Numerical Methods and Computation

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Lecture No # 14

Solution of a System of Linear Algebraic Equations (Continued)

Now in the previous lecture we were discussing about the decomposition method.

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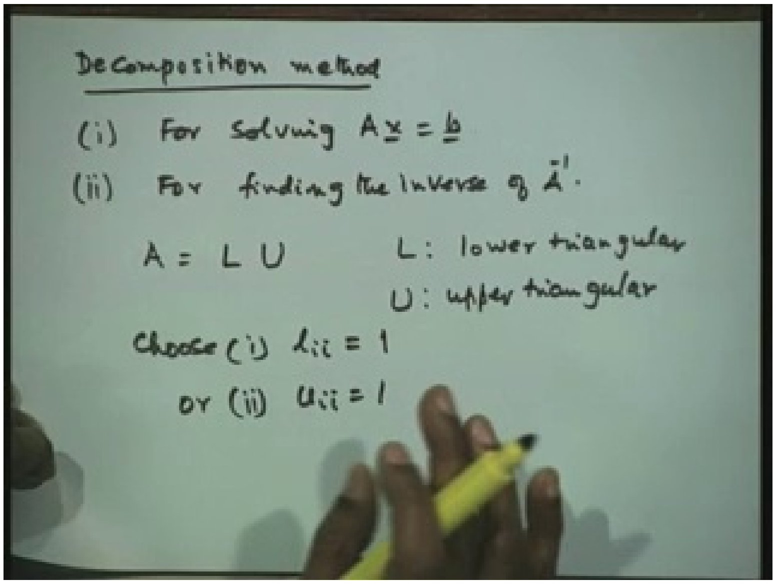

So we are discussing about the decomposition method. We mentioned that we can use this method for solving the system of equations Ax is equal to b. We can solve the system of equations; alternatively we can also find the inverse of a given matrix, A inverse. Now for this purpose we had written A as the product of L and U; L is a lower triangular matrix and U is an upper triangular matrix. Now we have also shown that if you multiply L and U and compare it with A, we have got an arbitrary parameters and hence we said chose either l_{ii} is equal to one that is the diagonal elements of L as one or we chose u_{ii} as one that is all the diagonal elements of U is equal to one. Then the computation is simplified enormously because you can just get all the elements of L and U by simple forward substitution, if we use this particular way of writing it. Of course there are other ways of writing it; however that will not simplify our computations. Now let us see how we can implement this in our computations.

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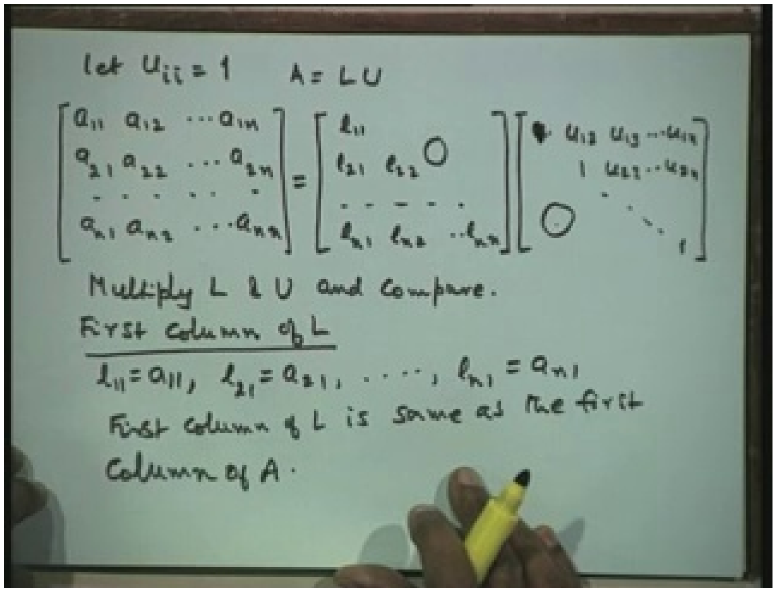

So let us take u_{ii} is equal to one. Therefore my system of equations A into A is equal to L into U would look like a_{11}, a_{12}, a_{1n}, a_{21}, a_{22}, a_{2n}, a_{n1}, a_{n2}, a_{nn}; this is our given matrix A. Then I write the product as l_{11}, l_{21}, l_{22}, so on, l_{n1}, l_{n2}, l_{nn}. These all are zeros. Then I have here U; we are taking U_{ii} is equal to one. So I will put one here, U_{12}, U_{13}, U_{1n}, 1, U_{23}, U_{2n}, so on as one. We have taken all the diagonal elements of U as one and we have set up this equation. The procedure in any case is, we multiply these two matrices and compare these elements. So we will say multiply; multiply L and U and compare. Interestingly we can find these elements of L and U in a particular sequence. We will see that we can first find all the elements in the first column of L. Now we multiply the first row of L, first column of U; that is simply l_{11}. So l_{11} is equal to a_{11}. Now I am going through the first column of L; so I will take the second row, multiply by the first column of U, so this is l_{21} into one; remaining are all zeros. So l_{21} is a_{21}. Similarly so on I will have the last row multiplied by the first column l_{n1} into one will be equal to a_{n1}; that means the first column of L is same as the first column of A. In other words there is no computation involved here. We just replace the first column of L by the first column of A simply.

