Numerical Methods and Computation

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Lecture No # 20

Solution of a System of Linear Algebraic Equations (Continued)

In our previous lecture we have defined the convergence of iterative methods. We have also obtained the optimal relaxation factor for the SOR method and if we can determine this value of the optimal relaxation factor accurately then the amount of computational cost reduced or the saving that we have made would be enormous; almost to the order of factor of ten or could be more also in a particular problem of a big system. Now let us take a simple example of how we are going to determine this optimal relaxation factor for a given method.

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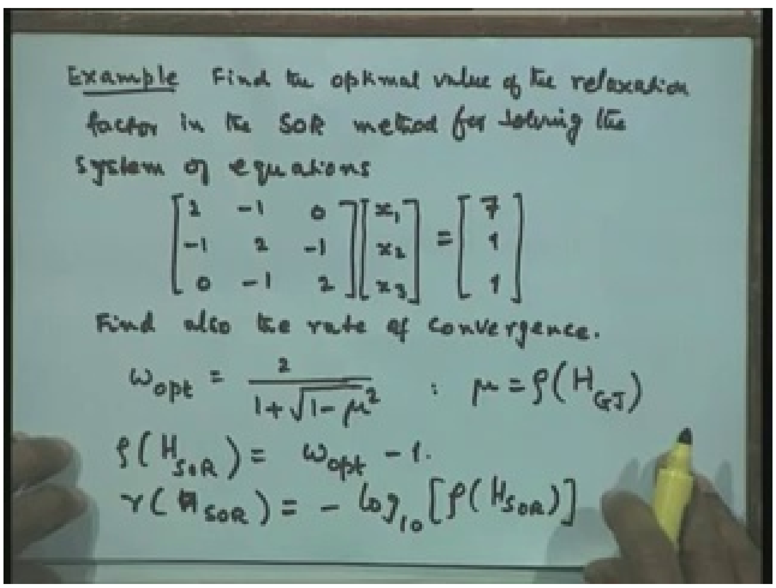

So let us take it this as an example. So we'll say find the optimal value of the relaxation factor in the SOR method. For solving we'll give a simple system of equations. For solving the system of equations two minus one zero minus one two minus one zero minus one two x one x two x three is equal to seven one one. Find also the rate of convergence. Now we are given two formulas for the finding the optimal value and the rate of convergence for the SOR method. And we are given the formula that the value of the relaxation factor which is optimum is given as two upon one plus under root of one minus of mu square, where mu is the spectral radius of the iteration matrix of Gauss Jacobi scheme. Therefore I would write down the iteration matrix for the Gauss Jacobi scheme finds its spectral radius, use that and find that omega optimal. Once I find the omega optimum, then I can write down the spectral radius of the SOR scheme as omega optimum minus one, then I can define the rate of convergence of this as minus log of; therefore I will have the rate of convergence of SOR scheme or we can just simply rate of convergence of SOR scheme is minus log base ten of spectral radius of H SOR. Therefore I need to find these three quantities to complete this particular problem. Now let us first of all write down what is our iteration matrix for the Gauss Jacobi scheme. So let me just keep it as it is.

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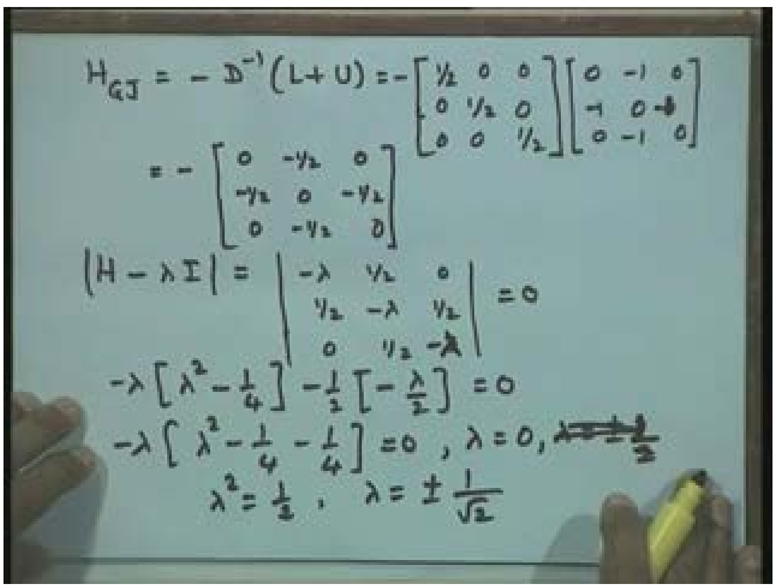

Now for the Gauss Jacobi scheme we have minus D inverse of L plus U. Now the diagonal matrix is two, two, two; therefore I can invert it immediately as one by two one by two one by two. So I will have here one by two, zero, zero, zero, one by two, zero, zero, zero, one by two. Then L plus U is the remaining part in the matrix; therefore what is left out would is, I will write zero minus one zero minus one zero zero zero minus one zero. So the diagonal has been picked out already. So what is left out is L plus U the remaining elements. Let us multiply it out. This gives us; this is zero minus half, zero, there is a minus one here, there is a minus one here.

Therefore I will have a minus half here, zero minus half here, zero minus half and zero. Now I will find the Eigen values of this. So let us just find out; let us drop the suffix +and simply write H minus lambda I determinant. Therefore let us observe this minus sign inside. So all these are positive quantities and I am subtracting minus lambda I; therefore I will have minus lambda plus half zero plus half minus lambda plus half zero plus half zero. This determined is equal to zero. Now let us apply the simple formula. Lets write down the expansion of this. This gives you minus lambda minus lambda that is lambda squared minus one by four. This is minus half and we are multiplying these two, so we will have lambda by two is equal to zero and I can minus lambda, I can take out common factor. So we will have lambda square minus one by four minus one by four. I have taken minus sign out, I will be left out with minus one by four again. Therefore I will have the values as lambda is zero lambda square is one by four that lambda is equal plus minus one by two.

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Let's write it here, lambda square is equal to one by two; let's write down three root lambda plus minus one upon root two.

