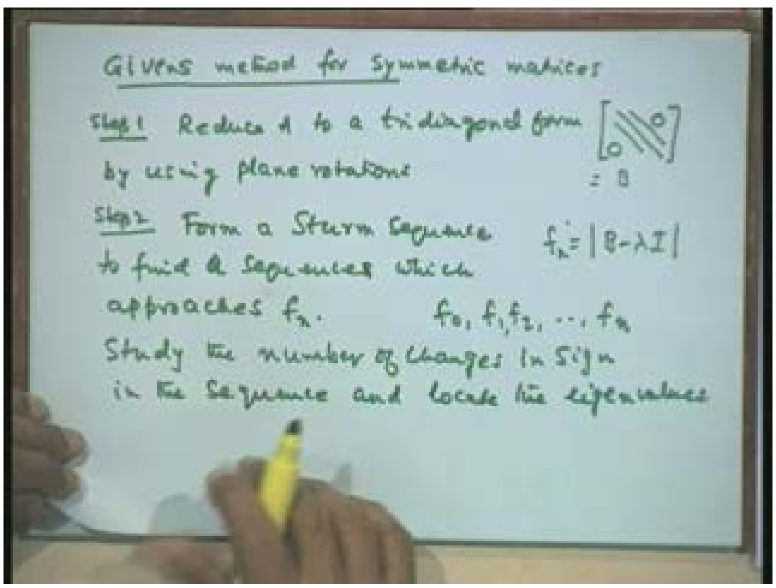
Numerical Methods and Computation
Prof. S.R.K. Iyengar
Department of Mathematics
Indian Institute of Technology, Delhi

Lecture No - 22
Solution of a System of Linear Algebraic Equations (Contd.)
Eigen Value Problems

In the previous lecture we have derived the Jacobi method for finding the Eigen values and the corresponding Eigen vectors of a symmetric matrix, however there is a disadvantage in the method and that is, 0 that has been created, can be destroyed in the later rotations therefore, we are not sure of the number of rotations that a particular problem would take. Therefore except if there is no alternative we will have, we can use that particular method. However we have some alternative methods which would see that, the 0 that have been created once, they are retained as zeros that means, 0 created once they are not destroyed. One such method is known as the Givens method.

(Refer Slide Time: 01:51)

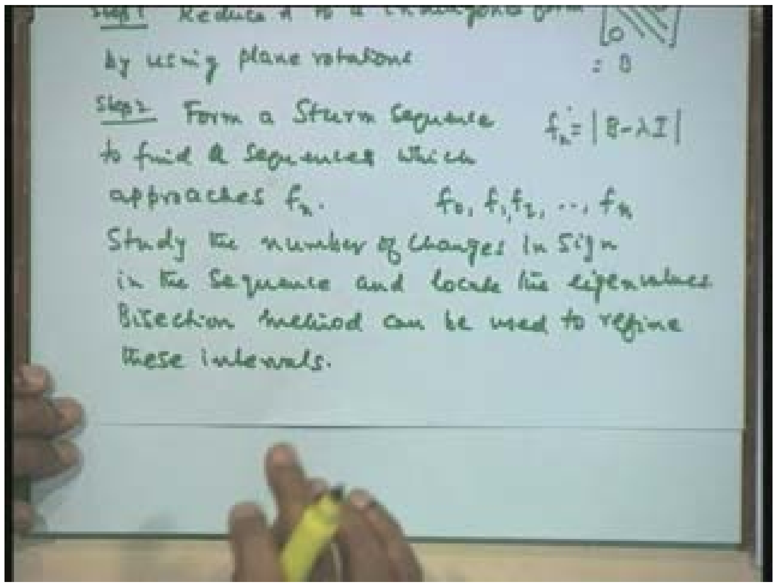


So let us describe the Givens method for symmetric matrices, now the method is derived in three steps, the method does not reduce the system A to one of the known three form we have, we have a matrices D L or U that is, diagonal lower or upper triangular, but this method reduces a matrices to a tri diagonal form, that means one main diagonal, one super diagonal and one sub diagonal. So the first step is reduce a matrix A, reduce A to a tri diagonal form, so we will now reduce the given matrix to the tri diagonal form to this, by using the plane rotations. The transformations we are using the same, as in the Jacobi method, by using plane rotations. Then in step 2, will form the Sturm sequence for finding the characteristic equation of this matrix. So if the given, the new matrix is B, if i call this as matrix B, i will find out the characteristic equation of this matrix as, B minus lambda I and denote it by f_n , suffix n, for the order of the matrix n.

So this is the characteristic equation of the matrix B. Now in step 2, i will now form a Sturm sequence to find this characteristic equation, so form a Sturm sequence, form a Sturm sequence to find a sequence, a sequence to find, to find a sequence which approaches f_n, which approaches f_n, that means we are going to derive a sequence f_0 f_2 so on and the last member of this sequence is our characteristic equation.

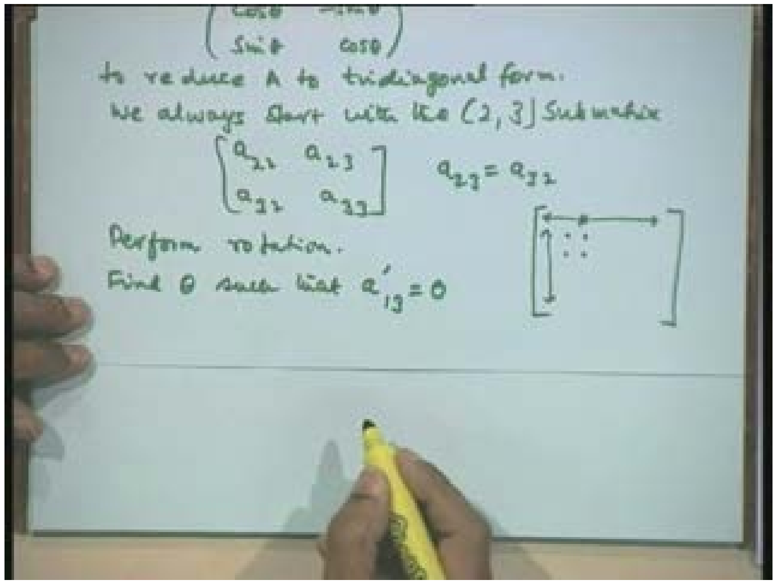
when once it is a Sturm sequence, i can find the number of changes in sign in any interval AB and then determine the number of Eigen values that are there in the interval AB. Now, study the changes in sign, the number of changes in sign in the sequence, in the sequence and locate the Eigen values and locate the Eigen values.

