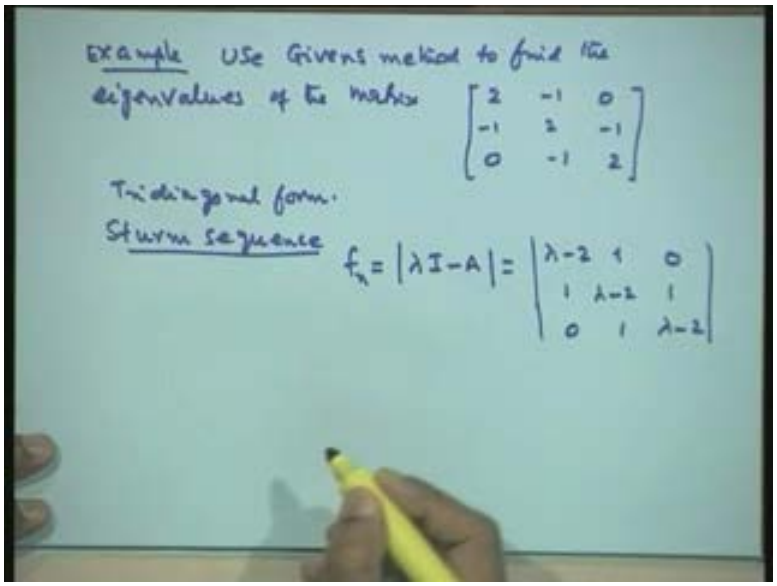
Numerical Methods and Computation
Prof. S.R.K. Iyengar
Department of Mathematics
Indian Institute of Technology Delhi

Lecture No - 23
Solution of a System of Linear Algebraic Equations (Continued)
Eigen Value Problems

In our previous lecture, we have derived the Givens method for finding the Eigen values and Eigen vectors of a given matrix, essentially the method reduces the given matrix into tri diagonal system, then we find the Sturm sequence whose last member is the characteristic equation that is f_n, is the last, is the last member of this sequence and that is the characteristic equation. Then we use the Sturm's theorem to locate the Eigen values in required length of interval, say 1 or less than 1, then when once you locate the Eigen values, we refine this Eigen values by using the bisection and lastly we can find the Eigen vectors if it is necessary. Last time we will take an example we, in which we reduce the given matrix to a tri diagonal form. Let us take now an example which we, from which we can find out the Eigen values using the Sturm's theorem.

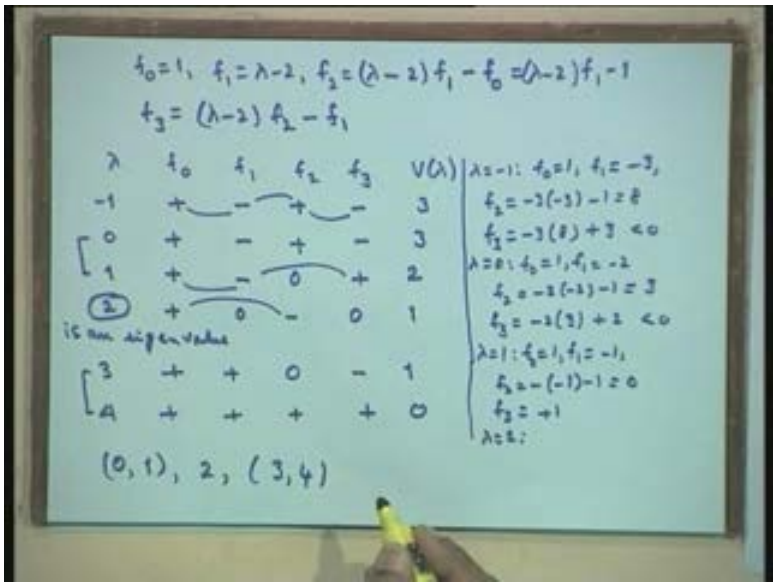
(Refer Slide Time: 05:59)



So let me take the example like this, use Givens method to find the Eigen values of the matrix 2, minus 1, 0, minus 1, 2, minus 1, 0, minus 1, 2. We consider this example earlier also for, we use this coefficient matrix in solving the system of equations in the SOR procedure. This problem is already in the tri diagonal form, so we do not have to reduce it to tri diagonal form, so what we are given is the problem is the tri diagonal form. Now i need to form this Sturm sequence, so let us write down what will be our Sturm sequence, now for this i need to write down our matrix f_n,

that is your characteristic equation lambda I minus A, that is equal to lambda minus 2, plus 1, 0, plus 1, lambda minus 2, plus 1, 0, plus 1, lambda minus 2. Now we want to form this sequence from here using this and let me use the next page.

(Refer Slide Time: 03:42)



So starting with f_0 is 1, 1 into 1 minor is lambda minus 2, therefore f_1 is minus lambda 2, now we want to use the formula f_r is lambda minus b_r f r minus 1 minus c r minus 1 square f r minus 2 or if you have called it as c_r then we will take it as c_r, we will take it as c_r. Now f_2 therefore is lambda minus b_2 that is lambda minus 2 into f_1, into f_1, this is the value is 1, therefore minus 1 square into f_0, f_0 is 1 therefore this is lambda minus 2 f_1 minus 1. Then f_3 is, we are expanding now about this element lambda minus 2 into f_2, minus square of this that is minus 1 f_1, now we will not simplify it further, we would try to compute this quantities and write down the changes in sign in the Sturm sequence, therefore what we have here is, let us put lambda sign f_0, f_1, f_2, f_3, and number of changes in sign we will call it as v of lambda.

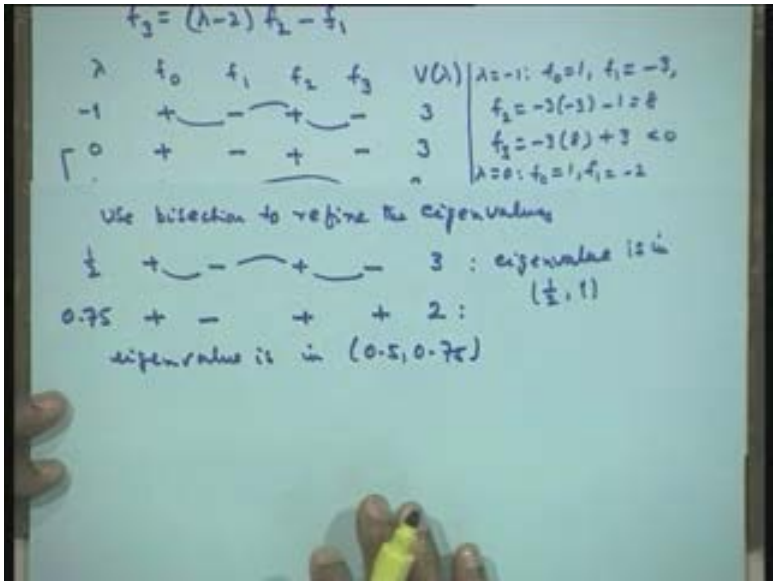
Now let us start at some number, let us start with lambda is minus 1, so when lambda is minus 1 f_0 is positive, f_0 is 1, f_1 is minus 1 minus 2 that gives us minus 3. Then i have f_2 is equal to lambda minus 2, that is minus 3 into minus 3 that is f_1 minus 3 and minus 1 that is equal to 8 and lastly f_3 obtained from this is minus 3 into f_2 that is 8 minus f_1 that is plus 3 and this is negative. So we have the signs of this as positive, f_1 is negative, f_2 is positive and f_3 is negative, so i want change your sign here, another change of sign here, another change of sign here, so i have three changes of sign at minus 1.

