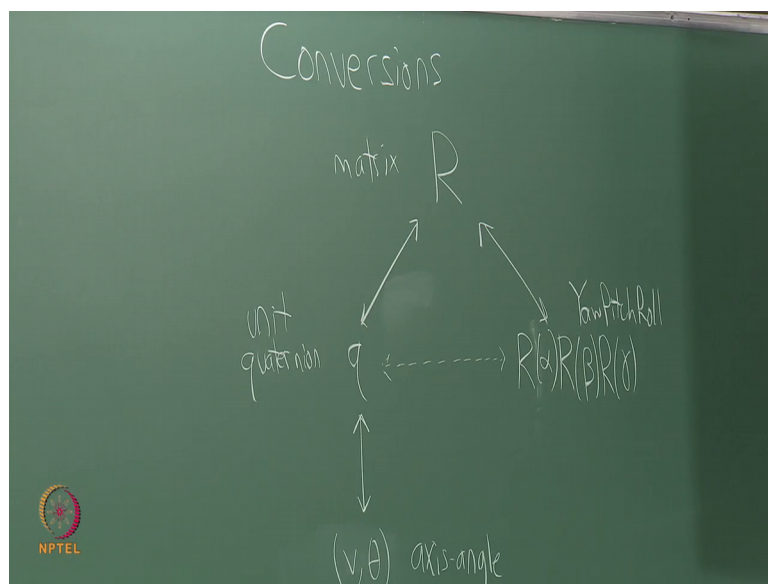


Virtual Reality
Prof. Steve Lavalle
Department of General
University of Illinois

Lecture – 05
Geometry of Virtual Worlds (converting and multiplying rotations, cont'd)

Good morning, let us continue for the next day of lectures. So, one thing I want to remind you of which I finished up last time.

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We were talking about 3 D rotations and we went through four different representations. So, there is the native representation of rotations which is the rotation matrix 3 by 3 matrix for the 3 D case and the first thing we gave was yaw pitch roll representation which we talked about this; these canonical rotations right.

We had yaw being looking like a no gesture we had pitch looking like yes and roll being like this and we also gave you unit quaternions as a better representation and the reason why was better is because it is very closely connected with another very natural representation of rotations, which is axis angle representation due to Euler's rotation theorem that every 3 D rotation can be represented as some amount of twist if you like about an axis through the origin and that is very natural and very helpful when you start doing algebra of rotations you start combining rotations.

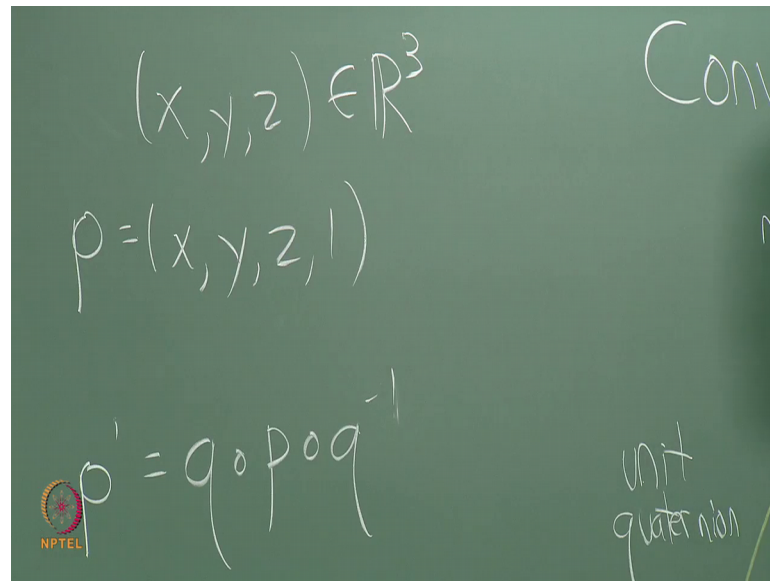
This; the algebra unit quaternions will combine the axes very well and. So, the transformation between these 2 was very simple right we had a very simple formula for that not difficult at all it is also computationally efficient transformations going back and forth between quaternion yaw pitch roll and rotation matrices are a bit more cumbersome there is a very simple matrix full of quaternion components it ends up being very simple expressions in the matrix. So, it is not too hard to go from quaternions to rotation matrix if you want to go from your pitch roll to matrix you just multiply your yaw pitch and roll matrices together. So, that is not very difficult either, but it may be time consuming or expensive in terms of computation you know try it by hand and you will see yeah.