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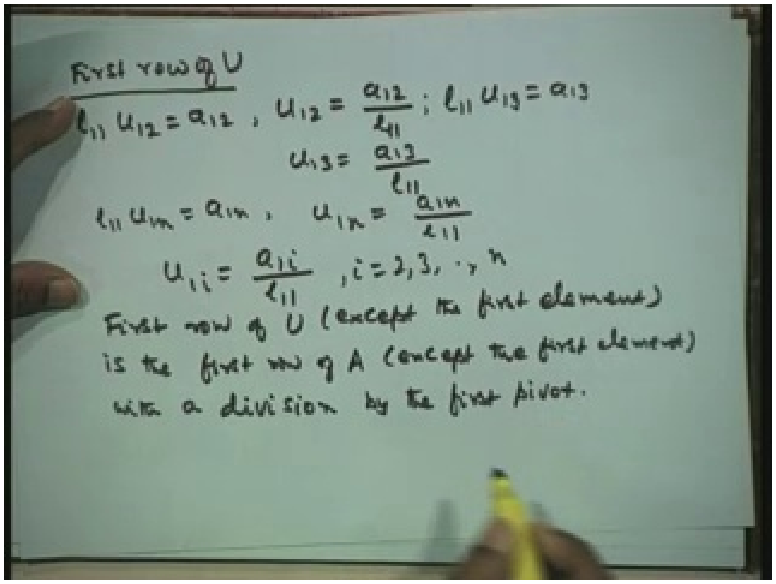

Then we go to first row of U. So in the next step I would now proceed on with the first row of U. Now proceed with multiplication; the first row, second column l_{11} into u_{12} and that will be a_{12}. Let's now solve for u_{12} is a_{12} by l_{11}. Then I multiply the first row with the third column that gives me l_{11} into u_{13}; that is your a_{13}. Therefore I have u_{13} is a_{13} by l_{11}. I proceed on with the first row multiplied by the last column, that gives me l_{11} into u_{1n} is equal to a_{1n}; therefore I have u_{1n} is a_{1n} by l_{11}. Therefore what we have here is u_{1i} is a_{1i} by l_{11}; i is equal to 2, 3 …In other words we can say that the first row also can be immediately determined; because the first row of U leaving the first pivot is nothing but the elements of A divided by the leading element of a; because we have proved that l_{11} is a_{11}. Therefore it is simply division of the elements of U by the leading element a_{11}. Therefore we can state here the first row of U except the first element is the first row of A, again except the first element with a division of a leading element by the first pivot. What we are saying is that you can take the elements of A divide each one of these elements by the leading element, that is a pivot and that will give me all the elements of u_{11}, u_{12}, u_{13}… u_{1n}. Now the first row, first column of L is complete; first row of U is complete. Now I go to second column of L, second row of U and so now let us find the elements of the second column of L.

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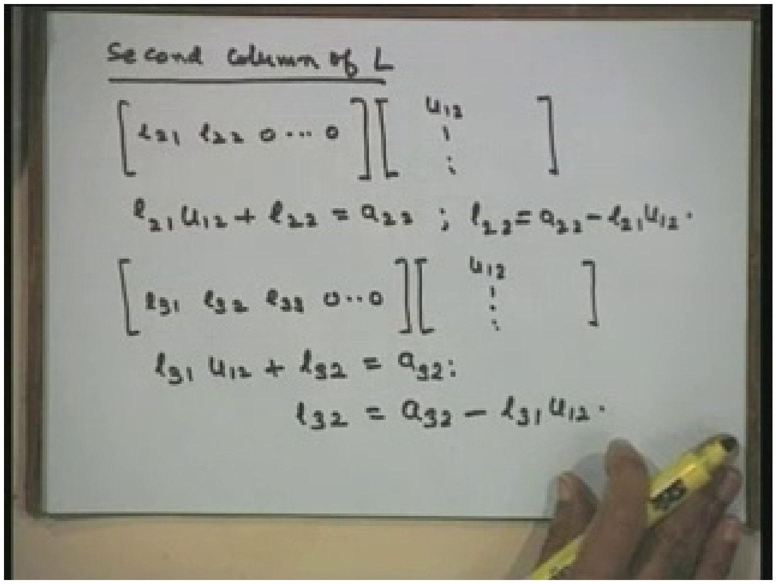

Now in order to determine the second column of L, I need to consider the product of the second row of L, second column of U and so on. Let us first do what we will get here. So the second row of L is this and the second column of U is $u_{12,}$ 1 and so on. If I multiply I would get $l_{21,}$ $u_{12,}$ plus l_{22} is equal to a_{22}. Therefore this gives us l_{22} is a_{22} minus l_{21} u_{12}. Now we have determined this element l_{22} here. Now I would take the third row of l and the second column of u; that means I will now consider the product $l_{31,}$ $l_{32,}$ l_{33} and so on, multiply by u_{12}, 1and so on which gives me l_{31}, u_{12}, plus l_{32} is equal to a_{32}. Therefore this gives me l_{32} is equal to a_{32} minus l_{31} u_{12}. Now we have determined the element $l_{32,}$ which means this element l_{32} has been determined.

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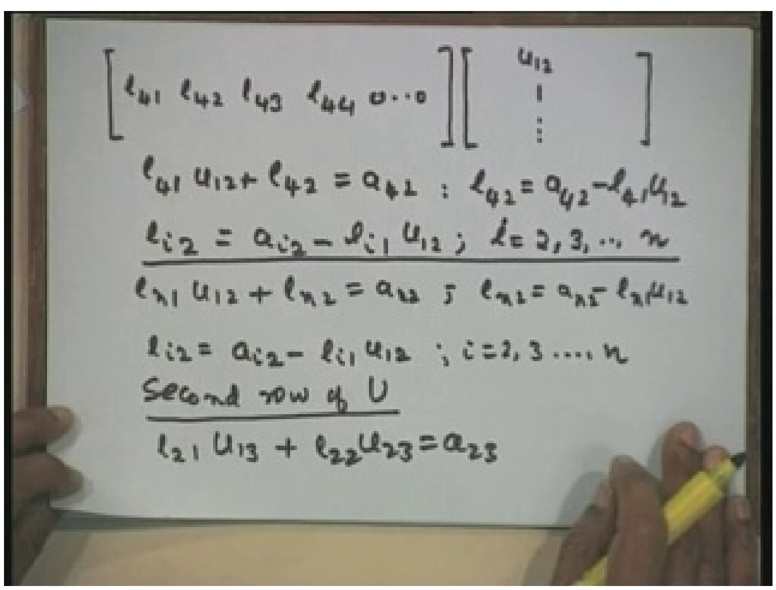