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Therefore I can write down the spectral radius of these Gauss Jacobi as one upon root two. Now I would substitute this in the value of omega optimum. Therefore omega optimum will be two divided by one plus under root of one minus mu square. This is our mu. Therefore this is one upon two, so let us take this, two up so I will have here two times root two. This is one plus root two, this is one and this is our root two and this is approximately one point one seven one six. I have rounded it off to the next digit, so this last digit I have rounded it to the next figure. Therefore now I will have the spectral radius of H SOR is omega optimum minus one. So I will have simply zero point one seven one six as the spectral radius of the SOR matrix. Hence I will have the rate of SOR method is equal to minus log base ten one seven one six which is the value of point seven six five six. Now you could just try to compare; write here as to how do we compare SOR with Gauss Jacobi itself. If I look at this we had written here the spectral radius of the Gauss Jacobi is one upon root two that is approximately point seven, this is one point four, one, four; ratio is about approximately point seven. Therefore in place of zero point seven we shall use zero point one seven, one, six. This gives us the value of the rate of convergence zero point seven six five six. The rate of convergence of Gauss Jacobi is much lower because we are having minus log ten approximately point seven is what we have written here. For the rate of convergence of the Gauss Jacobi and which is a much smaller quantity than this, as we approach zero the log of zero goes to infinity minus infinity. As we approach zero, it is going to bigger and bigger. So this is a bigger number so we will have the rate of convergence much smaller number. Therefore you can see the convergence rate here itself, such a huge difference in the convergence rate. Therefore this particular problem will converge; SOR will converge fast compared to the Gauss Jacobi scheme.

Now let us go to the other problem we mentioned about the Eigen value problems.

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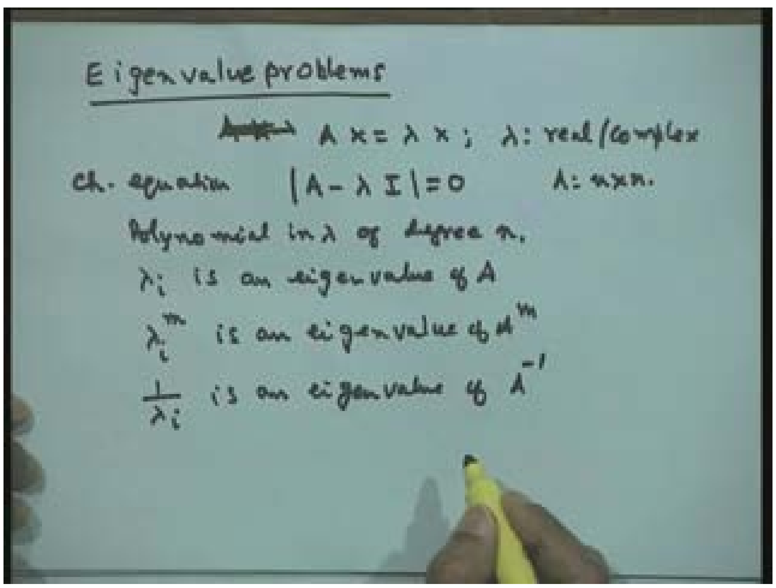

So let us discuss about the Eigen value problems. Eigen values problems is also very important problem in the areas of mechanical engineering, civil engineering and many other areas also. It does occur very often. Now we have given brief introduction on what is an Eigen value. Let us just remember what we have done there and little bit more so that we can use that for finding the numerical methods. So the basic problem is that we have Ax is equal to lambda x. We have Ax is equal to lambda x lambda is real or complex number. Then we have defined the characteristic equation as the determined; A minus lambda I, if I expand this, it is polynomial in lambda of degree whatever the degree of A. So this is a polynomial in lambda of degree n. We have taken A as a n into n matrix therefore we get a polynomial of degree n in lambda. The roots of these equations are the Eigen values of which may be real or complex. But we know the property that if lambda i is an Eigen value of A, then A square will have Eigen values lambda i square; A to the power of m will have Eigen values lambda i to the power of m, so the mth power of this is an Eigen value of A to the power of m or let us assume that A is non singular, that means the determent of A is not equal to zero. Then one upon lambda i is an Eigen value of A inverse.

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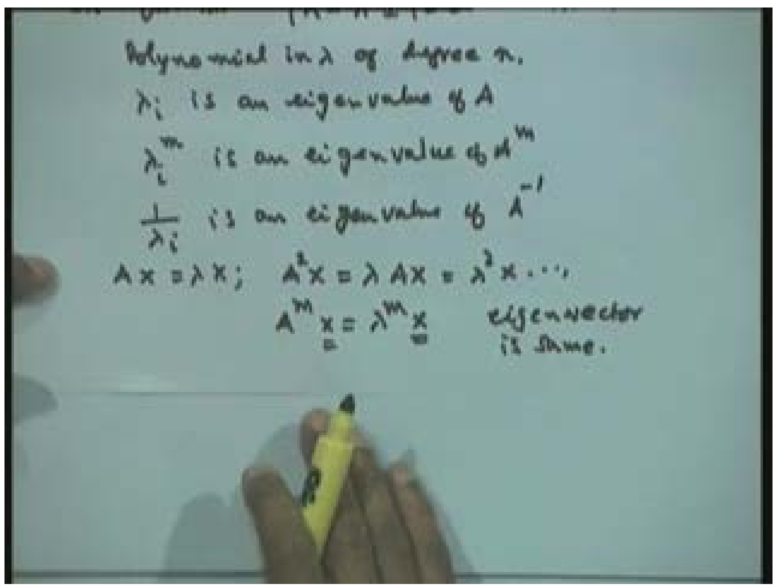

Now what is also important to know is about property to the Eigen vector. Let us just look at these two last two properties and see what will be the Eigen vector. If I start with Ax is equal to lambda x, I pre multiply by A. So this will be A squared x is equal to lambda into Ax lambda is a number so I can take it out. So this will be lambda square x, it is from these that we have concluded that lambda square is an Eigen value of A square. So if I continue all I get A to the power of mx is lambda to the power of mx. Therefore the Eigen vector is the same the Eigen value is lambda i to the power of m will be the Eigen value of A to the power of m but the Eigen vector is the same. Therefore this is important for us to know that the Eigen vector is same the Eigen vector doesn't change.

