Numerical Methods and Computation

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Lecture No # 19

Solution of a System of Linear Algebraic Equations (Continued)

Now in the previous lecture we gave some definitions of norm of matrix. Let me just illustrate through an example.

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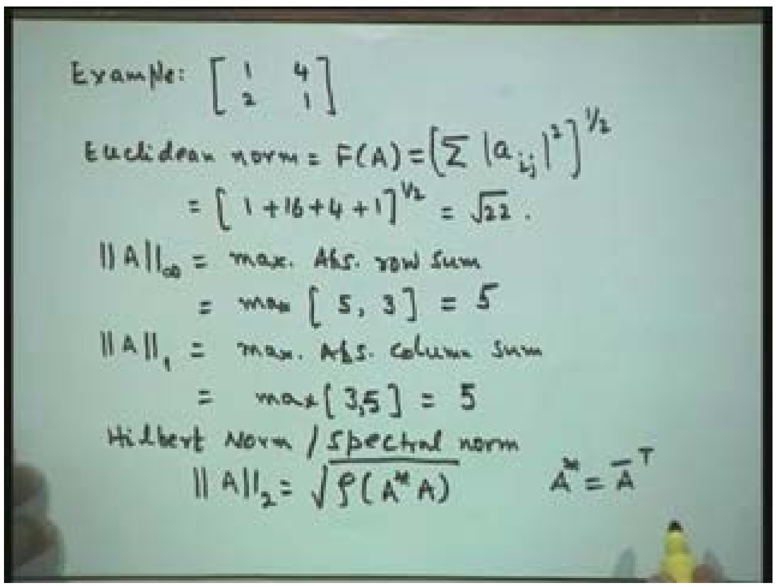

Of a simply two by two matrix, let us take this matrix; 1, 4; 2, 1. Now the first norm that we have defined is the Euclidean norm which we denote by F of A and this definition was summation of absolute values of all the elements, square of that, then we take square root of this sum. Therefore this will give us squares of all these elements, one plus sixteen plus four plus one to the power of half, so that gives us root twenty two as the measure if you take the Euclidean norm. Now we have taken the maximum absolute row sum and maximum absolute column sum also as a norm. So let us take those definitions, this is maximum absolute row sum. So that means maximum of absolute row sum one plus four, that is 5, 2, plus one that is three, so this is five. Then we have given the maximum absolute column sum also as a measure. So this is maximum absolute column sum and that is maximum of this; this is three, this is five, so this is three and five, that’s again equal to five. Then the last definition is given as the Hilbert norm or the spectral norm.

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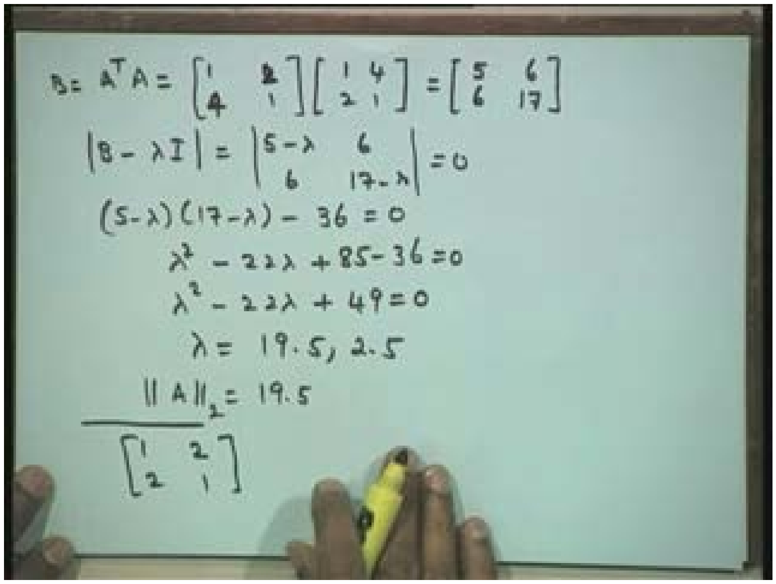

Let’s write down the definition of this. The definition of this was square root of spectral radius of A star A. A star is the conjugate transpose of A. Now in our case a matrix is not symmetric. Therefore we need to form this particular matrix. But here the matrix is real. Therefore conjugate is same as the given matrix so this will be simply A transpose. So what we need therefore is A transpose A that is 1, 4, 2, 1 and this is 1, 4, 2, 1. Therefore this product is one plus four that is equal to five; this is four plus two that is six; four plus two is six; sixteen plus one seventeen. Now I need this spectral radius, that is largest Eigen value in magnitude of this matrix. So I find the Eigen the values of this. Let us call this as B, so that I have B minus lambda I that is equal to five minus lambda six seventeen minus lambda, determinant is equal to zero. I can expand this and write this as five minus lambda into seventeen minus lambda minus thirty six is zero. I simplify lambda square minus twenty two lambda plus eighty five minus thirty six is zero which is same as lambda square minus twenty two lambda plus forty nine is equal to zero. Now I can find the roots of this quadratic straight forward. I will give you the values of this; this is 19.5 and 2.5. So that in this case the spectral norm is 19.5. Now if the given matrix is symmetric; for example, in this case maximum absolute row sum and maximum absolute column sum will be identical because it is symmetric matrix. Furthermore we mentioned that if it is symmetric then the A transpose is equal to A. Therefore we have shown that this will be simply equal to the spectral radius of the given matrix itself. So if A is equal to A star that is A is A transpose; if it is real then this is the square of the Eigen value, therefore under root will be this. Therefore if I take a symmetric matrix then everything will get simplified. For example like the matrix say let us take 1, 2, 2, 1.

