Numerical Methods and Computation

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Lecture No. # 15

Solution of a System of Linear Algebraic Equations (Continued)

In the previous lecture we have derived the decomposition method. The method can be used for finding the solution of a system of linear algebraic equations or we can use it for finding inverse of a given matrix.

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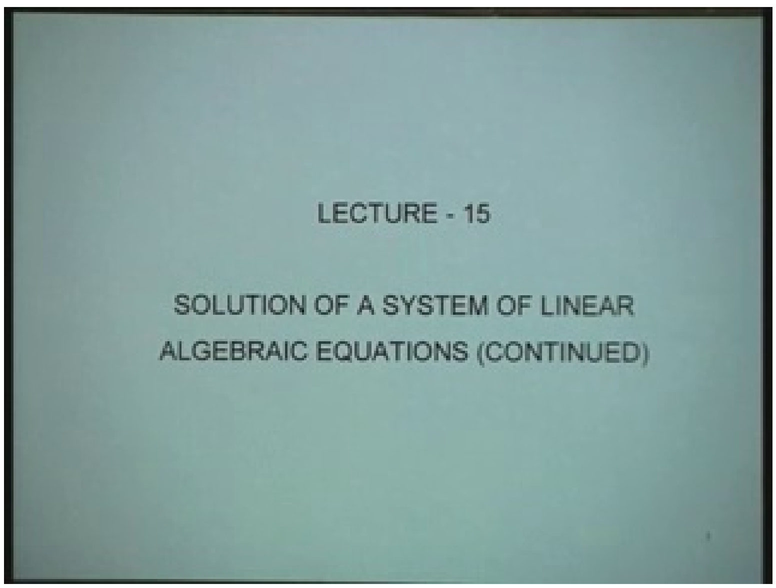

We have also mentioned that the decomposition method can be simplified if we can use the property of the given matrix. Now let us take this particular case when the given matrix A is a symmetric matrix. The method in that case has some special names.

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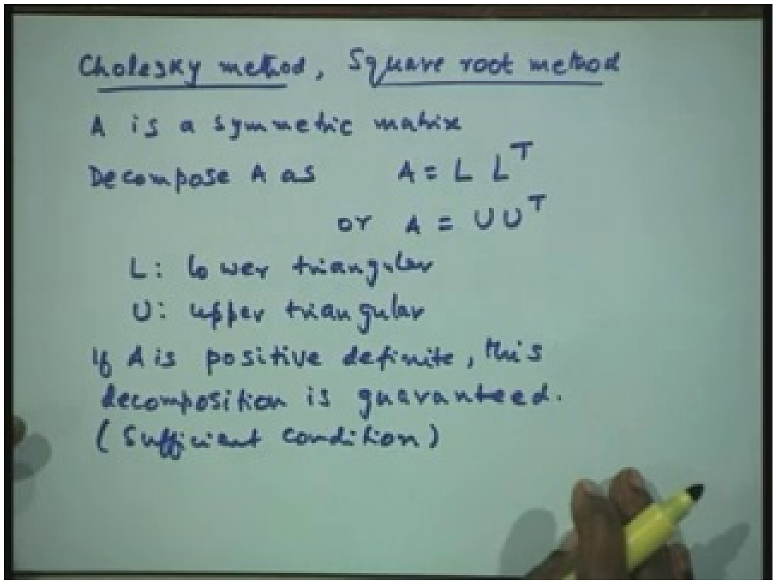

The method is called the Cholesky method or it is also called the square root method. The method was actually derived in two different places at the same time in different countries, so that's why the names of Cholesky has come and this square root methods have come; but they have been actually been derived almost at the same instant of time.

Now we start with the assumption that A is a symmetric matrix. Then we decompose A as, A is equal to L into L transpose or as A is equal to U into U transpose, where L is our lower triangular matrix and U is an upper triangular matrix. Any one of these decompositions can be used LL transpose or A is equal to U into U transpose. Now if A is positive definite, then this decomposition is always guaranteed. So if A is positive definite this decomposition is always guaranteed. As in the case of the LU decomposition method this condition is a sufficient condition and not a necessary condition. Now suppose that we have now decomposed A L L into L transpose. How do we use it further for this solution of problem.

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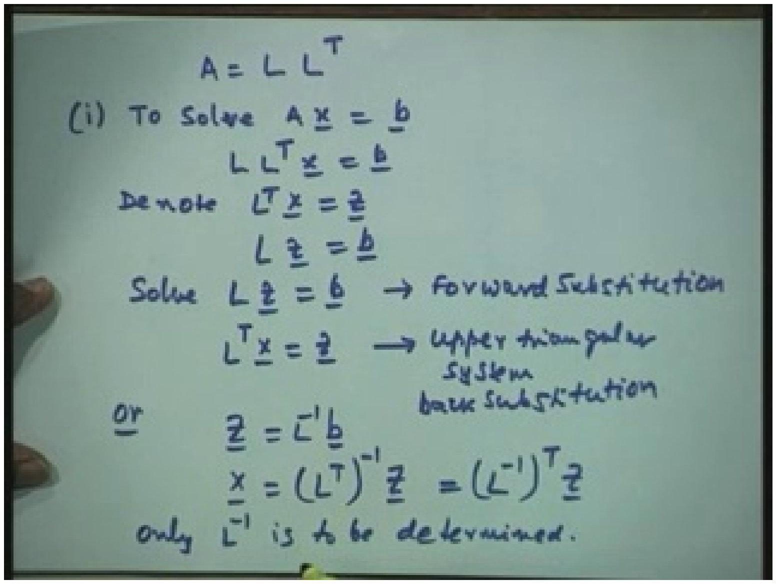

So let us take the case and illustrate for A is equal to L into L transpose. So in the first method we said that we can solve a system of equations or we can find inverse. So let us try to use this to solve the system of equations Ax is equal to b. We follow the same way as in the LU b decomposition, so we would write this as L into L transpose of x is equal to b. Then we shall denote this in a product inside; denote L transpose of x as some z, then this will read as L into z is equal to b. Then we shall say that we first solve the second set of equations, solve L_z is equal to b. Now L is a lower triangular matrix, therefore this can be solved by forward substitution. Then we solve the first set that is L_x is equal to z; L is a lower triangular matrix and therefore its transpose is an upper triangular matrix. So this will be an upper triangular system. Therefore we shall use the back substitution for solving this method.