(Refer Slide Time: 05:22)



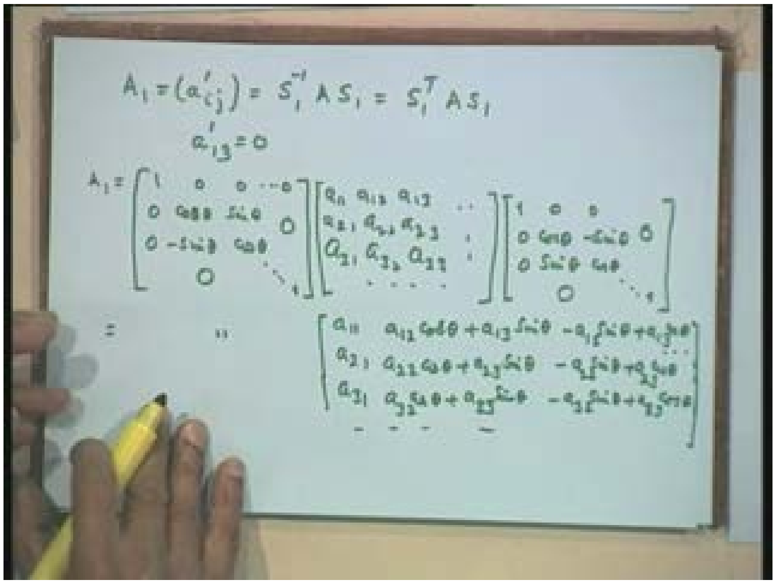
when once we locate the Eigen values in a particular interval, everything is now dependent only on change in signs of the sequence therefore, i can use the bisection method to reduce the interval further and further and get the required accuracy, because we are now using only this signs, now therefore bisection method is used, bisection method can be used to refine these intervals. Therefore, we can obtain the Eigen value to a required accuracy, whether you give the 4 plus 5 plus 6 plus accuracy, i can get the accuracy of this Eigen values by going on bisecting the interval in which the Eigen value lies. So it is just by changing the, looking at the number of changes in signs in this Sturm sequence and the third step is to find the Eigen vectors, step 3 is, find the Eigen vectors.

(Refer Slide Time: 06:19)



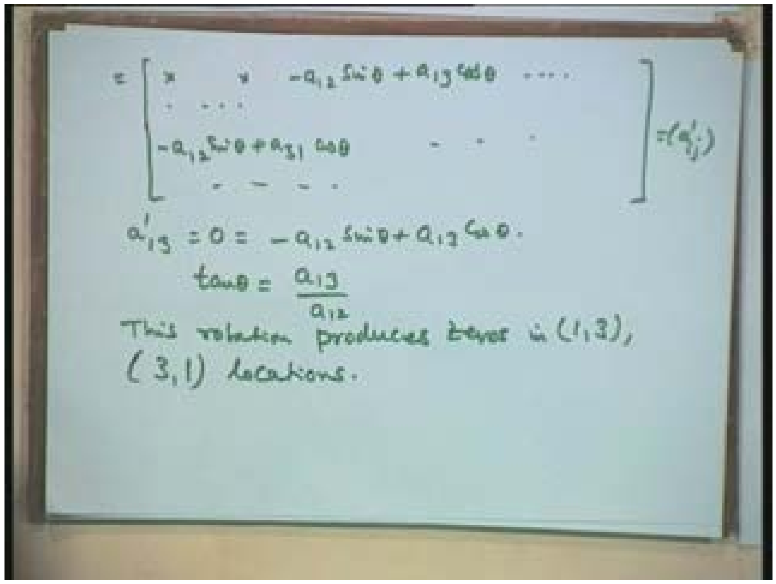
Now let us go back to the step one and see how we are going to achieve this tri diagonal form. Now we are using the same orthogonal matrix as we have done in Jacobi, so we use the same orthogonal matrix, orthogonal matrix, that was cos theta, minus sin theta, sin theta, cos theta to reduce A to tri diagonal form. Now the most important thing here is that, we always start from 2 3 sub matrix that is A_{22}, A_{23}, A_{32}, A_{32}, so it is always we concentrate on 2 3 sub matrix. We always start with the 2 3 sub matrix, that means our locations A_{22}, A_{23}, A_{32}, A_{33}, of course A_{23} is equal to A_{32} we are the symmetric matrix, that means we are now starting with this location, these two locations 2 3 and 3 2. We have not touched the first row, we are not touching the first column, then i perform the rotation, then, in Jacobi we set these two elements as 0, finding the largest half diagonal elements and then making this as 0 and then we are determining the theta, but here we will not do that, we perform the same rotation as we have in Jacobi but we find theta and set this element above as 0, that means we set the element A 1 3 as 0, not this element as 0, but we set the above element as 0, now perform rotation, perform rotation, find theta such that, let us call the new matrix as A prime, so A prime 1 3 is equal to 0, that means it is just above this element over here, now let us see what this gives us.

