numbers, here we are simplifying otherwise you could simply multiply the numbers. Now let us take another example for this.

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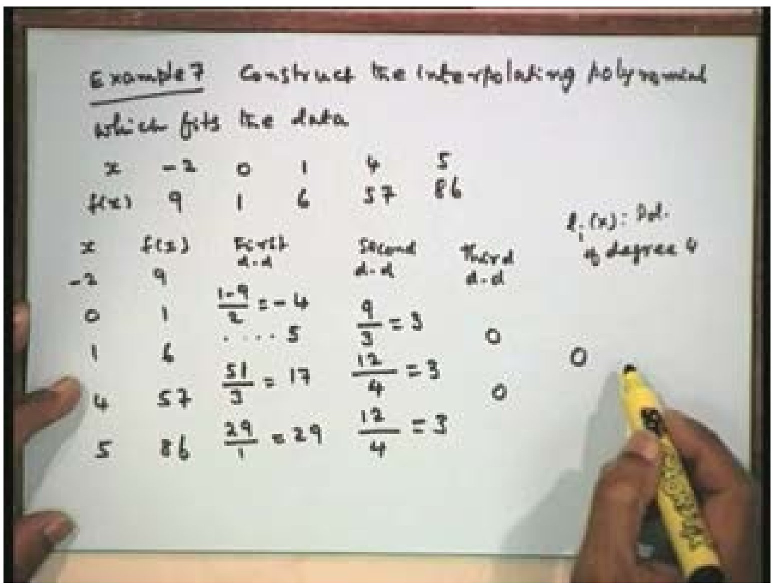

So I would write to construct an interpolating polynomial, construct the interpolating polynomial which fits the data x f(x) minus 2, 9, 0, 1, 1, 6, 4, 57, 5, 86. Now we note that there are 5 points here, therefore if I am using again Lagrange interpolating polynomial, all the $l_i(x)$ will be, that is your l_0 l_1 l_2 l_3 l_4, they will be polynomials of degree 4; these are polynomials of degree 4. We need to simplify the entire thing but we shall now again use the divided difference and let us write down the divided differences, x f(x) minus 2, 9, 0, 1, 1, 6, 4, 57, 5, 86. So will have this as first divided difference that is (1 minus 9) divided by (0 minus 2) that is plus 2, that is equal to minus 4. Then (6 minus 1) by 1 that is simply equal to 5, so let us put the value as 5. (57 minus 6) that is 51 divided by (4 minus 1) that is 3 that is equal to 17. (86 minus 57) that is 29 divided by (5 minus 4) that is 1 that is equal to 29. Now let us write down the second divided difference, this is (5 plus 4) that is 9 divided by (1 plus 2) that is equal to 3 that gives us 3. Then corresponding to this, I have (17 minus 5) that is 12 divided by (4 minus 0) that is 4, that is equal to 3. Then I have (29 minus 17) that is equal to 12 divide by (5 minus 4) that is equal to (5 minus

1) that is 4, that is equal to 3. Now the second divided difference have come out to be all constant and the same, therefore the third and higher order differences, divided differences will be 0, 0 and the fourth divided difference will also be 0. Now this is what I was mentioning last time, if the data given to us is not belonging to the polynomial of very high degree, we would have all the differences of after a certain order will be zeros, so that it will actually give us a polynomial lower degree, now we can see that the second divided difference only is non-zero, third divided difference is zero, therefore the second divided difference the coefficient is (x minus x_0) into (x minus x_1), therefore this is representing only a quadratic polynomial, all the higher the terms are 0 here. So let us now construct the interpolating polynomial for this.

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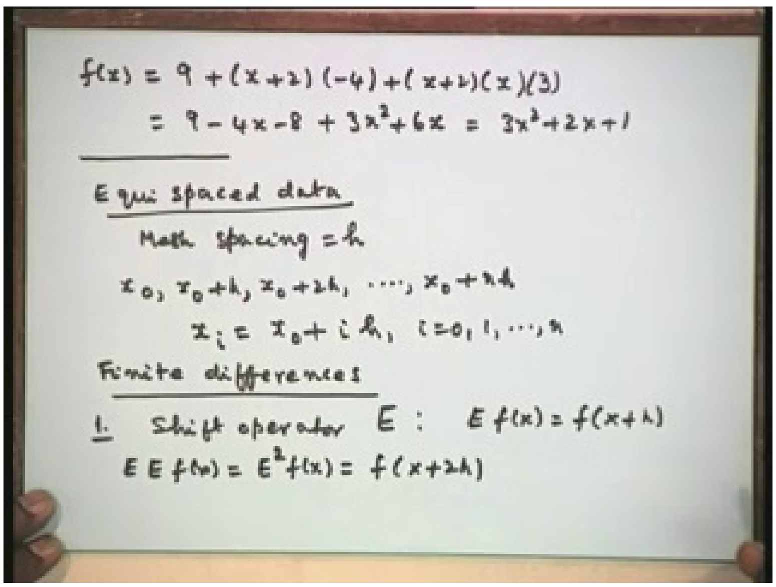

Therefore the polynomial will be given by f(x) is equal to f at x_0 that is 9 (x minus x_0) that is equal to (x plus 2) into the first divided difference, so I would now take these values as the values, plus (x plus 2) into (x minus 0) that is x into 3, that is second divided difference and all third and fourth divided difference are 0, therefore it does not give us any particular more terms from here. Therefore this is simply (9 minus 4x minus 8 plus 3 x squared plus 6 x) that is equal to (3 x squared plus 2 x plus 1). Therefore if the given data is a representing a lower degree polynomial, that would automatically be reflected in the divided difference table that we are constructing, so that we would immediately by looking at the table itself will shall be able to say, what is the degree of the polynomial that is representing this particular data.