Now let us now take the next row of l and the second column of u; that means we shall be considering $l_{41,}$ $l_{42,}$ $l_{43,}$ l_{44} into this second column which is u_{12}, 1and so on. If I multiply I get l_{41}, u_{12} plus l_{42} is equal to a_{42}. Therefore I get l_{42} is equal to a_{42} minus l_{41} u_{12}. Now from this I can generalize that l_{i1}. I can write down from here l_{i2} is a_{i2} minus l_{i1} u_{12}; i running from 2, 3 and so on n. Now the last element of this column will be obtained by multiplying the last row of l into the second column of u; that means we are talking of l_{n1}, u_{12} plus l_{n2} is equal to the element a_{n2}. This gives me l_{n2} is equal to a_{n2} two minus l_{n1} u_{12}. Hence you can see that for obtaining the elements of the second column of l, it is sufficient for me that I use this particular equation which I would write again as l_{i2} is a_{i2} minus l_{i1} $u_{12;}$ i running from 2, 3 and so on n. Therefore in one loop I will be able to get all the elements of the second column of l by just putting the range of the loop from i running to n; then I will be able to determine all the elements of the second column. Once we complete this we go to the second row of U that means I would now like to find out what will be the elements of this second row that is U_{23} so on up to U_{2n}. Therefore I now multiply the second row of L and third column of U which gives me $l_{21,}$ $u_{13,}$ $l_{22,}$ $u_{23;}$ u_{23} is a_{23}.

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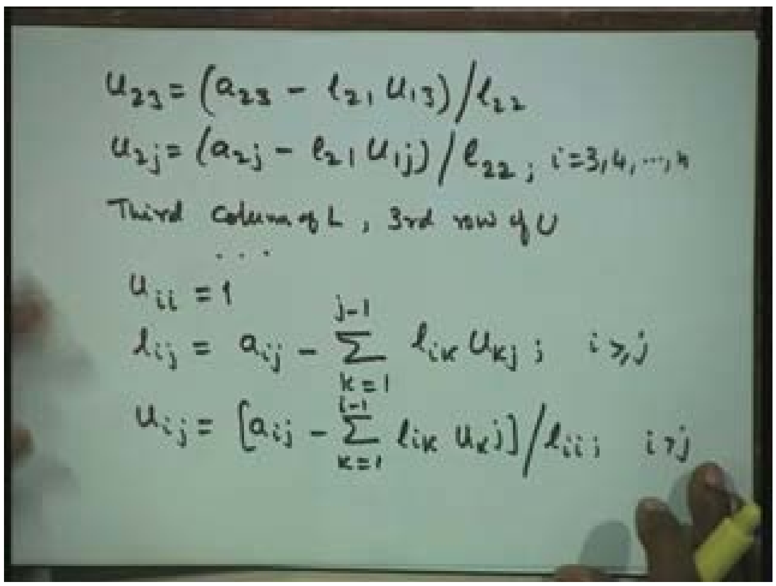

The first element of U that is your u_{23} is equal to a_{23} minus $l_{21,}$ u_{13} divided by l_{22}. So you can see that the elements of U would again require division by the leading pivot l_{22} determined in the previous step. Now we can generalize it and write it as u_{2j} is equal to a_{2j} minus l_{21} u_{1j} divided by l_{22} and i going from 3, 4 …n. Now the entire set that I had written here can now proceed on. We go to the third column of l, then third row of U and so on. So this is how we proceed on the actual computation. Therefore I can write down what all I have; let me write it. The method therefore is we start with u_{ii} is equal to one, then I can have l_{ij} is a_{ij} summation; I will put this in summation notation, k is equal to one to j minus one l_{ik} u_{kj} for all; of course i greater than or equal to j and these elements u_{ij} is a_{ij} minus summation of k is equal to 1, 2, i minus one l_{ik} u_{kj} and divided by the leading diagonal element that is your $l_{;}$ of course all i greater than j.
This is the notation in the summation notation. We can now write down all the elements of L and U in this particular form and what all we have written here is just followed from this one and we have just determined it in this.

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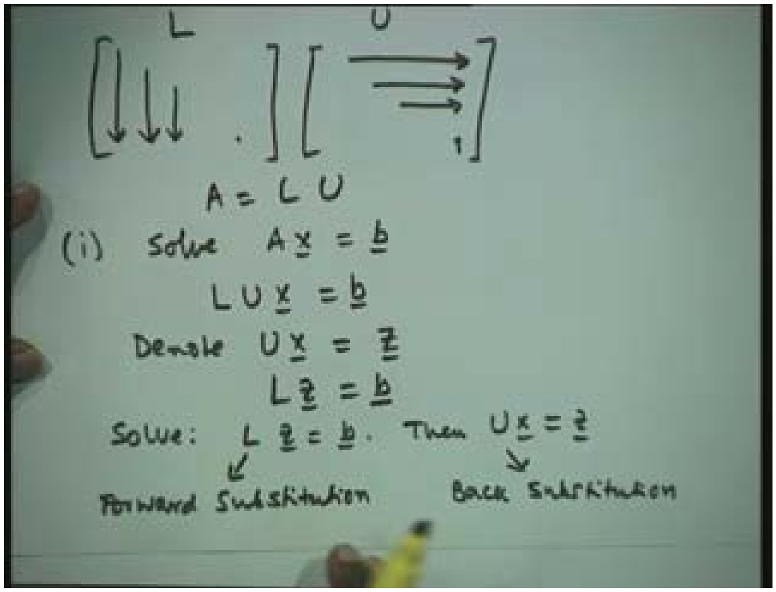