Now again as we have shown that we can solve here also by one forward substitution and one back substitution but the advantage will be not to use this but to use a slightly different procedure and that is we can have an alternative between these two. Let's write down the first one, take in L inverse of this. So I can write this as z is equal to L inverse of b and the second system I will write x is equal to L transpose inverse of z. We know that transpose and inverse can be interchanged, so this will be equal to L inverse of transpose of z. Therefore I need to determine only one inverse. L inverse is here, L inverse transpose is being used here. So I need to determine only L inverse. Now we have already shown it earlier that if L is lower triangular, its inverse is also lower triangular. Therefore I can use the forward substitution to get the inverse of L also. Therefore L inverse will also be obtained by using the forward or forward substitution only and we can then use the same L inverse in both the places and get the solution of the problem.

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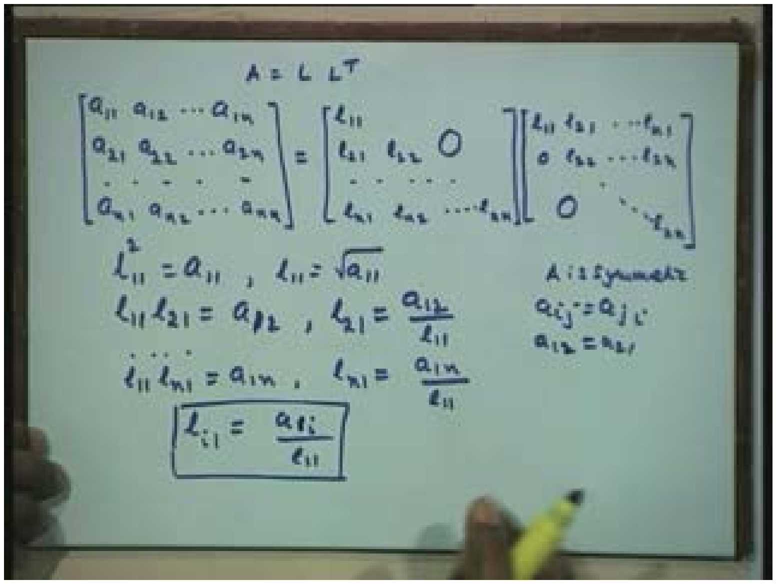

Now let us see how we are going to get a decomposition as A is equal to L into L transpose. So let us write it down our A is equal to LL transpose. Let's write down the full system a_1, 1, a_{12}, a_{1n}, $a_{21,}$ $a_{22,}$ $a_{2n,}$ a_{n1}, a_{n2}, a_{nn}. L is a lower triangular matrix, so let us take it as l_{11}, l_{21}, l_{22}, l_{n1}, l_{n2}, l_{nn} and we are multiplying by its transpose. So I can just transpose it and write this as l_{11}, l_{21}, l_{n1} that is a first column becomes the first row and the second column $l_{22,}$ l_{2n}, and so on. I will have l_{nn} over here. Now let us multiply it and then compare the coefficients. So if I take the first row and the first column I will have l_{11} square, so what I have here is l_{11} square is equal to $a_{11.}$ Therefore l_{11} is equal to square root of a_{11}. Now I complete the multiplication of the first row and the other columns, so l_{11} into l_{21} is $a_{12;}$ this is a product of this and this. Here a is symmetric, we have taken therefore a_{ij} is equal to a_{ji}; therefore a_{12} is equal to $a_{21;}$ a_{1n} is a_{n1} and so on. So these elements will be the identical elements here, so that we have here a_{12} is equal to $a_{21,}$ etcetera. Therefore l_{11} into l_{21} is equal to a_{12}. So I can get l_{21} from here as a_{12} by l_{11} from this. And so on the first row and the last column will give me l_{11}; l_{n1} is $a_{1n,}$ therefore l_{n1} is a_{1n} upon l_{11}. Therefore the entire set can be simply written as l_{i1} is equal to a_{i1} by $l_{11.}$ This is $l_{21,}$ $l_{31,}$ l_{n1} is a_{1n} by $l_{11.}$ So this is our first column of l that we have obtained from here.

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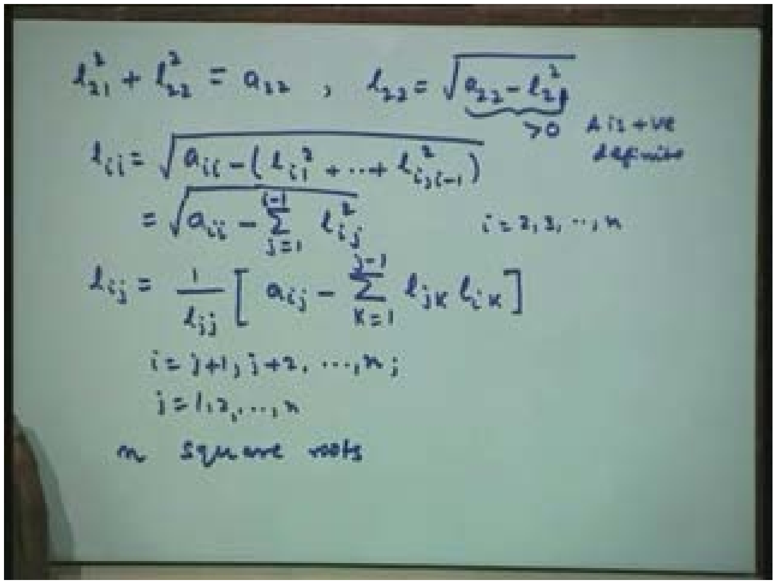

Now we go to the second row second column because we know don't need to multiply the first row because it is symmetric, so it is not necessary. So I will take the second row second column that is l_{21} square plus l_{22} square, that will be equal to a_{22}. So I will then have l_{21} square plus l_{22} square, that is a product of this second row second column is $a_{22;}$ therefore we take two to this side and take the under root of that. So I will have this as square root of a_{22} minus l_{21} square. Now it is here, we mention that if A is positive definite, our decomposition is guaranteed. If A is positive definite, this quantity is always strictly greater than zero. Positive definiteness would imply that your a_{22} l_{21} square is always strictly positive. We are now doing the entire arithmetic in real arithmetic but if you are going to do complex arithmetic, it is okay because then it is a negative number; then a square root will be a complex number but we are doing real arithmetic therefore I would like to have real values only, therefore this should be a positive quantity. Therefore in other cases this may turn out to be a negative quantity wherein the method would then fail. Now we proceed on and then compute all the other coefficients. I can now write down, if you see the diagonal element first, you take any row multiplied by the corresponding column; say i_{th} row and the i_{th} column then it will be l_{i1} square l_{i2} square…, l_{ii} square, that means I would get immediately l_{ii} is under root of a_{ii} minus of l_{i1} square plus so on l_{ii} minus one square or we simply remember it, so that we put this summation notation j is equal to 1, 2, i minus one, that is your l_{ij} square. Therefore these are all the squares of all the previous available elements. We are now on the diagonal element, so we are now taking the square of all the previous elements, that is this particular component here, which is subtracted from a_{ii}; that gives me all the diagonal elements, i is equal to 2, 3, …n. Now for the other elements, I will multiply by the method of matrix multiplication of this. Therefore I can immediately write down that l_{ij} is equal to the same. You can see that all these half diagonal elements would have a division by its pivot.