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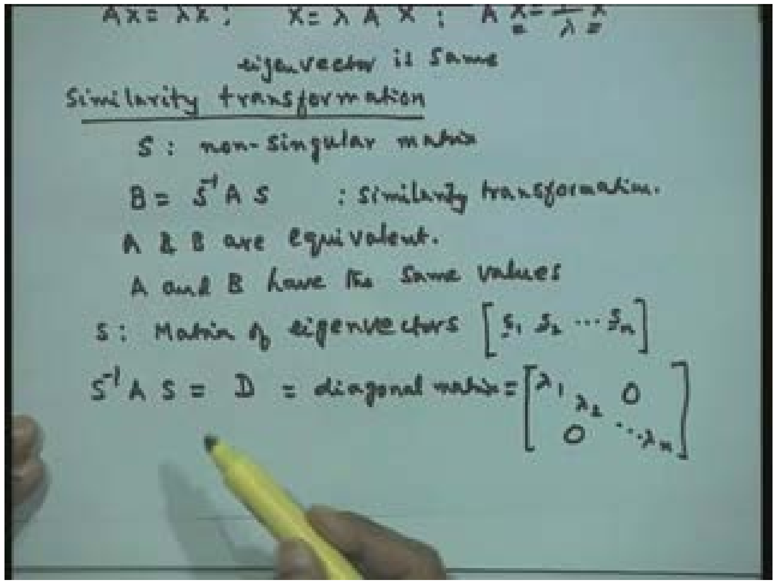

I want to find the property for the inverse so I again start with Ax is equal to lambda x pre multiplied by A inverse. So I will have lambda into A inverse of x then I bring lambda here in the denominator. So I will have here A inverse of x is equal to one upon lambda of x. I will have written right hand side first, this is one upon lambda of x. Therefore this is showing that one upon lambda is the Eigen value of A inverse and x is again the same; therefore the Eigen vector in this case is also same Eigen vector is same. Another important concept which you have done earlier is a similarity transformation. Let us define what the similarity transformation is. I take any non singular matrix S; S is the non singular matrix, then I form the matrix B, B is S inverse AS; S inverse AS; then I define this as a similarity transformation. Now we also say that A and B are equivalent matrices in this case sometimes we say that A and B are equivalent matrices. Now in what sense they are equivalent is that, that we can show from the property of the Eigen values that A and B as the same Eigen values. This actually implies that A and B as the same Eigen values, have the same Eigen values. The Eigen vectors are different, the Eigen values are the same Eigen values. Therefore the importance of the particular property is that that given a matrix A if I can use a particular non singular matrix to reduce A to simpler form then I can find the Eigen values of the new matrix easily and hence they are the same Eigen values of the original matrix. Therefore I can find the Eigen values of the original matrix. This is the basic concept that we shall use in all the numerical methods for the Eigen value problems.

Now the second property in this is we picked up S is any non singular matrix. Let us suppose that S is the matrix of the Eigen vector. Let us suppose it is a matrix of Eigen vectors. Let us put some S one S two vectors, these are vectors S one S two Sn. These are column vectors S one is a column vectors S two is second column vector Sn is the nth column vector. So this is the matrix of Eigen vectors, then S inverse AS would always reduce to a diagonal matrix. It will a diagonal matrix diagonal matrix and the Eigen values are placed on this. This result is obvious from us because we just made a statement that if A and B are equivalent, A and B have the same Eigen values. Now S inverse AS has been reduced to a diagonal matrix and hence the Eigen values of A and D are same and the Eigen values are the D are simply the diagonal elements itself. Therefore lambda one lambda two lambda n itself will be the Eigen values of this particular matrix. Further more in this case there is one to one correspondence between the position of this column and the Eigen values; what by mean is that S one here is the Eigen vector corresponding to lambda one S two is the Eigen vector corresponding to lambda two Sn is the Eigen vector corresponding to lambda n. That means in this position there is one to one correspondence between the columns of S and the Eigen values located on the diagonal.

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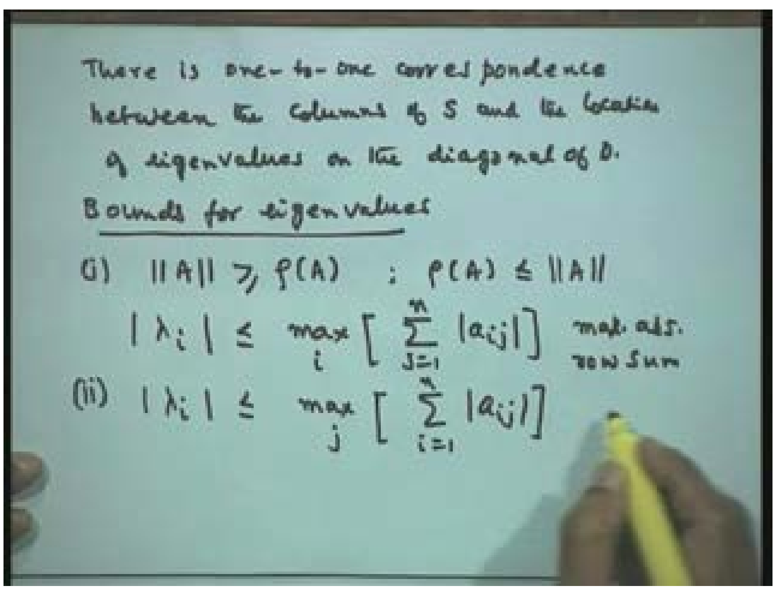

So there is one to one correspondence. There is one to one correspondence between the columns of S and the location of Eigen values on the diagonal of D. The method that we are going to describe just little later would concentrate on reducing the given matrix A to this diagonal form.

Hence I have the Eigen values of this particular problem and there automatically I have the Eigen vectors located as the columns of this matrix that I am producing it over here. Therefore I have both the Eigen values and the Eigen vectors in one particular step itself and I can give both of them. Now in many applications you may not really require all the Eigen values for a particular application you need to know, what is the largest Eigen value that this system has got or you want to have simply a bound of the Eigen value and not really the Eigen values to get a bound of an Eigen value is much more simpler problem then actually finding the Eigen values. So let us find how you can get some bounds for the Eigen values. Now one bound which we already done earlier while describing norm. If you remember we have shown that norm of A is always greater than spectral radius of A this property we have proved. I will rewrite it as spectral radius of A is less than equal to norm of A. Now the definition of spectral radius A is, it is a largest Eigen value in magnitude therefore the left hand side is magnitude of lambda i. So all Eigen values satisfies therefore the left hand side would imply that magnitude of lambda i is less than or equal to we use any norm that we know here that is the maximum absolute row sum.