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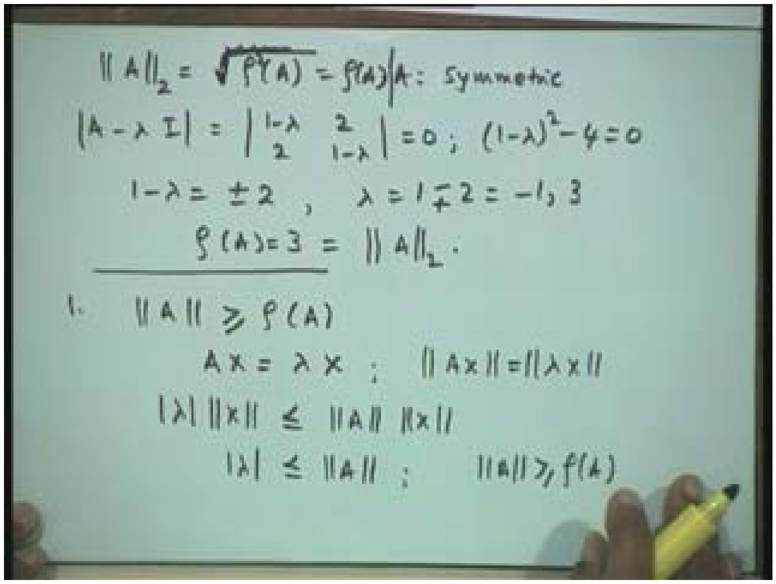

Now if I take this matrix 1, 2, 2, 1, lets find out what is your norm of A to that's Hilbert norm. Hilbert norm will be simply the spectral radius of A, because A is symmetric. Therefore I need only the Eigen values of A so I can find out A minus lambda I determinant of it. So I can find one minus lambda two one minus lambda is equal to zero, which gives me one minus lambda whole square minus four which is zero. We can take four to the right hand side and write one minus lambda is equal to plus minus two or lambda is one minus plus two that is minus one and three. The largest Eigen value in magnitude is three, therefore spectral radius of A is equal to three and in this case this is also equal to the spectral norm of matrix.

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Let' retain it and then put spectral radius of A, therefore spectral radius of A is equal to three, that is equal to norm of A two. Now we are talking of the Eigen values of a matrix and norm of a matrix. There must be some relation between the norm of matrix and Eigen values of the corresponding matrix and we have one result in which it says that the norm of a matrix is always greater than or equal to spectral radius of a matrix. Now the equality we have seen in the particular case in this example right here is, when the matrix is real symmetric the spectral norm is equal to spectral radius of A. This norm is equal to spectral radius of A. So the equality will be in that particular case but the proof is this just one line answer from the definition of a Eigen value of a matrix. So if we write down Ax is equal to lambda x, the definition of Eigen value problem I take norm on both sides. So I will write down norm of Ax is equal to norm of lambda x. Lambda is a number which is real or complex therefore it comes out as magnitude of lambda. So I will first write the right hand side so that is magnitude of lambda norm of x. The norm of a product, norm of A into x is less than or equal to norm of A into norm of x is this will be less than or equal to norm of A into norm of x. Norm of a matrix or a vector is a non zero positive quantity, therefore norm of x can be cancelled on both sides. Therefore we have magnitude of lambda is less than norm. This result is true for all Eigen values of A, therefore we have given the result that is norm of A is greater than or equal to spectral radius of A. So just reverse this; this is true for all lambda, therefore it is true for a spectral radius therefore norm of A is greater than or equal to spectral radius of A therefore the norm gives the upper bound for the Eigen values. Now we need one more definition.

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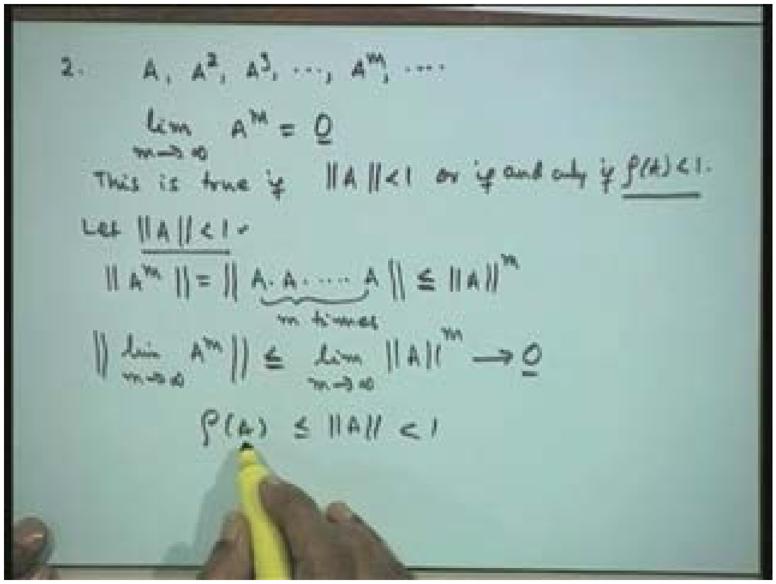

What we have here is a sequence of powers of a matrix A, A square, A cube, so on. We have A to the power of m, we would like to know when will this matrix or this sequence converge and converge to a null matrix. I want to know the conditions under which limit m tending to infinity A to the power of m is a null matrix. Now I would like to show that this is true if norm of A is less than one or if and only if spectral radius of A is less than one. Now the proof of this is just one line because, let us take that it is a norm of A is less than one. I will write down A into A so on into A; lets write down this m times norm, then I will use the result norm of A into B is less than or equal to norm of A into norm of B. So this will be norm of A into norm of A into norm of A m times. So this will be norm of A to the power of m less than or equal to norm of A to the power of m. Now let us take limit of m tending to infinity A to the power of m. Now apply the limit here, limit to be take it out so limit of m tending to infinity norm of A less than or equal to m. Now norm of A is given as less than one, therefore norm A to the power of m, m tending to infinity will tend to a null matrix. Therefore the required condition is that norm of A should be less than one and because of the reason that norm of A is greater than or equal to spectral radius of A will have the condition that its only if I leave it. We need to keep less than one spectral radius is less than one. So norm of A is less than one norm of A is greater than or equal to one. So we have spectral radius is less than or equal to norm of A, less than one. Here we strictly require norm of spectral radius of A less than one, then it would always converge to a null matrix. If the norm goes to zero its measure is going to be zero.

Let us take any definition of norm. For example, Euclidian norm; now this norm should go to zero irrespective of all the components. Therefore the only possibility is that each one of this component should go to zero. Now if you take absolute row sum, absolute column sum, even if any one of the element is stationary or it is not changing but all elements are going to be zero. Then the maximum absolute row sum is going to be a finite constant. Therefore if the norm is go to zero, every element should tend to be zero, that means it would finally imply that the matrix itself goes as a null matrix.