So first row will be l_{11}, second row l_{22} and so on, if we are doing for l_{ij} will give you $l_{jj.}$ So we will have the division by l_{jj} and we will have here a_{ij} minus summation. This does not have any square roots, so it will be simply k is equal to 1, 2, j minus one again l_{jk}, l_{ik}; i is equal to j plus one, j plus two, so on, k. Of course j is equal to 1, 2, 3 …n.

In this summation when the upper suffix is smaller than the lower suffix then that summation is ignored. For example, if I take j is equal to one here; I will get here summation 1, 2, 0, so there is nothing is there. Therefore that particular case will land into this particular case; this will land into the first one. So if I take j is equal to two, then I will start getting the next elements that we are getting here. So this is the general loop under which all the elements of l_{ii} and l_{jj} can be obtained. Now you would notice why the method was given a square root method. We have got square roots for each pivot, so all the pivots have to be done by square root. Therefore there n square roots are to be used, n square roots are to be obtained from here.

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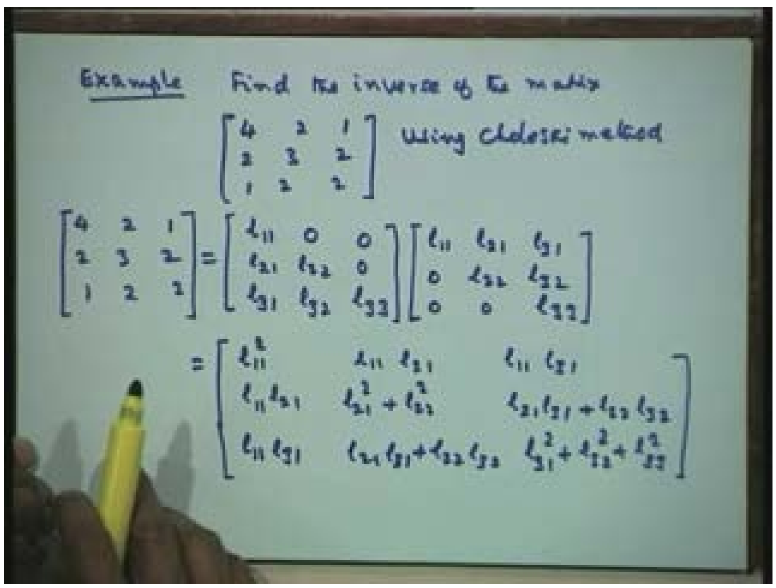

Now let us illustrate this by an example. I will try to find the inverse of a matrix 4, 2, 1; 2, 3, 2; and 1, 2, 2 and let's also write using Cholesky method. So let us just write down the three by three matrix again; 4, 2, 1; 2, 3, 2; and 1, 2, 2. The lower triangular matrix l_{11}, l_{21}, $l_{22,}$ l_{31}, l_{32}, l_{33}. We will have this as l_{11}, l_{21}, l_{31}; 0, l_{22}, l_{32}; 0, 0, l_{33}. Let us simplify right here; let's multiply it out l_{11} square, $l_{11,}$ l_{21}, l_{11}, $l_{31.}$ So we'll have $l_{11,}$ l_{21}, l_{21} square, l_{22} square, then we have l_{21}, l_{31}, l_{22}, l_{32}. That is a product of second row and third column. Then we'll have $l_{11,}$ l_{31} and then we have here this l_{21}, l_{31} plus l_{22} l_{32}, which is the product of third row second column and lastly we will have l_{31} square, l_{32} square, l_{33} square. Now given matrix is symmetric; we have written out l as l and l transpose, therefore these will be symmetric. Therefore this element has to be this element; this element will be this and this element will be this because of the structure of the symmetry that we have used here constructing L and L transpose. Now I just compare these elements and write down the solution of these elements.

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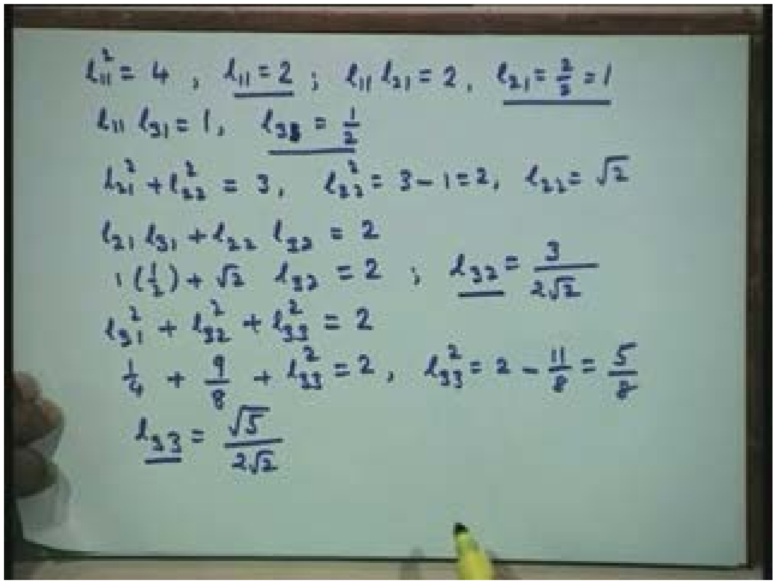

We will have here l_{11} square is equal to 4; therefore we have l_{11} is equal to two. Then the next element l_{11} l_{21} is equal to two.