So far we have discussed about the case when the data is not equispaced, it is arbitrary data, if you look at this example also, the difference between these two values is 2, this is 1, this is 3, this is equal to 1; therefore the spacing between the abscissas was arbitrary. Now if the spacing between the data points is equal, that is equispaced data, then we can still further simplify the Newton divided difference formula and get a formula which is much more convenient than even the Newton divided difference. As it is Newton divided difference can also be used for the equispaced data, however as I said we can construct still simpler formula then the Newton divided difference.

Let us see how we get that one and further we shall define some operators and use those operators to derive the formulas. So let us take the case of equispaced data, so let us take the mesh spacing as some h, we shall take mesh spacing as h that means we are talking of the points as x_0, x_0 plus h, x_0 plus 2 h and so on x_0 plus n times h. So we are talking of the points as x_0 x_1 x_2 x so on that means we are talking of the points as x_i is x_0 plus i h, i is equal to 0, 1 so on n. Now in order to construct the interpolating polynomials, I would like to define few operators; these are called finite difference operators, so we call them as finite differences. Now these operators are very useful for us in solving the ordinary and partial differential equations wherein we shall use numerical methods and the numerical methods are based on the finite differences or the finite element methods, so mostly we can use, we shall be using the finite differences. So let us define 3 operators, the first operator we shall define, is called the shift operator and we denote this by E. What is the effect of this operator is, if I operate it on any function, it will shift it to the value at the next point, it is the shift operator. So if I apply E on f(x), it will shift to the value at the next argument that is x plus h, so it will go to the, shift it to the value at the next point, that is why by definition shift operator, so it will shift the value of the function at the next nodal point. I can repeat this shift operator once more, E of E of f(x), which I would call this as E square of f(x), that will be f of (x plus 2 h), that is E of f of (x plus 2 h) again it will shift to the next value and so on.

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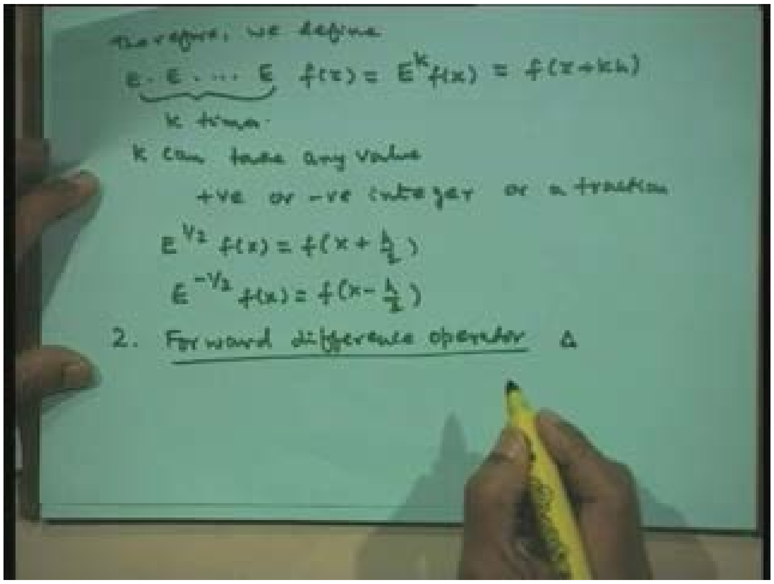

Therefore, we define E into E into E k times operated on f(x) is E to the power of k f(x) is equal to f of (x plus k h), here k can take any value, it may be a positive integer, it can be a negative integer or it can be even a fraction, therefore it can be positive or negative integer or a fraction. Now for example, I can define E to the power of half of f(x) is f of (x plus h by2), similarly I can define E to the power of minus half of f(x) is f of (x minus h by 2) therefore, any fraction can be taken and we shall define it as f of (x plus k h). Now we have defined the shift operator, which is one of the most important operator that are useful in constructing many difference formulas in the interpolation approximation in numerical integration and the other, and numerical differentiation also. Let us now define one more operator which is extremely useful for us and that is known as the forward difference operator, so let us call this as forward difference operator. We shall give the notation of this delta for the forward difference operator.