(Refer Slide Time: 09:36)



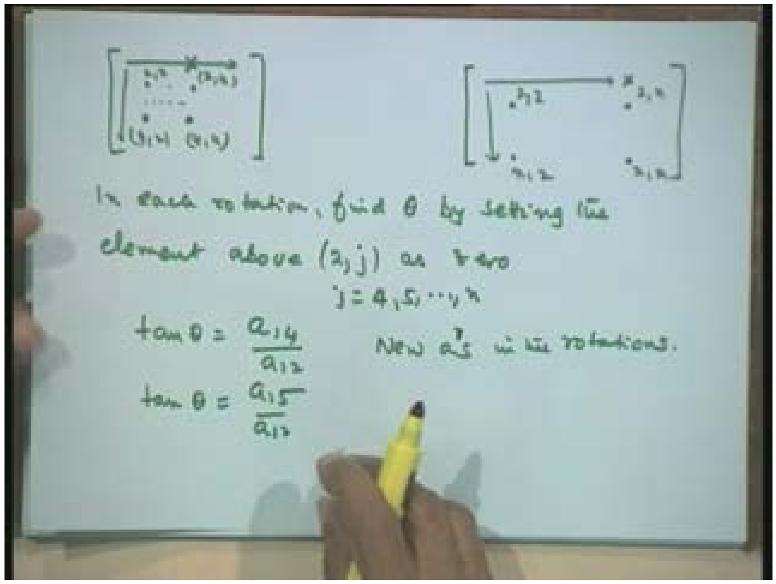
so the transformation would be, let us call it, we will call this as A prime ij the new matrix, as S_1 inverse AS_1, its orthogonal, so S_1 transpose A S_1 and in this new matrix we are saying that set A prime 1 3 is equal to 0, that is what we are saying. Now let us write down this matrix A_1, the first row is not changed so it retains as it is, as 0, cos theta and this is transpose of it, therefore i will have a sign theta over here, 0, minus sign theta, cos theta. So as inserted the matrix in the 2 3 sub matrix position 2 2, 2 3, 3 2, 3 3, then i will write down our matrix a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} and i do not need these elements, so i will just retain the elements as it is. Then i have again 1, 0, 0, 0, cos theta, minus sin theta, 0, sin theta, cost theta, these are all zeros, we have all zeros, here also we have all zeros. Now let us retain this matrix as it is and let us write down this product, so because this is 1, 0, 0, the first column remains as it is a_{31}, it remains as it is. Then let us go to first row second column, this is a_{12} cos theta plus a_{13} sin theta, a_{12} cos theta plus a_{13} sin theta, the first row third column gives minus a_{12} sin theta plus a_{13} cos theta. Now i need these two elements also, this is a_{22} cos theta a_{23} sin theta, this minus a_{22} sin theta plus a_{23} cos theta and similarly a_{32} cos theta a_{33} sin theta, minus a_{32} sin theta plus a_{33} cos theta. Now in this product, i want to set 1 3 location as 0, so i will now write down in this product only a_{13} location and corresponding, symmetry, therefore both the location are the same, so let us write down the location, the third location from here.

(Refer Slide Time: 13:26)



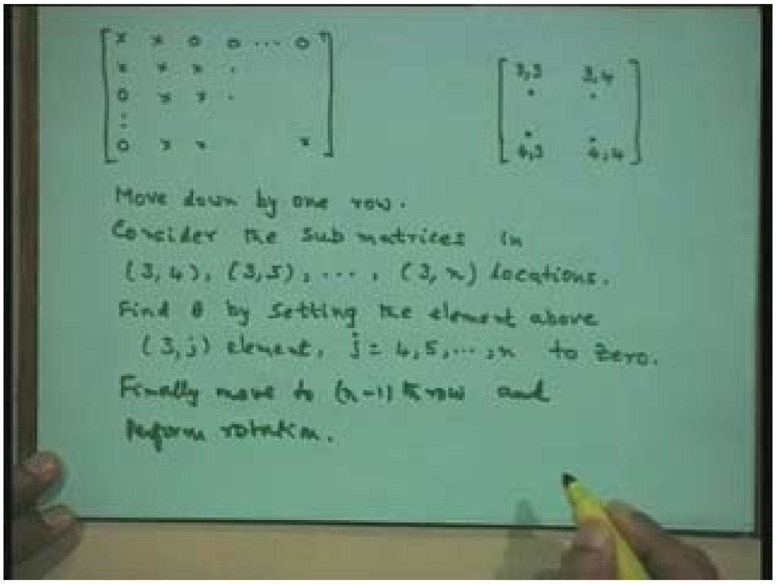
So this will give me some element here, some element here, now one is multiplying this, so it will simply give me minus a_{12} sin theta plus a_{13} cos theta, so first row, i am multiplying the first row, first row is simply 1, 0, 0, so i am multiplying this with this one. Now i would, if you look at this third element here, again i will have here minus a_{12} or a_{21} sin theta plus a_{31} cos theta. We do not need any other elements, we need only this element, because of symmetry we know these two locations are same a1 1 3 location, 3 1 location will identically be same. Now i want to set this element as 0, therefore i will set a prime 1 3, this is our prime matrix, this we are calling it as a prime ij, so a prime 1 3 0 will give minus a_{12} sin theta plus a_{13} cos theta. Therefore these gives me tan of theta is equal to a_{13} by a_{12}, this value of theta produces zeros in the location (1, 3) and (3, 1), so this rotation produces zeros in (1, 3) and (3, 1) locations. Now we would like to move along this row, the first step we are doing the rotation in this sub matrix system, 2 3 system, now what we do is, i now move to the 2 4 sub system, 2 4 means 2, 2 4, 4 2, 4 4 so we move along the row to the right, so we can see now, move to the right along the second row, what we mean by this is, therefore it means, we consider (2, 4), when after we complete (2, 4), we go (2, 5) after we complete (2, 5) i will go to (2, n) the entire row, so we move along the row each one, but leave the pivot of course, it is 2 4, 2 5, 2 3, 2 n and so on. So we move along the particular row and every time we set the, find the value of theta such that the element, as we have done here, the element above this is 0, so when i go to the next element here, 2 4 sub system, then i will now make this as 0, that means let us just try what it is.

