Radiation Heat Transfer
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Lecture - 36
Approximate Methods of Scattering: 1

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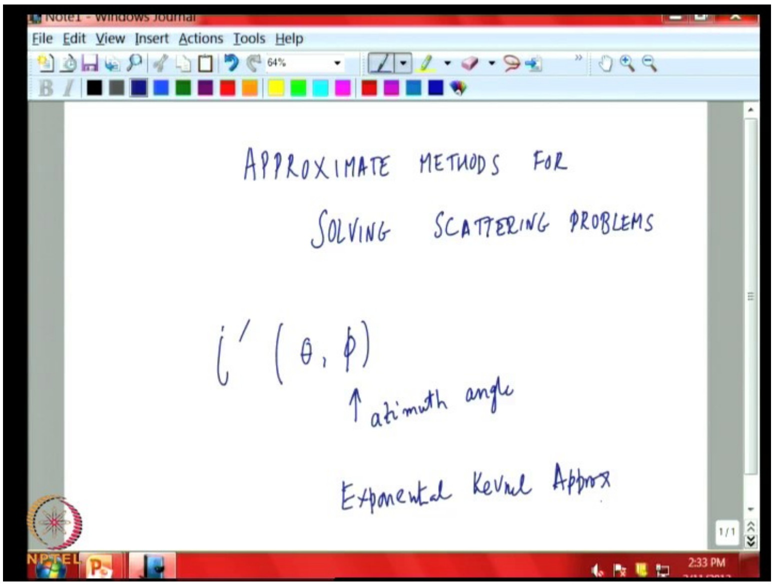

In this lecture, we continue our discussion on approximate methods for solving scattering problems. If we recall, the earlier problems we solved without scattering were much simpler, because we could use simple exponential kernel approximation to convert an integro-differential equation into a different equation; but when we include scattering, the problem gets complicated by one order of magnitude, because a ray traveling in any one direction is linked to rays travelling in all other directions through the scattering process. In scattering problems, you have to worry about how intensity of the rays depends on angle. If we recall, our intensity i prime is a function of both theta and phi. This is the azimuth angle.

If we recall that, we did not pay a lot of attention to the importance of angular variation, in the case of problems involving absorption, emission. In most cases, we were able to make the approximation of diffused isotropic emission that made the problem much simpler. The angular variation was accounted through exponential kennel approximation, which turned out to be quite adequate to deal with the problems involving emission and absorption; but when it comes to scattering, these problems cannot be ignored; angular variation is a critical factor in the case of scattering problems and you need to carefully account for the direction in which the rays are going.

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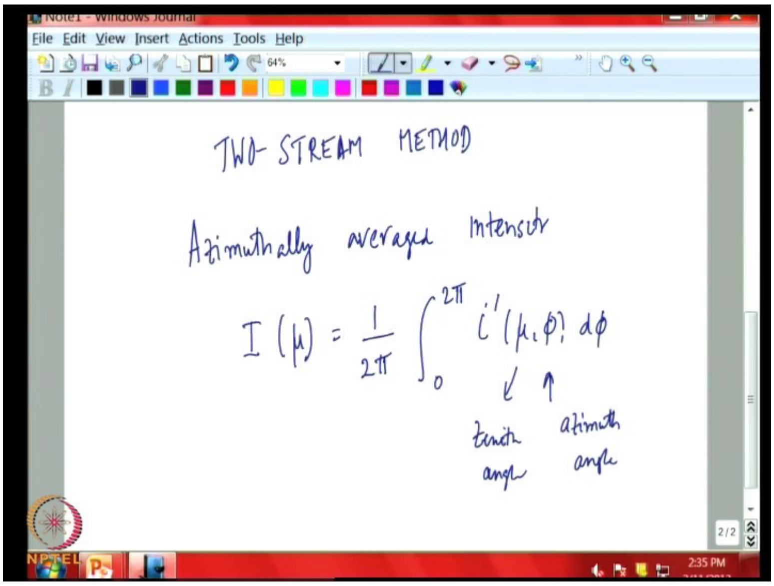

Now, in order to explain how scattering problems are dealt with, we take the simplest example of the two stream method, which we discussed earlier and we continue the discussion today. So, what you are doing is, you are essentially converting the problem which involves multiple directions to essentially two main directions up and down, and you thereby try to simplify the complexity of the original problem.

One direct way is to look at, azimuthally averaged intensity; that is we define an I, a function mu, that is cos theta, as 1 over 2 pi, 0 to 2 pi I prime of both mu and phi, d phi. We average the intensity over the azimuth angle, this direction, and only account for the zenith angle. This refers to azimuth angel.