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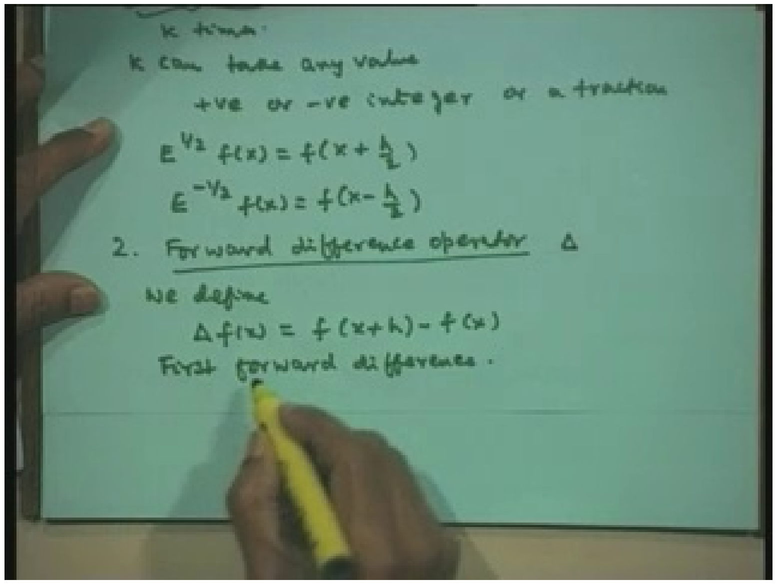

We define delta operated on f(x) is f of (x plus h) minus f of f(x). We shall call this as the first forward difference, first forward difference. Now you can see why this name has been given, this is the difference in the forward direction, the difference between two successive points that is the value at next point minus value at the present point and that we are calling it as the first forward difference. Similarly we can define higher order forward differences that is the second forward difference and third forward difference and so on. Let us see what is the expression for this second forward difference.

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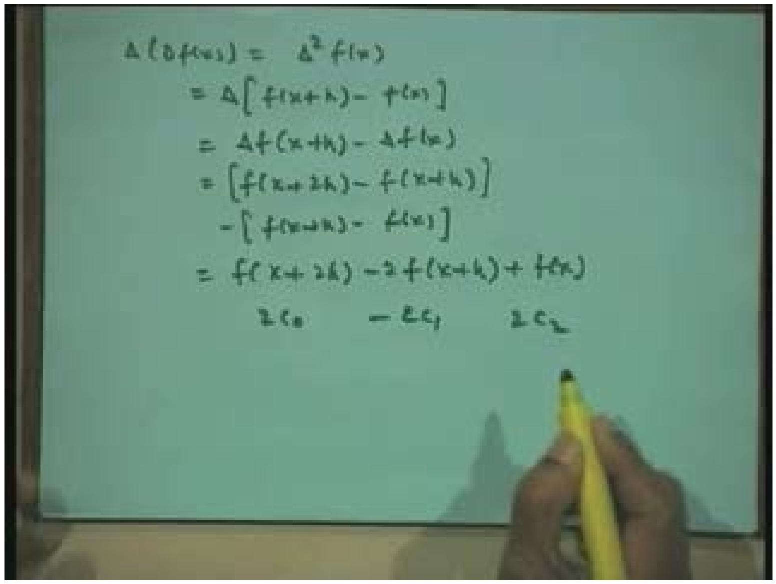

I write delta of delta f(x) is delta squared of f(x) that will be delta of f of (x plus h) minus f(x), I open it up delta f (x plus h) minus delta of f(x) but this expression is f of (x plus 2 h) minus f of (x plus h) and the second expression is f of (x plus h) minus f of x, which simplifies to f of (x plus 2h) minus twice f of (x plus h) plus f of x. This is the expression for delta squared f(x). The coefficients are $2c_0$ minus $2c_1$, $2c_2$.

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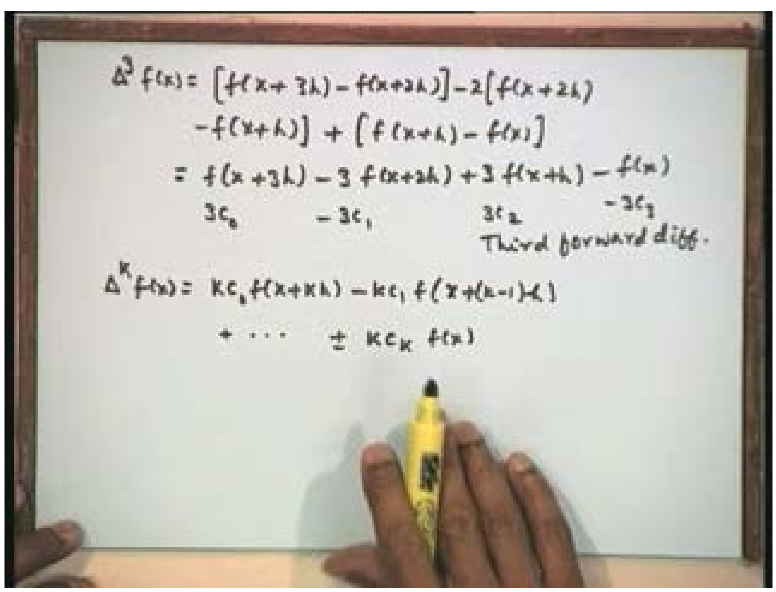

Now if you proceed further and write down by next difference using this particular values, I can write down delta cubed of f(x) the third forward difference, now I can, let us write straightaway from here, operate on this so I will have f of (x plus 3 h) minus f of (x plus 2 h) that corresponds to this term, minus 2 times corresponding to this I will have f of (x plus 2 h) minus f of (x plus h) that corresponds to this and corresponding to this we have f of (x plus h) minus f of f(x). Now let us simplify, this is f of (x plus 3h), this is minus 1 and minus 2, this is minus 3 f of (x plus 2 h), this is plus 2 plus 1 that is plus 3 f of (x plus h) and this is minus f(x) and this is our third forward difference, this is our third forward difference.