Now let us take 0, lambda is equal to 0; we have f_0 is 1 again, f_1 is when lambda is 0 it is minus 2, f_2 is equal to when lambda is 0 minus 2 times f_1 minus f_0 that is minus 1 this is plus 4 minus 1 that is equal to 3, F_3 is lambda minus 2, that is minus 2 again into 3 minus f_1 so this is still

negative, so i have again f_0 is positive, f_1 is negative, f_2 is positive, f_3 is negative, so there is no Eigen value between minus 1 and 0.

Now let us take lambda is equal to 1, when lambda is equal to 1, i have f_1 is, i am putting lambda is my 1, therefore its minus 1, f_2 is equal to lambda minus 2, that is minus 1 into f_1 that is minus 1 and minus 1 that is equal to 0 and f_3 is, now f_2 is 0, so simply minus of f_1 that is equal to plus 1, therefore i have at 1, this is positive, f_1 is negative, f_2 is 0, so we will continue the sign of this, so i put 0 over here and i have a plus sign here for f_3, so i have one change of sign here and another change of sign here, therefore this is 2, therefore there is a Eigen value between 0 and 1, there is a Eigen value between 0 and 1. Now i repeat it, let me take lambda is equal to 2; then i can straight away write it lambda is equal to 2, this is positive, this is a 0 and lambda is equal to 2 this is a 0, therefore this is a minus sign and lambda is equal to minus 2 is a 0, therefore minus of f_1 is 0, now f_3 is equal to 0, f_3 is 0, is our characteristic equation itself, therefore if i produce a 0 that means that is an Eigen value. Therefore this 2 is, even though there is a one change of sign it is showing that is 1 change of sign, but since f_3 is equal to 0, this is an Eigen value, this is an is, this is an Eigen value, 2 is an Eigen value. Then i give the, i will have the signs, i will give this signs 0, minus ,there is 1, so there is no nothing here, at 4 i have got, all become positive here and i have a 0 here, all of them are positive and therefore there is an Eigen value between 3 and 4. Therefore the, we have the Eigen values located in 0 and 1, 2 is an Eigen value, 3 and 4, these are the three Eigen values for this particular problem.

(Refer Slide Time: 10:00)

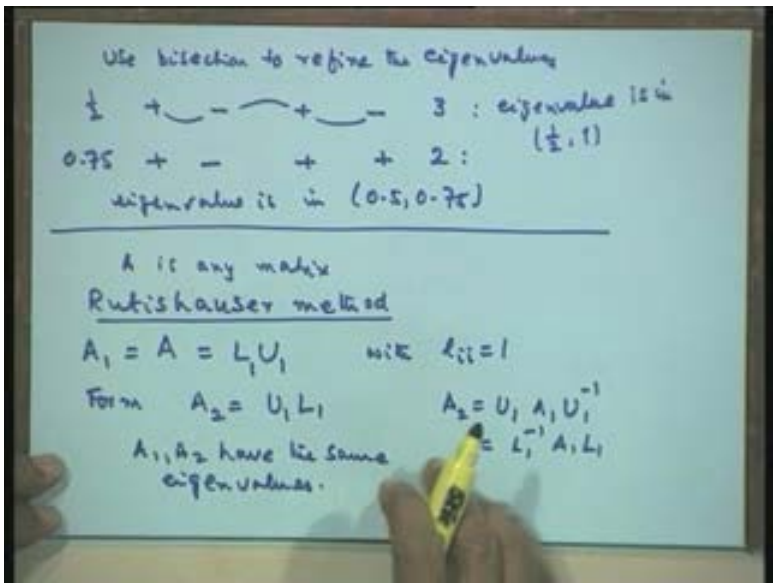


Now let us try to refine the Eigen value in 0 and 1, so we said, we know use bisection, now use bisection to refine, to refine the Eigen values. So between 0 and 1, i will take the value as half and then compute our signs of this and i will give the signs that we have here, plus minus plus minus, so i have here 1, 2, 3, so there are three changes of sign, then i can go back and then just

look at this particular 0, there are three changes of sign, there are three changes of sign, therefore there is no Eigen value between 0 and half but the Eigen value is between half and 1, therefore from here we can say the Eigen value is in half and 1.

Now i bisect it further, so i can take point 7 5 as the next quantity to be checked, so i would again go about and then i find that these are the signs for this, now there is a, between half and point 7 5, this is 3 changes sign, this is 2 changes of sign, therefore the Eigen values lies between point 5 and point 7 5, therefore Eigen value is in point 5 point 7 5. Now this is how we can go on bisecting it and get it the required accuracy of the number of decimal places whether this is 3 4 5 decimal places, it just a matter of computation and finding the signs of this sequence and then find taking the Eigen value. Now as i mentioned the Givens method is one of the popular methods and its variations that is available in the software. There are further modifications for this, there is what is known as house holder method, which is in a sense more powerful in some situations in which the rotations are not in the plane, so far we are using the rotation in the plane but the house holder method uses the rotations in the three dimensional plane that is, it is, it also takes a reflection. So it will be rotation in the plane and then rotation in the three dimensional planes, so it takes what is known as reflections also but however we shall not take that method. House holder method is also one of the most powerful methods and its variants, variants, many variants are there which are available on the software but Givens method as well as the Jacobi method is also available and quite popular in the software.

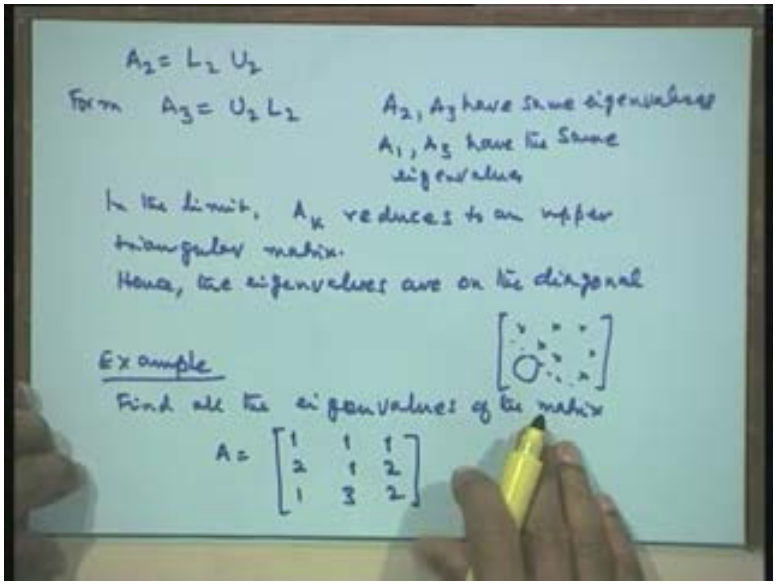
(Refer Slide Time: 13:03)