(Refer Slide Time: 17:06)



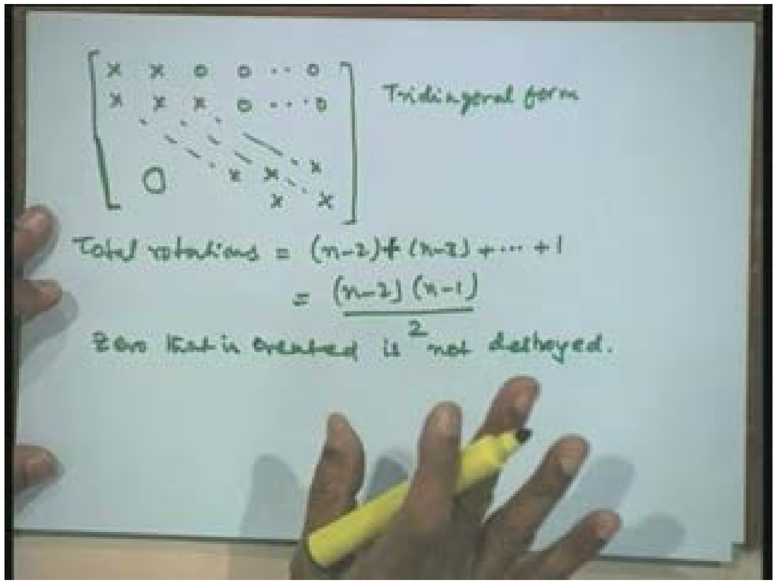
Now this first row stands as it is, the first column stands as it is, now i will have 2 and 4, so this is a, we do not consider this, we do not consider this, then i consider this, i will consider this, this is our 2 2 location, this is your 2 4 location, your 4 2 location and 4 4 location. So i would consider these 4 elements and then make this element as 0. The value of theta, that is determined by setting this as 0, so in the final stage i would have this, this column, this 2 2 element and then i go to 2 n element, this is 2 n, this is 2 2 and i have now here n 2 element and n element. I will now set last element as 0. Now in each rotation, in each rotation find theta by setting the element above a 2 j as 0, where j is equal to your, now you have taken it as 4 5 and so on n, so in each rotation we find theta by setting the element above, that is the location 2 j as 0, for j is equal to 4 5 6 and so on n. Now what happens is in each time when we are setting this, the value of tan theta would be, say that is your element a_{14} by a_{12}, this a is not original a, the new a in the rotation, these are all new a's in the rotations, we have performed the first rotation, then we are calling the matrix again as a elements, so we are using with the new ab's. So in the next step it will be, tan theta will be a_{15} by a_{12} and so on and lastly will have tan theta is a_{1n} by a_{12}.

(Refer Slide Time: 19:56)



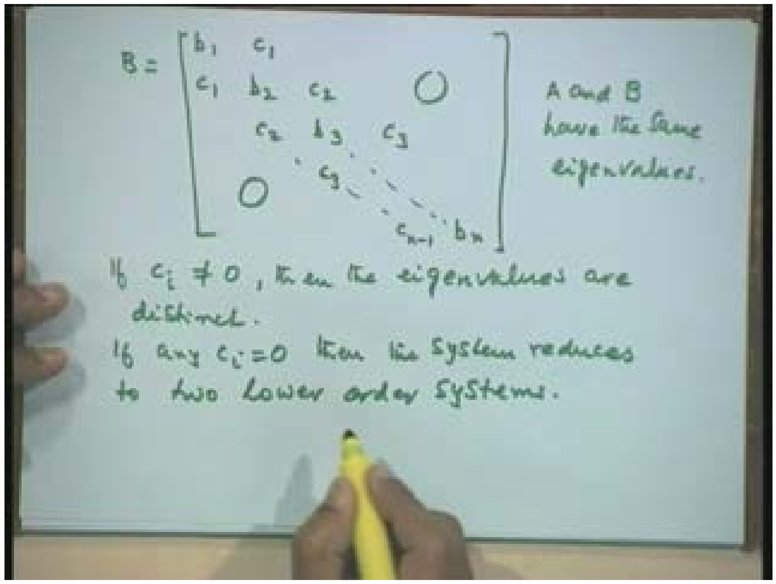
If i perform this rotation ,then what we have done here is, we have produced now, we have left the first two elements, then we made these, all these elements as 0, the element above has been made, all of them 0 and then, here the elements below also have been made as 0, correspondingly symmetric, now all these elements are now available to you, these are all nonzero elements, this is the, what we have done in this step, now we move down, but to the third pivot, now we started with the pivot 2 2, now i go down to third pivot, that is your, we start with 3 4, therefore i will start with 3 4, 4 3, 4 4, again set the element above as 0, therefore every time we move down a row, a row, then find theta such that the element above is 0, so we will now move down one row, move down by one row. Now consider the sub matrices in (3, 4) then (3, 5) so on (3, n) locations, that means we are considering, when we are talking of 3 4, we are talking of your third pivot, 3 3 pivot, this is your 3 4 and this is your 4 3 and this, this is our 3 3 location, 3 4 location, 4 3 location, 4 4 location. Set the element above this as 0 and find theta, so find theta by setting the element above (3, j) that is j is equal to 4 5 so on n to 0, find theta by setting the element above this 3 j element, that is 3 4, 3 5, 3 n to 0. Then this will reduce, give us zeros in the second row starting from the forth element and correspondingly this one. Finally move to n minus n^{th} row and perform rotation, move to n minus 1^{th} row and perform rotation.

(Refer Slide Time: 23:11)



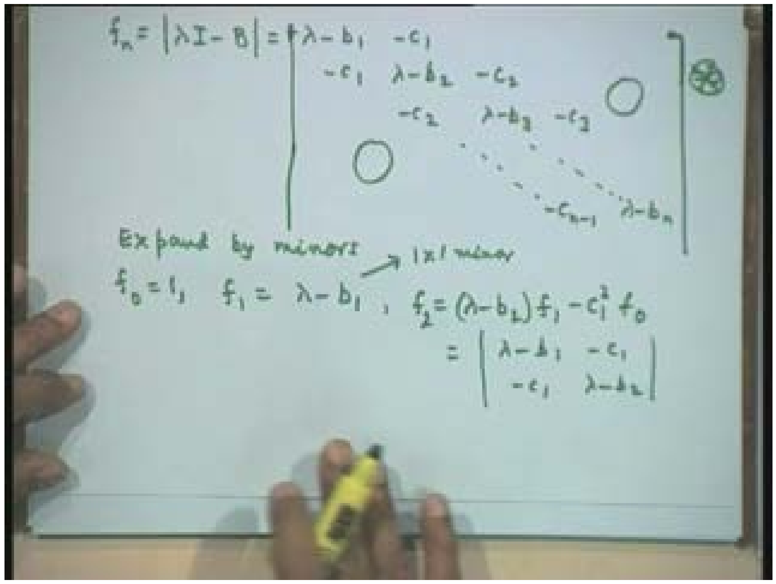
If i do this, then the matrix has been reduced to element here, element here, we have now made all these as zeros, element here, element here, element here, all these have been made zeros, so what i would therefore have is, element here, will have an element here, element here, element here, which is nothing but the tri diagonal system, so this is reduced to a tri diagonal form. Now the number of rotations we can count here, there are n minus 2 to be set in the first row, n minus 3 in the second row and 1 here, therefore the total number of rotations, total rotations is n minus 2 plus n minus 3 so on plus 1, therefore this is n minus 2 into n minus 1 by 2. Now this is the exact number, because 0 that is created once, 0 that is created is not destroyed, that means 0, once you create 0 it will remain 0 right through the all the other rotations, therefore this will be the exact number of rotations, so we are able to say that i need exactly n minus 2 into n minus 1 by 2 rotations to bring a matrix to a tri diagonal form. So this is something very good because, we are able to say exactly the number of rotations that is required and there is no, there is no option on a particular problem or it is dependent on a particular problem to have the number of rotations varying.