Now you can see the coefficients, this is 3 c_0 plus, minus sign change in sign 3 c_1, 3 c_2, minus 3 c_3. Therefore, these are all the binomial coefficient that come over here, a as the coefficients of this one. Therefore if I want any forward difference say, delta k f(x) I want, I need to construct start with k c_0 f(x plus k times h), k c_0 f(x plus k h) minus k c_1 f of (x plus (k minus 1) of h) and so on and finally will have here plus or minus depending on the value of k, even or odd this will be plus minus k c_k of f(x). So if it is even, you can see they will get a positive sign, this is plus minus plus minus, so will have a positive sign when it is even, when it is odd we will end up with a negative sign. Therefore any forward {difference} ((Refer Slide Time: 00:26:09 min)) can be straightaway written by using this binomial coefficients over here.

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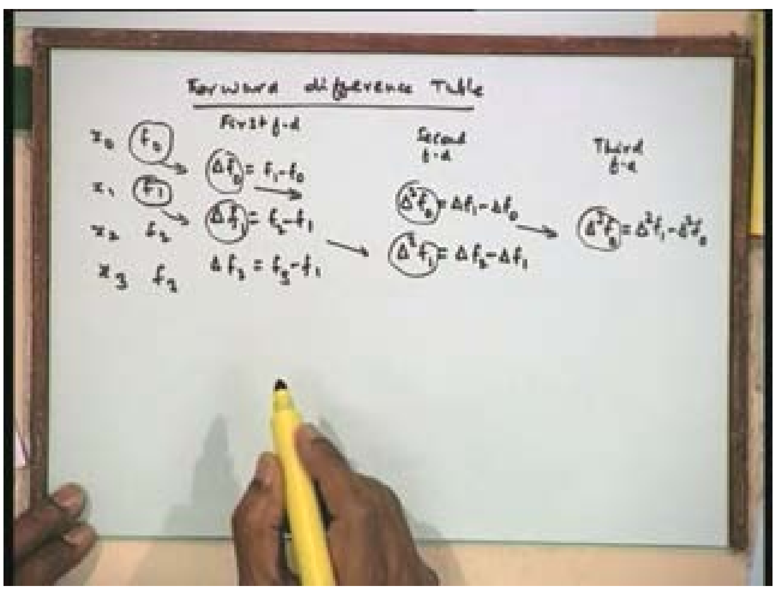

Let us make it forward difference table x_0 f_0, x_1 f_1, x_2 f_2, x_3 f_3, let me take only 4 of them and then illustrate it. So you will have here first forward difference, this is delta f_0 that is equal to f_1 minus f_0. Now, later on we will fill something; leave some space over here, when we do the next operator we would like to fill in those gaps. Now I use these two points to get delta f_1 that is f_2 minus f_1. Delta f_2 is equal to f_3 minus f_1. Now I will write down here my second forward difference, so I would define here delta squared f_0 that is delta f_1 minus delta f_0, delta f_1 minus delta f_0 this is second difference. This is delta squared of f_1 that is delta f_2 minus delta f_1, so this is delta squared f_1 is this minus this. Then I have third forward difference here and that is delta cubed f_0 is equal to delta squared f_1 minus delta squared f_0. Now let us see how the forward differences are positioned in a difference table that is delta f_0 delta squared f_0 etc. These values, they are all going on any forward direction, take any particular value, if you look at for example f_1, delta f_1, delta squared f_1, they would all be slanting downwards and they will be having the forward differences in this direction, slanting direction. Okay we will come back to this, just write down the table. Now I would like to define another operator called the backward difference operator.

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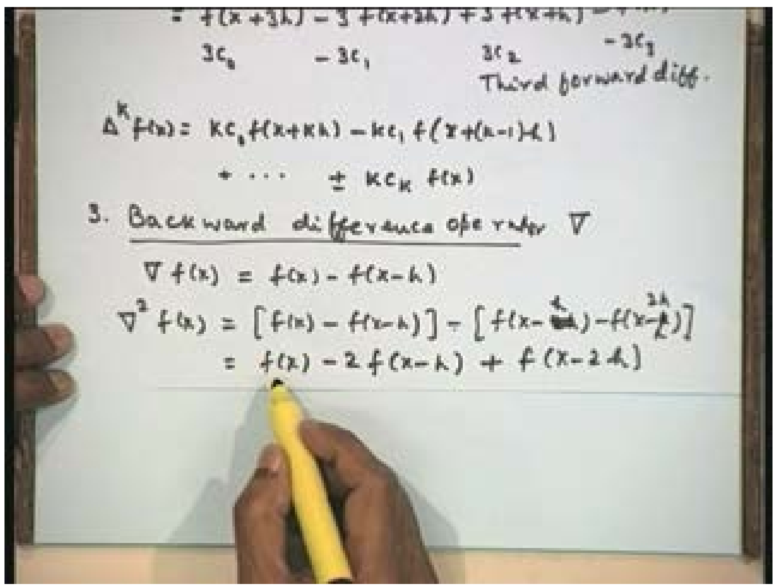