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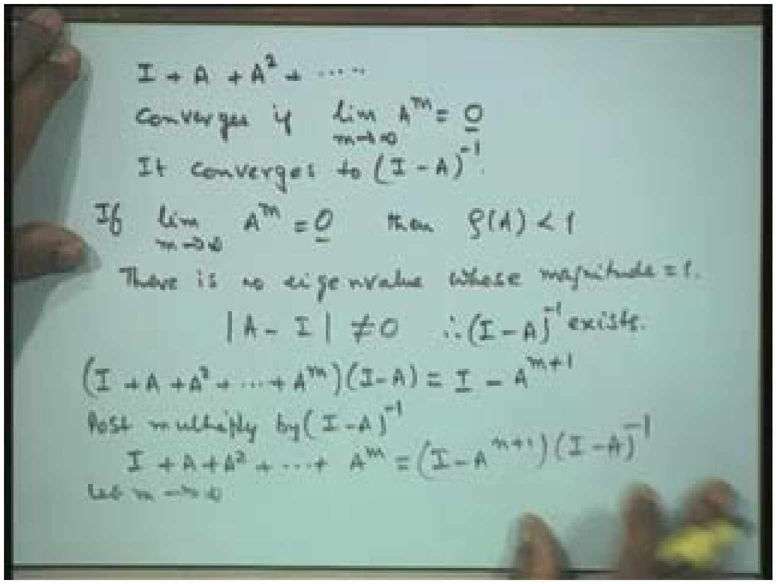

I got the infinite series I plus A plus A square plus so on. Now I would like to show that this converges. If limit of m tends to infinity A to the power of m is null; secondly it converges to I minus A inverse. You know this is very similar to the binomial series, this is if you have a binomial series one plus x plus x plus so on. We know that it converges in magnitude of x is less than one and it will converge to one minus H to the power of minus one or one upon one minus x. So the result is very similar to that; that result holds if limit of A to the power of m is null. Now let us prove this is just one line. So let us take if this is true if limit m tending to infinity A to the power of m is null then from the previous result spectral radius of A is less than one. I just showed that spectral radius is less than one. That means there is Eigen value which is equal to one; largest Eigen value in magnitude is less than one, therefore there is no Eigen value which is equal to one in magnitude. Therefore there is no Eigen value whose magnitude is equal to one. This would imply that A minus I detriment is not equal to zero. I set lambda is equal to one, a minus lambda I; so set lambda is equal to one. So A minus I but lambda is equal to one is not an Eigen value; therefore the detriment of A minus I is not equal to zero. What does it imply? It implies I minus A inverse exists. Say if a matrices is non similar; detriment of b is not equal to zero, its inverse exists. Therefore we are showing that I minus A inverse exists. Therefore under the condition given, condition limit m tend to infinity A to the power of m is null. I minus A inverse, all that exists.

I will take this as A to the power of m into I minus A. Let us just multiply it; this I into I that I have I this is I into minus A plus A into I that is minus A plus A cancels; similarly minus A square plus A square cancels except the last term A to the power of m plus one; minus A to the power of m plus one is only surviving element. Now what we do since I minus A inverse exists I post multiplied by IA inverse. So I have post multiply by I minus A inverse then this will simply read I plus A plus A square plus A to the power of m is equal to I minus A m plus one I minus A inverse. Now let m tend to infinity. Now if m tends to infinity A to the power of m tends to the zero, therefore this tends to zero, this goes to infinity and right hand side is I minus A inverse.

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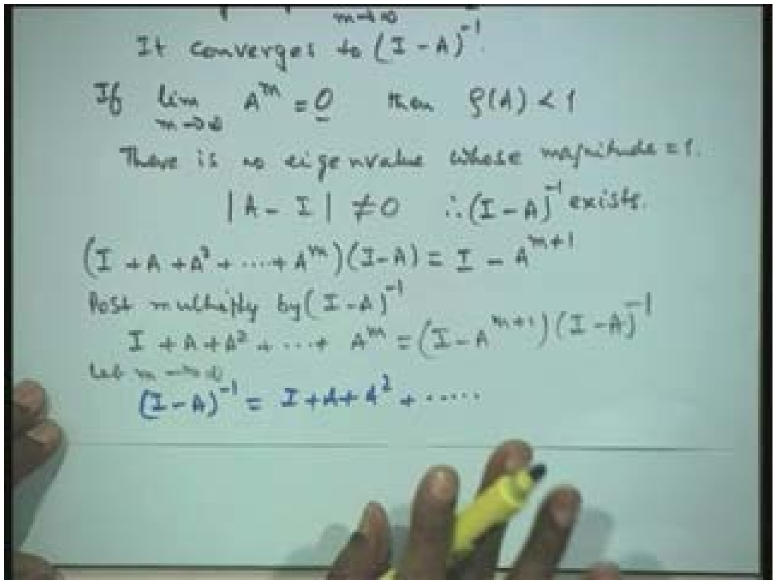

Therefore we are required result I minus A inverse is equal to I plus A plus A square plus so on. Therefore the only condition for convergence of the right hand side is norm of A is less than one or spectral radius is strictly less than one; then this sequence then this infinite series converges always. Now let us use this for proving our convergence of our iterative methods. Now let's take this as a result or a theorem.

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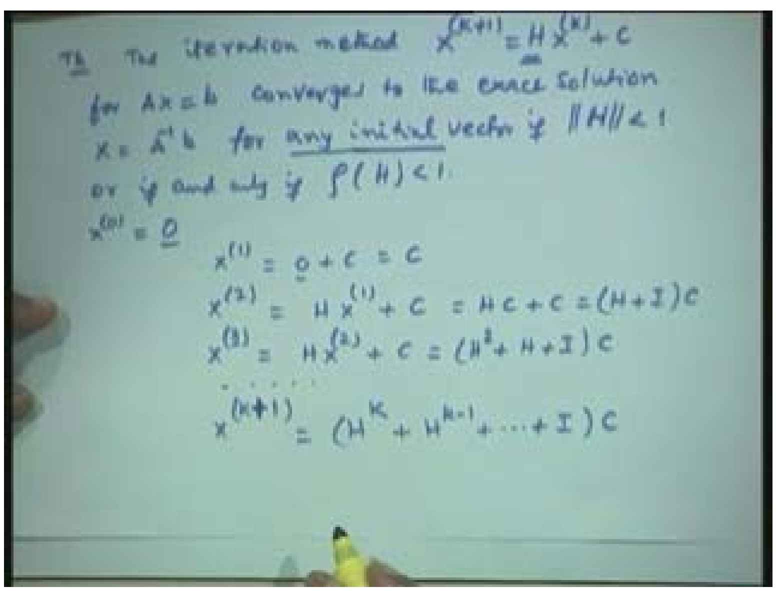