Let us take this as maximum absolute row sum that is maximum with respect to i of summation of all the elements in the row that is we have called it as maximum absolute row sum. So this is a bound for our Eigen value and I will use this as second value. I will take the maximum absolute column sum as my second bound magnitude of lambda i will less than or equal to maximum with respected to j i is one to n aij. So this will give as maximum with respect to j this is maximum absolute column sum. Now these are the two obvious bonds that we get from here but these are very rough bounds but we would now like to get which are some more sharper than these ones and for this we would like to look at our basic problem.

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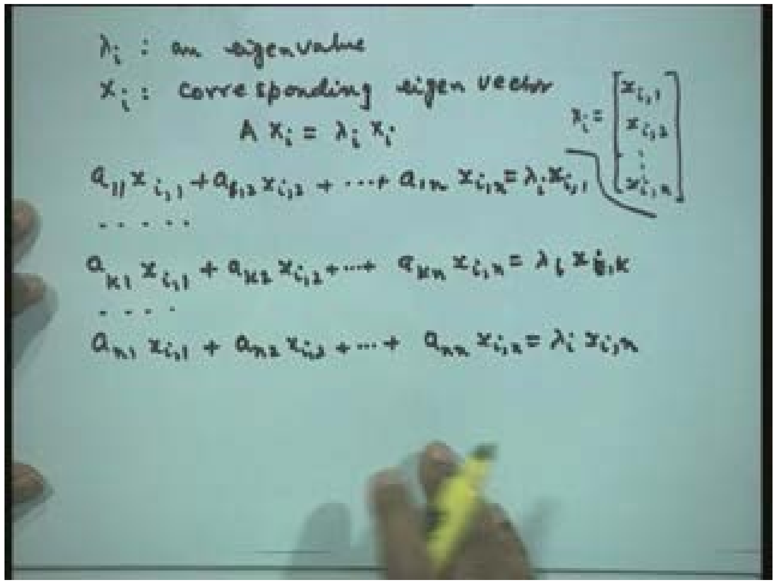

Ax is equal to lambda I, let us pick one Eigen value lambda i is an Eigen value. Let us take a particular Eigen value and let us take the corresponding Eigen vector, this is the corresponding Eigen vector; that means we are talking of A xi is equal to lambda i xi. Now I need the components of xi, so let us write down xi, its components say xi one xi two so on xin. So let us write down the components of the vector xi is with suffix as i one, i two, i three, so that will be the components of the vector xi. Now let us open up A xi into this one, therefore a one one into this plus a one two into this and so on. So let us write down this system a one one xi one ai two xi two a one two a one n xin is equal to right hand side is lambda i xi one. In right hand side is lambda i xi, so lambda i multiplies this first one xi minus one, so I need the kth one ak one xi one ak two xi two akn xin lambda i xk kn xin that is equal to lambda xk xi, I should have put here xik. This is xik and the last element is xn one xi one an two xi to ann xin lambda i xin. Now I will take a particular equation arbitrarily any equation; let me pick this equation as the kth equation. So this is my kth equation. I will take this as my kth equation, what I do here is of all these components in this line xi one xi two xi three xin, I will pick up that particular largest element in largest component in magnitude.

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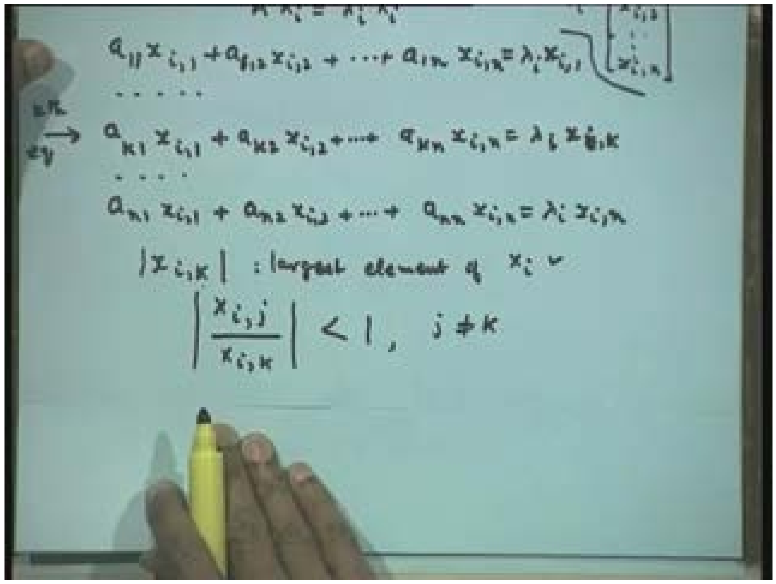

So xik a particular this one I will take this as the largest element that means what I am really doing here is I have taken this xi, I find the largest element in the magnitude in this components and I find that xik is largest element in magnitude. So I have found that xik is largest element in magnitude of all these components. Now I will pick up that particular equation to tackle it and find the bound and if that xik is largest element in magnitude I will pick up the corresponding equation that the kth equation that I have here. So once I pick this equation which is the largest element in magnitude, what does this imply; the ratio of xi any j divided xik in magnitude is going to be less than one. Because xik is largest element in magnitude, if take the ratio of other components j naught equal to k, then this is always going to be less than one; that is the definition of our largest element in of xi.