Now as we mentioned earlier we have L and U. When we are computing the elements of this I am first finding the elements of L, then I am completing the row of U, then I am completing this second column of L, then I am completing the second row of U. So we proceed in this particular manner until we reach the last element here the element was already one. So we reach the last element of one. Now the computer implementation has been made quite effectively. This particular method as I said is one of the most popular methods than the Gauss elimination method. Various strategies of actually writing the program to make the computations optimal has been done; one particular thing is this storage, that is in which you don't have to ask for storages for A, L and U. You can over write the elements of A with elements of L and U so that we know that the elements u_{ii} is equal to one. We can over write the elements of A by the elements of L and U as the computation progresses on. Therefore we don't need matrices of order n into n for A, n into n for L, n into n for U. So the optimality can be obtained in the way one can do the storage and the other aspects of the problem. Now let us assume that we have obtained our A is equal to L and U. The next step is we have given two problems in the starting and that is one to solve your system of equations Ax is equal to b. Now we substitute for A that is LU into x is equal to b. Now I denote the product U into x is equal to some vector Z, then this reads L into Z is equal to b. Then we shall write down the method as; we shall solve the second one, LZ is equal to b, then Ux is equal to Z. First solve the L into z is equal to b, then solve U into x is equal to z. Now we know L is lower triangular, therefore this is forward substitution. So we are solving this by forward substitution. We are solving these first set of equations using forward substitution and whereas this is U the upper triangular matrix and therefore this is solved by back substitution. So I can solve this by back substitution. Therefore besides the decomposition of A into L into U, I need one forward substitution and one back substitution to get the complete solution of the problem. Obviously the major the work or the bulk of the work is only decomposition of L into U. We can give the exact number of operation that takes place because these are simple forward substitutions.

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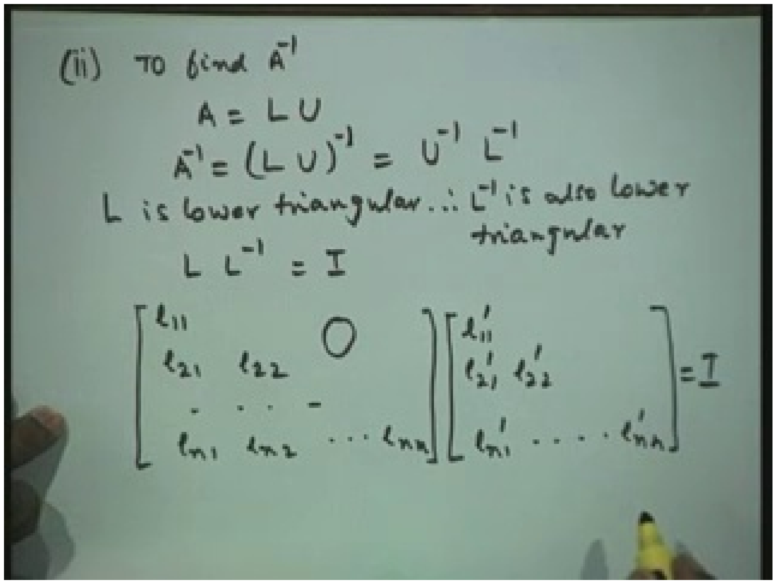

Now let us see what we would do if I need inverse of a matrix. Now if I need the inverse of the matrix we start with A is equal to L into U. Let us inverse both sides; A inverse is equal to L into U inverse, so that is equal to U inverse L inverse. Now we know that L is lower triangular, therefore L inverse is also lower triangular. Knowing L, I want find L inverse. Therefore I can find the elements of L inverse using the identity L into L inverse is equal to I that means I will write down l_{11}, l_{21}, l_{22}, l_{n1}, l_{n2}, l_{nn} and then the inverse is to be determined. So I will call this as l_{11} prime, l_{21} prime, l_{22} prime, l_{n1} prime, l_{n1} prime, l_{nn} prime and this is equal to I identity matrix. Now you can see that if I multiply these two I can get l_{11} prime; first row, first column gives me l_{11} prime.

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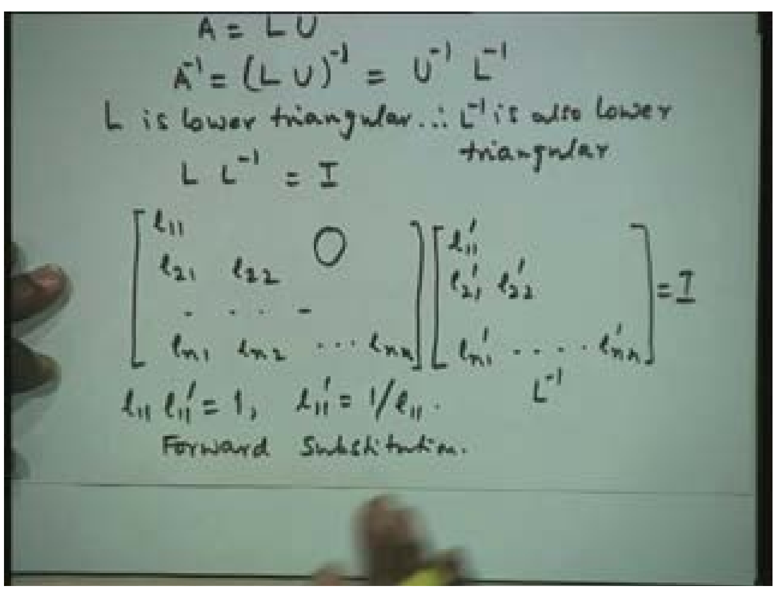

For example l_{11} into l_{11} prime is one. We are equating to identity matrix, therefore l_{11} prime is one upon l_{11}. Now I can proceed to find other elements and again this is a forward substitution. So again by forward substitution I can get all the elements of l, all the elements of l inverse. All the elements of l inverse can be determined by simple forward substitution using this. One particular point that had to be noted is that, this is a lower triangular. Therefore these diagonal elements will be just the inverse of the diagonal elements; we need not do that computation because l_{11} prime is upon $l_{11,}$ l_{22} prime will be one upon $l_{22,}$ l_{nn} prime will be equal to one upon l_{nn}. So these diagonal elements need not be computed, they can just be given the values of this and then we can proceed on to find the other elements, all by forward substitution. Now I need the U inverse, so again I would start with the concept that U is upper triangular.