Now we will go to, how to find the Eigen values of a arbitrary matrix not a symmetric matrix, so let us now take that A is any matrix, now this is a very difficult problem, the one of the earliest methods is known as the Rutishauser method, it is known as the Rutishauser method, this is one of the original methods, few methods that were started for the arbitrary matrices and then variants and this modifications and refinements of Rutishauser method has been made so that the method become very powerful but the idea behind Rutishauser method is very simple and very

straight forward and that is, we take the matrix A we decompose it into L into U we decompose into just products of your matrices L and U, so let us call this as L_1 and U_1 and in order to put it in a program we can, let us write down A_1, we write it, denote it as A_1 like this one, with of course our l_{ii} is equal to 1, then what he does, we form, i will call this A_2 as U_1 L_1, that means i interchange the order of this matrix here, the, we have now decomposed it as a L_1 U_1, change the order in this decomposition and write it as U_1 L_1. What is this A_2, let us write U_2 is U_1, L_1 by definition if i pre multiply by U_1 inverse i would get my L_1, therefore what we have here is A_1, i am removing U_1 inverse so i will have here U_1 inverse and writing for L_1, so i am post multiplying by U_1 inverse so i will have A_1 U_1 inverse or we could as well write this as alternatively we could written this as U_1 is equal to L_1 inverse A_1 L_1, i can look at it in both the ways, therefore A_1 and A_2 are similar matrices because this is similarity transformation, therefore A_1 and A_2 have the same Eigen values, so even if i write, interchange this one, from this i have got A_1 A_2 have the same Eigen values, have the same Eigen values. Now what i do, i multiply this U_1 and L_1, multiply this U_1 L_1 and again decompose it.

(Refer Slide Time: 15:52)

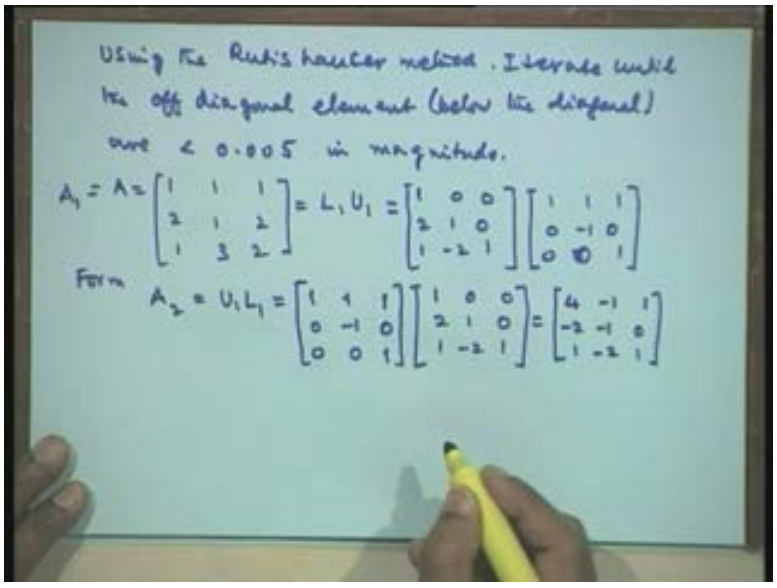


So i would again write this A_2 as L_2 U_2, so decompose again this one, now again form A_3 is U_2 L_2, now by definition A_3 and A_2 have the same Eigen values, A_2 A_3 have same Eigen values that means, A_1 A_3 have the same Eigen values, have the same Eigen values, now we proceed on in the limit, in the limit, this A_k reduces to an upper triangular matrix, reduces to an upper triangular matrix, hence the Eigen values are on the diagonal, hence the Eigen values are on the diagonal, therefore finally it reduces the matrix into this particular form, these are all zeros and we have these values, therefore the Eigen values are now located on the diagonal itself, now immediately you can see that the method would fail if you have complex Eigen values because, complex Eigen value cannot occur on the diagonal because, they are complex pairs, so in that

particular case the method will fail, therefore it will work when you have real Eigen values and the, all the Eigen values are located on the diagonal of the matrix.

Now this is the simple and straight forward method, a very trivial method but this is the precursor for the later on methods which have been developed based on this ideas and the, the further modifications or what is known as cure algorithm and other algorithm, wherein it is reduced to, not to this form but to one more diagonal below called the hessenberg form, so we use this matrix in to upper hessenberg form, so that the complex Eigen values can also be located, if there is, see you make ask, i mean before making that comment you may ask what would happen if i use Rutishauser method, but it has a complex at pairs, then what happens is, these we are now force, we are now saying that in the limit its goes to the upper triangular matrix, now there will be some elements here which will never go to 0, that means even though you go on doing this decomposition the elements on this one diagonal below will never go to 0 and that is a an indicator, that yes what you have, the Eigen values of the matrix have some complex Eigen values, therefore it is not going to 0, so it is appearance of these numbers here would tell that there is complex Eigen values, to take care of this one, it is reduced to not upper triangular matrix but in upper hessenberg form where one more diagonal will be available here but the eliminations are used in the Givens rotations other rotations are used, reduced upper hessenberg forms and then find the required Eigen values, those are the refinements that is done for this problem but we will not take up those detail, other methods but let us take an example for the Rutishauser method. So let us take this as an example, let us say, find all Eigen values of the matrix, let us take the matrix A as 1, 1, 1, 2, 1, 2, 1, 3, 2 [Student: sir this method will work for the symmetric matrix as well] but we have powerful methods for symmetric matrices, yes it would be, but we have more powerful methods for symmetric matrices.

(Refer Slide Time: 20:14)

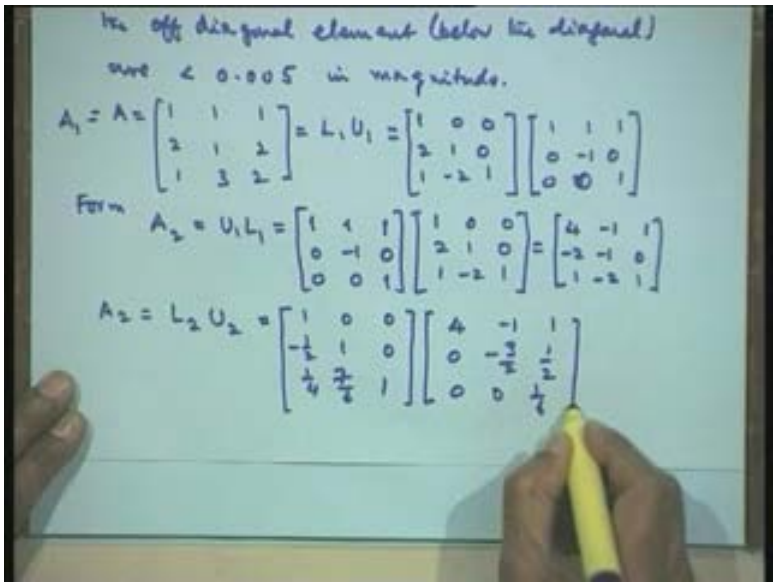


I using Rutishauser method, let us also put the criteria for a stopping iterate until the off diagonal elements below the diagonal off course, below the diagonal or less than some, let us put some

tolerance 0 0 point 5 in magnitude. Now it is necessary for us to give this particular criteria because we are saying in the limit this is going to reduce to a to a upper triangular form, therefore these three elements are going to go to 0 and since we are on computer we have to say what is our tolerance that we are giving for this Eigen values, depending on that number of iterations that will take or number of decompositions will depend on that one.