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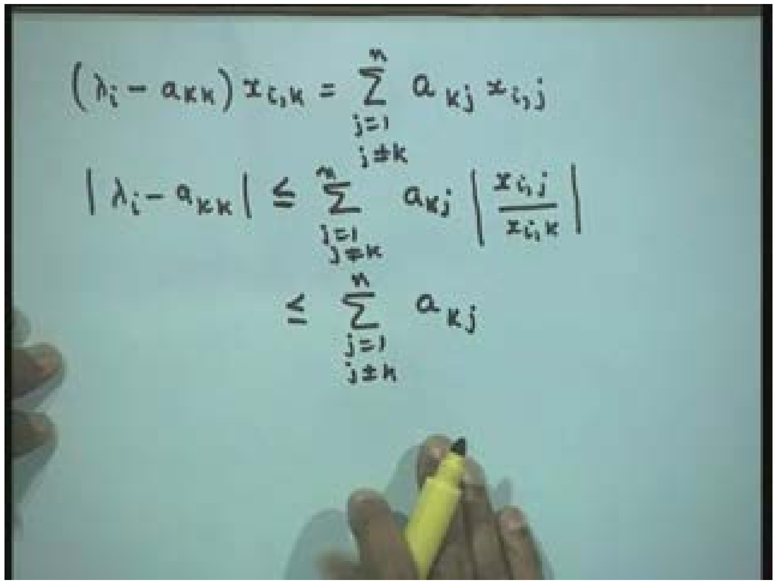

Now let us take from this xik; this element that is here to the right hand side, here this is your akk element will go to the right hand side and combine with this one so what I will have here is lambda i minus akk lambda i minus akk and let's retain our xik. Also further moment I have taken to the right hand side, then I will write this in summation notation, this is remaining elements of your this particular root; that is your akj xij j is one to n j naught equal to k. I have taken all the elements written here except the term corresponding to j naught equal to k. Now I divide by this xik and then take the magnitude of it so I can write down magnitude of lambda I, this is equal to summation of j is one to n. Now once if I take the magnitudes inside I shall then insert less than or equal to this that is equal to xij by xik but these ratios they are all strictly less than one. Therefore this would imply that this is less than equal to simply j is equal to one akj, modular's of akj. If you just compare what we have taken the absolute row sum of which one element as gone to the left hand side as combined with the lambda i and these are the remaining elements here and these are the some of the magnitudes of this remaining elements. Now but however we do not know what is k; we have picked up but in practice we do not know, therefore what we do is we will take the union of all these inequalities.

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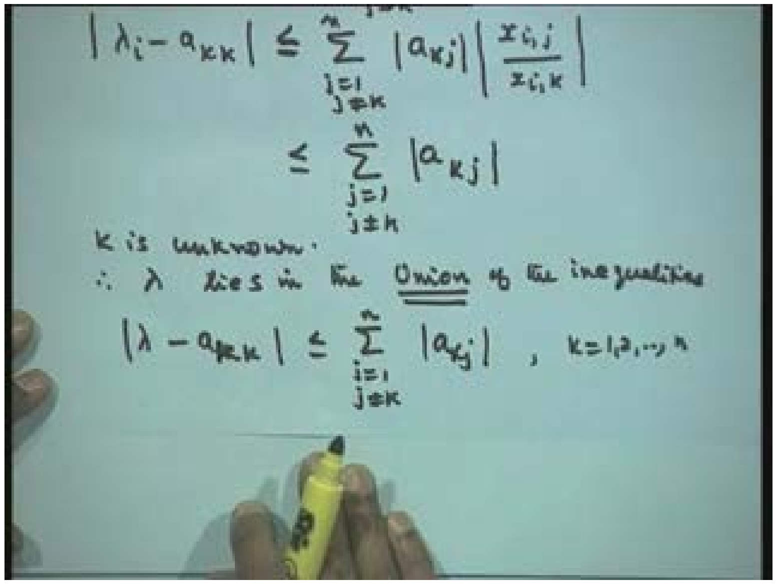

So k is unknown, therefore lambda lies in the union at the moment. Let me write down inequalities but we will latter give a another name to this; inequalities lambda minus aii or akk you can written akk akk less than or equal to j is one to n j naught equal to k akj k is one two three so on n. Therefore there are n inequalities of this particular type and we will get n of them and the union of all this will be the required bound. Now this we will take as the third bound that we have done.

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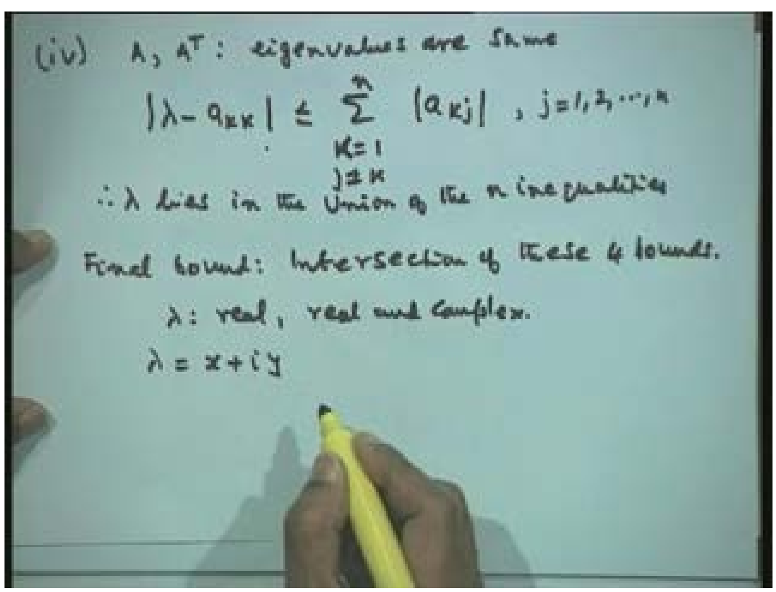

Now we know that the Eigen values of A and A transpose are same. So we know if I take the A and A transpose for both of them the Eigen values are same. Therefore this inequality would simply become magnitude of lambda minus akk is less than or equal to summation i. We are taking k here, so let's retain k k is equal to one to n magnitude akj j naught equal to k for all j; that means we are forming same thing with respect to the columns and the union again lambda lies in the union of the n inequalities. Now we have got here essentially four bounds, the four bounds are these; one is simply the maximum absolute row sum, this is maximum absolute column sum; then we have obtained this third bound as this and this the fourth bound. But all these bounds are independent bounds. Therefore if each one of them giving a particular bound and all of them are independent the intersection of these bounds should be the final required bound. Therefore the final bound will be intersection of these four bounds, these would give us a quite good estimate of this. But we know that the Eigen values lambda, they are real or real and complex or real and complex. Now if they are real and complex that means let us take lambda is equal to x plus iy suppose they are complex lambda is x plus iy.