(Refer Slide Time: 25:31)



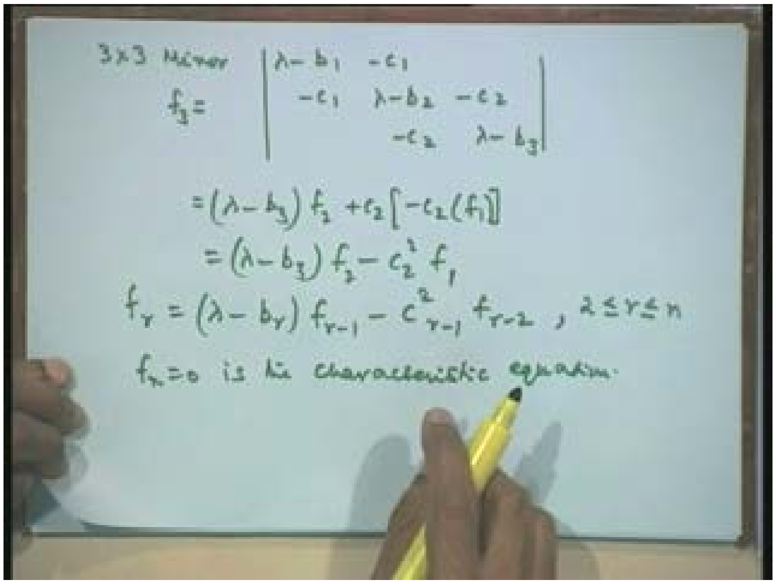
Now when, once i created this, so let us call this as a matrix B, so i have now produced the tri diagonal system b_1 c_1 symmetric; therefore this must be c_1, this is b_2, c_2, so it will be c_2, b_2, b_3, c_3 and so on b_n, will have this as c_n minus 1, so the matrix is going to look like this. Now these are similarity transformations which you have done, therefore A and B have the same Eigen values, have the same Eigen values. Now there is a theoretical observation on the Eigen values of a tri diagonal matrix, what it states is, if all the off diagonal elements are not 0, then the Eigen values of this are distinct, so that is the property of a tri diagonal system. If c_i is not equal to 0, then the Eigen values are distinct, then the Eigen values are distinct, now you can see that, if anyone of then is 0, it actually degenerates into two systems, let us set, suppose c_3 is 0, so there is a c_3 here, therefore there is a c_3 here, if i have c_3 as 0, this 0, this is 0, now this does not contain anything belonging to this system and this does not have this b_1, b_2, b_3 or c_1, c_2, c_3, therefore if anyone of the c_i is 0, then this system disintegrates into two sub systems, so the Eigen values of each smaller system can be determined in that case, so if c_i is not equal to 0, then that does not happen, so if c, if any c i is equal to 0, then the system reduces to two lower order systems, then the system reduces to two lower order systems.

(Refer Slide Time: 28:14)



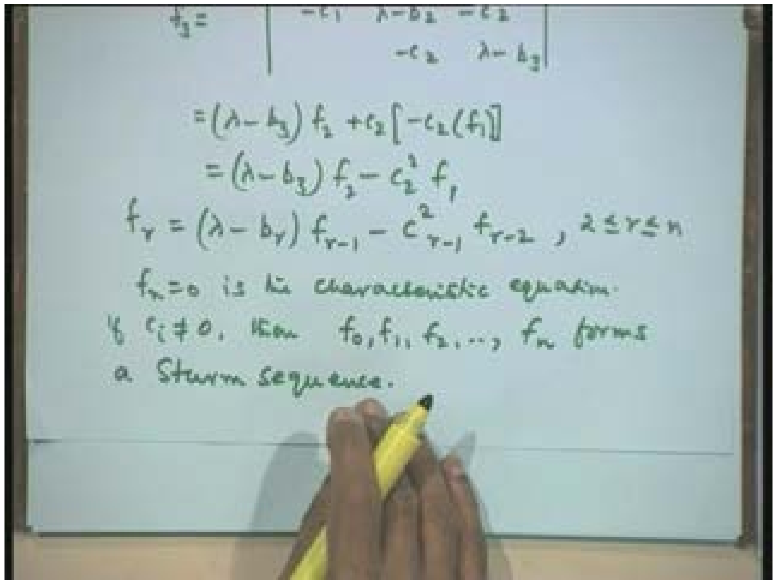
Now forming the Strum sequence from here is very simple, we would define the, our characteristic equation f_n as determinant of either lambda I minus b or b minus lambda I, so this will lambda minus b_1, minus c_1, minus c_1, lambda minus b_2, minus c_2 and then i will have minus c_2, lambda minus b_3, minus c_3, i will have this lambda minus b_n, minus c and minus 1. This is a determinant, this is the determinant, this is our, you can set it equal to 0, that is our characteristic equation but i want to have the Strum sequence for f_n, so this is our characteristic equation, so let us drop this 0 for the moment. I define, expand this by minors, expand by minors, we define f_0 as 1, so with that i take 1 into 1 minor, that is simply lambda minus b_1, denote that by f_1, that is lambda minus b_1, this is a 1 into 1 minor. Then i will take 2 by 2 minor, that is the determinant of these two but i have always expand by the last row or last column, so I will expand it by the last row, so I will have lambda minus b_2 multiplied by lambda minus b_1, lambda minus b_1 is f_1, so i will write lambda minus b_2 into f_1, minus c_1 square into 1, so i will write this f_2 as lambda minus b_2 into f_1 minus c_1 square f_0, i mean if you just want to have will look at this one, this is what we are considering, this, i am expanding it by the last row, so lambda minus b_2 into lambda minus b_1 and that is your f_1, minus, this location is minus, minus minus plus, into this, this product, therefore minus c_1 square always into 1, now this is your 2 into 2 minor.