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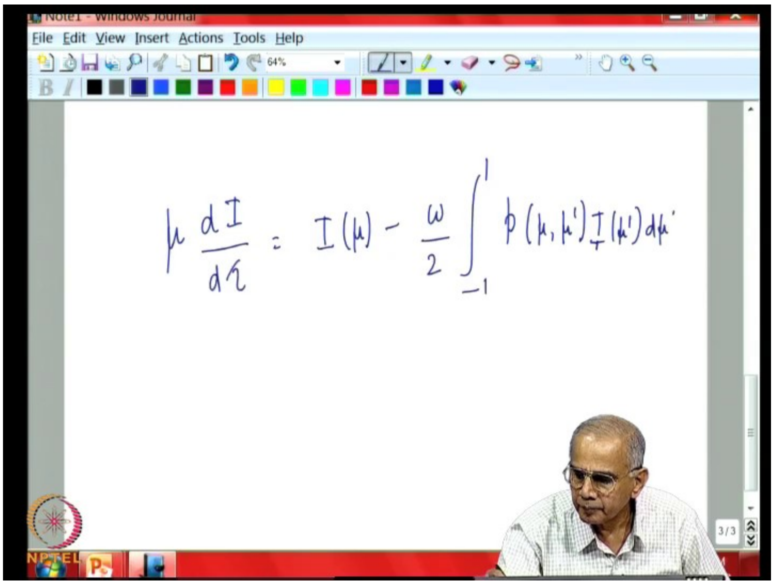

We average over azimuth angle, because in most cases, the azimuthal variation is not important as variation with zenith angle. For example, take the example of direct solar radiation impinging on a surface. The variation with azimuth angle is not as critical as the variation with zenith angle. That is the problem. Once you do that, the basic problem is scattering, neglecting emission, boils around to writing a simple equation, mu d I by d the optical depth, I which is now a function only of mu, not mu and phi, minus this single scattering, minus 1 to plus 1, the probability phase function, it depends on mu and mu prime, and I of mu prime, and d, d mu prime

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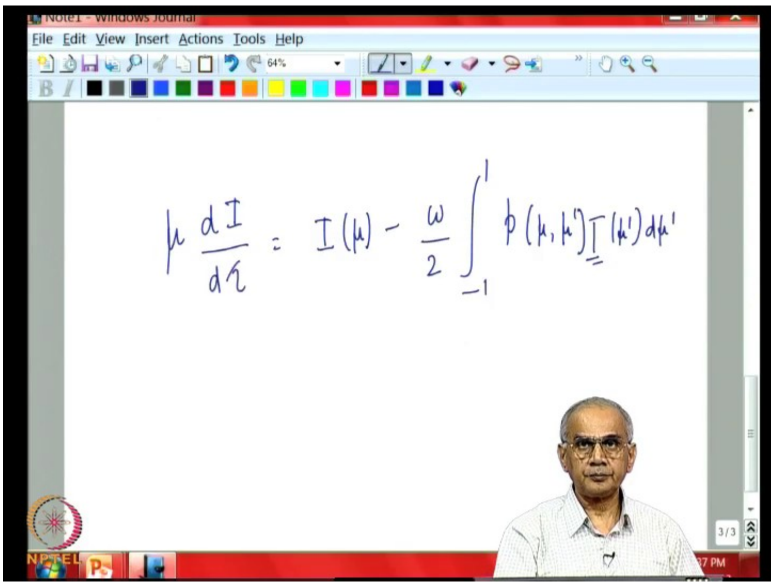

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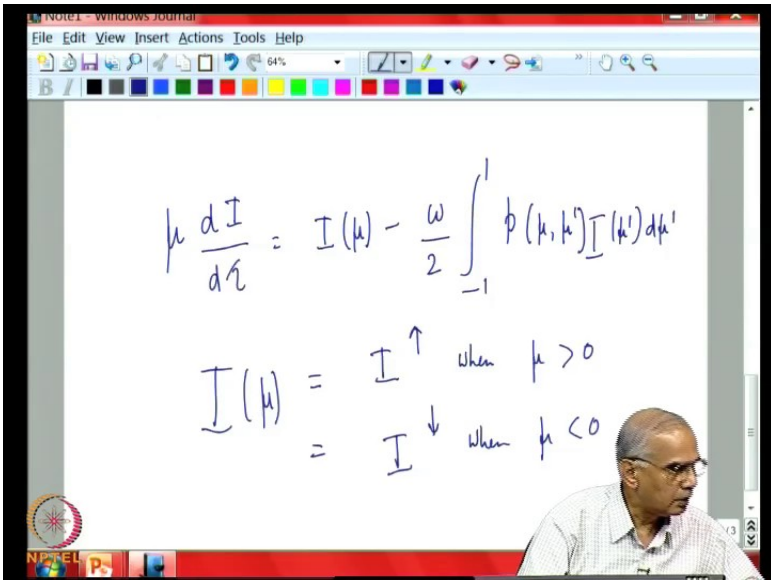