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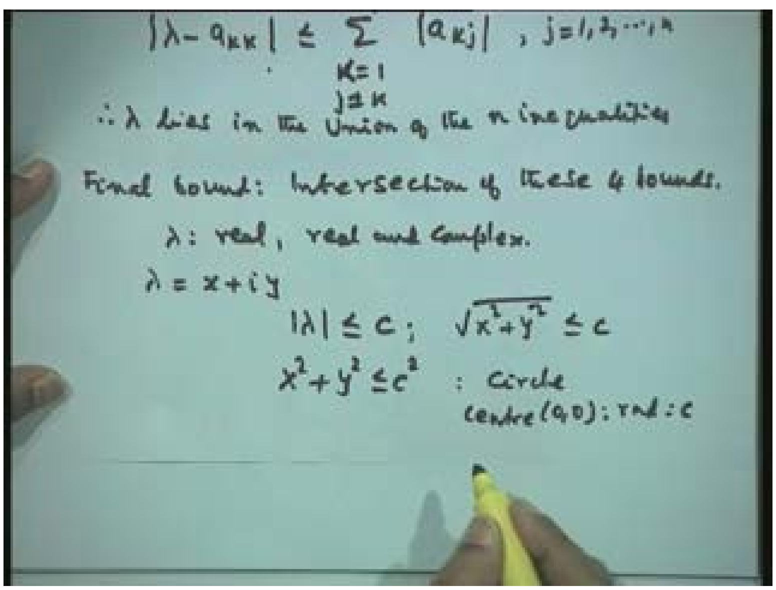

Then all these the first two bounds are stating that magnitude of lambda less than equal to sum constant c sum number which is a number if it is complex this would imply this under root x square plus y square is less than equal to c or which implies x square plus y square is less than c square which is a circle with center at zero, zero and radius c; therefore all the bounds that we have got here at that time we have written it as inequalities. Now we can show that all of them are circles when once the Eigen values become complex they all becomes circles therefore these are called the intersection of these n circles.

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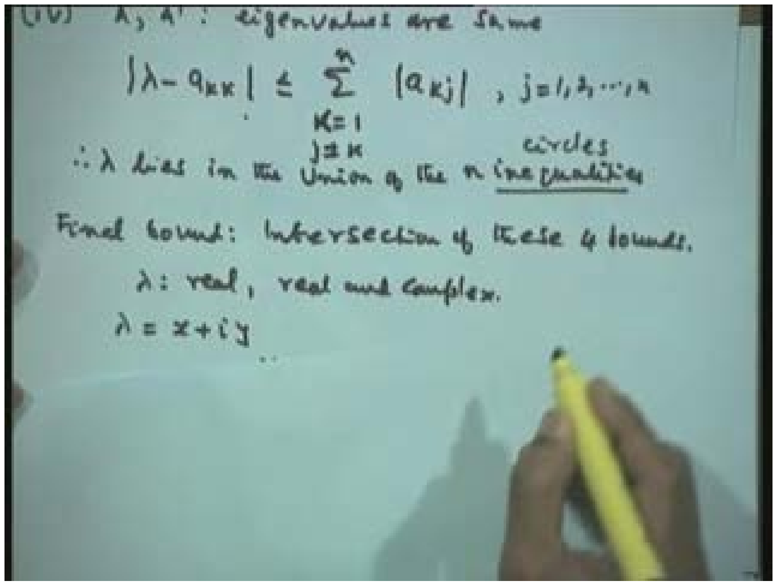

Now in place of these inequalities I can now write down that these are n circles. Similarly in the other case and these are all called Gershgorin circles.

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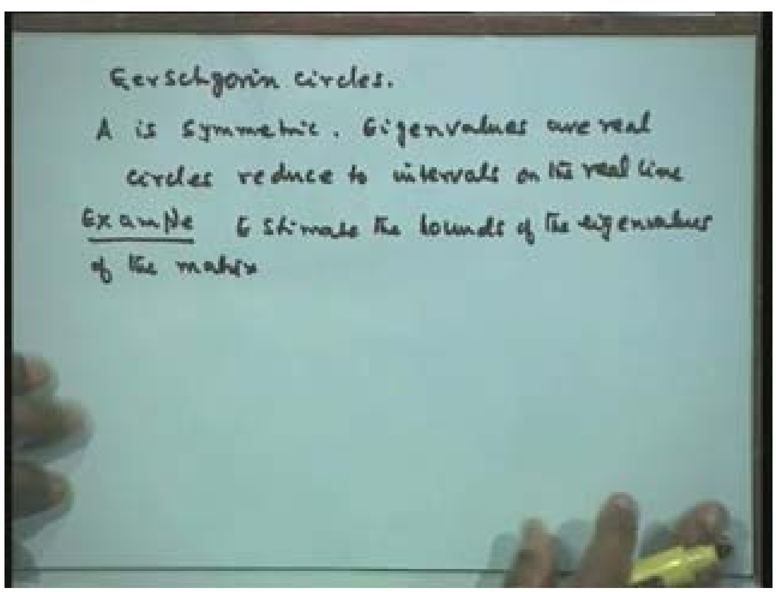

They are called a Gershgorin circle that means we are talking of intersection of circles. In the complex plane you have got x and ah the real part is x, imagine the part is y so we go to the complex plane and in the complex plane we can draw the circles and from the circles will find out what is the bounds of this for the complex Eigen values also. But a particular case is when A is symmetric when A is symmetric the Eigen values are no longer complex they are all real. So we know that Eigen values are real therefore these bounds would no longer be circles, it will be just simply intervals on the real line. Therefore they will be intervals; the circles reduce to intervals on the real line. Now let us just take an example how longer we are going to find this Gershgorin circles, let us taken an example of this.