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When we are talking of a square root we'll be talking of only the positive square root. Either you use right throughout the positive square root or you use throughout negative square root. I can also use the negative square root; it will also give me the same solution. So we are using either as a positive root but it is conventional to use square root as a positive square root. Therefore we have l_{11} is equal to two; therefore l_{21} is equal to two; divided by two, that's equal to one. Then we have the third element l_{11} l_{31} is equal to one. Therefore l_{31} is equal to one by two.

Now the three elements have been determined. The first column has been determined. Now I go to the second. So we will take the middle element l_{21} square plus l_{22} square that is second pivot is given as 3. Therefore let us find out l_{22} square from here, which is three minus l_{21} square; l_{21} square is 1, therefore I have 2. Therefore l_{22} is equal to root of two. Then we go to this last element $l_{21,}$ l_{31} plus $l_{22,}$ l_{32} is given to us as two. Now let us substitute the values l_{21} is one, l_{31} three one is half, l_{22} is root two, l_{32} is to be determined, and that's equal to two. So I can take half to this side which produces three by two, therefore l_{32} is three by two root two. Then the second column is complete. So we go to the third column that means the third element. So we will have to take the last element over here that is l_{31} square, l_{32} square l_{33} square is given as two. Now let us substitute the values l_{31} square is one by four, l_{32} square is nine by four into two, that is eight plus l_{33} square is equal to two. So let us take this to the right hand side, we will have this as eleven minus eleven by eight, which is five by eight. Therefore we have l_{33} which is root five by two root two. So all the elements are now determined.

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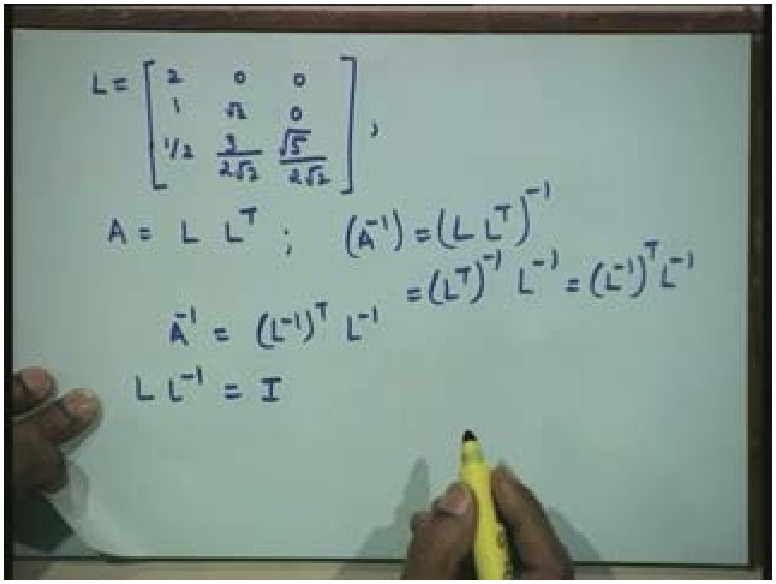

Now let us substitute the elements and then write down what is our L. Therefore from here I can write down the matrix L as 2, 0, 0; 1, root two, 0; that is l_{22} is root two and this is one upon two, three upon two root two, root five upon two root two. Now what is it we want, we want the inverse. We have written A is equal to L into L transpose, I invert it, so I will write down A inverse is equal to LL transpose inverse. Now in inverting the product it gets interchanged, L transpose inverse into L inverse. Now transpose and inverse can be interchanged. So I will write down L inverse transpose L inverse. Therefore our A inverse is simply L inverse transpose L inverse. Therefore I need to get L inverse and then transpose it here, multiply to get my A inverse. So L is lower triangular, therefore it's inverse is also lower triangular. Therefore I will use the idea that L and inverse must be equal to I, product should be equal to I identity matrix.

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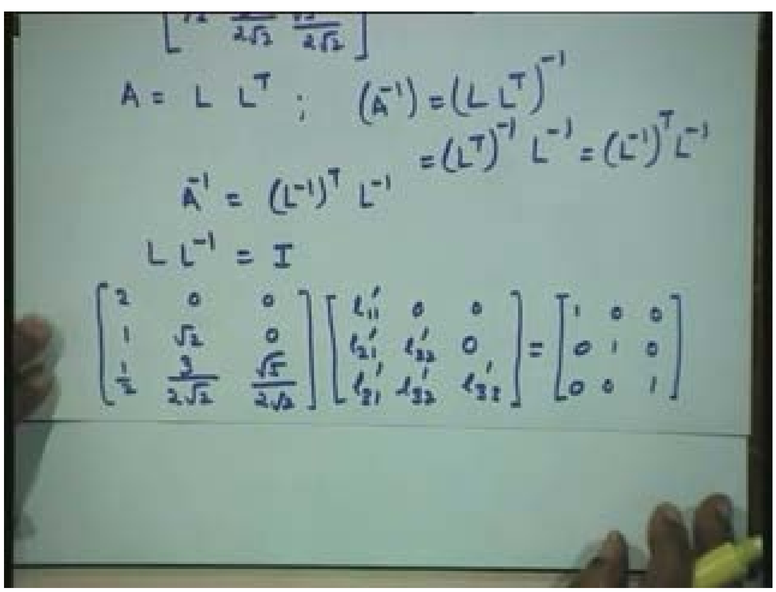

Let’s write down our L and L inverse. So we will have here 2, 0, 0; 1, root two, 0; ½, three by three root two, root five by two root two into L inverse. Now let’s call this as L_{11} dash, 0, 0; l_{21} dash, l_{22} dash, 0; l_{31} dash, l_{32} dash, l_{33} dash. So let’s denote the elements by the dash terms. We can make one observation; some of the elements are trivial because this is a lower triangular matrix, same is true for upper triangular also. The diagonal elements of this will be inverse of the diagonal elements. In fact we don’t need to compute them; l_{11} dash has to come as one by two; l_{22} dash will be one upon root two and l_{33} dash will be two root two by root five. But of course in our computation we can proceed on; but that will be a check on doing the correct solution or not.