This is I; This is the simplified form of the radiation heat transfer equation in which we have ignored the, or average over all azimuth angle, so that, the only account for the zenith angle variation. In the two stream approximation, we have a very simple assumption, that I is the function of mu is written as, equal to 2 values I up, when mu is greater than 0, and equal to I down, when mu is less than 0. After averaging over all azimuth angles, we are making a further approximation, that the intensity is not varying with mu in the upper and the lower hemisphere, but the intensity upwards and downwards are not the same. We want to fundamentally account for the fact that intensity depends on whether it is going up or going down, but within each hemisphere.

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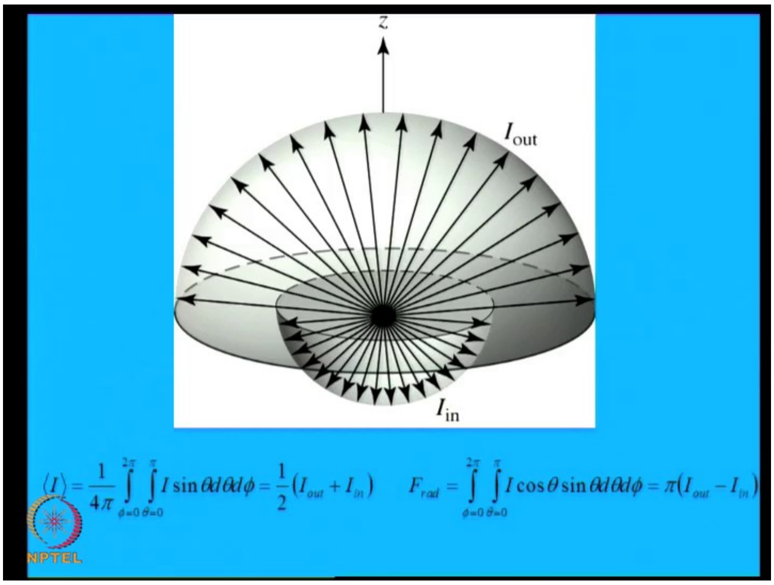

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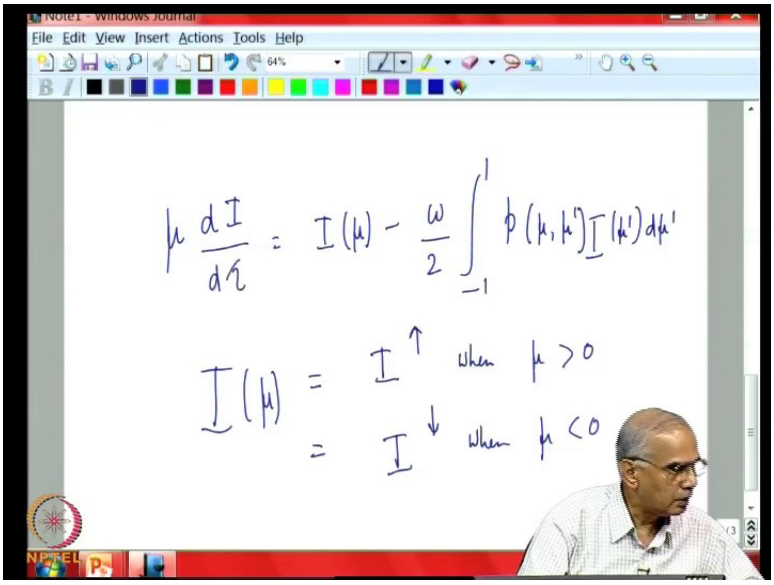

Now, this can be illustrated by a picture here, which we will see now. We are going to have a simple model in which the upper and downward hemispheres are different. This is a simple model in which, in the upper hemisphere, intensity is independent of angle and in the downward hemisphere also independent of angle. And, you have only 2 values I up and I down; that is the whole problem. We are going to do that integration here, shown as follows. Everything simplifies in terms of I up and I down, or I out and I in. Once we do that, the previous equation now, becomes very simple.