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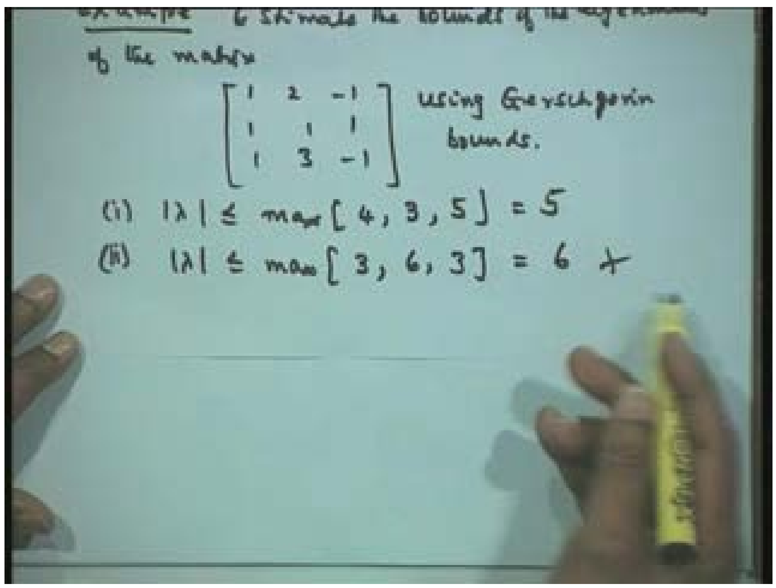

Estimate the bounds of the Eigen values of the matrix is one two minus one one one one one three minus one using Gershgorin bounds. See an immediate application of this would be let us suppose that the in our iteration procedure of Jacobi or Gauss Seidel or SOR. Let us suppose the components of the coefficient matrix is point two point five point one minus point two minus point three minus point three; now I can immediately find the spectral radius of it because I know each Eigen values would be lying within this bound circle. So I can immediately say the norm of this is less than one, so I am guaranteed that this iteration is going to converge. I just don't have to do any other proof; I can straight away get the bound for the Eigen value form the matrix itself and then say our convergence is for the iterative procedure is guaranteed so that is immediate application of this Gershgorin bounds.

Let us write down all the four bounds magnitude of lambda is less than or equal to maximum of absolute row sum; absolute row sum is two plus three plus one four one plus one plus one that is three one plus three plus one that is five, so that is equal to five. The second bound is magnitude of lambda less than or equal to absolute column sum, this is three, this is six, this is three, this is equal to six. Now you can see that these are all independent bounds, therefore you can immediately throw this away because magnitude of lambda this is a circle with smaller radius than this is the bigger radius and we want finally the intersection, therefore this circle is not to be considered because it is within this point.

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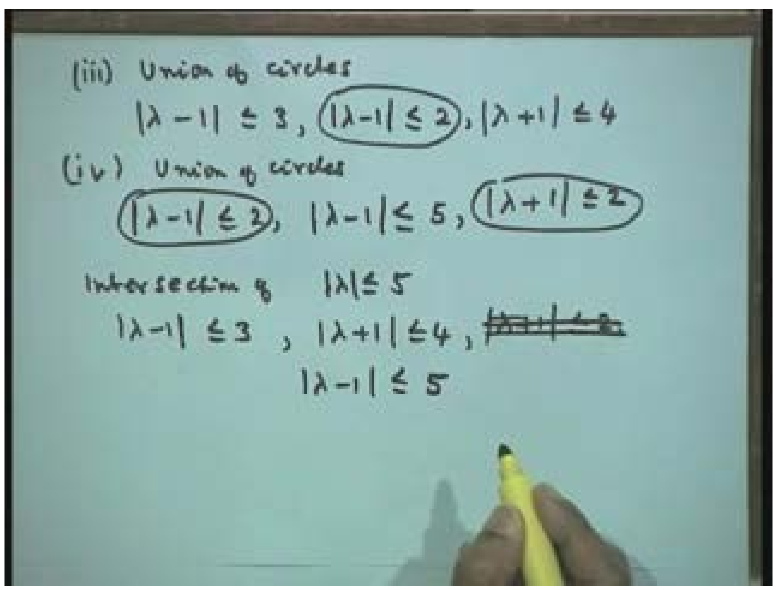

So let us take the third one, what is important here is union of circles. Now lambda minus A one one that is lambda minus one that is I am taking first element lambda minus A one one will be less than or equal to sum the absolute sum of the remaining that is two plus one three. I have taken the second pivot as one and the half diagonal element as one and one so I will have this as two and from the third row I get lambda plus one is less than or equal to four. Now before I proceed to the fourth one we can look at whether anyone of these bounds is contained in the other one. Now let us just see that this is a circle we centre it one radius three centre it one radius two this circle is inside this circle. So we don't need to consider this particular one but these two are intersecting, so they are we have to consider both of them. Now the fourth bound is union of circles. Now let us take the with respect to column; so first column gives magnitude lambda minus one is less than or equal to two; the second column gives us magnitude of lambda minus one is less than equal to three plus two, that is five and the third column gives lambda plus one is less than equal to two. This middle one if you see this circle was centre it one radius five, this is centre minus one radius two, this is centre one radius two both of them are contained in this one. So these two bounds we need not even consider them because they are contained in this. Now finally we need intersection of all this four. Now this is a much larger circle, this is the largest circle which we have excluded. Now we finally have to write down the intersection of bound is intersection of the first bound magnitude of lambda less than or equal to five that is the first bound that we have got from there. Then from the second union this union of circles magnitude lambda minus one less than or equal to three magnitude lambda plus one less than or equal to four and magnitude lambda plus one less than or equal to two. Sorry here this is magnitude of lambda minus one less than or equal to five, I have taken this picked up this one this one and this one. Now it would be interesting for us to just look at how it is going look like.

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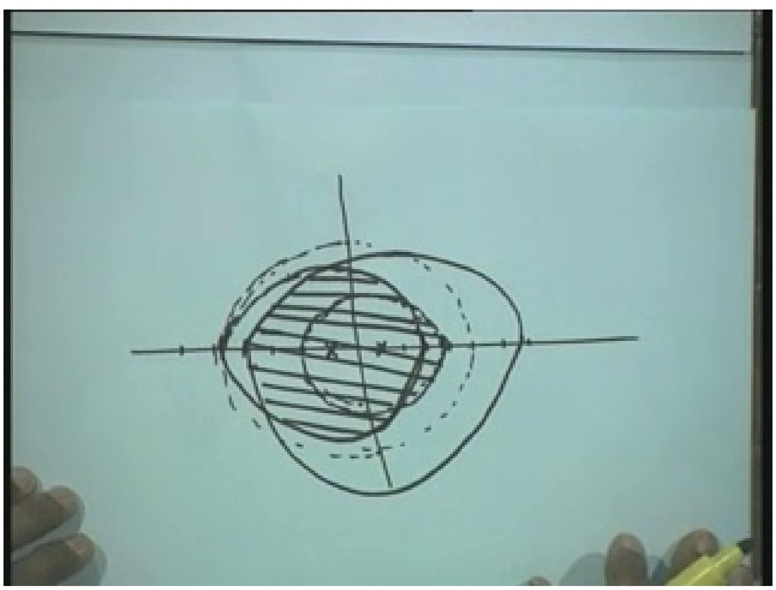

