Numerical Methods and Computation
Prof. S.R.K Iyengar
Department of Mathematics
Indian Institute of Technology Delhi
Lecture No # 16
Solution of a System of Linear Algebraic Equations (Continued)

In our last lecture we have derived the partition method to find the inverse matrix. We have also shown the need to have such a method in order to tackle the large matrices.

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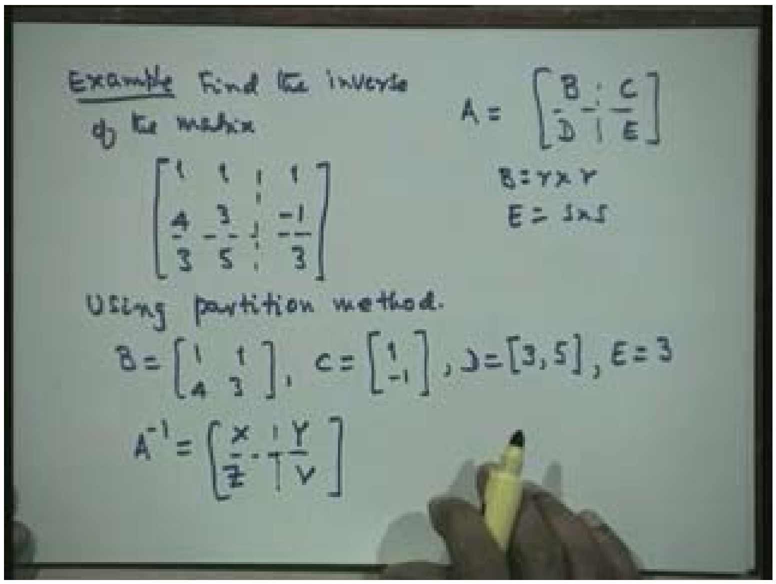

We have partitioned a given matrix into some suitable form for us as some B C D E. We have shown that the inverse of this matrix A can be obtained by finding the two inverses of lower order matrices where we have taken B as r into r matrix and E as s into s matrix. Now by finding the two inverses, one of r into r and another of s into s we can find the inverse of the matrix A. Of course we will have to do little bit more of matrix multiplications also in order to produce the final result. However there is substantial savings in the total computation cost. Now let us illustrate this through an example. So let us take this as an example. Now let us say, find the inverse of the matrix. Let us just take a three by three matrix, so it is easy for us to illustrate this using partition method. Now that means we are using this partition. Therefore we are partitioning it as B is equal to 1, 1; 4, 3. C is the column vector. In this case one minus one D is a row vector, three and five D is this row vector and E is one into one matrix, which is simply three. So this is E, that is one into one matrix. Now in order to find the inverse by the partition method let us also partition A inverse, we will partition this as X Y Z and V.

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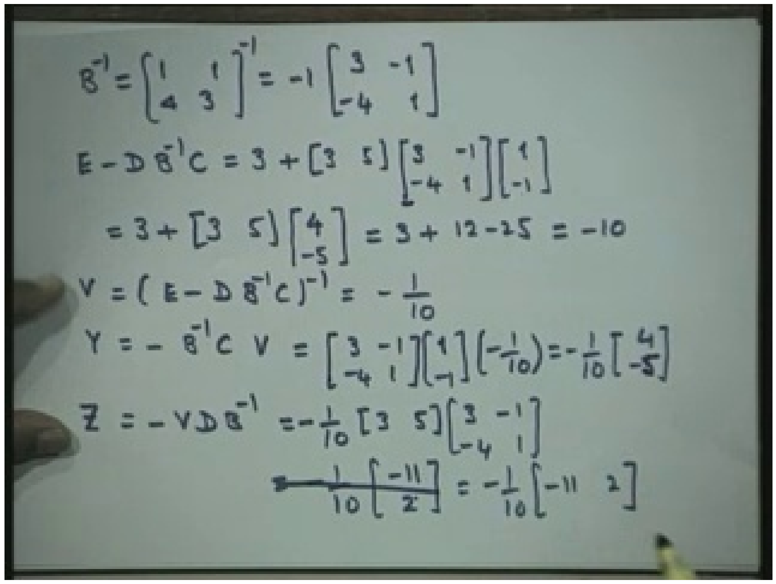

To start with the procedure we need the inverse of B. So let us start with inverse of B. So I need B inverse. So let us find the inverse of this matrix, 1, 1, 4, 3 inverse. So let us do it straight away. Its determinant is minus one, so I will have a minus one, so I will have the element three here; I have the element one here and its cofactors and transpose of it. So I have the inverse simply of B as this. Now the next matrix multiplication that I need, I will pick up from what we have done last time. I need E minus D B inverse C. I need this quantity to determine the next inverse. This is three and we have B inverse with a negative sign, so we can put a positive sign here and D is this row vector 3, 5 so let us write down 3, 5 as your row vector. B inverse is here that is three minus one minus four one. We observed this minus sign here already and then you are multiplying by C and C is this column vector one minus one; so this is your one minus one. So let us multiply it out that is three plus three five. Now I multiply this matrix and the vector that is three plus one four minus four minus one minus five and that is three plus twelve minus twenty-five. So that is fifteen minus twenty five, which is equal to minus ten. Then we have shown that this partition part V, this matrix is equal to one into one matrix. Therefore it is an element that is equal to E minus D B inverse C (inverse of this previous part). So here it is a number therefore we simply get one upon minus one upon ten. Then we have shown that Y is equal to minus B inverse C of V. Now let us substitute B inverse here, so we can remove this minus sign as plus sign. So I will make this as three minus one minus four one and C is the vector that we have here, one minus one into V (V is this minus one by ten). Therefore this is minus one upon ten and this is three plus one equal to four (that is minus four minus one) and hence minus five. Y is also not determined. Then we had shown that Z is equal to minus V D B inverse. Now V is minus one by ten, so I can absorb this one upon ten; D is the row vector three and five and B inverse is three minus one minus four, one. Now we have a minus sign, so let us put one more minus sign over here, so that this is minus one upon ten; this is nine minus twenty; that is minus eleven; minus five minus three plus five is equal to two. Let me write it again. This is minus one upon ten; this is minus eleven and two. So this Z is a row vector minus one by ten minus one by eleven and two.