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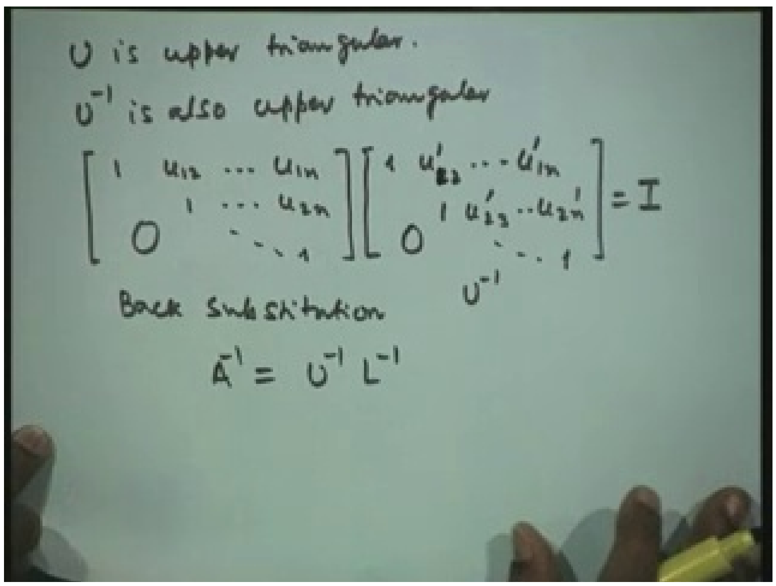

U is upper triangular, therefore U inverse is also upper triangular. Now U is 1, u_{12}, u_{1n}, 1, u_{2n}, 1; therefore its inverse will be 1. I am writing the solution of the elements because if I write u_{11} prime, I know one into u_{11} prime will be equal to your one. So I can write down one itself, that is u_{12} prime, u_{1n} prime, 1, u_{23} prime, u_{2n} prime, so on and one is equal to I. Now I can work backwards and get all the elements of U. Therefore the back substitution would give me all the elements of matrix U inverse. Once we determine this L inverse and U inverse then we can write down required A inverse as U inverse into L inverse.

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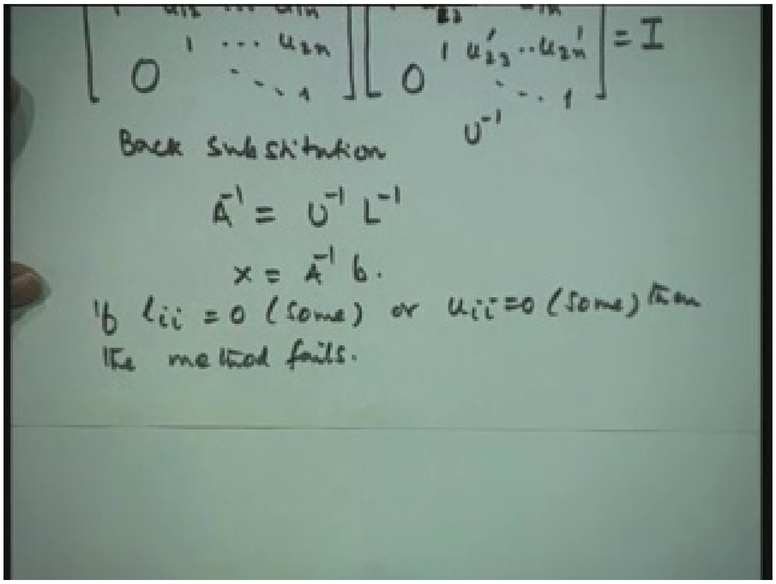

Now this can be used to just keep the inverse of the matrix or it can be used also for finding the solution in system, if you are solving a system of equations. We can also use this as x is equal to A inverse of b, if we need the solution of the equations Ax is equal to b. Now as given here, we discussed in Gauss elimination also that if the pivot is zero or very small then the method may fail. Here also we will have a difficulty if any one of this pivots l_{ii} or u_{ii}, whichever procedure we take, becomes zero. So it will fail only if l_{ii} is equal to zero, (not all, only some) or u_{ii} is equal to zero, then the method fails. But of course there is always the way out for this one. One can introduce other techniques of avoiding this pit fall of having a pivot as a zero. However if the given matrix is positive definite matrix, then none of the l_i's will be zero or u_i will be zero and decomposition is guaranteed.

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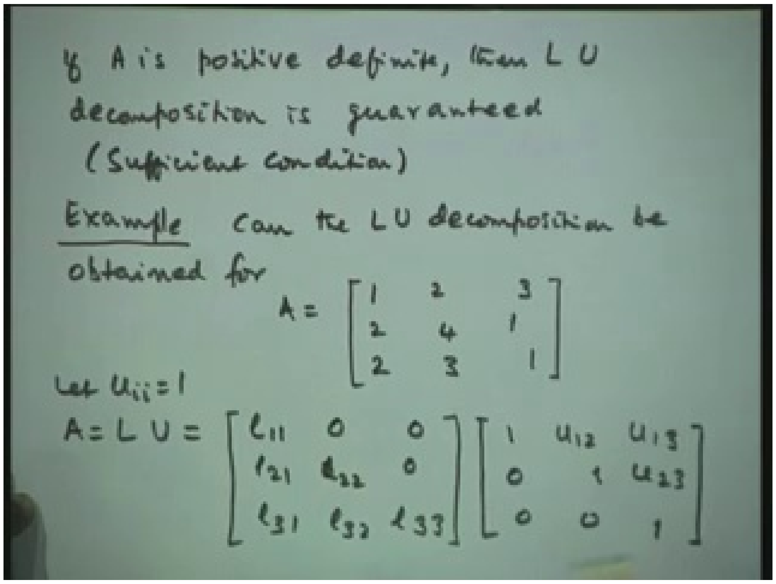

So if A is positive definite then L U decomposition is guaranteed. We have given all the leading minors are positive or the leading minors of matrix is there; that is the determinant of one into one is greater than zero, determinant of two into two is greater than zero and so on. That's the way of testing that it is a positive definite matrix, but this is a sufficient condition not a necessary condition. In the other cases we would use the other techniques of avoiding this pivot being a zero. Let us first take a simple example wherein our procedure may fail. Let us just take that example where it may fail. Let's write the question like this; can the L U decomposition be obtained for the given matrix; can be obtained for A. Let's take it as A is equal to 1, 2, 3 and take this as 2, 4, 1and 2, 3, 1. Now this is a three by three matrix. So let us straight away write down our A is equal to L into U and again we'll take u_{ii} is equal to one. Let us take u_{ii} is equal to one. So I will have here is l_{11}, 0, 0; l_{21} 0, 0; l_{22}, 0, 0; l_{31}, l_{32}, l_{33}; 1, u_{12}, u_{13}, 1, u_{23}, 1.