However if you want to go from R down to these, then you have to do considerable more work, it usually involves some equation solving and then there are sometimes numerical issues because you are essentially finding numerical roots, but it all has closed form expressions there is code for doing that it is not so bad, but you just have to pay attention, if you write such code yourself that there are some subtle numerical issues going from R down to here. So, generally because of the way these workout people very often prefer to remain in this part of the space and not do other conversions right.

So, if we can work directly in there, then there are a lot of advantages and that is one of the reasons why I gave you quaternion multiplication as an alternative to the matrix multiplication. So, that you do not have to convert to matrices you can just stay completely; completely and quaternion land and do your algebraic computations there. In fact, you can go further and do your quaternion transformations there.

So, if you want to just sorry I should say you can do your rotations directly using quaternions. So, I just want to give you a quick expression for that I am not deriving the algebra of that here I would take considerable more time, but suppose you have a point X, Y, Z that you would like to transform I will extend it by one dimension.

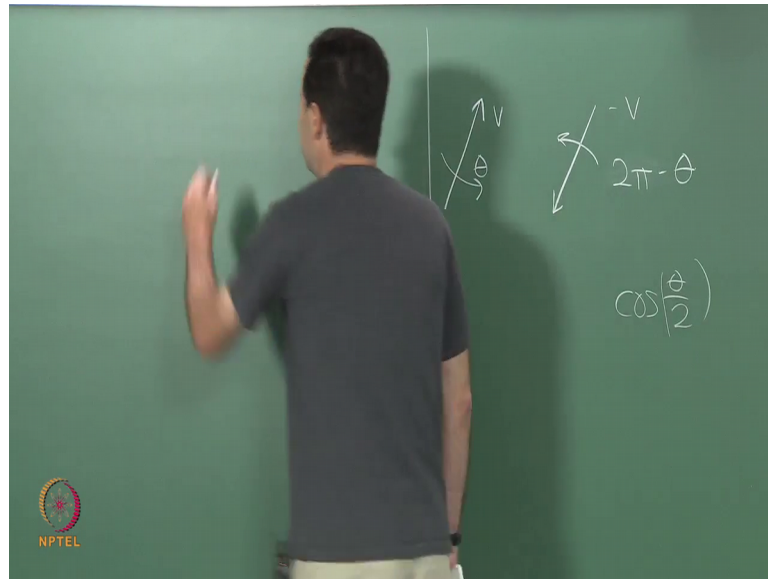
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So, I will just write it like this $X, Y, Z, 1$ and I will pretend that that is another quaternion and now if I want to rotate this point X, Y, Z that is let us say perhaps it is in well just say anywhere on \mathbb{R}^3 if I want to rotate this, then I get some p prime that I calculate as q quaternion multiplication with p quaternion multiplication with q inverse and this q is representing the rotation that I want to apply to the point X, Y, Z right that makes sense.

So, in other words, let me summarize give and q some rotation that I would like to apply it does of course, correspond to a rotation matrix, but instead of converting to that rotation matrix I can directly do this algebra and apply q to p using this chain of 2 multiplications in order to get out the results. So, the resulting p prime that I get, I just read off the first 3 components and that will tell me the rotated point. So, so the quaternion algebra has been nicely defined in a way that makes this work. So, that it gives you the same result, but I am not deriving it here you can look that up on your own does that make sense and you remember how easy it is to make you inverse right.