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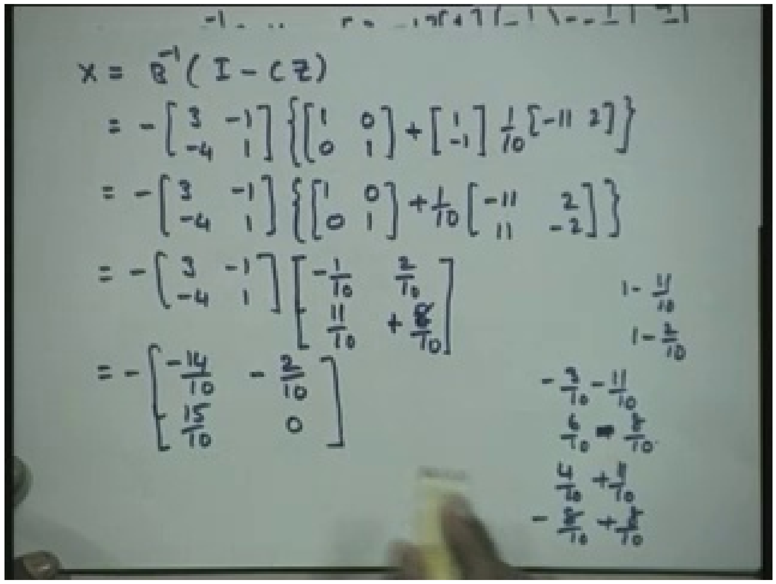

Now the last quantity to be determined is x. So the vector X is equal to B inverse of I minus CZ. This is the matrix, X matrix. Now let us again substitute B that is minus three, minus one, minus four, one and put this in bracket; I is one, zero, zero, one; minus C is one, minus one. For Z let's put this as plus sign, so that we have one by ten, minus eleven, and two. Now let's write one more step here, that is three, minus one, minus four, one; zero, one, zero, one plus one by ten. This is a two into two matrix, so this is minus eleven two eleven minus two.

Now I can simplify this one more step before I multiply it; three, minus one, minus four, one; this is one, I can put it here, one minus eleven by ten. So I will have here minus one by ten. Then I have here two by ten, lets written it as two by ten. This is one that is eleven by ten; that is one plus eight by ten. The last vector is one minus two by ten; therefore this is eight by ten. Let's multiply it out; there is a minus sign here. This is minus three by ten, minus eleven by ten; so this is minus fourteen by ten. Then we have (three into two) six by ten plus eight by ten, so this is minus eight by ten. So this will be minus eight by ten; that is minus two by ten. Then we have four by ten plus eleven by ten; so we will have fifteen by ten and lastly we have this product which is minus eight by ten plus eight by ten. So I have zero over here. Now we have found out all these four quantities. We place them in the proper order X Y Z and V, which will give us the inverse.

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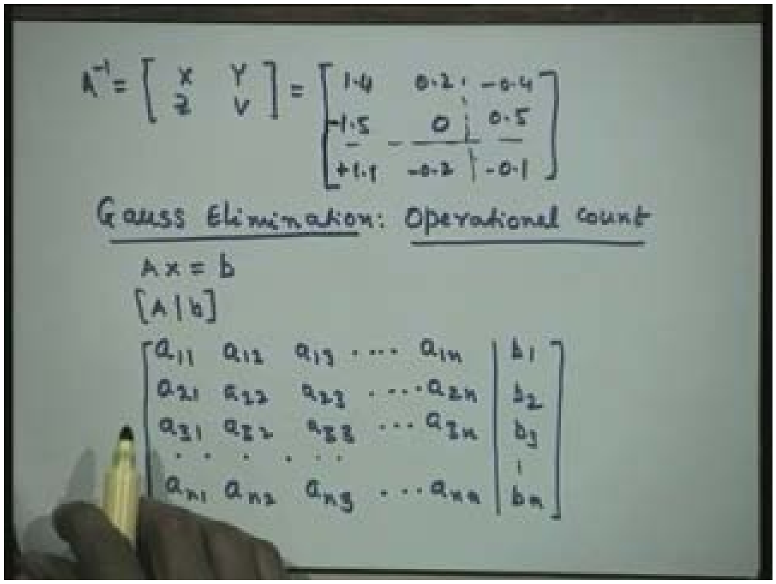

So A inverse is equal to X Y Z V and let's take minus sign inside and it gives you 1.4, this is 0.2, this is 1.5 and 0 with a negative sign. I am putting minus sign inside so I will have this as 1.5 and 0. This was your partition that we had got here and Y is here, that is minus one by ten into four by five. Therefore this will give you minus 0.4 and this is plus 0.5. Then Z is here, let us take it inside, so we will have plus 1.1, minus 0.2. This is eleven by ten and minus two by ten and finally the corner element is our V minus one by ten, so this is your minus 0.1. So therefore this is the final inverse of the given matrix and even though for the hand computation this looks little lengthy, however natural computation is much faster because we are only finding the inverses of two matrices of this orders and then remaining are matrix multiplications.