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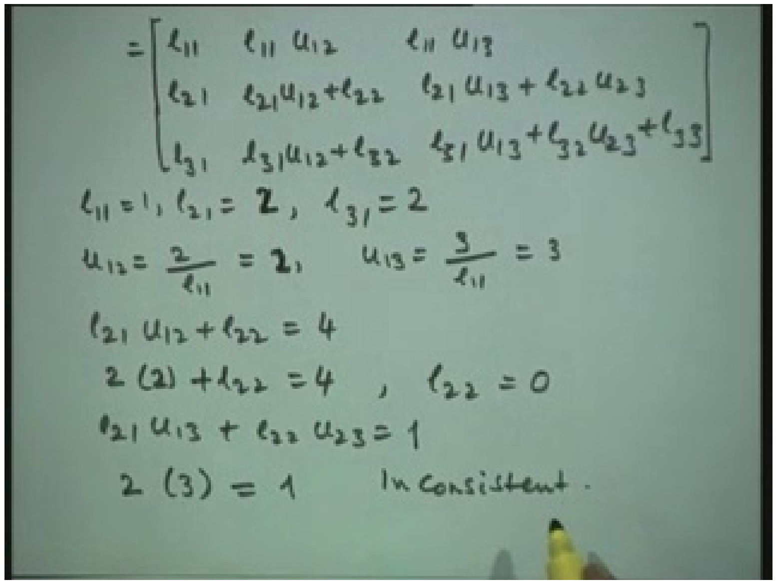

Now let us just multiply it out and write it here itself. This is l_{11} into $l_{11,}$ u_{12}, l_{11}, u_{13}, that is the first row; $l_{21,}$ l_{21} u_{12} plus l_{22} , l_{21} u_{13} plus l_{22} u_{23}, that is the second row; then $l_{31,}$ l_{31} u_{12} plus l_{32} , l_{31} u_{13} plus l_{32} u_{23} plus l_{33}. Now this is the product of our L and U which I would compare with the elements of A. So I know the l_{11}; the first column is the same, l_{21} is two, l_{31} is equal to two. Now let us determine these two elements; u_{12} is equal to two divided by $l_{11,}$ that's equal to 2; u_{13} is three divided by l_{11} that is equal to 3. Now we go to l_{21} u_{12} plus l_{22} is four. This element l_{21} u_{12} plus u_{22} is four. So let us substitute for $l_{21,}$ that is 2 into 2 plus l_{22} is equal to four, therefore I get l_{22} is zero. This pivot being zero will immediately bring us inconsistency. Now we can see the how it is giving inconsistency. Let us put in the next element l_{21} u_{13} l_{22} u_{23} that is this element should be equal to one. So let us substitute the values l_{21} is 2, u_{13} is 3; so this is equal to one; l_{22} is zero, so it is an inconsistency that is coming here. Of course in an actual computation on a computer it would divide, because we are doing it by hands so we are able to see that its zero and inconsistent; but in actual practice it is going to take l_{21} u_{12} to the right hand side divided by l_{22}, therefore division by zero comes and so the inconsistency would come from such a situation.

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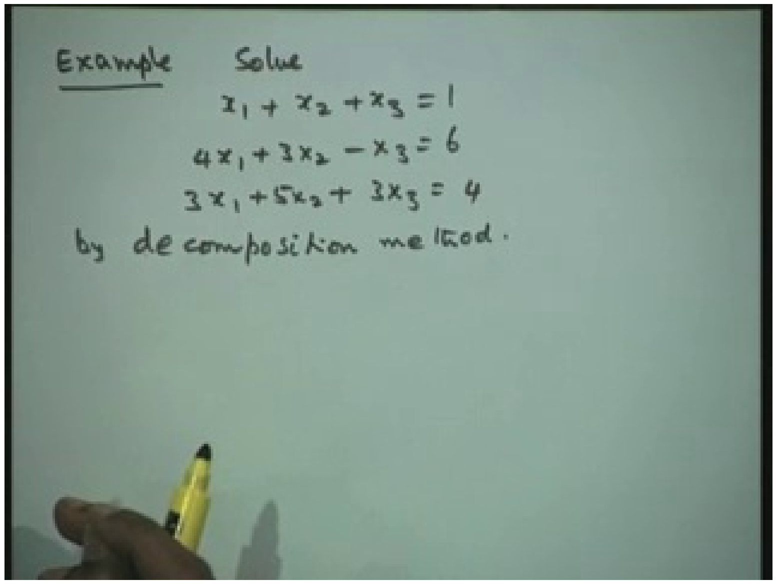

Now let me take another example in which we can give the solution of the problem itself. So let us take this example. Solve x_1 plus x_2 plus x_3 is one; $4x_1$ plus $3x_2$ minus x_3 is six; $3x_1$ plus $5x_2$ plus $3x_3$ is equal to four by decomposition method. Now we would take the help of the previous computation that we have done that the product of L and U is this. So we can straight away use this particular matrix and then save some time for rewriting them.