Backward difference operator, for this I shall use the notation inverted delta; I shall use the notation of invert delta. Definition is, we take the difference in the backward direction, the difference in the backward direction. Therefore this will be f(x) minus f of (x minus h), so the current value ordinate minus the ordinate at the previous nodal point, f(x) minus of f of (x minus h). So its a difference in the backward direction and therefore I can now operate it once more, delta squared f(x) is equal to, if I operate on the first one, I would get [f of x minus f of (x minus h)] minus, I operate now on this [f of (x minus 2h) minus f of (x minus h)]. Therefore f of x, this is minus f(x), this I have made a mistake here, this is f of (x minus h), the current ordinate minus the previous ordinate, minus of (x minus h) minus f of (x minus h), minus twice f of (x minus h) plus f of (x minus 2h). Again you can see that these coefficients are again binomial coefficients, 2 c_0, minus 2 c_1, plus 2 c_2. Therefore they would again follow the binomial coefficients only and therefore let us write down the next coefficient.

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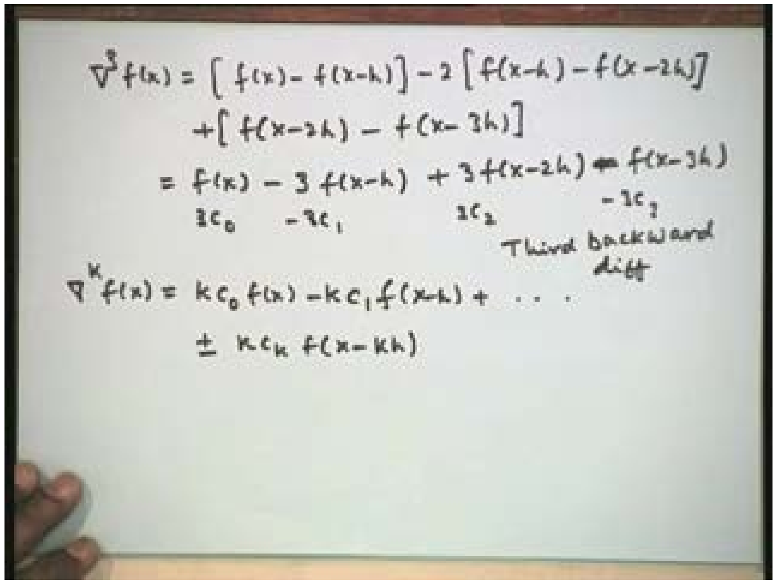

Delta cubed of f(x) and then we can generalize it, let us apply on this, this is, with respect to this factor will have [f of x f of (x minus h)] minus 2 times [f of (x minus h) minus f of (x minus 2h)] that is corresponding to this factor. Corresponding to the third factor will have [f of (x minus 2h) minus f of (x minus 3h)], which I can simplify and write this f(x) minus 3 times f of (x minus h) plus 2 plus 1 plus f of (x minus 2h) minus f of (x minus 3h). So this we shall call it as the third backward difference, this is the third backward difference. Again this is 3 c_0, this is minus 3 c_1, 3 c_2, minus 3 c_3, which is the same as the binomial coefficient which we had in the forward differences. Therefore, except in that we are going in the backward direction the coefficients are going to the same thing. Therefore if I am writing a k^{th} backward difference, I will start with k c_0 f of (x) minus k c_1 f of (x minus h) plus so on and lastly I have here plus or minus k c_k f(x minus k h). The last one will be f of (x minus k h) and I will write it as plus minus of k c h.

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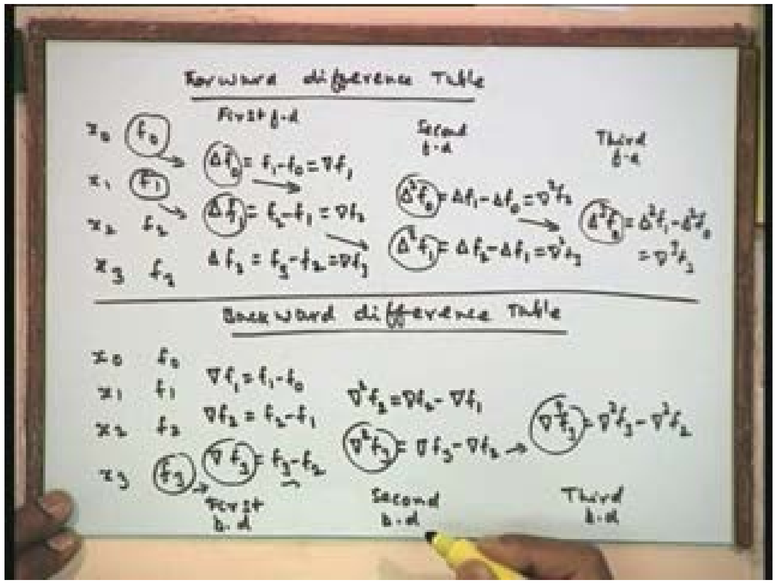

Now let us go back to this table and let us fill here, the backward difference table and then I will entry there, let us write down the backward differences. So let us take the same 4 values x_0, x_1, x_2, x_3 so we have f_0, f_1, f_2, f_3. Now our backward difference, write it here, first backward difference that is equal to f_1 minus f_0, delta f_1 is f_1 minus f_0. Then this will be delta f_2 is f_2 minus f_1, this is f_2 minus f_1 backward difference. This is delta f_3, f_3 minus f_2.