This is how the partition method can be applied which is a direct method. We can apply it for finding the inverse of a very large system by reducing it in to a smaller system. Now we have mentioned earlier that for all the direct methods we can definitely find an operational count, what is the total amount of operation that we are performing, so that you would be able to tell exactly how much computed time that we will take if you are using a particular computer; because we know the time that we will take for a multiplication and division and the time it takes for addition and subtraction. Once you know the number of additions, subtractions and the number of multiplication and division that a method has got, we would able to say exactly if given a method, a hundred by hundred matrix or a fifty by fifty matrix, how much computer time is required for us to solve this particular problem.

So let us illustrate it for one of the methods that we have done. The other methods can be done easily. So let us take the Gauss elimination and perform what is the operation count for Gauss elimination. So let us take the Gauss elimination and do the operations count. We shall call this as operational count. Let us remember what we have done in Gauss elimination. We have the system of equation; equation Ax is equal to b. Then we have written the augmented matrix Ab. So let us write it a_{11}, a_{12}, a_{13}, a_{1n}; a_{21}, a_{22}, a_{23}, a_{2n}; a_{31}, a_{32}, a_{33}, a_{3n}; a_{n1}, a_{n2}, a_{n3}, a_{nn} and the right hand side vector b_1, b_2, b_3, b_n. Now the method that we have given for the Gauss elimination was we said take the first pivot divide by the first pivot throughout, so that we can produce a_1 here and then we eliminate all these elements below. Therefore let us first start with the divisions which are less.

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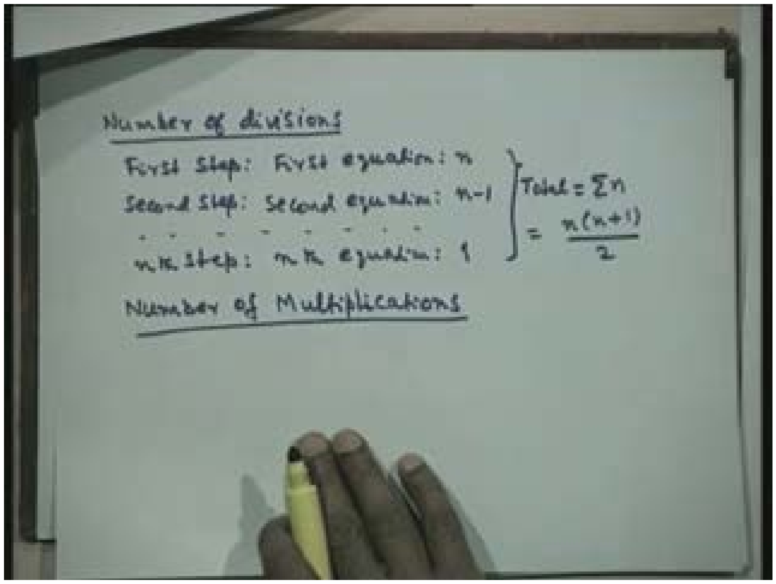

So let us take what is the number of divisions. Let’s count them separately. Now let us take the first step. Now in the first step I have taken the first pivot and divided all these elements. So the first equation requires division of this n minus one elements plus one element. So we need a total of n elements for division of the first step. So we let us write the first equation for which we need n of them. Now the remaining part does not have any division. So let us proceed on, let us go to the second step. In the second step we take this pivot. This second pivot uses this pivot to eliminate all the remaining elements. So for that we are dividing all the remaining elements in this row by this pivot. So I am dividing this n minus two elements plus one element; n minus one element in the second step. So this will be our second equation, there are n minus one divisions in the second step. Now you can see that when we go to the third step I am now in the pivot at this location. Therefore I am now using this pivot to eliminate all of them. So I am now dividing this n minus three plus one that is n minus two elements and so on. So let us go on and do it and let us take n_{th} step, that is your n_{th} equation. Now when we have reached n_{th} step, we have reached this location of this pivot. Now this pivot divides this to produce the starting solution, so it has got one division in the nth step that gives you the last value of xn which the value of this one. Now let’s write this is as total. Total is one plus two plus three plus so on n, so that is your summation of n, one plus two plus three so on n. So this is the summation of n that is equal to n into n plus one by two. Now the divisions have all been carried out, all the pivots have been brought as one. So when we back the substitute, there are no further divisions because the elements have already been made one. So there are no divisions further. Therefore the total number of divisions in our Gauss elimination is only n into n plus one by two.

[Conversation between student and Professor – Not Audible (00:18:57 min)]

There are two ways given; one was to divide by a_{11} and multiply. Alternatively we have given a procedure, that you take the pivot, divide by the pivot first so that the diagonal element becomes one. So in this procedure I divided by a_{11} throughout, now the pivot has become one. Now the second pivot, now these are all zeros. Second pivot is now used and we divide throughout these equations, so the second pivot has become one.