Let us just draw a nice diagram. et us try to draw a diagram of this one and lets it make it very small; one two three four five six of them one two three four five six. The first one is circle with centre at the origin radius five, so this is our circle that the first circle. Now this second one is centre at one radius three centre is at one and radius is three of them; so this goes through this is the centre it goes through this and it goes to this it goes through this. So I will have, circle with it centre at minus one zero radius four, so the centre is at this and it passes through this and for this. So I draw this diagram again and finally centre at one radius five; so this is x and through this for this is a huge circle that comes like this. I want intersection of this all of them. So what is common for all of them is only this; it should lie in all of them, all the independent bounce it should like the first; we have to take the union of this two; this is a group. So this group you have to take together then intersection with this union of these two and intersection with this. So union of this two would give us the union of this union of this and the other circle and intersection with this final circle. So this part will be excluded because this goes out in the intersection, so this will be the at the bounds for the Eigen values.

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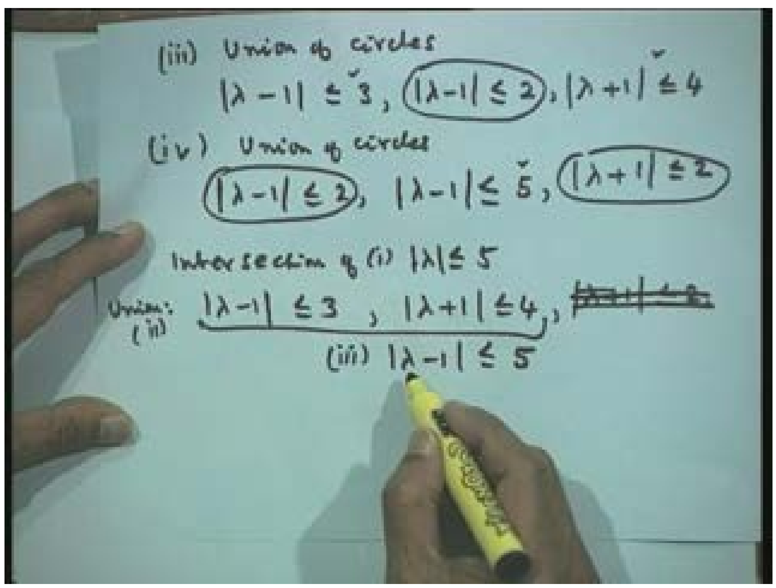

So you have to take the ah its important is this is a union of these two now it is not the intersection of these four it is the intersection of this circle to the union of this circles and three this circle. Therefore these are the three groups that we have and intersection of this two three groups will be the required bound for the Eigen value. So this how we can draw the location of the Eigen values and then find out that the the Eigen values lie in this particular region. Now let us take a simple example for a symmetric matrix, let us take this example.

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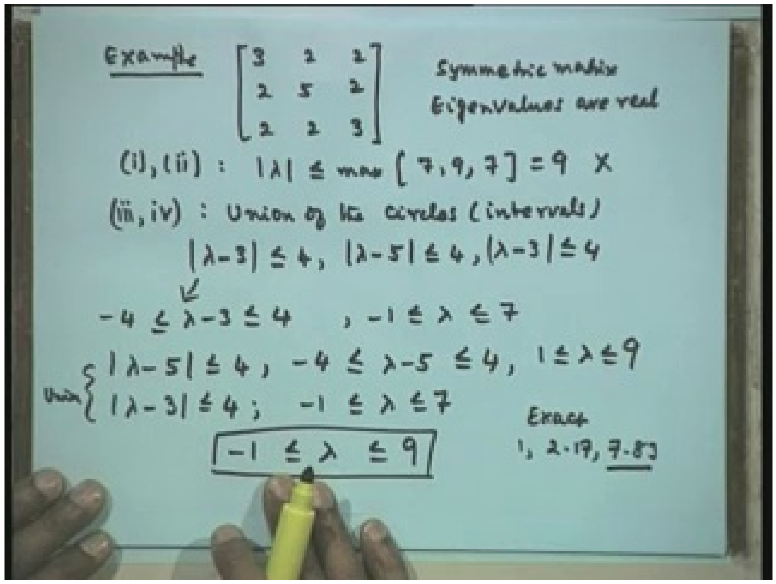

Let me take this as three two two two five two two two three. Now this is a symmetric matrix, therefore Eigen values are real Eigen values are real, therefore it will be no longer be circles it will be only intervals. Now since it is symmetric the bounds one and two will be the same because absolute columns sum will be same as absolute rows sum. So both of them would give you the same bound that is magnitude of lambda is less than or equal to maximum of that is equal to seven nine seven that is equal to nine and the third and fourth both of them would again give the same one that gives you union of the circles; these are not circles these are now intervals. They become intervals that is equal to magnitude lambda minus three is less than or equal to four lambda minus five less than or equal to four lambda minus three less than or equal to four. Now we can see that this is too big therefore the intersection we can through away this particular bound because this is much smaller than these bounds. Now let us open it up, the first one let us write it this is minus four less than lambda minus three is less than or equal to four so is an interval. So let us open it up, this gives you minus one less than or equal to lambda less than or equal to seven, so I have taken it this side. The second one gives us lambda minus five less than or equal to four that is minus four less than or equal to lambda minus five. Lets write lambda minus five less than or equal to four, I bring it to this side so I will have one less than or equal to lambda less than or equal to nine. Then have the third one lambda minus three less than or equal to four is same as the first one, so this would also give you minus one less than lambda less than or equal to seven. Now I need the union of this circles, I want the union of this two the union would be I want the union of this; that will be minus one less than or equal to lambda less than or equal to nine. Therefore this is the bound for the Eigen values of this system and the exact Eigen values of this I will give you the values this is one two point one seven seven point eight three. These are the exact Eigen values of this particular three by three system which you can see that we have got very good estimate of the Eigen value. This is bound a seven point eight three and we have got here is nine as the interval.

Now you have might have noticed one thing that this one and two has indeed not given any bound in this one; therefore this is a very rough upper bound compared to this other union circles that is Gershgorin circles. So usually this is the three bounds; three and four are the best bound that will be available. The intersection of them would be the required bound.