Now the second backward difference will be, this will be delta squared f_2 is delta f_2 minus delta f_1, this minus this is delta squared f_2 and this is delta squared f_3 that is equal, delta sorry this is, we start with, tikh hai, that is all right, that is equal to delta f_3 minus delta f_2 and finally the third backward difference is delta cubed f_3 is delta squared f_3 minus delta squared f_2. Now why I had written on the same page to illustrate is, that given constructed a table; I can recognize any element of a column as a possible forward difference or the same as a backward difference also. Now look at this entry from these two tables, f_1 minus f_0, this is f_1 minus f_0, therefore the first forward difference can be identified as a backward difference delta f_1. If I want to construct a forward difference polynomial or a backward difference polynomial, I can take an element of this here and then recognize as the suitable forward difference or a backward difference also. This is same as backward difference delta f_1 and you can see that this is same as backward difference of delta f_2. Whereas this was forward difference of f_1 and this is also same as backward difference of f_3, that is f_3 minus f_2, this f_3 minus f_2, so these are both are same.

Similarly now when I proceed further, now this is nothing but the backward difference with respect to f_2, this is the backward difference f_3 and this is the backward difference with f_3. So that means I can, if I take a value f_3 here, delta f_3, delta squared f_3, delta cubed f_3 and we are going in the upward direction, in the ascending way we are going, so all the backward differences will ascend and all the forward differences will descend, so that is how one can recognize in this table, this one. Any element in this table, you can call it either as a forward difference or a suitable backward difference. Therefore we need not construct separate tables for the forward difference and backward differences; we can identify it from the definition of the forward and the backward differences. Now even though we have defined the forward backward and shift operators, all of them are related, we can derive one from the other by definition; Let us just look at what these relationships are.

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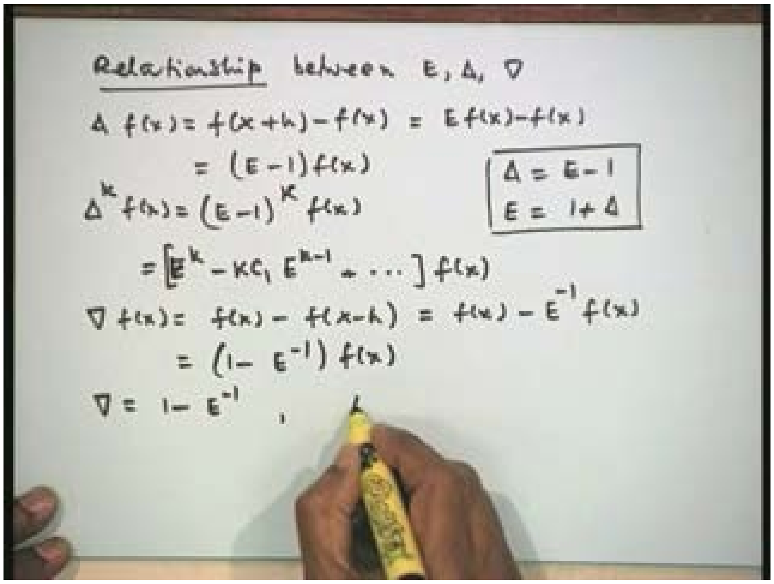

Relationship between E forward difference and backward difference, now let us write down what is our definition of delta f(x) that is f of (x plus h) minus f of x. Therefore this is equal to E of f(x) minus f(x), by definition E of f of (x plus h) is E of f(x) minus f of x, that is equal to (E minus 1) of f(x). Therefore the operator delta is equivalent to operator (E minus 1) or alternatively E is equal to (1 plus delta), I can bring minus 1 to this side and I can have this as the relationship between this. Now you would see that getting the higher order difference is very simple for us here, if I want delta k here f(x), it will be operating (E minus 1), operate on (E minus 1) so on k times, so this will be (E minus 1) to the power of k f(x), so (E minus 1) to the power minus 1, to the power of k. Now you see; if you write the binomial expansion symbolically, this is nothing but you are going to get all your binomial coefficients, so this will be simply [E to the power of k minus k c_1 E to the power of k minus 1 and so on] operate on f(x). So whatever we have derived earlier, those binomial coefficients should automatically turn up and I can view the forward differences that we had written there by using this particular we also. Now let us write down what is our backward difference; that is f of(x) minus f of (x minus h) that is f(x) minus E to the power of minus 1 of f(x). So we said k can be positive or negative or a fraction, so we take k as minus 1, so E to the power of minus 1 of f(x) will be f of (x minus h), therefore this is 1 minus E to the power of, will take E to the power of minus 1 is inverse operator, E to the power of minus 1 f of (x). Therefore our delta is 1 minus E to the power of minus 1. Now I can bring it to this side, E minus 1 is 1 minus delta, I bring it to this side and this and invert it again E is equal to 1 minus delta to the power of minus 1. So if I want delta in terms of E, I will use this, if I want E in terms of backward delta, I will use this particular expression. Now all the backward differences, as I mentioned in the case of the forward differences, I can get from here also.