(Refer Slide Time: 31:17)



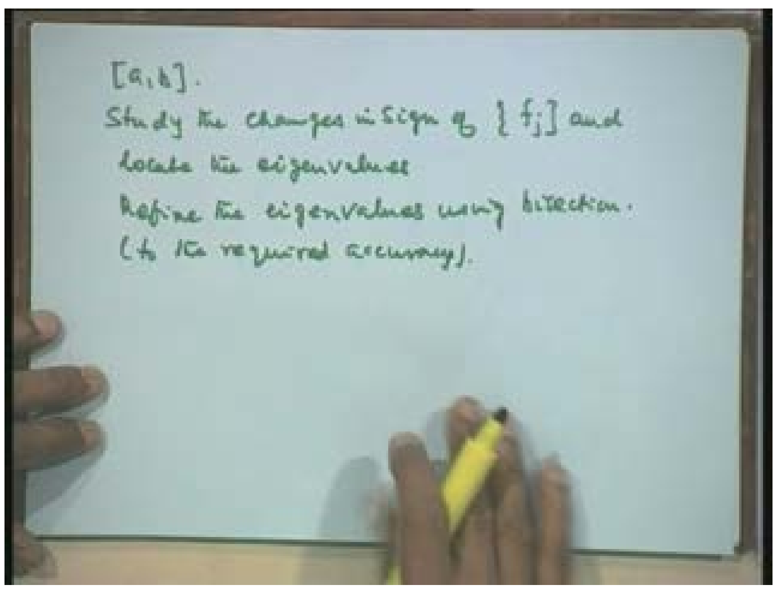
Now i take the 3 into 3 minor and that is lambda minus b_1, minus c_1, minus c_1, lambda minus b_2, minus c_2, minus c_2, lambda minus b_3, now this, i will call this as f_3, this i will call it as f_3. Now let us expand again by the third row, lambda minus b_3 into this determinant, which is already f_2, so this is lambda minus b_3 into f_2, then minus of minus, that is plus c_2 and we have removed this column and this row, so i will have here minus c_2 into lambda minus b_1 that is f_1, that is minus c, i have removed this, this particular column and this row, so i am expanding the determinant, so this will be minus c_2 into lambda minus b_1 lambda minus b_1 was f_1, so this will be lambda minus b_3 f_2 minus c_2 square f_1. Therefore i have the general formula, lambda minus b_r f r minus 1 minus c r minus 1 square f_r minus 2, for all values of r lying between 2 r and n, so all of them can be retain in a reconciliation form, this is simple reconciliation that will go, so with the reconciliation, we can immediately compute all the values and the last one f_n is our characteristic equation, therefore f $_n$ is equal to 0 is the characteristic equation.

(Refer Slide Time: 33:58)



Now if none of these c_i's are 0, which you have assumed earlier, then this sequence f_0, f_1, f_2, f_n forms strums sequence, this proof i will not take it, but will give this result. If c_i not equal to 0, then f_0, f_1, f_2, f_n forms a Strum sequence.

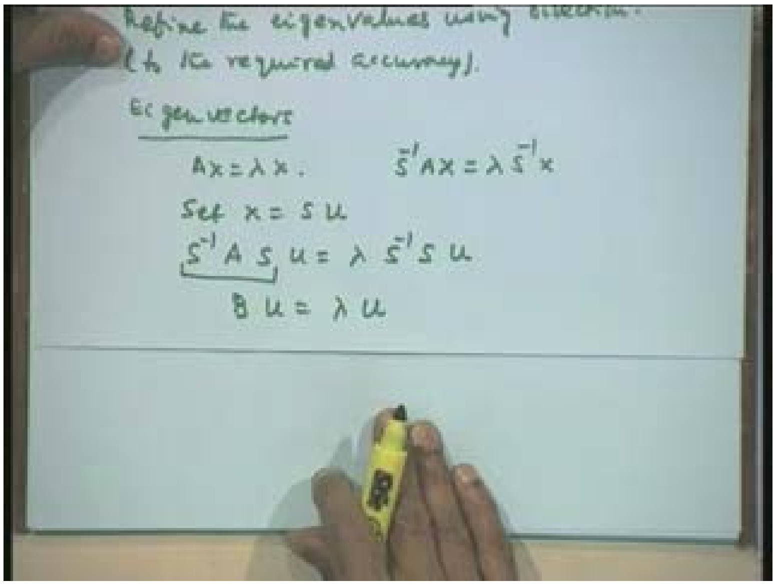
(Refer Slide Time: 34:27)



Now i can choose an arbitrary interval, any interval (a, b), let us suppose, i take an interval (a, b), then i can study the, if f_n is 0, it any one of the end points, obviously it is an Eigen values, otherwise i can study the number of changes of sign of this Strum sequence in the interval (a, b) and then locate the Eigen value in the interval (a, b), if it is there or it is not there. Therefore study the changes in sign, study the changes in sign of f_j and locate the Eigen values, and locate the Eigen values. Now we shall use bisection to refine this Eigen values to the required accuracy, refine the Eigen values, refine the Eigen values using bisection, using bisection, obviously to the required accuracy.