We retain it as xk plus one is equal to Hxk plus c for Ax is equal to b converges to the exact solution x is equal to A inverse b for any initial vector. If norm of H less than one or if and only if spectral radius of H is less than one. In some of the examples that we have done earlier using the Gauss Jacobi, Gauss Seidel methods we have determined the iteration matrix H. Once I pick up this iteration matrix I can find its norm or go to the spectral norm. I can find the largest Eigen value of that matrix and then see whether the largest Eigen value is less than one in magnitude or not. One important thing here is we said it for any initial vector so arbitrary initial vector so if nothing is available for us we can start with this zero. So let this solution be zero, we can start with solution also zero and even then it is going to converge. Now let us prove this result a just few lines. Let us put k is equal to zero one two onwards so let us take x zero is equal to zero for any initial vectors. So let us take zero as initial vector then x one will be equal to zero plus c, that is equal to c. I am substituting x of zero is equal to zero, so I will get here as c. Lets write down x two is equal to Hx one plus c; but x one is equal to c, that is Hc plus c that is equal to H plus I into c. Now we proceed further, x three is Hx two plus c so H into this quantity. So I will have H square plus H plus I of c this is H into H square H and this I coming from here and so on. If I take the k plus oneth iterate, this will become H to the power of k one less than this plus H to the power of k minus one so on plus I c. Now take limit take k tending to infinity.

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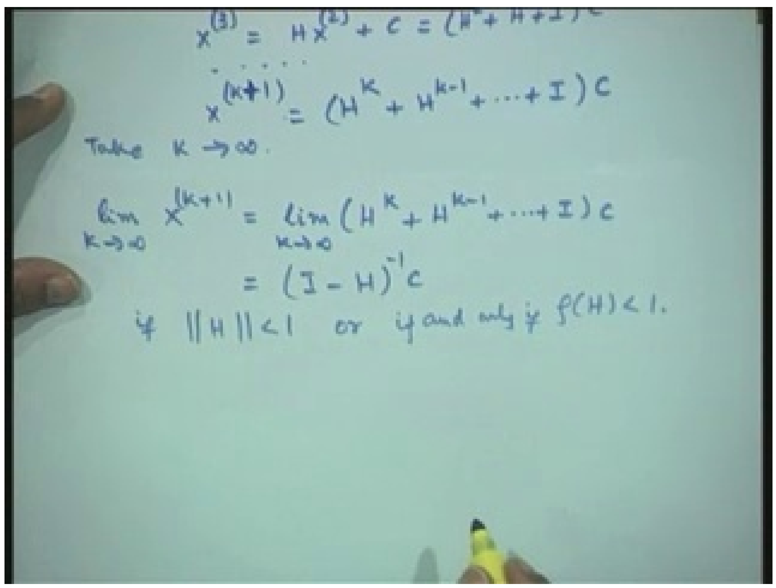

So the left hand side would be limit of k tending to infinity xk plus one. Now this result we have just now proved that this c, this convergence if norm of H is less than one or spectral radius is strictly less than one. Therefore this is equal to limit k tending to infinity H to power of k, k minus one c. This will be equal to I minus H inverse of c. If norm of H is less than one or if and only if spectral radius of which is less than one.

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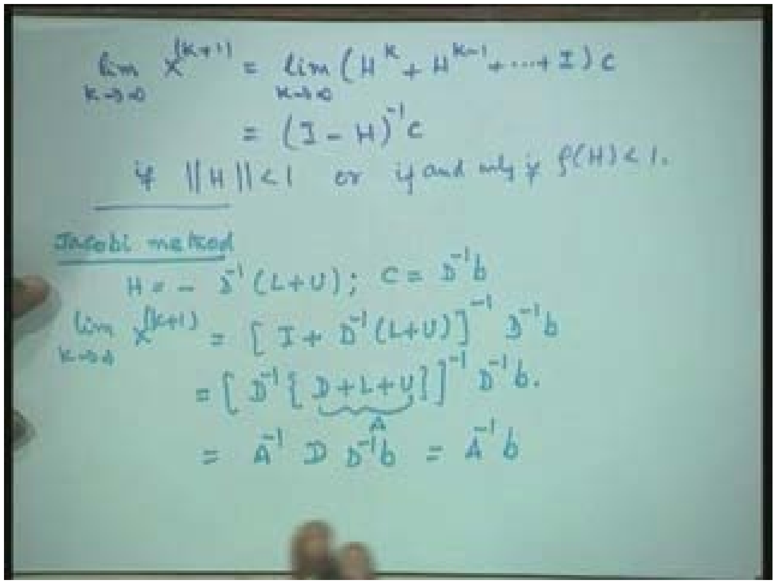

Therefore this is the convergence proof for all the iterative methods that the iteration matrix will be obtained and we will find its norm or will find its Eigen values and find this particular radius and if this spectral radius is less than one than the iteration converges. If the spectral radius is greater than one it is going to diverge very very fast. Let us take the Jacobi method. Now H is minus D inverse of L plus U minus D inverse of L plus U and this quantity that we have here, this limit, let's write down limit k tending to infinity xk plus one is I minus this. Therefore I plus D inverse of L plus U inverse of it into c was I can write down c also here c was D inverse of b c was D inverse of b. So now I will take D out from here. So I will write down D inverse of D plus L plus U inverse. So I will take the common factor on the left. So D inverse of D from here L plus U inverse D inverse of b this is equal to A this is A inverse or the product will be A inverse D inverse of inverse that is D and this is D inverse of b. Therefore this is A inverse of b though the limit is indeed went to A inverse of b, when we have this the iteration matrix for H a Jacobi this. And the right to the contribution the right hand vectors c is D inverse of b, so this will always converge and we are showing that will converge for any initial vector arbitrary initial vector and as I said if nothing is available for you, you can start of with a zero. However even if you know some values of this solution x one x two or anyone of them, it will reduce few iterations. If you are starting with the zero vectors there are few more iterations which will be taken for convergence; but if you know some approximate estimate you can reduce the number of iterations for required convergence.