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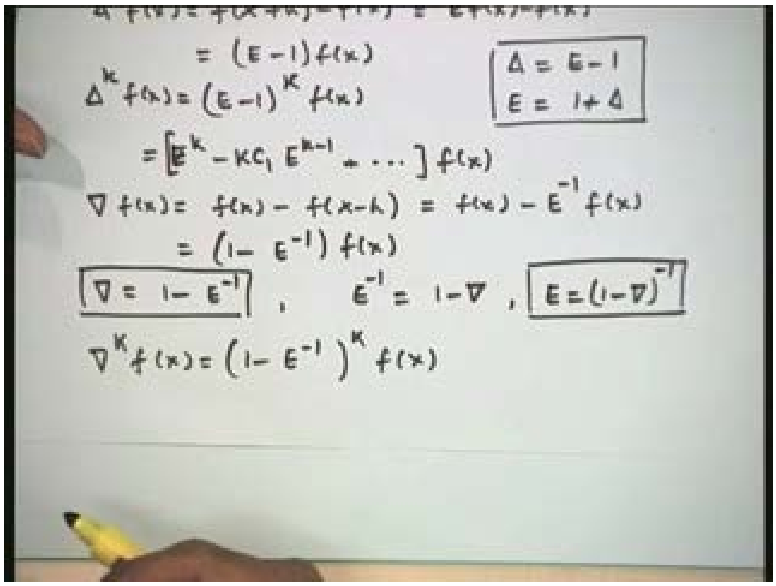

This will be simply delta to the power of k of f(x) will be simply (1 minus E to the power of minus 1) to the power of k of f(x). So I can now open it up in a symbolic way by using the binomial expansion and I can operate it on f(x) and get all the forward differences, therefore again, they are again binomial coefficients as we can see here. These operators are all inter linked, therefore that was the reason why we were able to recognize in the difference table a particular element, as either a forward difference of a particular value or a backward difference with respect to another backward value.

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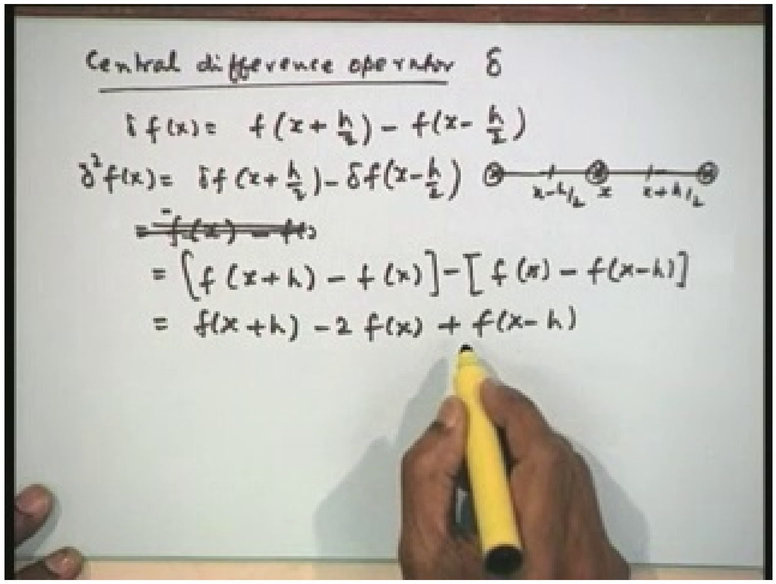

Now one more important operator that we would like to have is called the central difference operator, the notation used for this is, this delta. Now it is a central difference, therefore it is a symmetric difference. So if I operate on f(x) what it does is, it takes the value at the half the next point minus f(x) minus half. This is symmetrically placed about the x point that we have taken, this is our x and this is your x plus h by 2 and this is your x minus h by 2, so it is symmetrically placed. However the interesting thing to note is, that these (x plus h by 2) (x minus h by 2), they are not the nodal points because we have got x_0, x_1, x_2, x_3, x_4 but (x_0 plus h by 2) is not a nodal point. Therefore this difference is not using the nodal point but if I now go the next difference, this will be delta of f(x plus h by 2) minus delta of f(x minus h by 2).

So if I now apply on this, it will take the next value as f(x) minus f of, okay I will write one more step here, this is delta of f(x), now operate on this, this will become f of x plus h by 2 plus h by 2 that is (x plus h), f of x plus h by 2 minus h by 2 that is f of (x), minus x minus h by 2 plus h by 2 that is f(x), minus f of x minus h by 2 minus h by 2 so that is f(x minus h). So this is f of (x plus h) minus twice of f(x) plus f of (x minus h). All these three are nodal points and now they are symmetrically placed; now we are using these points. Now there is a big advantage which we shall see a moment later, as the central difference operators would be much more accurate than the other differences, the forward and backward differences. We will define what we mean by better approximations, because they are symmetrically place, the error in this case is smaller than in the forward and backward differences. Now I can therefore construct a central difference table also for this.