[Conversation between student and Professor – Not Audible (00:19:32 min)]

We don't divide a_{11} by a_{11}. So when we divide a_{11} by a_{11} we know that it is going be one. So we are not going to divide a_{11} by a_{11}, thereby we introduce an error in the round off error. So we assume that you are dividing this element by this element it, so it will be one. So therefore the count will be leaving that particular pivot you need to do. So now we need to do the number of multiplications. So let us find out what is the number of multiplications.

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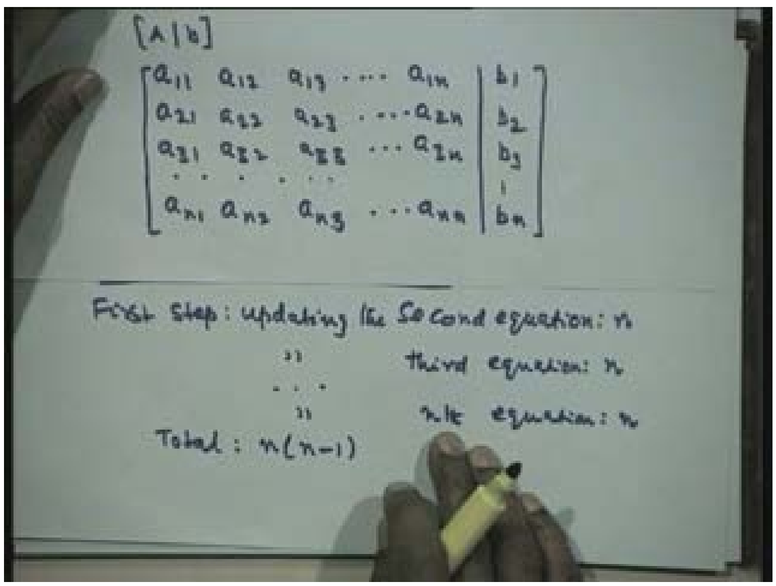

Now let's write down the number of multiplications. Now let's go back to the first step. So let me write down the first step. Now we have divided by $a_{11,}$ so I have the elements. Now we are producing zero and multiplying by a_2 and subtracting all the elements. So what I am doing is I am multiplying whatever the quantity by a_{21} and subtracting; multiply a_{21} and subtracting; a_{21} subtracting and multiplying by a_{21}. So I am now multiplying n minus one in plus one; that is n times I am multiplying in the first step. So in the first step I am updating the second equation. We are updating the second equation and that requires total of n multiplications. The first step requires that this element be made zero and this element also be made zero and all of them. So updating the third equation; I am now multiplying by a_{31} and subtract; a_{31} and subtract; a_{31} into this and subtract. Therefore I am still continuing with n multiplications; and so on for updating the nth equation, I need n of them. If total numbers of equations updated are second to n, (there are n minus one equation that is updated) therefore the total at this for the first step is n into n minus one. There are n minus one equation that is updated starting from second to n. Therefore n minus one into n will be the total multiplication that will be required. Now let us go to the second step.

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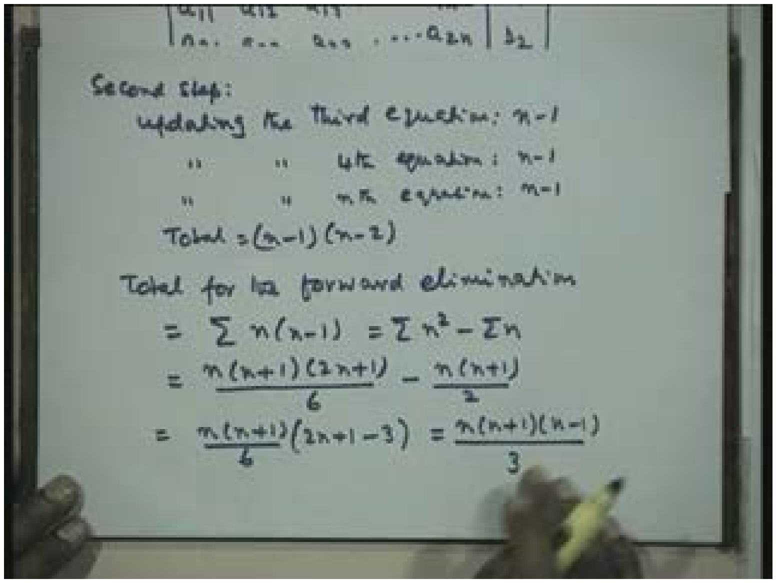

Now when we are at the second step, we are now updating the third equation onwards. Now let's have a look at this one, where I am using this pivot here. I am multiplying this by this a_{32} and subtract; multiply this quantity by a_{32} and subtract; multiply the quantity here by a_{32} and subtract. So I have got here n minus two plus one, that is equal to n minus one element are updated. So I have now updated n minus one element, now one less than in the previous case. So similarly I am updating the fourth equation, the same number. So finally updating the n^{th} equation; I have n minus one. Now the total equations that are updated are from third to n^{th}, therefore n minus two updates are there, therefore n minus two into n minus one. Therefore the total here is equal to n minus one into n minus two. Now we can generalize and then get the total for this forward elimination. We have eliminated in the forward direction. So we'll write down the total for this and we can take this quantity and that will be summation of n into n minus one. The next one is n minus one n minus two; the next one will be n minus two n minus three and so on. Therefore this will be simply summation of n into n minus one.