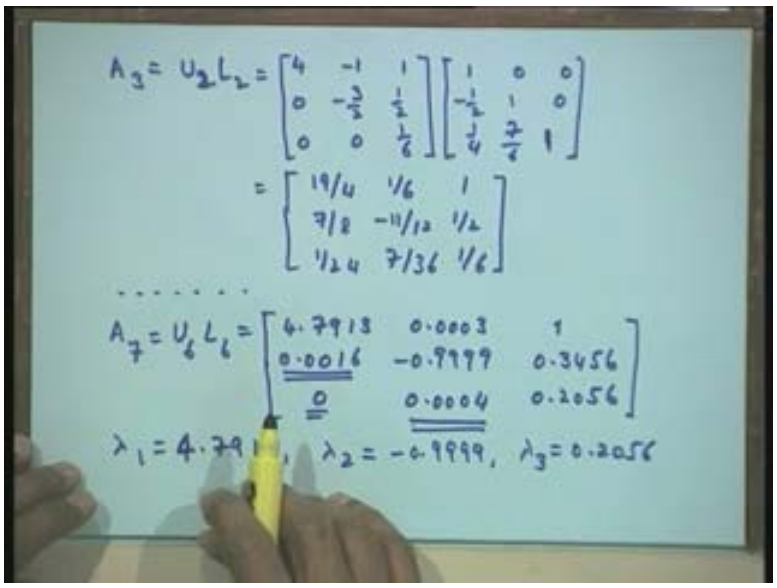
So let us start therefore with writing A is equal to 1, 1, 1, 2, 1, 2, 1, 3, 2 and decompose this as L_1 into U_1, decomposes this as L_1 and U_1. Now this is straight forward one which we already have done it, let me give the result for this one, this is decomposition is very simple, so i will take this as 1, minus 2, 1, this is 1, 1, 1, 0, minus 1, 0, 0, 0, 1, this is the decomposition now this, then we are saying form the matrix A_2, this we are calling it as A_1, so i can write this as A, A_2 is equal to U_1 L_1 so i interchange this product, so i will write this as 1, 1, 0, minus 1, 0, 0, 0, 1 and 1, 0, 0, 2, 1, 0, 1, minus 2, 1. So i am just interchanging these two products, then i find the product of this, this gives me 4, so i can multiply it out, this comes out 4, minus 1, 1, minus 2, minus 1, 0, 1, minus 2, 1.

(Refer Slide Time: 23:01)



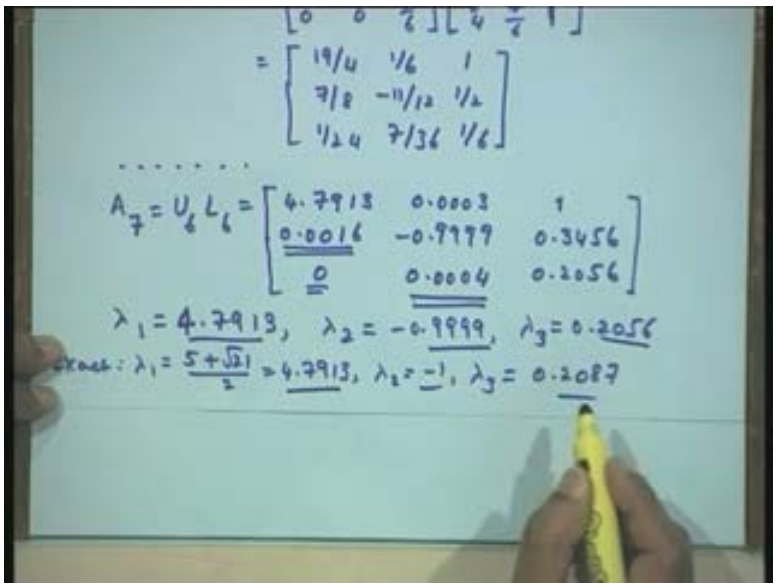
Now i must decompose this again, so that means i must write down A_2 is equal to L_2 U_2 and this is, the values of this as 1, 0, 0, minus half, 1, 0, 1 by 4, 7 by 6, 1 and this is 4, minus 1, 1, minus 3 by 2, minus half, 0, 0, 1 by 6. So this i decomposed it, product of a lower triangular matrix and upper triangular matrix and we have the lower triangular matrix in this form and the upper triangular matrix in this particular form. Now in the next step, i must interchange them and multiply again.

(Refer Slide Time: 23:55)



So i would now define this as A_3 U_2 L_2, so that means i will write this as, 4, minus 1, 1, 0, minus 3 by 2, half, 0, 0, 1 by 6 and L_2 is 1, 0, 0, minus half, 1, 0, 7 by 6, 1. Now i can multiply this and i will have this as, this is 19 by 4, 1 by 6, 1, 7 by 8, minus 11 by 12, half, 1 by 24, 7 by 36, 1 by 6. Now i need number of these decompositions pass and i would leave this and give you the results, so that you can verify this, i need to reach the stage of A_7 and i will have U_6 L_6 and this comes out to be 4 point 7 9 1 3, 0 point 0 0 3, 1, 0 point 0 0 1 6, minus point 9 9 9, 3 4 5 6, this terms to be 0, 0 point 0 0 4, 0 point 2 0 5 6, after repeating it for the six times, this decompositions, this is the value that i produce at the value U_6 L_6 that is your A_7. Now we have been given the accuracy, tolerance of point 0 0 5, so you can see that the, we have reached the required stage, this is smaller, the previous iteration was greater than 0 0 5, but this iteration you can see that these three values shall be compared with the tolerance and when once this tolerance is reached, we assume that this has now reached the upper triangular form, so therefore the Eigen values will be located on the diagonal, therefore we pick up the Eigen values as, lambda 1 is 4 point 7 9 1 3, the second Eigen value is minus 0 point 9 9 9 9 and the third Eigen value is 0 point 2 0 5 6, these are the three Eigen values.

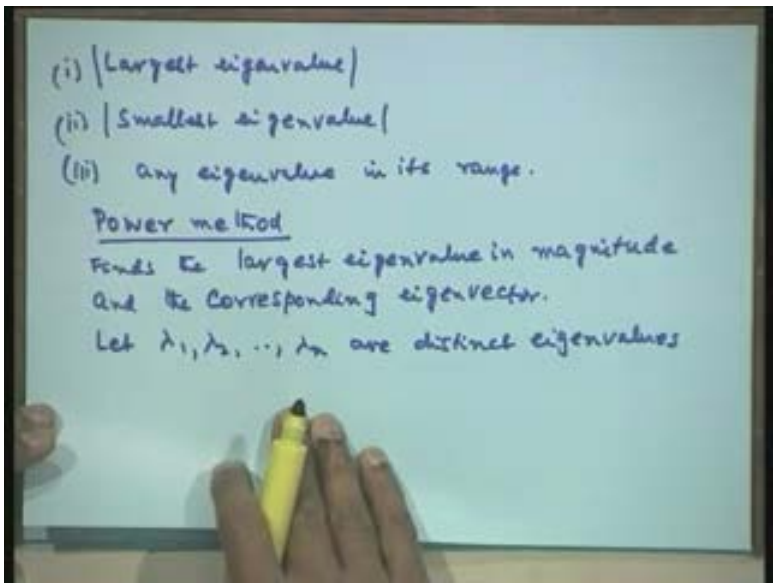
(Refer Slide Time: 26:49)



The exact Eigen values are lambda 1 is, 5 plus root 21 by 2 that is 4 point 7 9 1 3; lambda 2 is minus 1 and lambda 3 is the pair with respect to this, 5 minus root 21 by this that is 2 0 8 7. Now you can see that the, quite accurately the Eigen value as been obtained these, this is minus 1 and these are the Eigen values. As i said, Rutishauser method is a good method provided your Eigen values or all distinct, so that this reduction is possible much more faster, in the case of the complex or repeated Eigen values, there are some problems and therefore the refinement that were taken place were because of that and as i said, the software contains the refinement of this as a cure algorithm.