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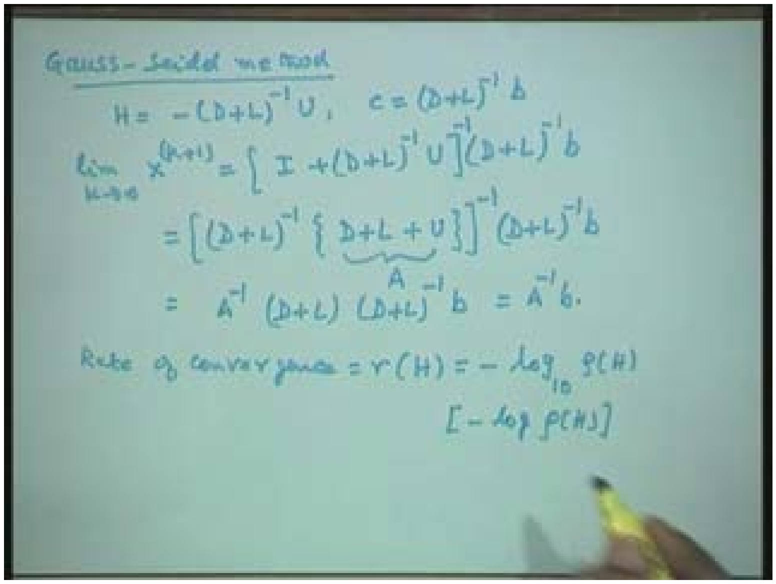

Now let us also see what would happen to Gauss Seidel. Now in the case of the Gauss Seidel method we have H is minus of D plus L inverse of U and c was D plus L inverse of b. Therefore let us repeat what we have done, their limit of k tending to infinity xk plus one is I minus H that is I plus D plus L inverse of U D plus L inverse of b. Now we do this same thing. Let me take out the factor D plus L inverse on the left, so that I is replaced by D plus L D D plus L D plus L inverse is I plus U inverse D plus L inverse on b. Now again this is A, therefore I take the inverse of this product. So I will A inverse D plus L D plus L inverse of b that is again A inverse b. Now once we have defined the convergence that all these methods are going to converge provided the norm of the iteration matrix is less than one or spectral radius is less than one, we need to define how fast are give a measure of the rate of convergence. Now we defined the important number that is a rate of convergence of any iterative method rate of convergence are usually referred as r of H. That is r rate of convergence h is the iteration matrix that is defined as minus logarithm base ten spectral radius of H. Some books do use base e also.

Now we can see that spectral radius is a quantity less than one therefore logarithm of that number is a negative number so this becomes a plus sign. Now let us apply this and then whatever we done the examples earlier let us see what is the rate of convergence methods that we have done there. I will just take the example, show it you.

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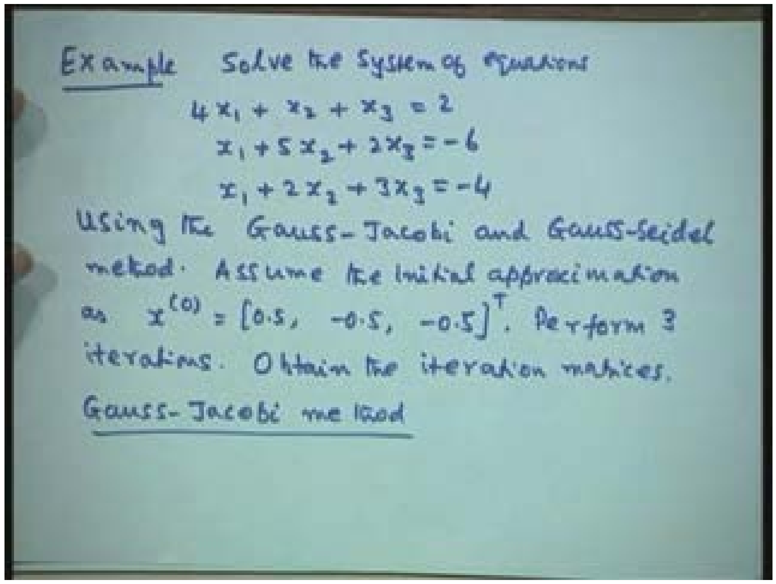

We solve this particular example using the Gauss Jacobi, Gauss Seidel and we also said perform three iterations and obtain iteration matrixes. So we have done this example in which you obtain the iteration matrixes for Gauss Jacobi and Gauss Seidel. So based on this I will pick up the iteration matrix from here.

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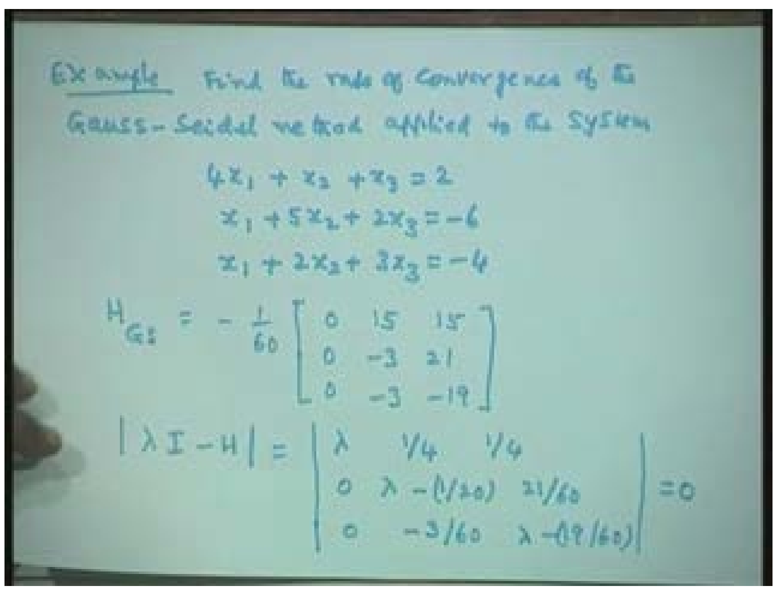

But let us write down an example based on this. I will take this as find the rate of convergence find the rate of convergence of the Gauss Seidel method of the Gauss Seidel method applied to the system. I will take this as four x one plus x two plus x three is equal to two x one plus five x two plus two x three is minus six x one plus two x two three x three is equal to minus four.