Let us simplify this, let us open this up; this is summation n square minus summation n. Summation of n square is known to us that is n into n plus one into two n plus one by six and summation n is n into n plus one by two. Let us combine them n into n plus one by six into two n plus one minus three. So this is two n minus two, so I will take out two outside. So I will have n into n plus one into n minus one divided by three. So the total I require for forward elimination is this many of them multiplications. Now we have completed the forward elimination so the back substitution is now to be taken care of. Let's now take what will be the back substitution.

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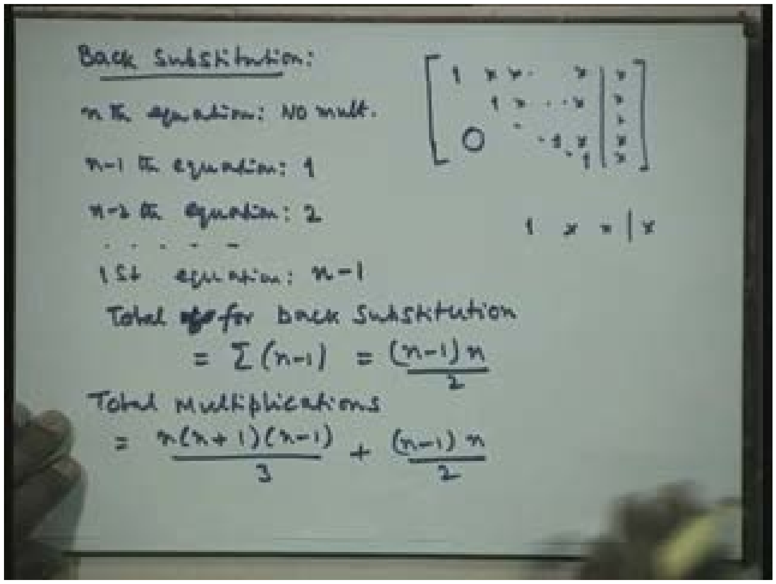

Now let us just have a look at this one. When we have completed the procedure what we have here is the upper triangular matrix. So let's write down how it is going to look like. It is going to be an element like this one element, like this and so on and I have a one here. We have divided it by the pivot, so we are having 1, 1 here and we have got elements over here. This is how the matrix is going to look like after we have completed this elimination in the forward direction. Now the n^{th} equation does not have any multiplication because this is equal to this as the solution. Therefore in n^{th} equation, there is no multiplication; just this xn is equal to the right hand side. So there is no computation involved in the last equation. Now let's go back to the n minus 1^{th} equation. Now look at this n minus 1^{th} equation, this is containing one here, element here and element here. So I am now substituting this value that we have obtained xn here multiplying taking to the right hand side or take to the right hand side and multiply. So I have a multiplication here to be done with this particular element and a subtraction. So I have n minus 1^{th} equation, I have one multiplication to be done. Now go to the n minus 2^{th} equation.
The n minus 2^{th} equation is looking like 1, element, element, and element here. Now the solution of xn minus one and xn were determined. Therefore I am multiplying this, multiplying this, taking to the right hand side and subtracting. So I have got two multiplications that have to be done here. So I have two multiplications to be done here and so on I go to the first equation.

Now let's look at the first equation. I have to multiply these n minus one element, take to the right hand side and add them. Therefore I have got n minus one multiplication to be done here. So this requires n minus one multiplication. Therefore let us do total for back substitution. Total for back substitution is summation of n minus one, so that this is n minus one into n by two. Now we have completed the count, so let us now write down the total multiplications. The total multiplication here is the total for forward elimination plus total for back substitution. So I will have here n into n plus one n into n plus one n minus one by three plus n minus one into n by two. So this will be the total multiplications in our Gauss elimination procedure.

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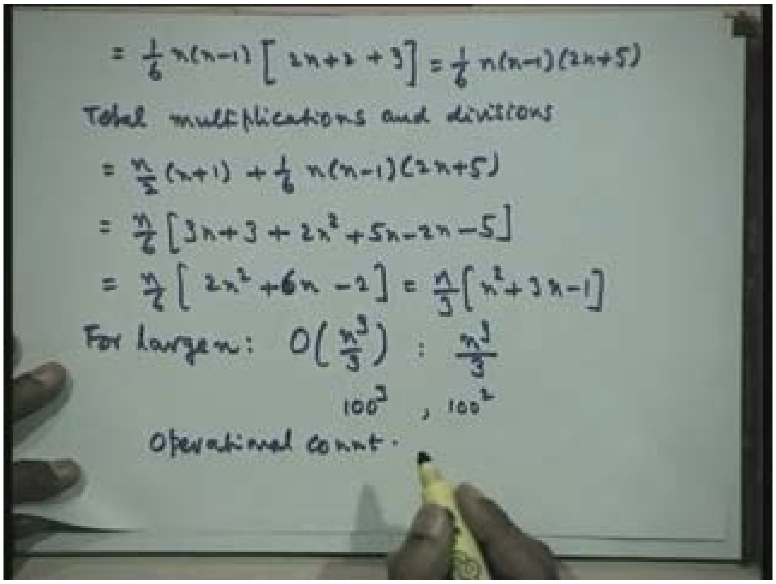

Now let us simplify this. I will take this quantity here. So let us take six outside and n into n minus one outside, so that I have here two times n plus two. I have taken n into n minus one by six. So I will have two n plus two plus three, which comes from three. This is one by six n into n minus one into two n plus five. Normally operation count would imply the total number of multiplication and divisions; the reason being that the time taken for subtraction or addition is a much smaller factor compared to the multiplication and division. So let us take what is the total multiplication and divisions. Total divisions we have as n into n plus one by two plus one by six n into n minus one into two n plus five. So this is our total multiplication and divisions.