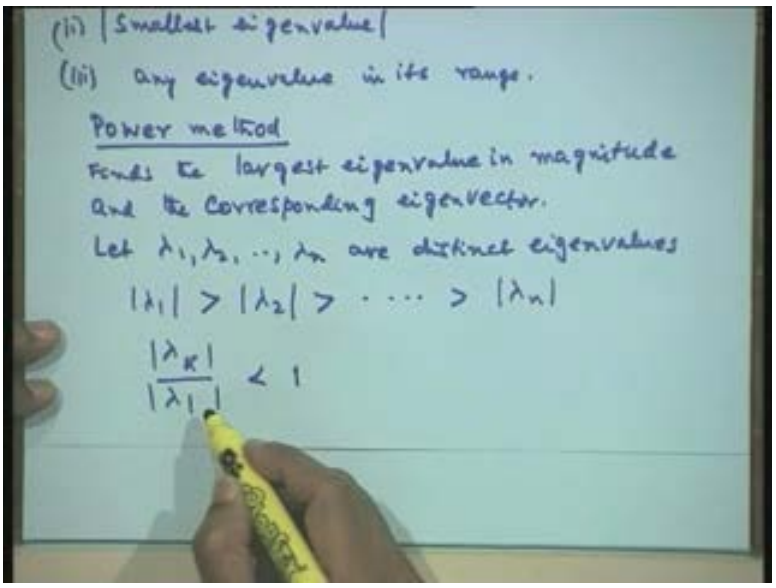
Now what we would like to see next is that, if suppose you want a single Eigen value of a matrix, what are the occasions where do we need it, for example if you are talking of the Gauss Seidel iteration, Jacobi iteration or SOR iterations or if you want to find the optimal omega in SOR, we wanted the spectral radius of the Jacobi matrix or if i want the rate of convergence, i need the largest Eigen value in magnitude that is spectral radius, that means i want one Eigen value of the large system which we would be considering, may be few hundred of the equations, now it is not necessary for us to use Jacobi method or any other method to find all the Eigen values and then find the largest Eigen value in magnitude, that will be very very expensive to find the rate of convergence or whether is converging or not, therefore we must have some methods wherein, i can locate or find the Eigen value, a particular Eigen value required that is a largest Eigen value in magnitude and in many practical applications, engineering applications, you need smallest Eigen value in magnitude, so we need smallest Eigen value in magnitude, largest Eigen value in magnitude or a particular Eigen value somewhere in the range of the Eigen values. So we will now give methods which will locate the, either the largest Eigen value in magnitude.

(Refer Slide Time: 29:17)



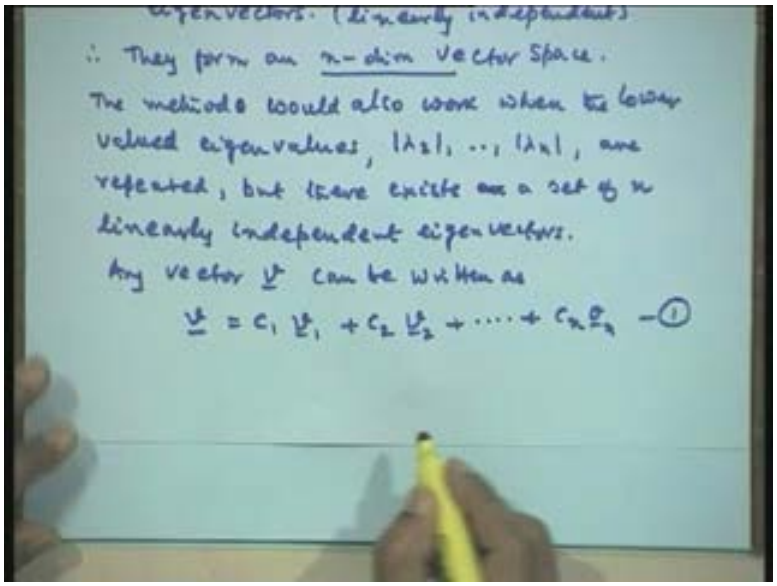
So we want the largest Eigen value in magnitude or we want the smallest Eigen value in magnitude or the third case is that, we need any Eigen value in its range, for example you may say that, i know that this matrix has an Eigen value close to 3, find Eigen value but that will not be the largest or the smallest, it is a particular Eigen value, so i can also find the, this particular Eigen value for this and the method that is used for finding the largest Eigen value is known as the power method, this determines a largest Eigen value in magnitude and the corresponding Eigen vector, so this method finds the largest Eigen value in magnitude, in magnitude and the corresponding Eigen vector. Now let us assume that we have got distinct Eigen values, so let us assume that lambda 1 lambda 2 lambda n are distinct Eigen values.

(Refer Slide Time: 31:12)



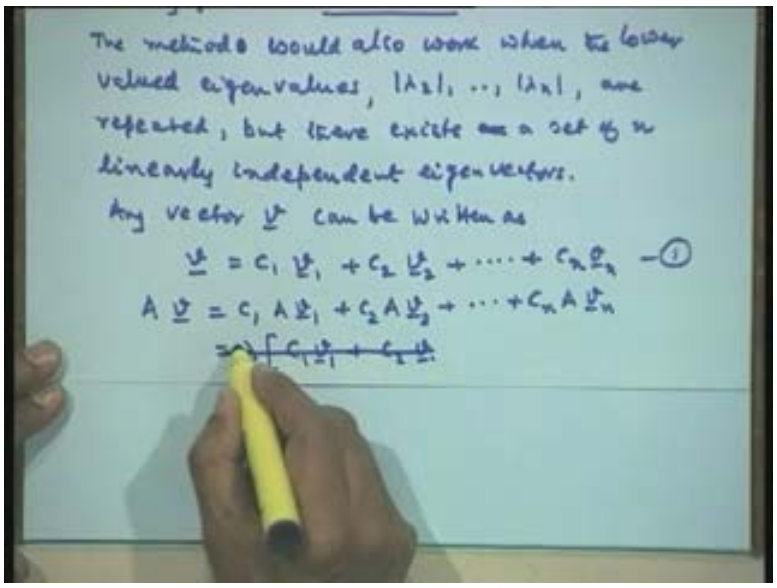
We shall state slightly later that this method will work, if the largest Eigen value is distinct but there will be repeated Eigen value elsewhere, like for example you may have a matrix in which the Eigen values are 3, 1, 1, 1 but i must have the complete system of Eigen vectors, so a 3, a 4 by 4 matrix should have 4 Eigen vectors, then it is a complete system of Eigen vectors, a 3 by 3 matrix should have a 3 Eigen vectors, linearly independent Eigen vectors and that forms a complete system. You have, you have number of matrices wherein the matrix may be order 3 but you may have the Eigen vectors less than the required number that is 3, if some of them are repeated Eigen values, say if you have Eigen values are lambda is 1 and 1, you may have only one independent Eigen vector not two independent vectors, then that will not form a complete system of Eigen vectors but when once we assume that distinct, then the Eigen vectors are also linearly independent, therefore we are starting with this but we will generalize it and say that it will work in the other cases also wherein, you have complete system of Eigen vectors. Now since these are distinct Eigen values, let us order them according to their magnitude, so let us assume that magnitude of lambda 1 is greater than magnitude of lambda 2 so on, greater than magnitude of lambda 1. What this would really imply is, that magnitude of lambda k by lambda 1 is less than 1, so these are arranged in the order; therefore this is the largest Eigen value in magnitude, therefore the ratio is going to be less than 1, so this we are going to use in our method.