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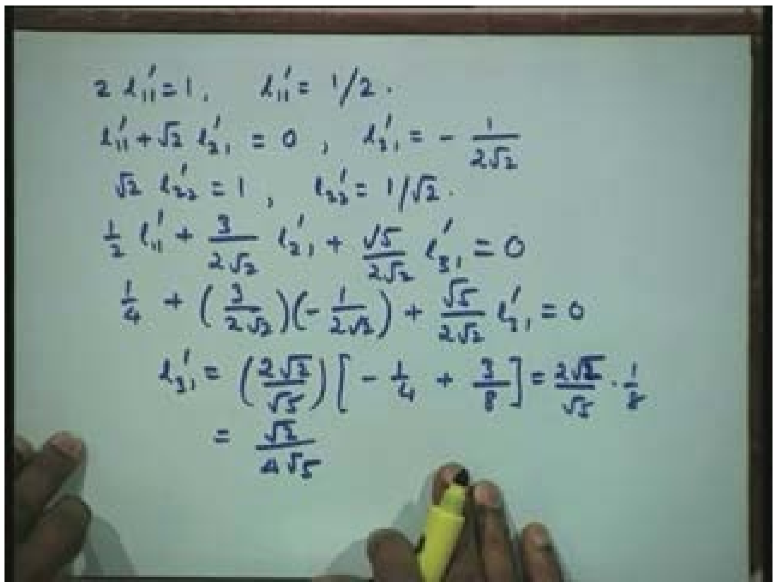

So these elements are trivially known, therefore we can see the if I multiply the row by this column, what I am getting is two times l_{11} prime is one, that is your this element. Therefore l_{11} dash is equal to one by two. So this is the element that is true, then we multiply the second row and the first column. So that I will have l_{11} dash plus root two, l_{21} dash; I then multiply the second row first column, which should be equal to zero. So this should be equal to zero, therefore l_{21} dash. Now l_{11} dash goes to the right hand side, so I will have minus one upon two root two. Second row second column just gives you root into l_{22} dash that is equal to one, so l_{22} dash is equal to one upon root two. The observation which you made that the diagonal elements be of this will be inverse of this that is one upon root two. Then I can multiply this row and this column. So I will have half l_{11} dash three upon two root two l_{21} dash plus root five by two root two l_{31} dash is equal to zero.

Now let us substitute these elements here, that is equal to half into half, that is equal to one by four; three by root two, l_{21} dash is minus one upon two root two and this we retain it as this, two root two, l_{31} dash is equal to zero. So let us take everything to the right hand side and get the element l_{31} dash that is two root two by five and then this will be minus one by four plus three by eight. That gives us two root two by root five and this is equal to one by eight. So that is equal to root two by four root five.

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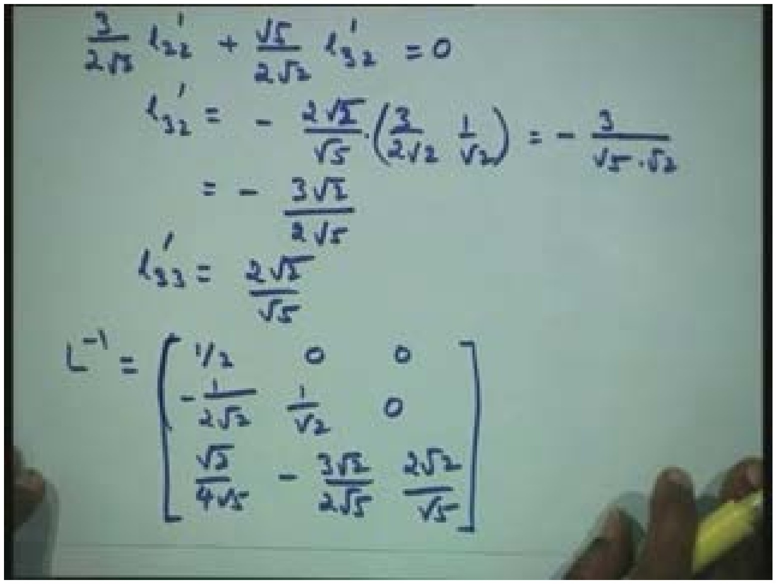

Now we have determined the element l_{31} dash. Now to get the next element I need to multiply by this third row with the second column. So if I do that, it gives you three upon two root two, l_{22} dash plus root five by two root two, l_{33} dash. I will just have a look at this; here we are multiplying this row this column, so what I would get here is three upon two root two into l_{22} dash plus root five upon two root two, l_{32} dash and these elements should correspond to this, that is equal to zero. Therefore we can find out l_{32} dash is equal to minus this factor, let's write it; first two root two, divided by this three upon two root two, one upon root two. So these gives us minus three by root five or let's write it as three root two by two root five. Lastly we have the product of this row and this column. So that will be two root two by root five which is the inverse of this element. Therefore we have determined our L inverse; L inverse is equal to l_{11} dash that is equal to one by half. I will just write these elements; ½, 0, 0; minus one upon two root two, one upon root two, zero; root two by four root five, minus three root two by two root five and two root two by root five; so this is our inverse.

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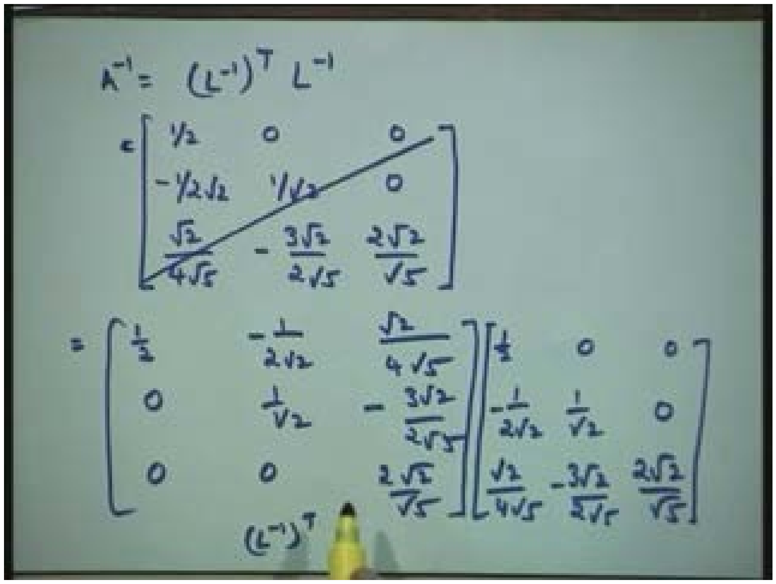

Now I write down the solution of the problem as A inverse is equal to L inverse transpose into L inverse. So I will invert this and write it as first matrix that is equal to one by two,0, 0; minus one upon two root two, one upon root two, zero; root two by four root five, three root two by two root five and two root two by root five. I should have written the transpose of the matrix L inverse. Let us avoid writing one extra step. Let us now write the transpose of L inverse; one by two, minus one upon two root two, root two upon four root five; zero, one upon root two and three root two by two root five; zero, zero, two root two by root five. This is our L inverse of transpose. This matrix ½, 0, 0; minus one upon two root two, one upon root two, zero; root two upon four root five, three root two by root two five, two root two by root five. So this is the matrix.