Let us simplify this. I can take n by six common, so that will become three n plus three plus n by six. Let us multiply it out. This is two n square plus five n minus two n minus five. So that is n by six, which is equal to two n square plus six n minus two and we can remove two from here and write this as n by three n square plus three n minus one. This is the exact number of multiplications divisions that Gauss elimination has got. Therefore we normally say for large n Gauss elimination is of order of n cubed by three or simply we write down as n cubed by three. Now the reason why it is so obvious is because n square is much smaller compared to n cubed, when n is large. For example, n is say hundred so this is what we are talking of; hundred cubed compared to hundred square. So this gives you ten to the power of four, this is ten to the power of six. So this is much smaller compared to this one, therefore for very large n we usually say that the numbers of operations of Gauss elimination is of the order of n cubed by three. However we can give the exact number of total multiplication and divisions that are required. As I said this is usually called the operational count. First of all this is called the operational count.

[Conversation between student and Professor – Not Audible (00:33:36 min)]

The time that is taken on any computer for division and multiplication is same and also the time that is taken for additional subtraction is identically same. But the time that is taken for addition subtraction is much factor less than the time that is taken for multiplication. That’s why multiplications or divisions are always taken together and addition subtractions are always taken together. Now we can count very easily the total additions and subtractions.

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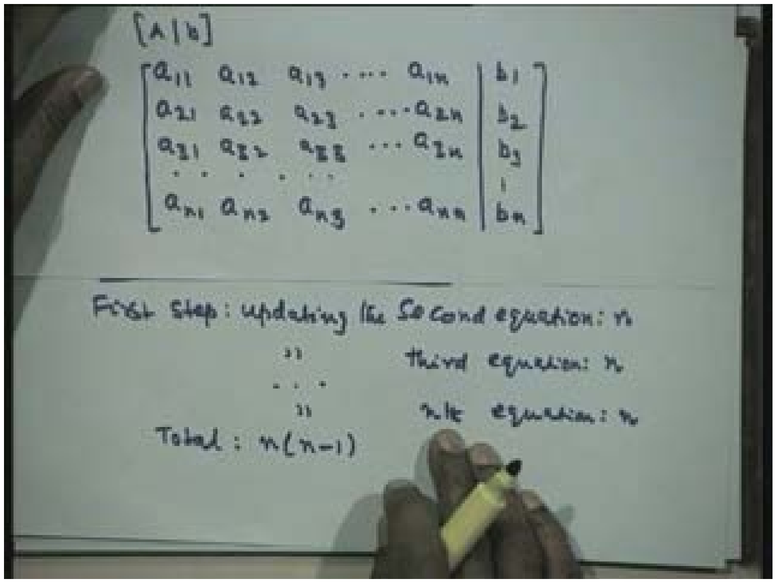

Let's start with our multiplications. For every multiplication that we are doing, there is an addition subtraction involved. Let's now go to the first step. I am now multiplying, subtracting it there, multiply, subtract, multiply, subtract, multiply and subtract. Therefore if you look at the first step, the numbers of additions subtractions are identically same as the number of multiplication that we have because for each multiplication we have this. So for all the steps we require it. Now you go to the second step. It is identically the same story; the second pivot; now we are multiplying this by a_{32}, subtract it, again we have one subtraction with this multiplication; I have a subtraction with this multiplication, I have a subtraction. Therefore the entire Gauss elimination procedure, the forward one that we have gone through has got all of them identically with same addition subtraction. Now let us go back to substitution. If I go back to the substitution, this last element has no subtraction because this is equal to this. Now the n minus 1^{th} equation, I have multiplied, taken it to the right hand side and then subtracting. Therefore with this multiplication is attached one subtraction, so n minus 1^{th} equation has one subtraction. If I go to the previous equation I have multiplication here and multiplication here. There is an addition here; go to the right hand side, there is an addition here. Therefore along with the two multiplications, I have got again two addition subtraction. Therefore the same thing if you go to the first equation, there are n minus one multiplication here and n minus two additions here and one addition to the right hand side; so n minus one additions here. Therefore along with each multiplication we have one addition associated with it and hence the operational count for the addition and subtractions will be the grand total multiplications that we have.

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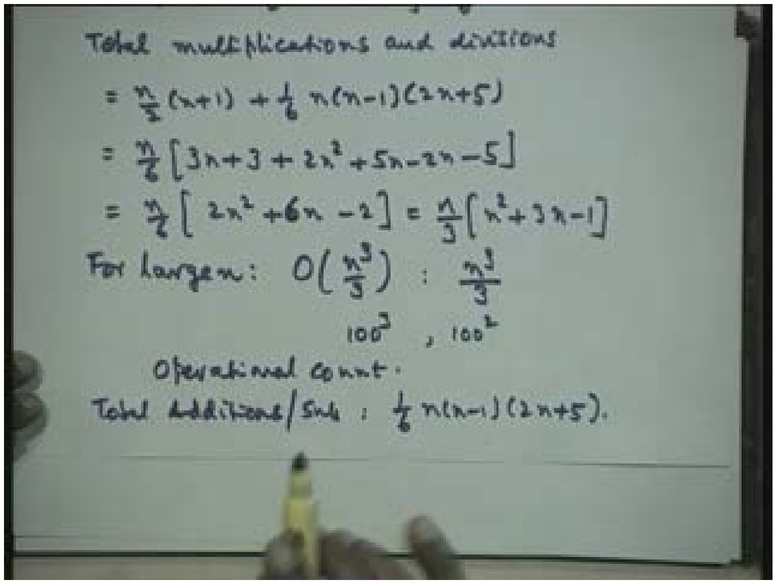