(Refer Slide Time: 33:03)



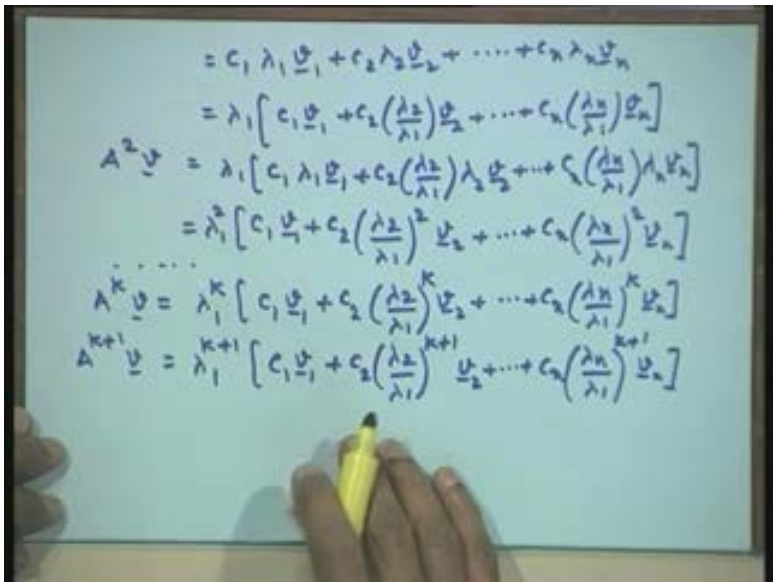
Now we shall say, let the corresponding Eigen vectors be v_1 v_2, let us put under score, so that we have this as a vector, let v_1 v_2 v_n be the corresponding Eigen vectors that means, v_1 Eigen vector corresponds to lambda 1, v_2 Eigen vector corresponds to lambda 2 and so on. Let us take a two dimensional case, let us just illustrate what we want to express it here, if have two independent Eigen vectors for a matrix A, then these two vectors can be taken as basis of a two dimension coordinate system, for example if you have the x y coordinate system, i can take any other coordinate system, any other vector as a, for example a i b j and some a star i plus b star j as a new coordinate system, so i can always have a new coordinate system into two dimensional space, therefore if i have a 2 by 2 matrix and i find 2 i linearly independent Eigen vectors that will form a basis for a two dimensional space, similarly if i take a 3 by 3 matrix, i produce 3 linearly independent Eigen vectors, it will form an three dimensional space, similarly since these are corresponding Eigen vectors and these are linearly independent Eigen vectors, therefore they form an n dimensional space, vectors space. Therefore these are the corresponding Eigen vectors but they are also linearly independent, these are all also linearly independent. Therefore these Eigen vectors, they form an n dimensional vector space. Now we would like to make this statement here, that even if the low Eigen values, lower Eigen values that is lambda 2 lambda 3 lambda n, any of them is repeated any number of times we do not bother as long as it produces an linearly independent Eigen vectors, that means the method would also work, would also work when the lower valued, Eigen values, that is we are talking of magnitude of lambda 2 so on magnitude of lambda n are repeated, but there exists a set of n linearly independent Eigen vectors, the reason is that we want to form an n dimensional vector space. Now if i take any vector in this space, i can write it as a linear combination of this, so any vector v can be written as sum, c_1 v_1, c_2 v_2, c_n v_n.

(Refer Slide Time: 37:03)



Now what i do, it is the, power method is very simple, we just pre multiply this by A, so pre multiply by A, so c_1 A v_1 plus c_2 A v_2 so on c_n A v_n but by definition Av_1 is lambda v_1, Av_2 is lambda v_2, A v_n lambda v_n, so i can write this as, since lambda is there, let me take out lambda common and write this as c_1 v_1 plus c_2 v_2, sorry, i will write down in the next page, i need the next this one.

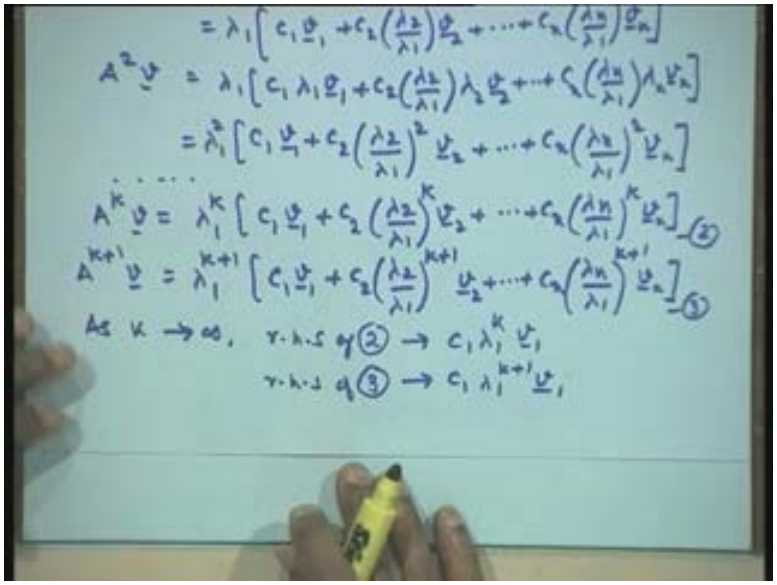
(Refer Slide Time: 38:04)



Let us write down one more step here, c_1 is equal to lambda 1 v_1 plus c_2 lambda 2 v_2 plus c_n lambda n v_n, so A v_1 lambda 1 v_1, A v_2 is lambda v_2 because they are corresponding Eigen vectors. Now i will take out lambda 1 common from here, c_1 v_1 lambda 2 by lambda 1 v_2 plus c_n lambda n by lambda 1 v_n.