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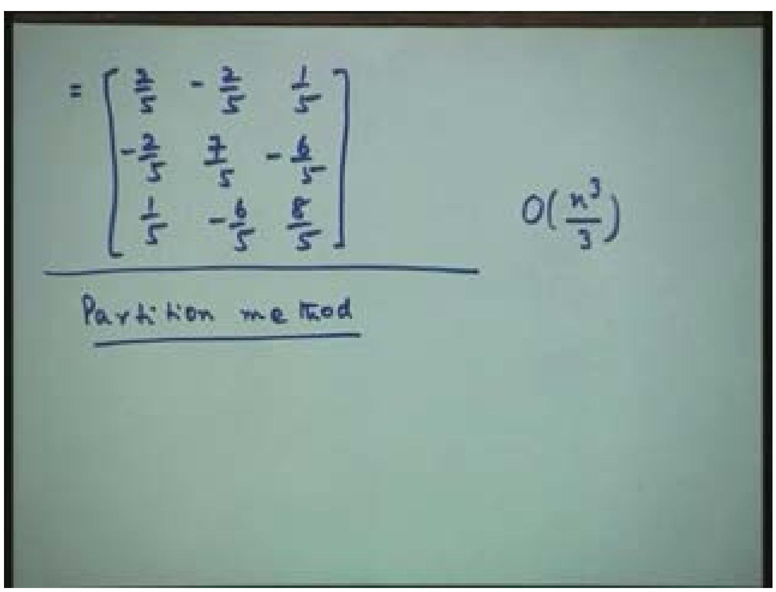

Now it is a very simple thing. Just multiply it out and simplify it, I will give the value for this so the value that comes out is simply equal to 2/5, -2/5, 1/5; -2/5, 7/5, -6/5; 1/5, -6/5, 8/5. This is obtained by just multiplying the two matrices that we have given there. Now the application of this would be that you have determined the inverse in a particular problem and if I have more than one right hand side, this will be advantageous for us to find the inverse. We are finding only one L inverse in this particular procedure. Now these two direct methods that is the LU decomposition method and its version that is Cholesky method for symmetric matrices and the Gauss elimination are the most powerful method that are used in our software. Even though there are more variants of the decomposition procedure than the Gauss elimination procedure, even today these are the most power methods for solving the system of equations, when the system of equations is small. If the system is too large of millions of or lakhs of equations, we just cannot load them on the computer. Therefore there is no way that we can use these direct methods to solve such huge systems.

But there are some other practical problems. Let us try to solve a practical problem like this. You are solving a particular problem within a given domain by a finite difference method (some set of differential equations) and it gives you system of linear equations, lets assume it is a linear equation. Now you find that the region that you have taken is not sufficient and you want to have a bigger region. Now you again solve by the same method, you produce again a system of algebraic equations. Now the idea is can we use the computation that has already been made? That means you have solved this system of equations by decomposition, and you have A inverse for the system available. Now I solve the problem by extending that problem further. Can I use this computation that has already been used rather than waste the computations. The answer is yes, the particular application in fluid mechanics wherein the condition for solving a particular differential equation is given as a condition like, as x tends to infinity it should have this value. So a condition as extending to infinity is given to us but infinity is not defined in a particular problem, infinity could be nine in a problem and in other problem it could be hundred, so you want to solve it numerically and so you fixed a finite length for that. You find that you have not achieved U is equal to U naught in the problem, so you have not positioned your infinity properly. So we have to move the infinity further. So infinity could be achieved, now you solve the problem. What happens to the resultant is that the original system of equation stays as it is. If you look the rows and column of Ax is equal to b, what we have really done is we have added few rows and few columns to the original system and got a new system.

Now the idea is we want to use the inverse that has been determined earlier in the previous problem to solve the current problem and reduce the tremendous amount of computations. The other application that we would have is suppose you have a system of hundred by hundred equations to be solved. Now I prefer to solve two fifty by fifty system of equations and some more computations rather using hundred by hundred systems.

We are going to show later that the computation efforts for the Gauss elimination or the decomposition are order of n cubed by three for n large; that means if I solve the problem hundred by hundred and I solve a fifty by fifty, the amount of computation is at least of the order of magnitude of four times. So if one problem takes two minutes, other problem takes another six to eight minutes, so the order of the time that is taken up by doubling a system is much higher.

Therefore what I would prefer in that case is I would like to use if possible the smaller systems and in that direction we have what is known as partition method, which will partition our matrix A and obtain the inverse of this matrix using the inverse of the smaller systems.

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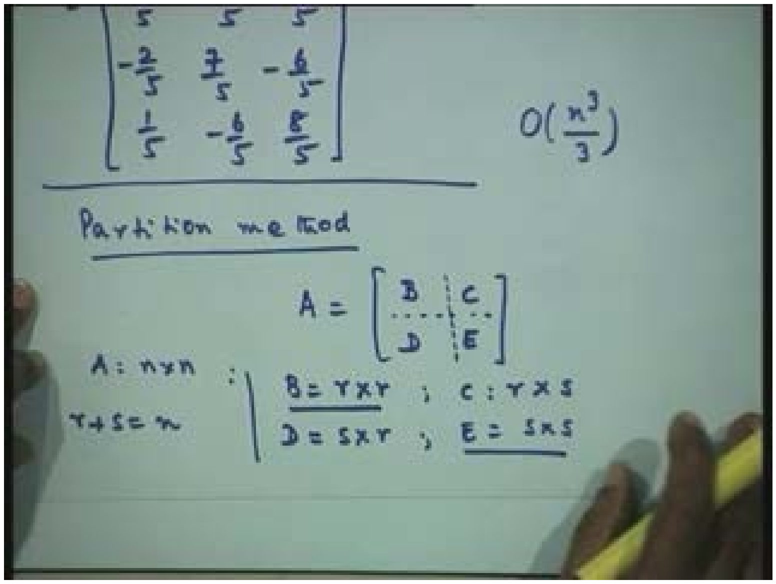

As I said this has got very important applications in many areas of engineering. So what we do is given this matrix A, I would conveniently decompose it, separate them as partition as this and then name this as some matrices B C D E. Now let us take A as n into n matrix, so I can take B is equal to some r into r matrix and C is same rows, therefore C is r into s, D is equal to s into s and E is equal to s into s. So r plus s is equal to n. So these two matrices B and E are square matrices and these are rectangular matrices. This is r into s, this is s into r but this is r into and r s into s. These are square matrices and these will be rectangular matrices.