We have got this part, this is your total divisions and this is your total multiplications and divisions. Therefore this part will be our total additional subtractions in our operational count. Therefore the total additions or subtractions will be one by six n into n minus one into two n plus five. Now with this you would now be able to say how much exactly a particular problem is going to take time because I know the number of addition subtractions. I know the time that is taken for each addition subtraction. I know the time for this major operation, so I can just multiply the total number and give exactly what is the computer time that it will take once you know the clock speed of the CPU speed. Once you know the CPU speed and time taken for this, you can immediately count it and then say this is going to take these many seconds of time or this many minutes of time for this particular problem. So this is the advantage of the direct method that we shall be able to give the exact time that will be taken for a particular problem. Now we can do the same analysis for the other methods also. I would leave this as an exercise but will give the solution for this.

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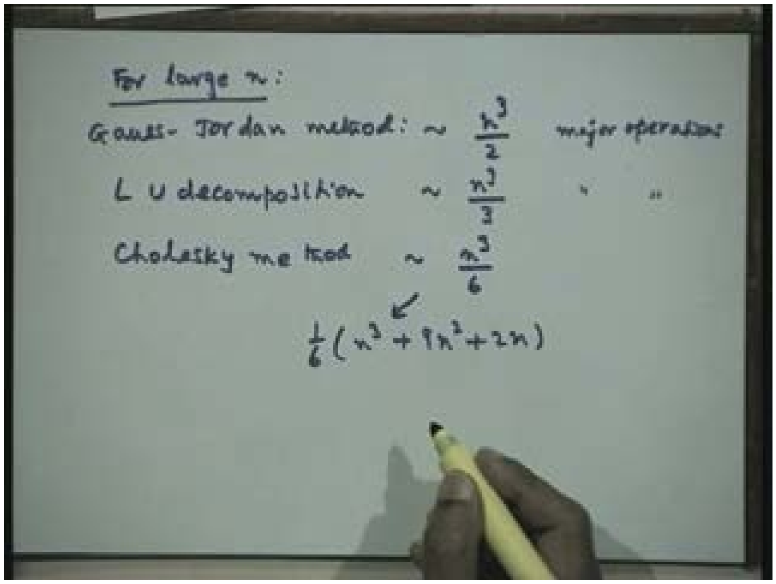

For large n the Gauss Jordan procedure, if I take the second pivot we are doing elimination below and elimination above. So when I go to this element, I am now eliminating below as well as above. So I now need to add to this thing that is going to take for making this upper triangular part also as zero. Therefore when we were solving the example we said that Gauss Jordan is much more expensive than the Gauss elimination procedure. However we shall use it for finding inverse of a matrix which is a very convenient procedure and the number of operations is n cubed by two. These are again the major operation that is your multiplications and divisions. Therefore it is n cubed by two whereas Gauss elimination is n cubed by three. Then LU decomposition is identically same as the Gauss elimination procedure. So this also takes n cubed by three major operations. The other method that we have given is the Cholesky method that is the LL transpose decomposition which we called it as Cholesky method. Now here we would like to take advantage of the symmetry of the matrix that is even in the computations as well as in the storage. Normally when we do the storage for Cholesky we store either the upper triangular part or lower triangular part and do the operations and the operations that is n cubed by six that is almost half of the of the Gauss elimination procedure. Now I can give the exact number for this. This is one by six of n cubed plus nine n square plus two n; this is the number of operations.

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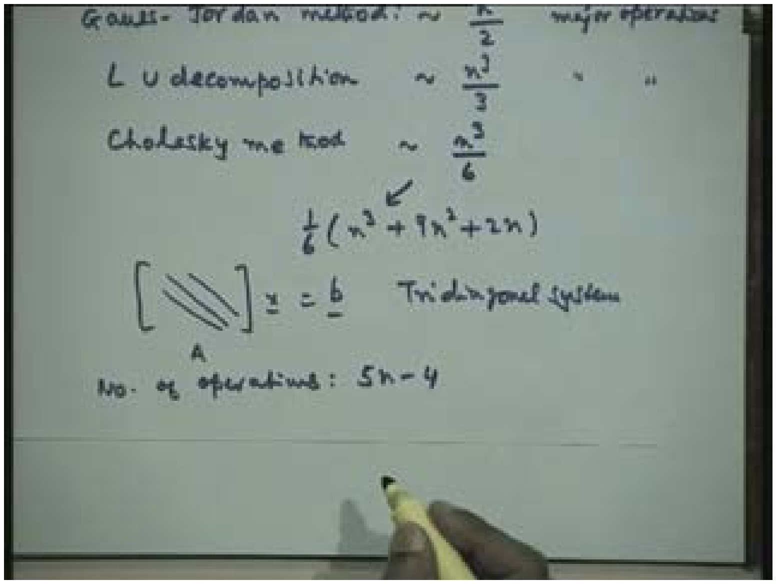

Now very often when you are solving the ordinary differential equations or partial differential equations by the finite difference or finite methods, we come across tridiagonal system of equation. This is a very important system of equation that arises in the practical applications. The system is that your A is going to have only three diagonals, that is your x is equal to b. So we shall call this as a tridiagonal system of equations. Now interestingly the number of operations for this tridiagonal system is five n minus four. That means if I am solving hundred by hundred tridiagonal system, it is going to take only five hundred minus four that is your four hundred and ninety-six major operations and nothing more than that. That means it is going to be few milli seconds. I mentioned that in a practical problem you come across millions of equations. When you are solving the millions of equations, when you are solving that system of partial differential equations or system of ordinary differential equations, you use the method in such a way that you produce a simpler system possible; say a tridiagonal system or I have a five diagonal system or something like that.