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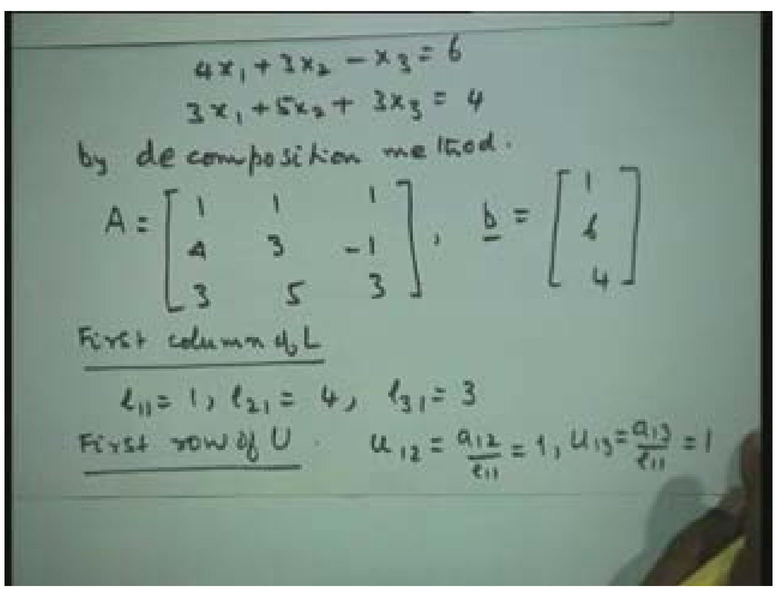

We know that the first column of L is same as the first column of A. A is equal to 1, 1, 1; 4, 3, -1; 3, 5, 3 and the right hand side vector b is 1, 6 and 4. So we will go by the first column of L. Therefore this is same as the first column; l_{11} is 1, l_{21} is 4, l_{31} is equal to 3.Then I can get the first row of U, we said that the elements of A divided by the pivot; therefore we will have here u_{12} is equal to a_{12} by $l_{11;}$ a_{12} is one and l_{11} is one, therefore its ratio is one; u_{13} is a_{13} divided by the pivot; a_{13} is 1, again pivot is 1, so I will have this as 1.

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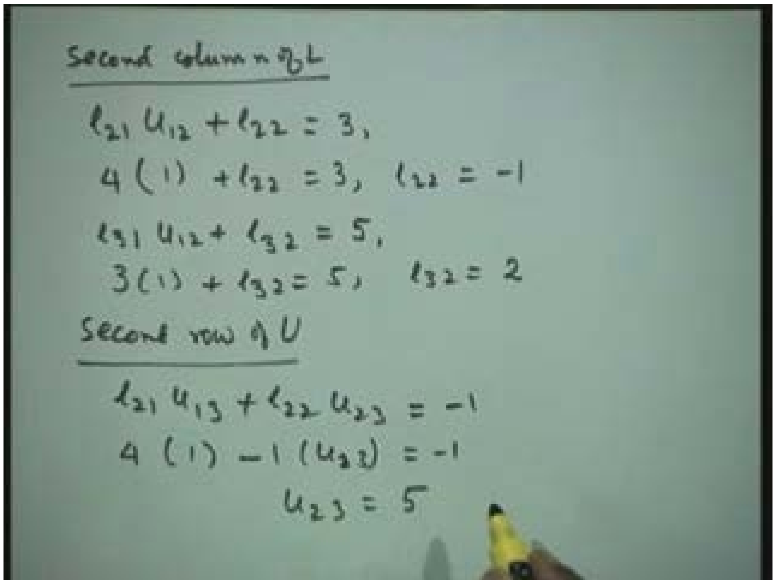

Then we go to the second column of L. So that is I would take this element as l_{21} u_{12} plus $l_{22,}$ that is this element and the diagonal element of this is 3. So the diagonal element of this is three. I will now substitute the values of l_{21} that is 4, u_{12} is 1 plus l_{22} is equal to three. Therefore I will have here l_{22} is equal to minus one. Then I need the second column, so the next element of l, the next element of the product is this. So I will take l_{31} u_{12} plus l_{32}; so I will have l_{31} u_{12} plus l_{32} and this must be five. Now let us substitute l_{31} is 3, u_{12} is 1plus l_{32} is 5, therefore l_{32} is equal to 2. Now we have completed the second column of U. Therefore I need second row of U.

Now the second row of U comes from this element that is l_{21} u_{13} plus l_{22} u_{23}. I am writing l_{21} u_{13} plus l_{22} u_{23} and that is equal to minus one. Now let us substitute the values of this, l_{21} is 4, u_{13} is equal to 1, l_{22} is minus 1, u_{23} the element to be determined is minus 1. Therefore 4 goes to this side, I will have u_{23} is equal to 5. Now the second row is complete.

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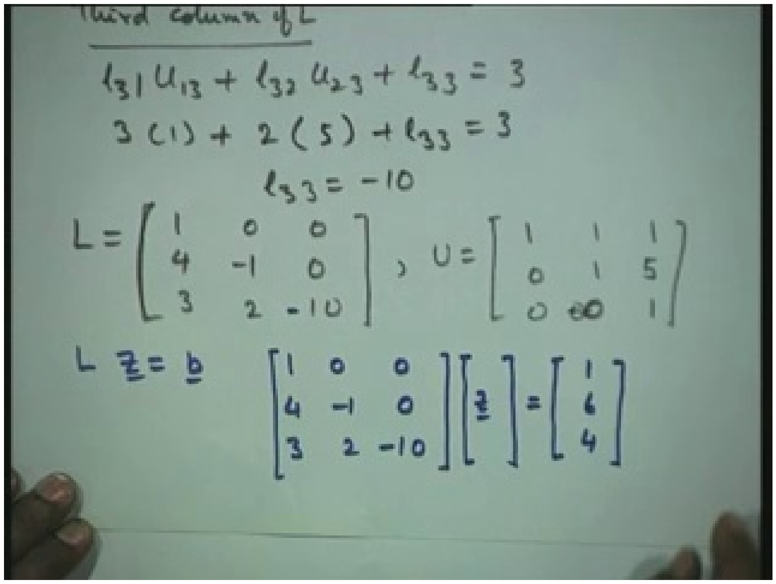

Now we go to third column of L. Now the third column of L means this is the last element that is to be done and this gives me this particular element that is your l_{31} u_{13}. So I can write from here this is l_{31} u_{13} plus l_{32} u_{23} plus l_{33} and this element is given to us as three. So let us substitute the values of these elements that we have over here. L_{31} is 3, U_{13} is 1. Now we have l_{32} is equal to 2 and u_{23} is equal to 5 plus l_{33} is equal to 3. Therefore l_{33} is equal to -10. Therefore let us write down what is our L and U; it is 1, 0, 0; 4, 3 is these elements; -1, 0, 2 10 and U is equal to 1, 1, 1; 1, 5, 1; 0, 0, 1. Now the next step to solve this L into Z is equal to b, therefore let's write down 1, 0, 0; 4, -1, 0; 3, 2, -10; and this is our vector z_1, z_2, z_3; and b is the right hand vector, 1, 6, 4.