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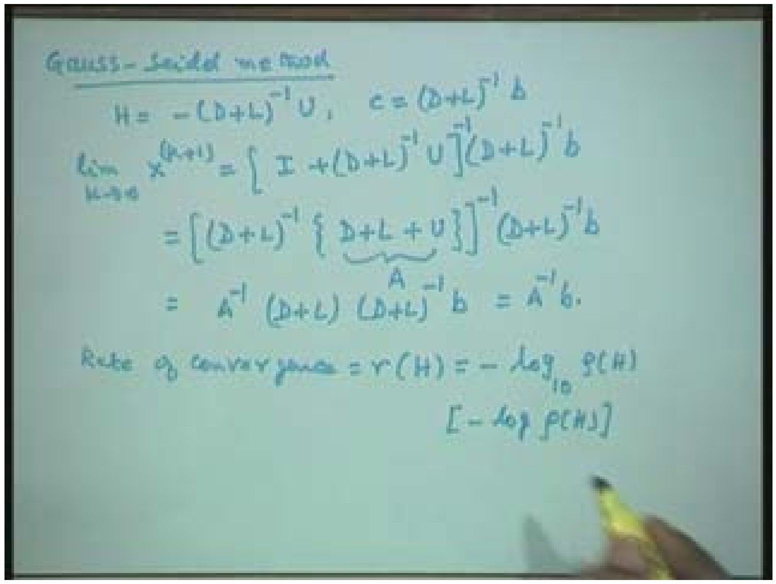

Now for this matrix, if you just look back your notes we have derived this particular iteration matrix for Gauss Seidel method and we have obtained minus D inverse of L plus U and then multiplied it and got this as a iteration matrix there. So let us pick up this iteration matrix and use it here so the iteration matrix for H Gauss Seidel will write down as a suffices Gauss Seidel that is equal to minus one upon sixty zero zero zero fifteen fifteen minus three twenty one minus three nineteen. Now once I find iteration matrix for finding rate of convergence I want the largest Eigen value of this matrix. So let us form the Eigen values for this matrix. Therefore we can write down lambda I minus H or H minus lambda I will use lambda I minus H because there is also minus sign so I can absorb that minus sign over here. So that will be equal to lambda. We take one by sixty inside, so I will have one by four one by four is one by four zero lambda, this is plus sign. Therefore this minus sign will be retained, therefore this is minus one upon twenty three by sixty. I will take it as one upon twenty and twenty one upon sixty. This minus sign is absorbed so twenty one upon sixty, so I will have this and this is minus three by sixty and one upon twenty and this is lambda minus nineteen by sixty.

Now the advantage when solving a three by three system for Gauss Seidel will be; we know this Colum is always going to be lambda zero zero, so the expansion will be very simple.

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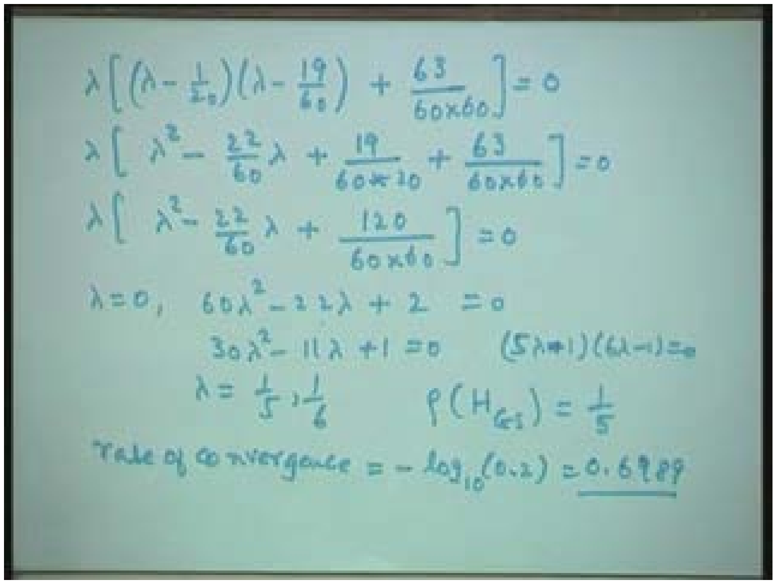

So let us expand it out so we will have lambda minus one by twenty lambda minus nineteen by sixty and this is plus sixty three by sixty square that's equal to zero. So let us simplify the interior that is lambda square minus this is a twenty two by sixty lambda this is plus nineteen by sixty into twenty sixty three by sixty into sixty. In all most all the cases these things do simplify a lot; before we write its general form so this we can write as a lambda two minus twenty two by sixty into lambda I multiply by this by three that is equal to fifty seven into six plus sixty three that is one twenty divided by sixty into sixty. So what I meant is when we have a factor outside this factor that is there the always get cancel to the minimum possible extent. Therefore this I can write down the Eigen values therefore lambda is equal to zero and sixty lambda square minus twenty two lambda; this is canceled that is equal to two is equal to zero. I have canceled a sixty is here, so this is a two here. So I will have a sixty lambda square minus twenty two lambda plus this or thirty lambda square minus lambda plus one is equal to zero. I find its Eigen values this is simply one by six and one by five this is equal to yes five lambda plus one minus one into six lambda minus one is zero. So lambda is equal to one by five and lambda is equal to one by six, therefore the spectral radius of H Gauss Seidel is one by five. The largest of this two that is point two or one point five. Therefore the rate of convergence is equal to minus of point two, this is approximately point six nine eight nine. Therefore this is how we can find the rate of convergence and then compare the rate of convergence of different methods like Gauss Jacobi or successive over relaxation or the Gauss Seidel method.

We mentioned earlier that when we are solving the Gauss Jacobi or the Gauss Seidel the order that is given to us is very important. If you interchange the order of the system of equations the Eigen values will change because you are interchanging the rows. Therefore the property of the convergence or divergence would completely be spoiled. Now let me just show an example a very trivial example.

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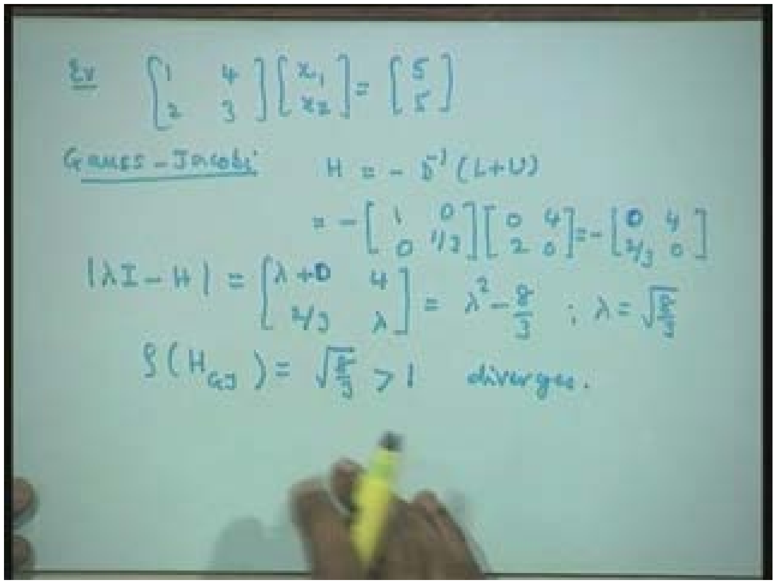

Let us take this example of one four two three x one x two is equal to five five. This is an artificial example which I had taken. This solution as one one, that is five and five so this is an example. Let us solve this by Gauss Jacobi. I want to see whether it is going to converge or not so let us write down H H will be equal to minus D inverse of L plus U. So minus let us do D inverse immediately in D is only one and three so its inverse will be one one by three zero L plus U zero four two zero; you remove the diagonal, so what is left out is this. Let us multiply it out this is one four two by three zero. Now let’s find out the Eigen values of this, so lambda I minus H is equal to lambda minus H that is lambda plus one. This is four, this is two by three lambda. I made a mistake here, this is zero this is zero this is a zero. So this is equal to lambda square minus eight by three, therefore lambda is equal to plus minus eight by three. Therefore spectral radius of H Gauss Jacobi is root of eight by three greater than one. This diverges definitely. Therefore if you apply the Gauss Jacobi method to solve this system it is always going to diverge.