Now i repeat the procedure, so i pre multiply by A again, so i will write down A square of v, pre multiply by A, so i will have A square of v, therefore this will be lambda 1, now again A v_1 will be lambda 1 v_1 plus c_2 lambda 2 by lambda 1, i have again A v_2, A v_2 is lambda 2 v_2 plus c_n lambda n by lambda 1, again A v_n is lambda n v_n. Again i will take lambda 1 out, so i will have lambda 1 square c_1 v_1 c_2 lambda 2 by lambda 1 whole square, again this is lambda 2 by lambda 1, this is lambda 2 by lambda 1, so this is lambda 2 by lambda 1 whole square v_2 plus c_n lambda n by lambda 1 whole square of v_n. Now let us suppose we have done it k times, so let me write down what will happen when we pre multiply k times, this will be lambda 1 to the power of k c_1 v_1 lambda 2 by lambda 1 to the power of k v_2 plus c_n lambda n by lambda 1 to the power of k and v_n. Now let the, next one also i require to give the final observation so will have lambda 1 to the power of k plus 1 c_1 v_1 c_2 (lambda 2 by lambda 1)$^{k+1}$ v_2 plus c_n (lambda n by lambda 1)$^{k+1}$ v_n.

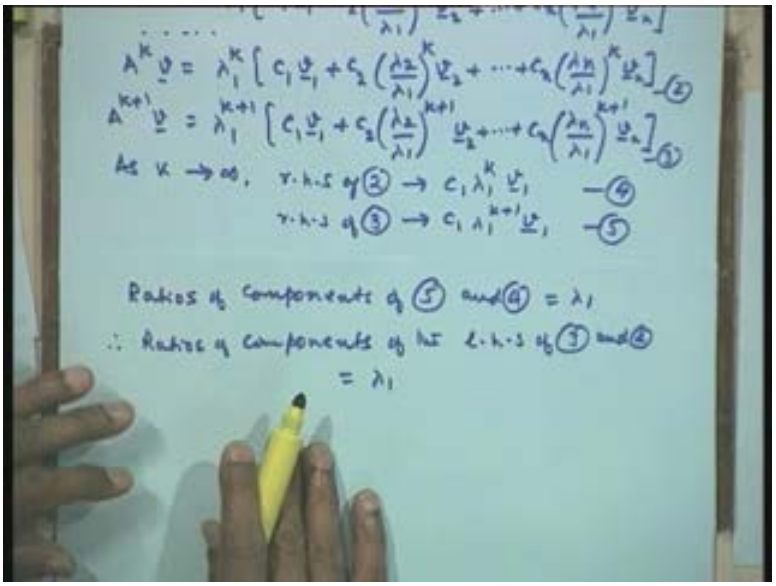
(Refer Slide Time: 41:16)



Now let us just number this as 2, this as number 3. Now let us assume as k tends to infinity what happens, as k tends to infinity the right hand side of 2, all this ratios in magnitude, magnitude lambda 2 by lambda 1 is less than 1, therefore this would go to 0, this would go to 0, so all of them would go to 0, the right hand side will converge to c_1 v_1 into lambda 1 k, therefore the right hand side of 2 tends to c_1 lambda 1 to the power of k v_1, similarly the right hand side of 3, all of them are again 0, so this will converge to c_1 (lambda 1)$^{k+1}$ v_1.

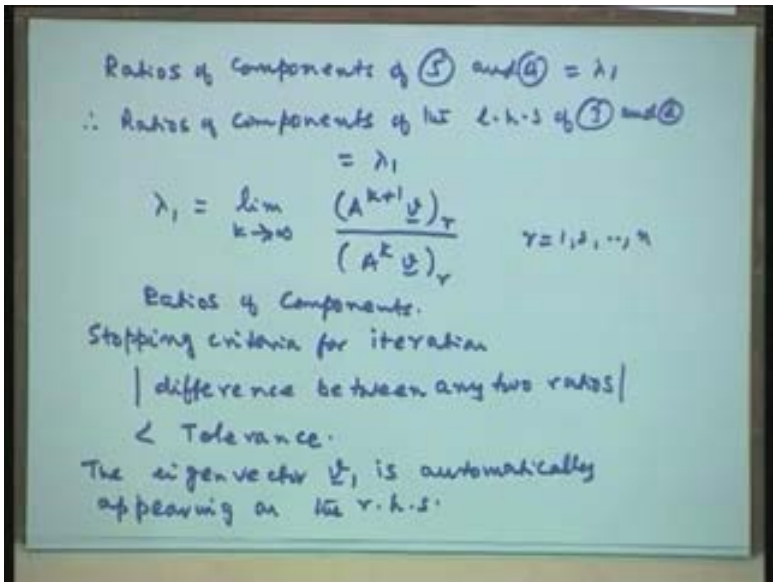
Now what we are looking at is, when we have performed sufficient number of iterations, this multiplications, what would happen to the right hand side, the right hand side this, any previous iterate will converge to this and the next iterate will converge this, now if i take the ratio of the components of the right hand side, this quantity, this ratio is lambda 1, so take this component of this divided by this, c_1 is a number cancels, now v_1 v_1 its component also cancels, the ratio of this is lambda 1, therefore the ratio of the right hand side of any component is lambda 1, therefore these are all also vectors on the left hand side, if i take the ratio of this components of these two vectors, it should be equal to the right hand side because their identity, this is equal to, so in the limit the ratio on the right side is lambda 1, therefore the ratios on the left hand side also should be will be lambda 1.

(Refer Slide Time: 43:16)



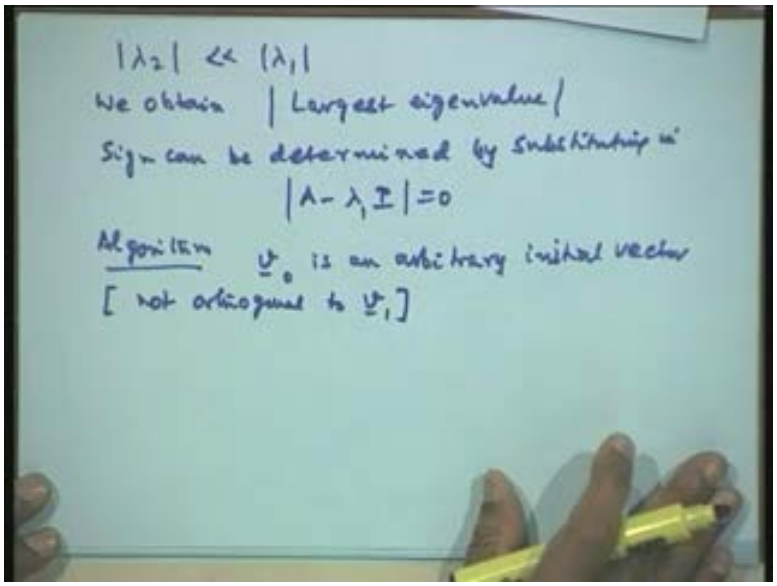
Therefore we have the required formula, you can say the ratio, ratios of components, let us call this as 4 and this as 5, of 5 and 4, the ratios of the components of 5 and 4 is equal to lambda 1, therefore ratios of the components of the left hand sides ((00:43:55 min)) of 3 and 2 is equal to lambda 1, we are talking of ratio of this 3 left hand side and the components of the ratio of the components of 3 and 2.