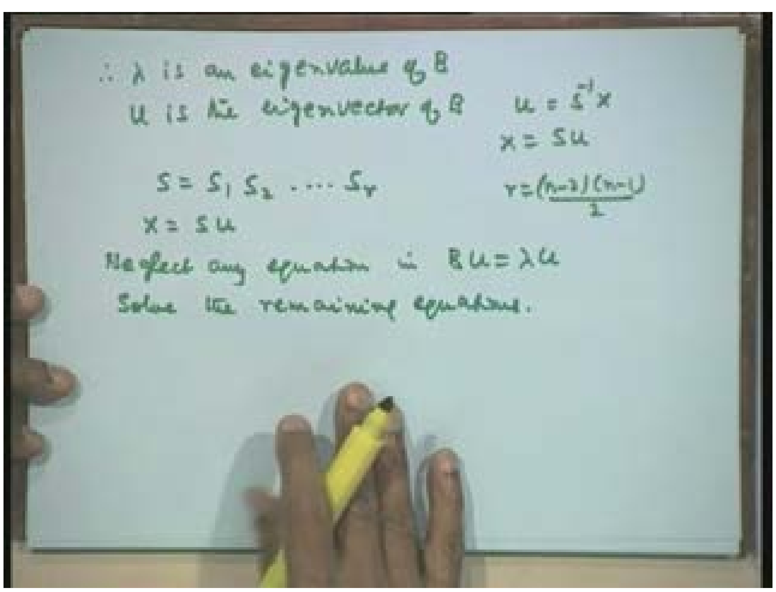
Now if you just want to find the Eigen values, the Givens method is the most powerful method, because the number of rotations are fixed and it works very fast because, it just changes of sign to be studied and it works very fast and it is one of the most powerful method for finding the Eigen values, if you want find Eigen values alone. Now let us see, how would you found Eigen vectors.

(Refer Slide Time: 36:28)



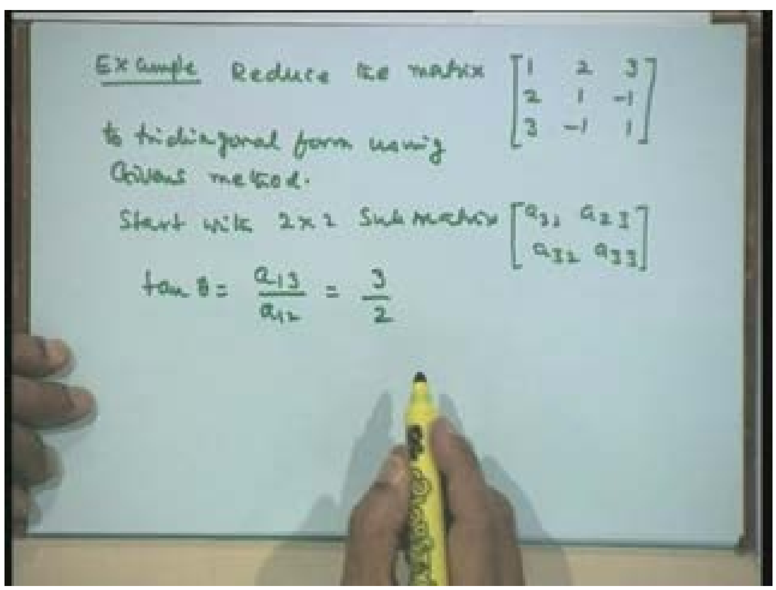
Now let us just remember what we have done, we started with our matrix A X is equal to lambda X, now we perform this operation, so what i do is, i first pre multiply this by S inverse, so i will have here S inverse A X is equal to lambda S inverse, lambda is constant, so i take it out. Now in order to bring your similarity transformation, S inverse A S, i will now set X is equal to some S into U, i will set X is equal to S into U, then this will read S inverse A S U is lambda S inverse S of U, the left hand side is what we are finally obtained B, that is your B of U, is lambda of U.

(Refer Slide Time: 37:28)



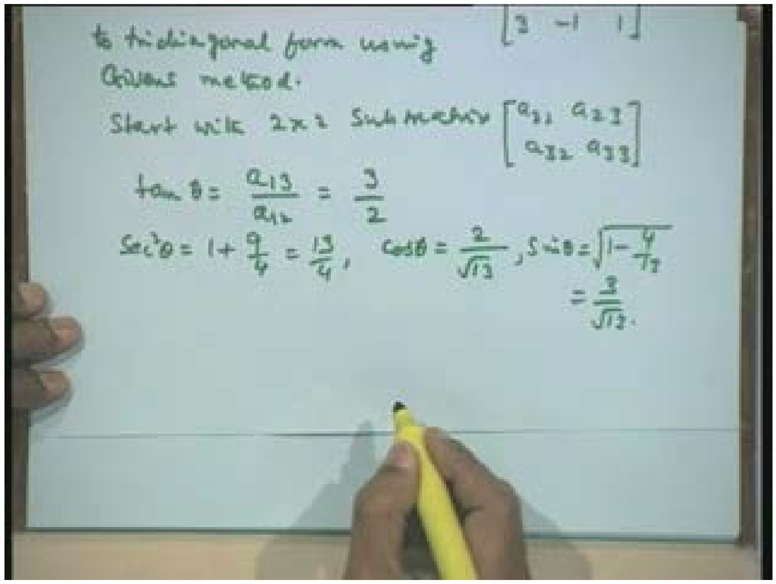
That means the Eigen values of A and B are same, that we know, therefore lambda is Eigen value of B, now U is the Eigen vector of B, now therefore you have lambda is an Eigen value of B and U is the Eigen vector of B, U is the Eigen vector of B, but what is U? we can just look back here, U is equal to S inverse of X, U is equal to S inverse of X or look at the other way round, X is equal to S of U, in other words if i, if i can find the Eigen vectors of the new matrix, tri diagonal matrix, then i can obtain the Eigen vectors of the original matrix by simply pre multiplying by the matrix of transformation S and that S is equal to, as in the Jacobi method this is simply s_r, s_r is equal to n into, r is equal to n minus 2 into n minus 1 by 2, we have exact number of rotation as this, so which i will call this as r, so S is equal to S_1, S_2, S_r and when once i determine this, i can now write down what is my S into U. Now if, now to, now to find out what is the Eigen vectors of B, the Eigen vectors of B is very trivial, because you know it is a tri diagonal system, you can drop any equation in, arbitrarily any equation in between, solve the system above, solve the system below, then it is by forward substitutions you will get the solutions and the Eigen vector U will be having 0 in the location where you have neglected the equation, suppose you have neglected i^{th} equation, then will set 0 in i^{th} location and the remaining elements of U will be the solution of the problem that we have done, because that is a straight forward solution of a problem, so for finding U the, neglect any equation, any equation in B U is equal to lambda U, so that is what we have, B U is equal to lambda U, neglect any equation in this, solve the remaining, solve the remaining equations. So if i^{th} equation is neglected, then U is equal to the elements that we have got and 0 in the i^{th} location and the elements that we have get over here, so we put 0 in the i^{th} location, that particular row that has been neglected and the solution remaining part is the solution that we have obtained and that will be my U, i mean once i find U, i know that X is equal to S into U, therefore this is a little bit extra computation in Givens method, if you want the Eigen vector also.