Now once you have such systems the number of operations are enormously less and hence we will be able to solve even millions of equations within a very short span of time. It may take few hours on this but still the problem can be solved. That is the reason why I am saying that the suitable method should be used for solving any particular problem, be it a differential equation or any other formula.

However if it is not possible for us to put it in this form, where it is possible for us? In this particular format there is no alternative for us to go for the iterative methods. Now if you have a system of equations of the order of ten to the power of six by ten to the power of six, it is almost full or it has got a band structure such that is the band width is very large for the matrix A and it is not possible for us to store this on the computer. All these direct methods requires storage of the entire structure into the computer so all the elements are stored. Now since all the elements are stored you need sufficient space to store the given system of equations and in the later part when you are solving them you also require sufficient space to use that. If it is not possible then the only alternative is that we use an iterative method where in you can load one equation at a time or hundred equations or thousand equations at a time, solve it and then load the next hundred equations or thousand equations and solve that. Therefore in such problems where the direct methods are not applicable that is the system is very large and the A is also little dense, then we would like to use the iterative methods.

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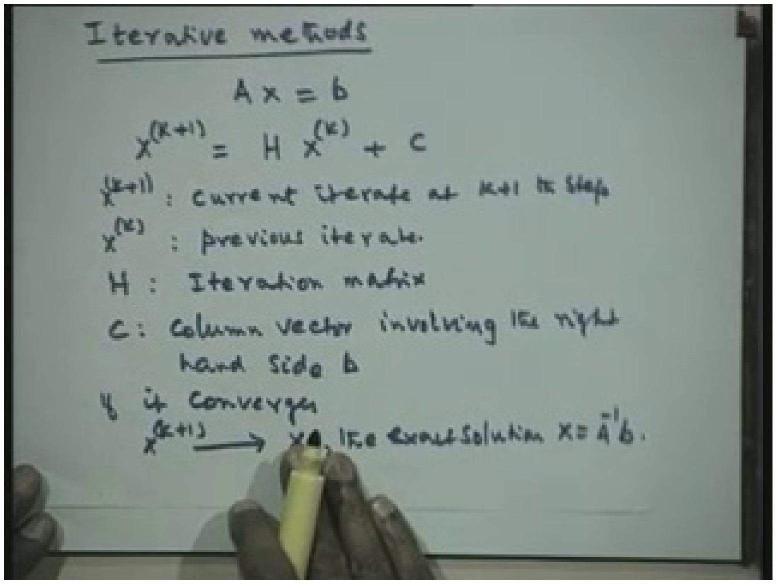

Let us define what is an iterative method for the solution of this problem. You see these methods are very important because in a particular when you have, for example, for designing an aircraft or designing a motor car, for the number of variables that are there at any particular point, you like to know the wind velocity at that point and the acceleration at that point, the stress at that point and so many variables are there. Each one will have three components. It's a three dimensional problem so we will have three components. So the number of components of a variable and the number of variables are so many, therefore the total number of components will be enormous. While solving those partial system partial differential equations the system that comes out of it is not a system which is amenable for the direct methods. So in such cases of complicated problems you would like to choose the iterative methods for the solution of this system of equations. Every iterative method connects the current iterate to the previous iterate. So if I write down a method like xk plus one, this is equal to Hxk plus c. Then I would call this xk plus one as a current iterates at k plus 1^{th} step and xk is the previous iterate.

Now the the method connects the current iterate with the previous iterate wherein this H is called the iteration matrix and c is a column vector involving the right hand side b. So we would like to derive an iterative method which would look like this; if the iterative method is converging then in the limit as H as k tends to infinity this will tend to x. Therefore if it converges then xk plus one or xk will tend to x, the exact solution. It will tend to exact solution in which x is equal to A inverse of b. The solution is, x is A inverse b; therefore if it converges then xk plus one would converge to this particular value, the exact solution where x is equal to A inverse b.

Now in our later lectures we will see as to the properties of this matrix H because this iteration matrix is the most important matrix. Now its convergence or divergence depends entirely on the behavior of this particular matrix. It is possible that even a two by two system may diverge if you don't write it properly. We will give an example next time that even if you take a two by two system and try to solve by an iterative method it may diverge. Therefore the properties of this iteration matrix are very important. When we solved in the direct methods we solved it in the given form Gauss elimination and we also found when the pivot is zero we had also used the partial pivoting, we have inter changing the equations and so on. Therefore the order in which the iterative method is applied is very important. If you change the order then the properties of the iteration matrix is going to change; that means if your inter changing first equation is tenth equation then the properties of this H is going to change. Therefore you may be making a convergent method as divergent method and a divergent method as convergent method if you are using this particular type of inter changes.