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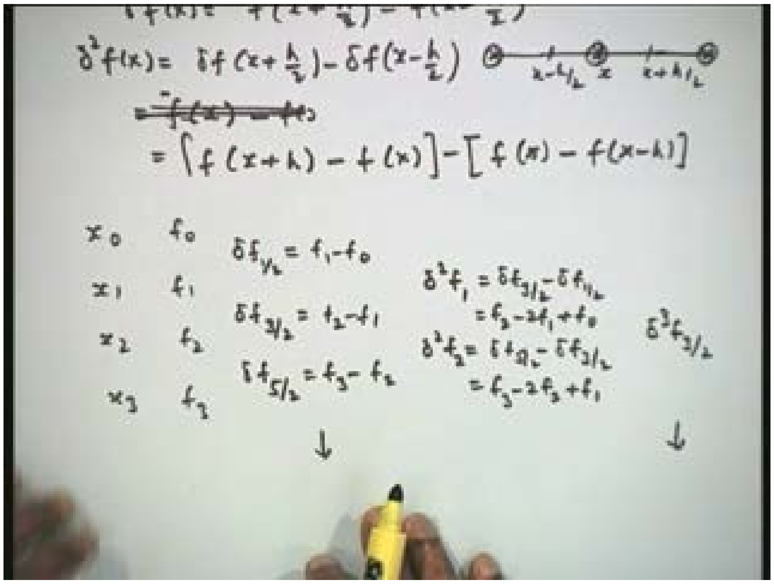

Let us just write down few points x_0, x_1, x_2, x_3, f_0, f_1, f_2, f_3. What I would write here is, I will take delta $f_{1/2}$ that is equal to, by definition f_1 minus f_0. Now if I write delta $f_{1/2}$ that is (x_0 plus h by 2) then it will be f (x_0 plus h) minus f (x_0), so this will be f_1 minus f_0. Then this will be delta $f_{3/2}$ that will be, that is 3 h by 2 plus h by 2 that is f_2, 3 by 3 h by 2 minus h by 2 that is f_1, that is f_2 minus f_1 and this is delta $f_{5/2}$ that is f_3 minus f_2, but these two combination will give us delta square of f_1 because that is from this definition that we have, that is delta square f(x) will be f(x plus h) minus 2 f(x) plus f(x minus h). Therefore delta square f_1 will be f_2 minus 2 f_1 plus f_0, that is equal to delta $f_{3/2}$ minus delta $f_{1/2}$ and which we can write this as f_2 minus 2 f_1 plus f_0. Now this is also, I can write down delta squared f_2, this minus this delta $f_{5/2}$ delta $f_{3/2}$ which would be equal to f_3 minus 2 f_2 plus f_1. Then using these two I would get delta $f_{3/2}$ that is 1 plus h by 2, that is x_1 plus h by 2 is x this. These odd differences would not be using the nodal points because this is all half distances f the, here the way in which we had written ((Refer Slide Time: 47:45)) f of, we are using this differences, all of them but we have written it as delta $f_{1/2}$, delta $f_{3/2}$ and then, therefore now we are constructed the table such that all of them belong to the data points only, but these are taken as the central difference of delta half.

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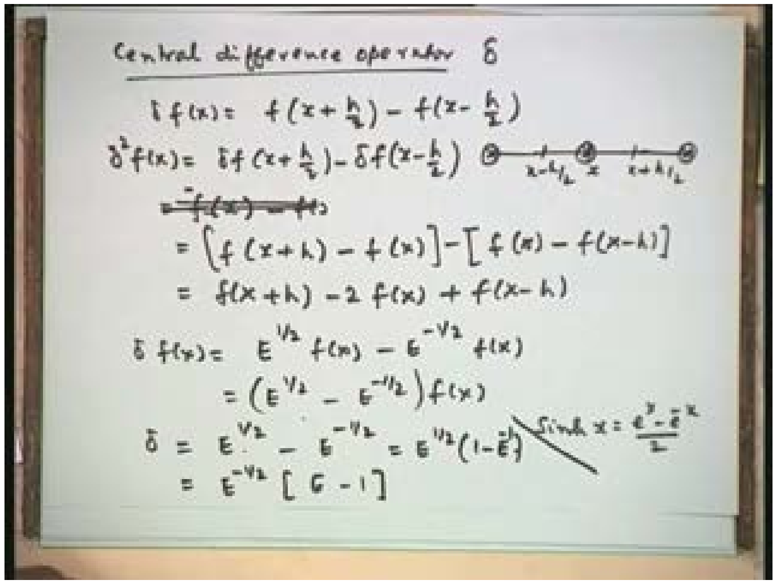

Now we will just close in a minute, let us see what is the relation between delta and E; so if I are write this delta f(x), I can write this as E to the power of half f(x) minus E to the power of minus half of f(x), (x plus h by 2) (x minus h by 2) so this is to the power of minus half and this, that is (E to the power of half minus E to the power of minus half) of f(x). Now we want to put this in a notation, if you remember sine hyperbolic of x is E to the power of x minus E to the power of minus x by 2, right. This expression shall be used when we get the equivalence of the delta operator with the differential operator d is equal to d by dx. Therefore our equivalence here is E to the power of half minus E to the power of minus half, which if you want to take E to the power of half, I mean so that you can recognize this operation that we have written it here, we can write this also as E to the power of half into 1 minus E, I take E to the power of half common, so this will be 1, sorry, E to the power of minus 1 or I can take E to the power of minus half outside and write this as E minus 1. Multiply it out, this is E to the power of plus half minus E to the power of minus half and this is E to the power of half E to the power of, so whichever way you want to look at it, I can write this operator in terms of this form and then use that particular thing. Okay we will stop with this today.