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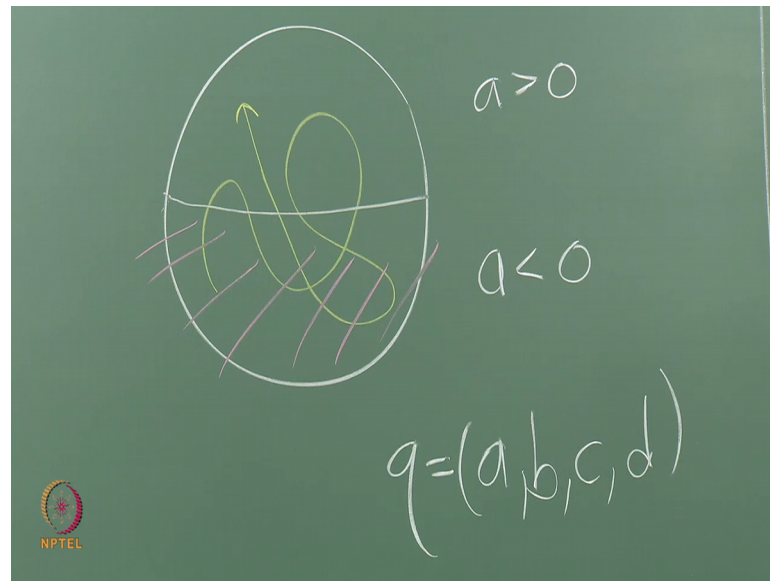


For example, you can just negate the last 3 components speaking of that we had an interesting question last time which I just want to cover we talked about the double representations of every 3 D rotation when you use axis angle and so if you have axis v and some amount of rotation θ then if you take minus v I said that it is equivalent to a rotation of minus θ and then when we look at the formulas for converting from quaternion to axis angle we notice that there was a sign problem well normally if I apply trig functions to some angle θ if I add 2π to it the trig functions yield the same result right.

So, it seems like it should not matter, but it turns out it does matter for this axis angle conversion formula because it has a term that says cosine θ divided by 2. So, the fact that you are dividing by 2 here means that if you do add 2π to it. It will give a different result right because you are dividing by 2 before applying it. So, it is better if you are using this formula to write it as 2π minus θ .

So, that is where that little bit of confusion, came in last time and it seemed like the sins did not work one more thing I want to say about quaternions and then we will get on to the next bit of topics, remember that we have this 2 different representations there are 2 different unit quaternions for every rotation what could we say about those quaternions where are they located on the unit sphere opposites right are called antipodal points.

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So, one thing that happens is suppose you are rotating your head around and you have some tracking algorithm and it is essentially moving through that space. So, this is let us suppose that sphere that sits in four dimensional space it is a 3 dimensional unit quaternion sphere and then as you move your head around you are essentially moving along the sphere somehow right is that an kind of visualization. So, imagine the quaternions changing. So, as a function of time you are moving around this is transformation moves along this sphere.

Now, I can in doing it as a sphere I can imagine it has an equator. So, that sometimes I am in the northern hemisphere and sometimes I am in the southern hemisphere what does that mean well above I could be saying it is a greater than 0 and below I could be saying it is a less than 0 where a remember a quaternion is a, b, c, d. So, I could go moving around and if I am not careful if I start doing lots of transformations like just going back and forth converting.

I should say lots of conversions if I do lots of conversions between rotations it may be the case that if I convert one way and come back I might all of a sudden jump to the opposite side of the sphere and I might conclude that something absurd has happened with my tracking algorithm right how did I end up on the other side it looks like a huge error, but in terms of rotation it may be exactly right because it just jumped to the opposite point.

So, that causes a lot of trouble and confusion sometimes encoding these algorithms and in dealing with these kind of tracking algorithms and a lot of transformations in these spaces. So, some people might like to say I will just prohibit the lower hemisphere for example,. So, I may just say stay away from here and so, that as soon as you come down the northern hemisphere as soon as a is changing and getting smaller and smaller if it hits 0 you could just jump to the antipodal point and then continue upward does that make sense. So, you can keep it trapped in the in the upper hemisphere the northern hemisphere in this case if you if you view north being a increasing.

So, you can force to change; however, there is still some subtleties right when you come exactly to the equator because it is its not it is not exactly clear what you could do what you should do exactly on the equator because you can still jump to the antipodal point along the equator itself and you may have to protect that and divide that in half and. So, forth as you go all the way down to the final remaining dimension. So, it is a bit complicated at the equator itself, but I just wanted to point that out this 2 to one representation that we have sometimes causes confusion if you make very simple tests like how far apart are these 2 quaternions using simple Euclidean distance.