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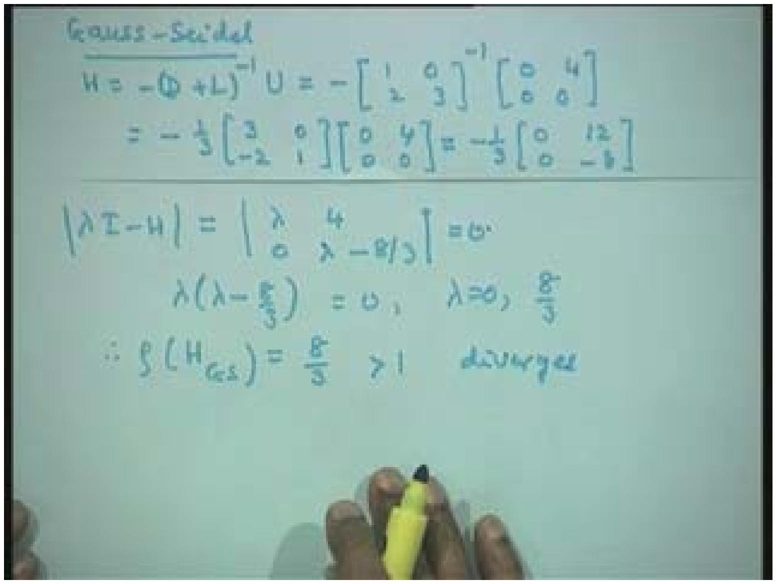

Now let us do the same thing and see whether Gauss Seidel is going to converge. We have H is equal to minus D plus L inverse of U minus D plus. L is the lower triangular part one two and three one two three zero inverse. The upper triangular part is simply containing four here zero zero. Now the value of the determinant is this is equal to three, so I will have this as one by three this is three zero one minus two. So this is opposite sign that is minus two three and one and this is zero four zero zero. Therefore I will have this as minus one by three zero that is zero three into four twelve this gives you zero this is minus eight. Now let's find the Eigen values of this. Therefore I will have here lambda I minus H is equal to lambda. This is equal to zero, therefore this gives you four a twelve by three. This is equal to zero and this is equal to lambda minus eight by three. Therefore this gives you lambda into lambda by three, therefore lambda is equal to zero and eight by three and therefore this spectral radius of Gauss Seidel is eight by three therefore it diverges. It diverges much faster than Gauss Jacobi; the Gauss Jacobi spectral radius was only root of eight by three whereas this is square of that number. Therefore the Gauss Seidel is going to diverge much faster than the Gauss Jacobi in this scheme.

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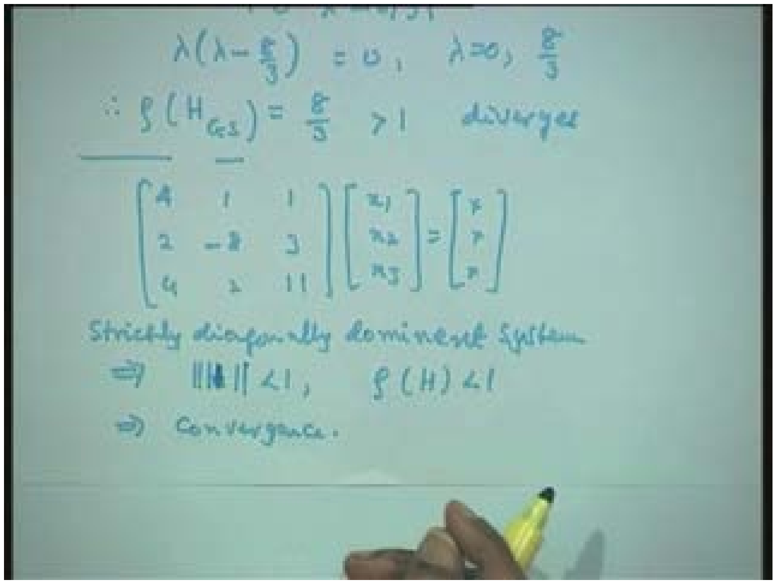

However we mentioned that the given system is diagonally dominant and it would automatically imply norm is less than one and spectral radius is less than one. Therefore in that case we don't have to bother about, for example, if I write a system like four one one, let us say two minus eight three and four two eleven. If I write down this particular system x one x two x three is equal to whatever the elements are, I can state that this system will converge for any method like Gauss Seidel successive over relaxation or if it is symmetric all of them will converge because four is greater than two; magnitude of minus eight is greater than five, eleven is greater than six. Therefore this is strictly diagonally dominant system, this is a strictly diagonal dominant diagonally dominant system. This would imply that norm of A H norm of H is going to be less than one spectral radius of H is going to be strictly less than one. Therefore the convergence is guaranteed in this particular case, therefore this always implies convergence. The only thing you would like to know is that what is the rate of convergence otherwise we are guaranteed of convergence in a diagonally dominant system.