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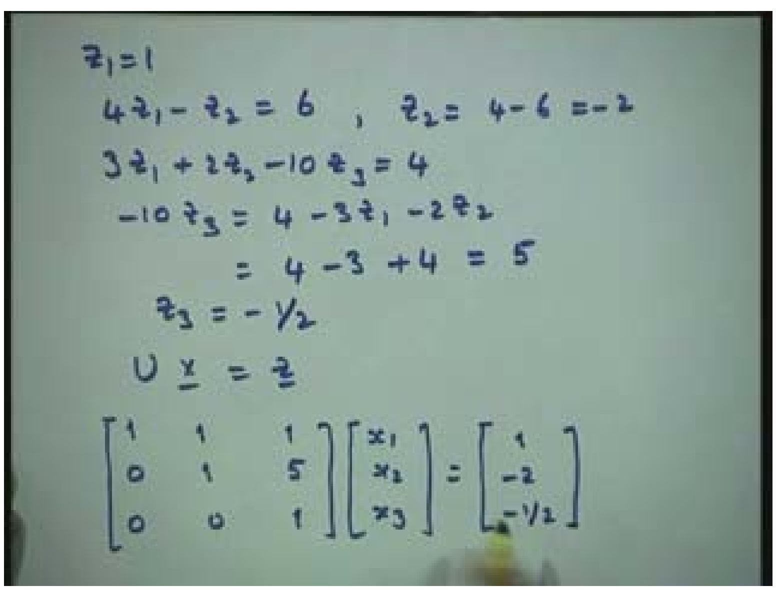

Now this gives the forward substitution, therefore z_1 is equal to 1. The second row $4z_1$ minus z_2 is equal to six. Therefore z_2 is equal to $4z_1 - 6$, that is equal to -2. I have written z_2. This is 4z with a negative sign, so four minus six that is equal to minus two. Then the third row gives you $3z_1$ plus $2z_2$ minus $10z_3$ is equal to 4 and I will have minus $10z_3$ is 4 minus $3z_1$ minus $2z_2$, so that is four minus three plus four that is equal to five. Therefore z_3 is equal to -1/2. Now the second step is we have solved l_z is equal to b. Now the solution comes from Ux is equal to z, where our U is this matrix that we have just obtained. So that is your 1, 1, 1; 0, 1, 5; 0, 0, 1and the variables are x_1 x_2 x_3 and the value of z that we have obtained, that is one, minus two, minus half.

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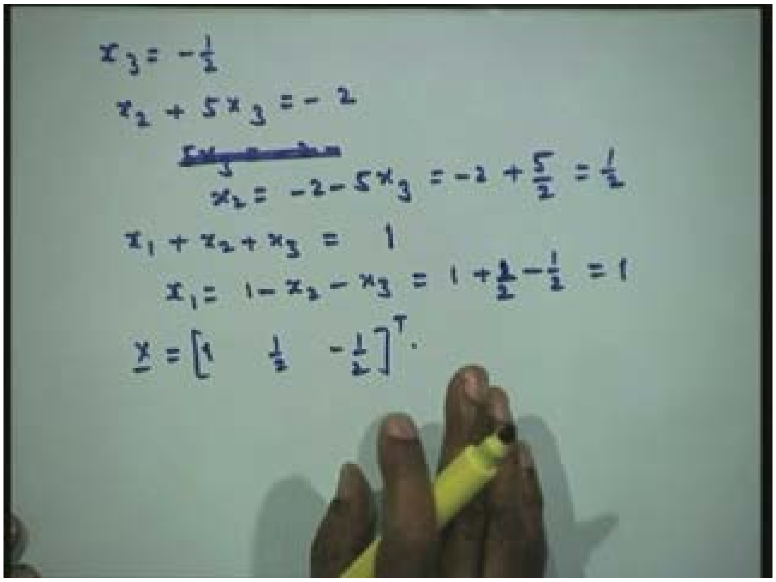

Now back substitution will give us the solution. Therefore x_3 is equal to minus half, which is the last row, then this x_3 minus half. Now I go to the second row, I will have x_2 plus $5x_3$ is equal to minus two. Therefore I will have x_2 is equal to minus two minus $5x_3$. So we will have minus two plus five by two, so this is equal to one by two. Then I go to the first equation x_1 plus x_2 plus x_3 is equal to 1, therefore x_1 is one minus x_2 minus x_3. Therefore we'll have one plus half minus half, which is equal to one. Therefore the solution vector finally is one half minus half. So the solution vector is given. Once the decomposition of A is done as L into U then one forward substitution and one back substitution gives us the complete solution of the problem. A comment here is that the amount of round of error that would occur here is much less than what it would be in Gauss elimination procedure.

Now very often when the matrix has some special property we must be able to use the property of that matrix in order to make the computations optimal. One such property is the symmetric matrix. For example, when you are loading a matrix on to the computer, since it is symmetric one would like to load only one part; upper part or the lower part to economize on the storage spaces also. Therefore we must be able to use this particular property of the matrix that is symmetric. Therefore the method that we have discussed, decomposition method, we must be able to modify it in order that the computations also will become economical. It is possible for us to take advantage that the given matrix is a symmetric, modify the decomposition method further to get a simpler method for computation for symmetric matrices.