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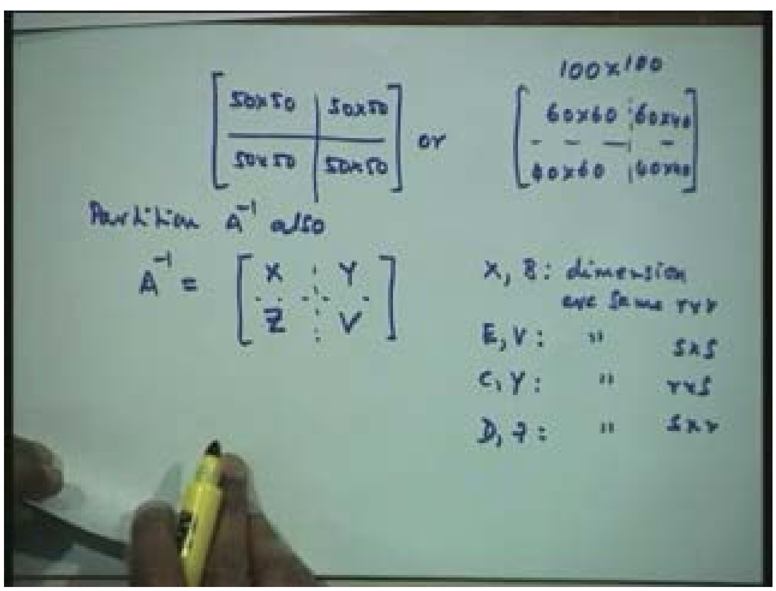

Suppose you have a system of hundred by hundred. I can decompose if it was given like this, I can take sixty by sixty here, I can take forty by forty system here; the two square matrices, then this will be sixty by forty and this will be forty by sixty. Or I could as well decompose it as fifty by fifty. Now what we were trying to say here is that if I want the inverse of this hundred by hundred matrix I would like to utilize the known inverse; for example, of this fifty by fifty matrix or I know the inverse of sixty by sixty matrix and thereby the amount of computation is reduced tremendously. Now in this case what we would do is we would also partition A inverse in this same way. So let us write A inverse is X Y Z V; partition also A inverse as Z and V, so that the dimension of X and B are the same. So similarly your E and V dimensions will be the same and your C and Y dimensions will be the same and D and Z dimensions will be the same. We have taken this as r into r, we have taken this as E and V; as s into s; this C is r into s; this is s into r.

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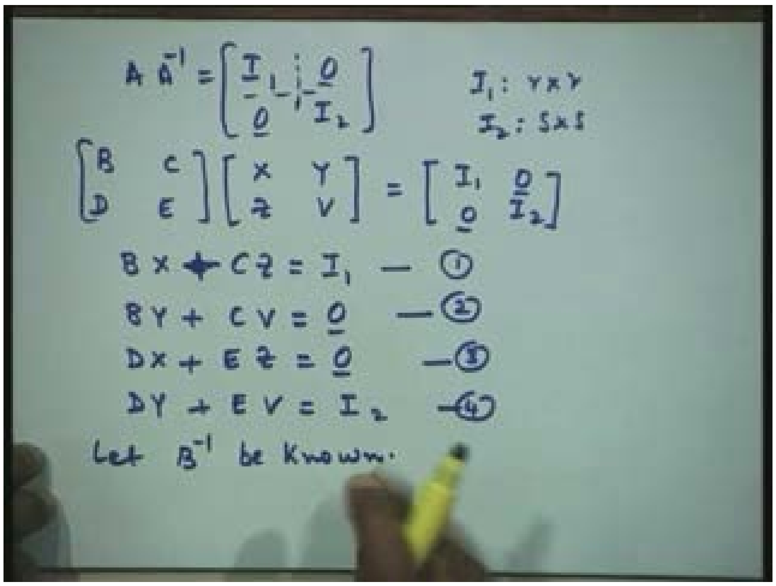

AA inverse should be equal to I, and this I will also partition it as $I_{1,}$ I_2. I_1 is of dimension r into r, I_2 is of dimension s into s. So I will partition it also into two identity matrices; I_1 and I_2. This will be the same dimension of our X and V and I_2 will be the dimension of V. Now let us put the values of A and A inverse, that is B, C; D, E; X, Y; Z, V and this is equal to I_1 and $I_{2.}$ Now let us just multiply out the two matrices. There is matrix multiplication. So I will have B into X plus C into Z equal to $I_{1.}$ So let us just number it as one. Then B into Y plus C into V is null, this is a null matrix. Let's number it as two. Then D into X plus E into Z is null. So i will have D into X plus E into Z is null. Then the last equation is D into Y plus E into V is equal to I_2. Now we have assumed that the inverse of the B matrix. One of the inverses is available for us either inverse of B is available or inverse of E is available. So let us start because as an application oriented problem, let us assume inverse B exists and so let B inverse be known. Now it is simply to solve these four equations for these four unknowns, so since B inverse is known let us start with equation two.