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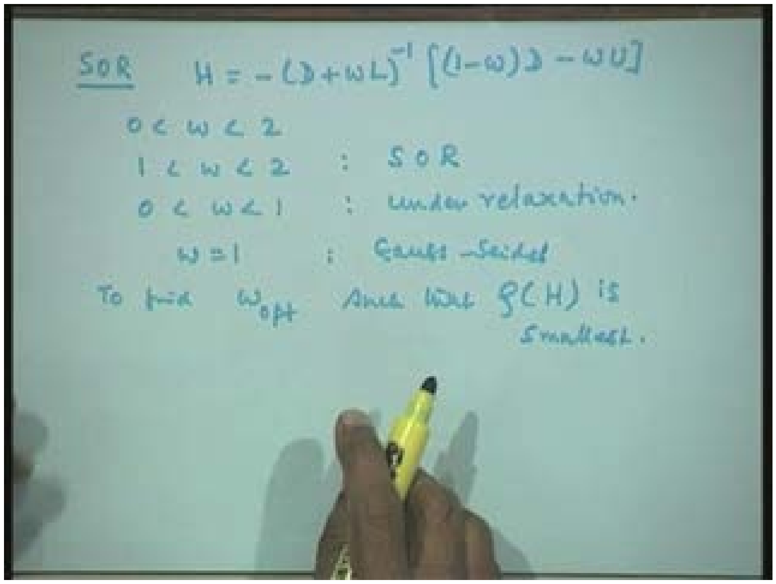

Now let's come to SOR, now we mention that in the SOR the H was minus D plus omega L inverse one minus omega into D minus omega U. We stated earlier that omega is a relaxation parameter and it lies between zero and two in order that the SOR scheme converges. We also said that if it lies between one and two, we call this as a successive over relaxation if it is lies between zero and one. We call this under relaxation but we also mention; in practice nobody uses under relaxation unless there is a specific reason for using it and omega is one it reduces to Gauss Seidel. Now we also mentioned that it is necessary for us to find out an optimal value of omega such that the rate of convergence is the best. That means we need to find omega optimum optimal value of omega such that spectral radius of H is smallest. If I want the rate of convergence largest spectral radius it should be smaller because it is spectral radius is less than one. So we are talking of rate of convergence is minus logarithm of spectral radius. So if this spectral radius goes closer to zero then the number is going to be log of that number is going to be very large. Therefore such that row of H is smallest, that means it is finding the spectral radius and minimization of that with respect to lambda; that's what the problem is. I will not give the detailed analysis of this but I will give the result. It connects the omega optimal; connects with the iteration matrix of the Gauss Jacobi and it's spectral radius.

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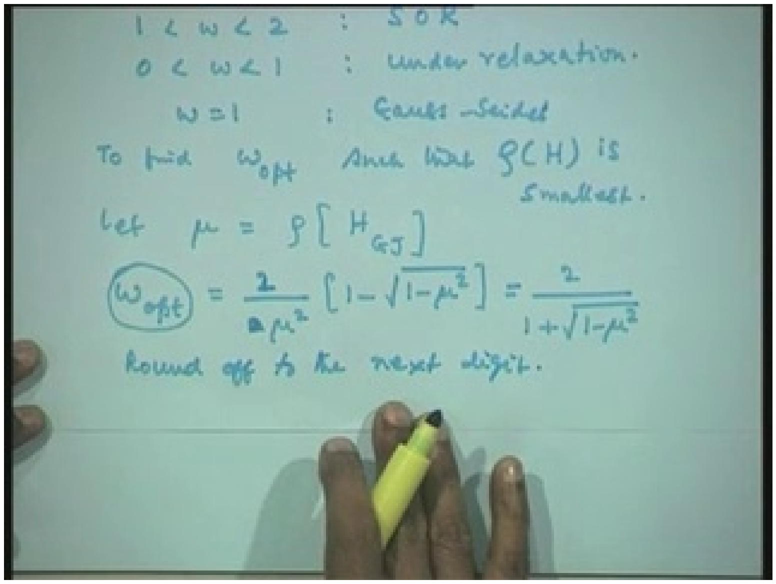

We will denote mu by the spectral radius of H Gauss Jacobi that means we can construct the Gauss Jacobi, find its spectral radius and that we call it as mu. Then omega optimum is given by this quantity two upon mu square two upon mu square one minus under root one minus mu square. Alternately we can rationalize it also I have to bring this to the denominator, then this will cancel off and I will be left out with only one upon one plus under root of one minus mu square. So either I use this particular form or use this particular form, therefore the omega optimum will be obtained by finding this spectral radius of the Gauss Jacobi scheme. Use that value and find this as omega optimum and we always round off to the next digit. We don't use the normal rounding this one round off to the next digit round off to the next digit. The reason being that if you try to draw the graph of the spectral radius of the SOR with respect to lambda in this, to the left of the omega optimum the slope of the graph is minus infinity. Therefore there is a possibility that you may land in to a non convergent sequence, therefore we round it to a next digit so that we are always safe in this case.

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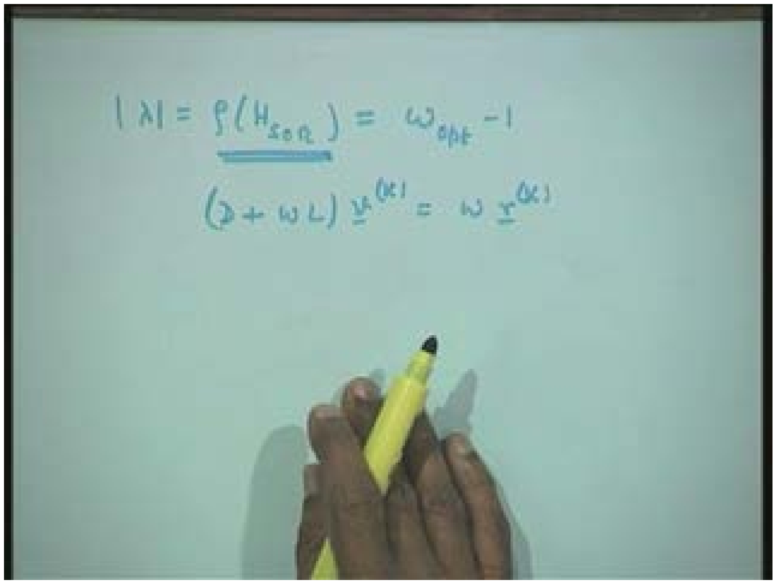

Now once this value is found then the magnitude of lambda that is equal to spectral radius of H SOR is given by omega optimum minus one. So I find the omega optimum and then minus one. We earlier said that omega lies between one and two for the SOR method this will be my spectral radius of H SOR. Therefore the finding of the omega optimum is not too difficult in the case wherever it is applicable. I need to find only this spectral radius of the Jacobi scheme, substitute it over it find the omega optimum and that omega optimum will give me the value that I should use in the scheme SOR scheme. If you remember we have used the scheme as D plus omega L vk is equal to omega rk. So we have used this error format for the SOR also. So we could use this itself and this is the value that we are going to substitute over here and solve particular problem.