They may be. In fact, very very close together, but they appear very far apart and attempts to fix that will help, but you have to be very careful about them. So, it helps to visualize it on a sphere I think and just remember that it is one dimension higher questions about that alright. So, I am going to move on to the next topic I am getting up the homogeneous transformation matrices today I want to remind you of a couple of properties of matrices before we get to that which I believe I mentioned last time, but I will quickly mention them again here if I want to for example, rotate some point X, Y, Z .

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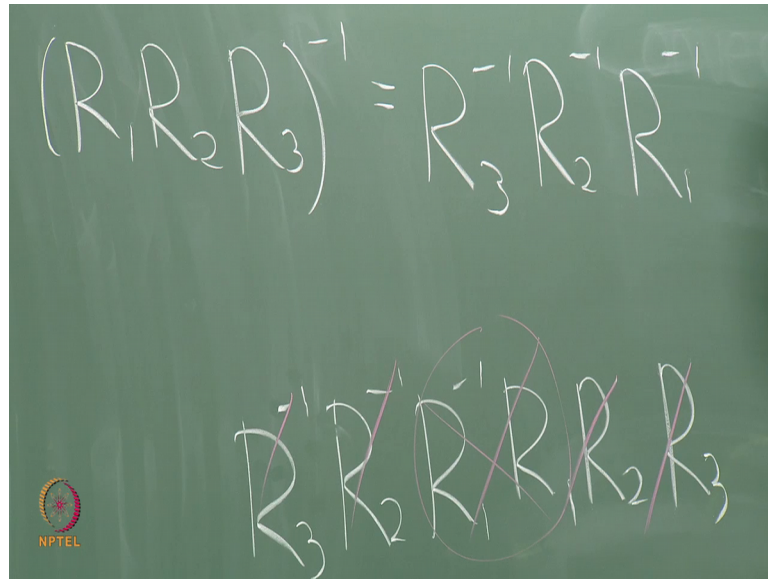
$$(x, y, z)$$
$$R_2(R_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix})$$
$$(R_2 R_1) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

So, I put it in vertical column form and then I apply a rotation matrix to it just write it as R this is what I get right. So, I do this multiplication and I get out the result if I want to then apply a second transform a second rotation to the result of that the rotation comes out here correct. So, if I wanted to first apply the rotations together by multiplying the matrices it is just important to understand right.

So, if I were instead were to first do these multiplications and then apply the result it is just important to understand that the first one happening is R_1 getting applied and the second one is R_2 . So, it goes in this direction from right to left and since in many languages, we are a very custom reading from left to right it annoys people, right.

So, it looks like it is going backwards. So, it is just something to pay attention to it is easy to get confused by that and remember the other thing is that if you want to make an inverse of a chain of matrices.

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The image shows a chalkboard with a green background. At the top, the equation $(R_1 R_2 R_3)^{-1} = R_3^{-1} R_2^{-1} R_1^{-1}$ is written in white chalk. Below this, the sequence $R_3^{-1} R_2^{-1} R_1^{-1} R_1 R_2 R_3$ is written, with red lines striking through the $R_1^{-1} R_1$ and $R_2^{-1} R_2$ pairs, and a red circle around the $R_3^{-1} R_3$ pair. In the bottom left corner of the chalkboard, there is a small circular logo with a star and the text 'NPTEL' below it.

For example, if I want to have R_1, R_2, R_3 inverse just remember and this is going to become important as we start chaining matrices together and if you start digging into the software R_3 inverse R_2 inverse R_1 inverse and this is very easy to see if I just take my R_1, R_2, R_3 and I apply this inverse next alright.

So, that is the next thing I apply. So, a state that goes on this side. So, if I apply this inverse next here I get R_1 inverse R_2 inverse R_3 inverse and then you can just follow all of these cancellations. So, R_1 times R inverse that should be the identity and then the whole thing collapses together and then these 2 cancel and then it collapses together and then these 2 cancel and you just get the identity right a matrix times its inverse should give you the identity. So, that is what we get right.

So, just remember that just basic property of shows up in group theory which is very general. So, this is the reason why it holds here because of non-commutativity, we have to be very careful, I will give you a great example of that in just a little bit alright questions probably seen these things before somewhere, but just emphasizing that they are becoming important now there may or may not have been immediately useful to you before.