(Refer Slide Time: 41:05)



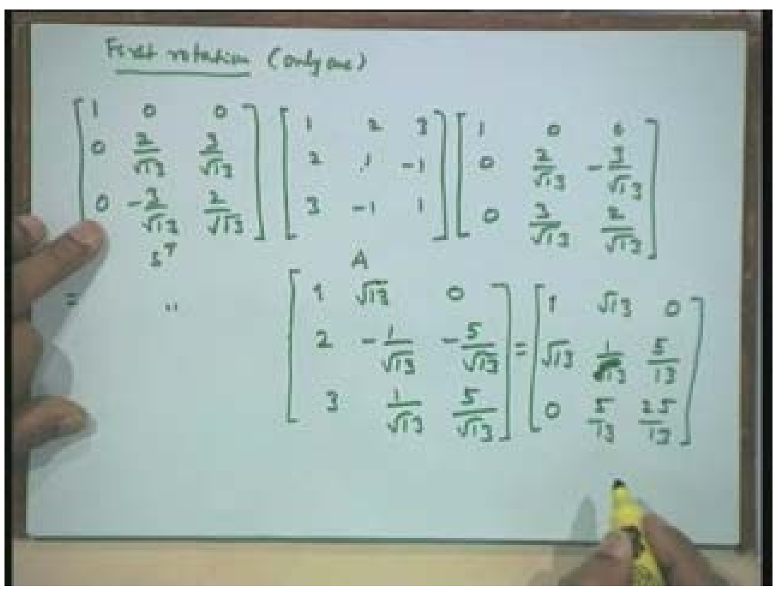
Okay, let us first of all take an example on this one, reduce the matrix that is, 1, 2, 3, 2, 1, minus 1, 3, minus 1, 1 to tri diagonal form using Givens method. Now we start with always 2 2 sub matrix, so we start with 2 into 2 sub matrix that is your, a_{22}, a_{23}, a_{32}, a_{33}. Now tan theta is, a_{13} by a_{12}, a_{13} by a_{12}, that is a_{13} is 2 and a_{12} is 2 that is 3 by 2.

(Refer Slide Time: 42:45)



Now let us do exact arithmetic, so that we have the, we do not have decimal numbers, so let us write it, let us write down secant square theta from here, that is 1 plus tan square theta, that is 13 by 4, therefore i can get cos theta from here, as a 2 upon root 13, then i can get sin theta as under root of 1 minus 4 upon 13, so that is 3 by root 13.

(Refer Slide Time: 43:25)



Therefore the first rotation will give me, first rotation indeed here there is only one rotation that is required because, we want the tri diagonal system, therefore i need the diagonal, i need one super diagonal, this sub diagonal therefore, there is only one element to be made 0 here, therefore there will be only one rotation, even the way i written first rotation, this is the only one, this is the only one rotation that we will have.

what will be the matrix that will have write down 1, 0, 0, 0, cos theta, this is transpose, so i will have here 3 by root 13, 0, minus 3 by root 13, 2 by root 13, this is your S transpose. Given matrix is, 1, 2, 3, 2, 1, minus 1, 3, minus 1, 1, so this is our matrix A, then i will have S matrix 1, 0, 0, 0, 2 by root 13, minus 3 by root 13, 0, 3 by root 13, 2 by root 13.

Now let us retain this matrix as it is and let us write down the product, so you will have this is, 1, 0, 0, at the first row gives me this, first row into 1 remains the same, then this is 2 into 2 4 plus 9 so 13 by root 13 so i will have root 13, this is 4 plus 9 so root 13, then here this is minus 6 and this is plus 6, so i will have 0 here. This row this column, again 2 into 1 is 2, then we are multiplying these two, this is plus 2 this is minus 3 so we will have minus 1 by root 13. Then this row this column, this is minus 3 minus 2, we will have minus 5 by root 13. Then 3 remains as 3, then this is minus 2 plus 3, therefore this is 1 upon root 13, this is plus 3 plus 2, that is plus 5 by root 13. Now, we are multiplying now, the first row, the first row remains the same, therefore this is 1, root 13, 0 remains the same because we are multiplying by 1 0 0. Now the second row first column, that is 2 into 2 4 that is 3 into 3 9 , so 13 by root 13, so i will have here root 13, then 2 into minus 1 that is minus 2 plus 3, so i will have 1 by root 13, 1 by root 13. Yes no root, yes no root, then i have this row this column, this is minus 10 and this is plus 15, so will have here 5, 5 by 13, then this row this column, this is minus 6 plus 6 this is 0, of course i can fill this elements but let us, it is symmetric so i can, without computing i can write it, but let us see that we have done the problem correctly. So if i multiply this and this, i will get here plus 3 plus 2 that is 5 by 13 and the last row gives me, this is plus 15 plus 10 so i will have here 25.

Now as i said that, in actual practice when you are doing, we, we need not do this element, this element because we know it is symmetric it has to come, when once you find this elements we can symmetrically write those elements, unless we have done some error in the problem, otherwise it will be the same. So we really do not have to waste the time in evaluating the other elements also, now this is the problem that we have been asked to reduce into the tri diagonal form, so we have, this is the required tri diagonal form, which has, we obtained in one rotation. Now the next step, if you want the Eigen values, i will take a simpler example, because numbers look to be little complicated here, i will take the simpler example and then show how we can obtain the Eigen values, using this Sturm sequence we can get it, but since, as i am saying the numbers are little big complicated, we need to, need decimals so will let take a separate example. remaining we will do in next class.