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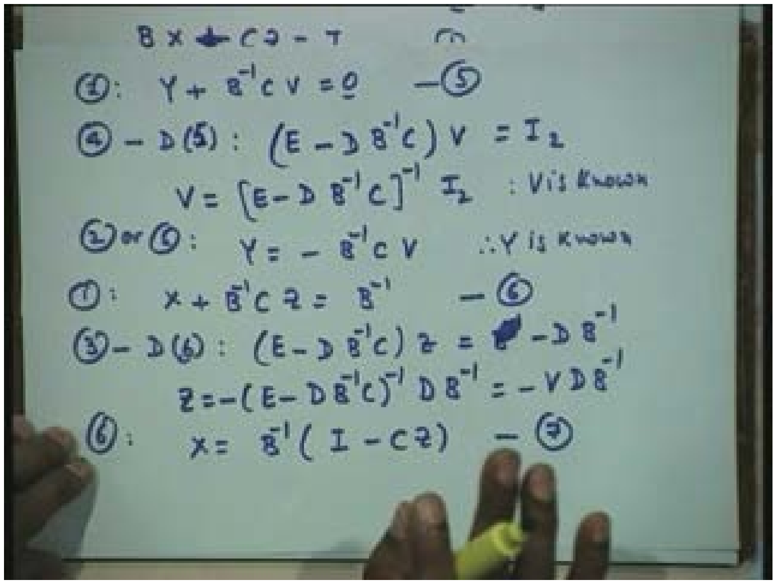

Let us start with the equation two. I pre multiply by B inverse, so this gives me B plus Y plus B inverse CV is equal to null. I have pre multiplied this by B inverse, so Y B inverse CV is equal to zero. Now I eliminate D, eliminate Y from four using this. So what i have to do here is I multiply this pre multiply by D and subtract from here. So what I will do therefore is I will take fourth equation minus D into equation two, and let's call it as five. From four I will pre multiply this by D and subtract from there. From this I am subtracting that is equal to E. Now we are multiplying this by D therefore this D Y is gone and I will have E minus D B inverse C. I pre multiply this by D, so this is gone, so D B inverse of C into V and this is equal to I_2. Now this goes to the right hand side, therefore V is equal to E minus DB inverse C inverse. This is I_2. I can throw away I_2 because I_2 is multiplied by identity matrix. Therefore V is known.

Now let's go to two or five. Now therefore Y is equal to minus B inverse C V (from five, Y is equal to minus B inverse C V). Therefore Y is known; therefore Y is equal to known. Now I have manipulated equation two and four, I will do similarly for one and three. So let us take equation one. Again pre multiply by B inverse, so I will have X plus B inverse of C Z is equal to B inverse. I pre multiplied this by B inverse. Now I want to throw away X from here, so I multiply this by D and subtract from there, therefore three minus D into let's number it as six. So from the equation three I multiply by D and subtracted from here, so what I will again have here is E minus D B inverse C of Z, that is your E here, I have multiplied this by D, D B inverse C that is equal to minus DB inverse. We multiplied by D and subtracted minus D B inverse, therefore Z is equal to E minus D B inverse C inverse D B inverse. Let's bring this minus sign over here but this E minus D B inverse C inverse is V. Therefore I can use this value of E over here and write this simply as V D B inverse. Now the solution is complete. Now use six; six gives me X is equal to B inverse, I can take out the common there I minus C Z I minus C Z. So I am using this, I am taking this side and taking the common on the left. B inverse I minus C Z. So this is your seven.

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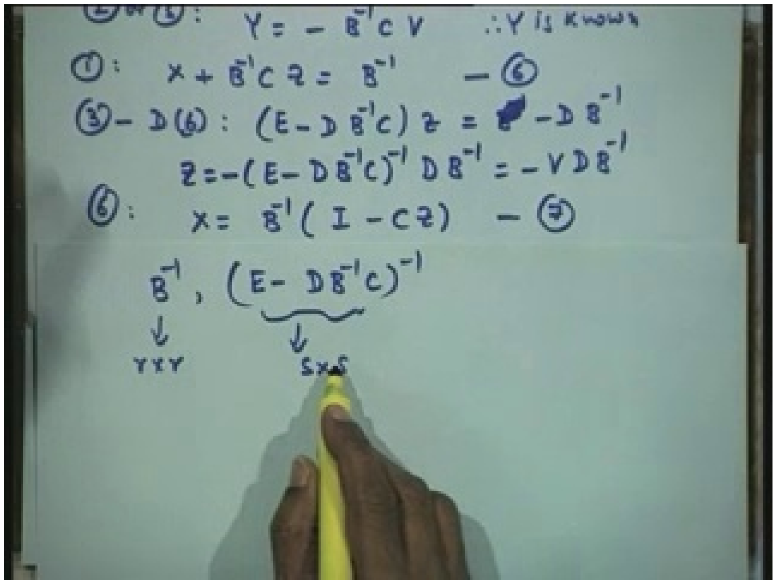

Now we have completed the solution of all the four. Now what we should see here is that in building this we have used only two inverses; one is B inverse, the other one that we used here is E minus D B inverse C inverse. These are the two inverses that we have used in the construction of the entire system over here. Now what is this order; the order of this is r into r, the order of E was s, therefore order of this is s into s. Therefore we are now using the inverse of only two matrices r into r, s into s.

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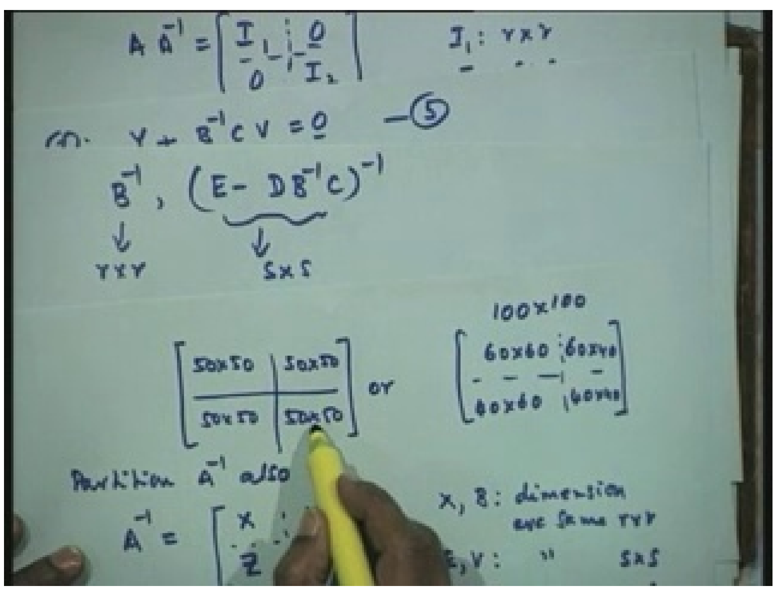

Now if you go back to the example which we had written earlier, if you are constructing the inverse of this particular matrix I need inverse of this and not a hundred by hundred system. But by using this or for example in this case I need the inverse of two fifty by fifty matrices and of course I need some multiplication of the matrices to be done; leave alone the extra computation. So we shall be doing enormous saving of computation in finding the inverse of large system of equations by using this particular concept that we can partition a given matrix into a suitable form. And then find out inverse of that particular matrix using the inverses of the lower order matrices.