(Refer Slide Time: 44:13)



Therefore we have the value for lambda 1, that will be equal to limit, we have taken k tending to infinity, k is tending to infinity, the ratio of the components A^{k+1} v, let us take r for component, divided by $(A^k v)_r$, r is equal to 1 2 3 n, therefore these will give you the ratios of components, these are ratios of components, there are n ratios in this, because this is one dimension, therefore there are n approximation for lambda 1, all of them converging to same number lambda 1, therefore this will automatically give us the stopping criteria, that these are n ratios, the difference between any two ratios should be less than tolerance, then all of them have converge to a particular number, therefore we can immediately write down what is the stopping criteria for iteration is simply magnitude of difference between any two ratios is less than some given tolerance, some given tolerance, therefore this gives us the Eigen value in magnitude. Now let us just look back to this slide once more, now in the limit the right hand sides of 2 and 3 are reducing to c_1 lambda 1 k v_1, c_1 lambda 1 k v_1, therefore automatically v_1 is following us, everywhere v_1 is there, therefore whatever this one we have got on the right hand side when we have taken this common factor out of this, when we have taken the common factor out of this what we have here is nothing v_1, if i multiply this, this will be constant multiple of the vector, which is same as vector, therefore whatever we have the right hand side is itself the Eigen vector corresponding to the this one, we do not have to do any more computation. Therefore the Eigen vector is automatically available on the right hand side, so the Eigen vector v_1 is automatically appearing, is automatically appearing on the right hand side. Now only thing you may needed, may be you would like to have it as normalized form or anything, we can do it normalized form and write down the Eigen vector.

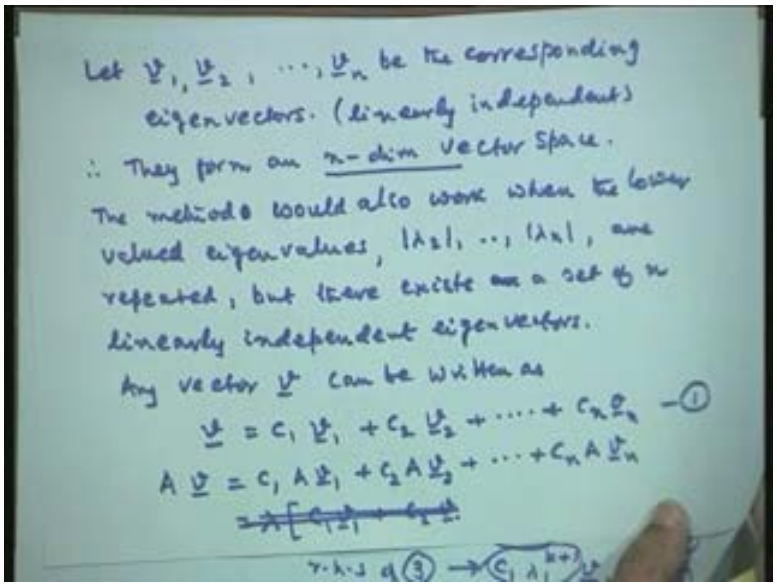
(Refer Slide Time: 47:16)



Now the fastest convergence will be obtained if the next Eigen value, smaller than the largest Eigen value is much smaller, that means if lambda 2, if the second value is much smaller than lambda 1, then it is going to converge very very fast. Now one important thing to note here is this, this is going to give us the largest Eigen value in magnitude not its sign, it gives the largest Eigen value, therefore we obtain in this method the magnitude of the largest Eigen value. Therefore when we are forming the ratios, we may forget its sign, we need not write the signs, because that, that sign is not going to be useful for us. Now if i want the sign of the Eigen value, i must put back in the determinant and then make it c is 0, that a characteristic equation is satisfied, so the sign can be determined by substituting in the characteristic equation, A minus lambda I is equal to 0, that means whether it is for the positive value it is getting satisfied or for the negative value it is going to satisfied, we just have to check whether it is satisfied, so that you, you are sure of the, of the sign of the Eigen value.

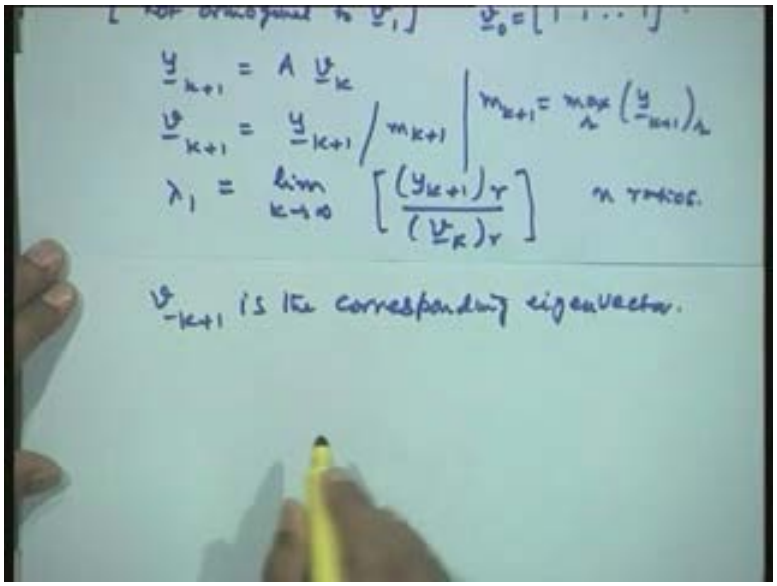
Now let us write down this an algorithm, simple algorithm, so let us call this as algorithm, now here when we are starting this procedure, when we have done it, we have written this particular step let v be equal to this one, so v is unknown to us.

(Refer Slide Time: 48:56)



Therefore the, if you have the actual linearly independent Eigen vectors i need a linear combination of this, therefore i take arbitrarily any Eigen vector as my initial vector, because nothing is known, i can take arbitrarily any Eigen vector, so we will take v_0 is an arbitrary starting vector, arbitrary initial vector. The only difficulty that can arise in an arbitrary initial vector is, if you have taken this Eigen vector as orthogonal to the actual Eigen vector then iteration will not converge, because as the only problem difficulty that can arise, it so happens that your arbitrary initial vector is orthogonal to the final vector then the method will not work, so not orthogonal to v_1. Probably you will never be able to write such an arbitrary initial vector which will really be orthogonal but we have to say that when it may fail, may fail and therefore normally we can, if nothing is known, we just away straight away start with 1 1 1 as the Eigen vector, you can safely start as 1 1 1 as the as the initial vector.

(Refer Slide Time: 50:12)



Then the algorithm is, let us write what we have, have to write down here is the, this step i want to write down, so what i would do is, i will denote by the vector y_{k+1} is equal to A v_k. Now it is important that in this computations, we do not know the actual values or numbers in the matrix A, so when you multiply by this v, it is possible that the round off errors may grow, therefore we would like to keep this round off error into control that means, we would see that none of the components of this v is greater than 1, that means we will normalize such that the largest component of v is 1 then, the round off error is completely under control and it is a simple tool for us to do it, what we do is before i put back, because next step is to pre multiply, before i multiply i will normalize v, so what I would do is, i will write down new one as y_{k+1} this is vector and divide this by m_{k+1}, m_{k+1} is a number, let us write it here, m_{k+1} is the largest element of this one, i want normalizing such that the largest element of v is 1, so this will be maximum of r of y_{k+1} of r, so that means whatever i have got here, this vector, i will take this largest element and divided it out, so that the largest element in magnitude is equal to 1, so then i would have this as m_{k+1} and therefore your lambda 1 is limit of k tending to infinity $(y_{k+1})_r$ by $(v_k)_r$, these are n ratios, these are n ratios and let us just close it here and therefore whatever is there available for us v_{k+1} is the Eigen vector, is the corresponding Eigen vector, corresponding Eigen vector. okay we would close for today.