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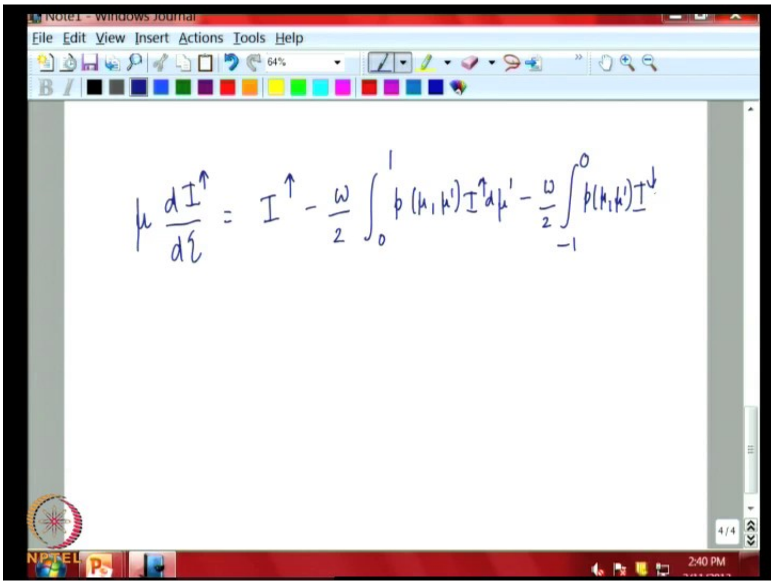

We now write equations for I up and I down, and we illustrate that now by showing that, mu d I up d tau is equal to I up minus single scattering by 2, 0 to 1, this is scattering phase function, is now function only of mu and not of phi; and We are doing 2 integrations, one from 0 to 1, and one from 0 to minus 1.

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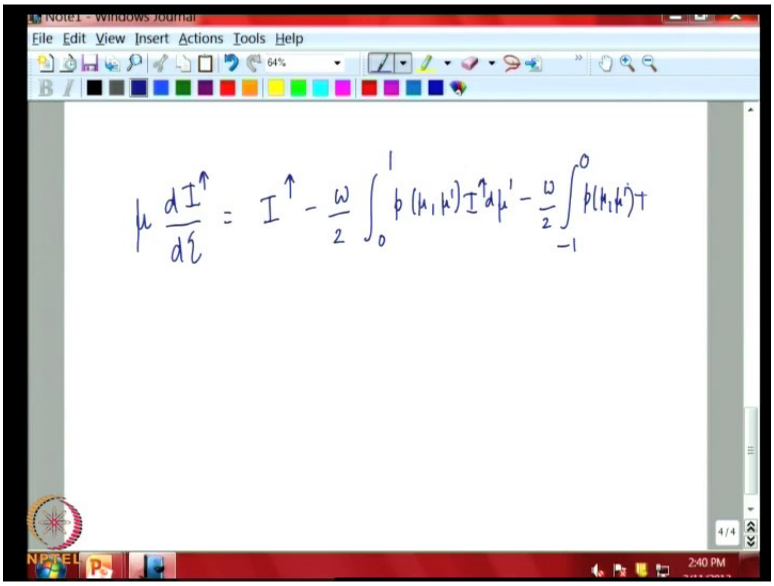

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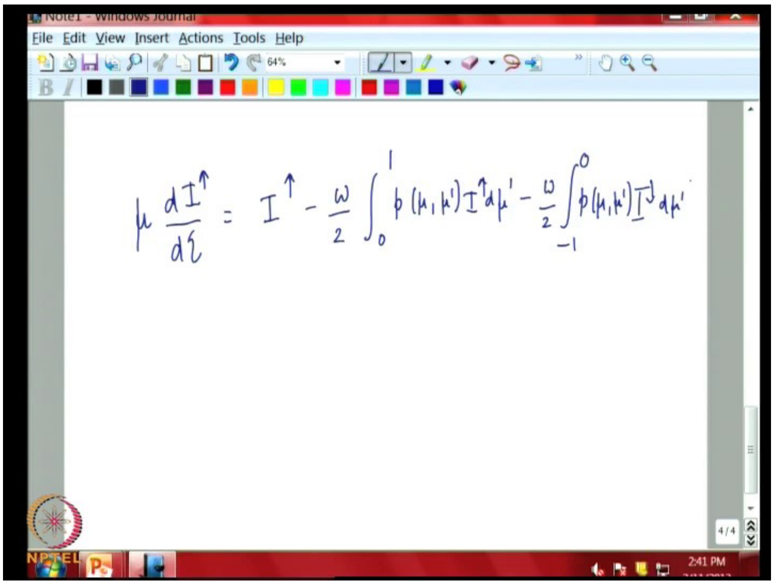

We write this phase function, function of mu and mu prime, I, this is now down, into d mu prime.

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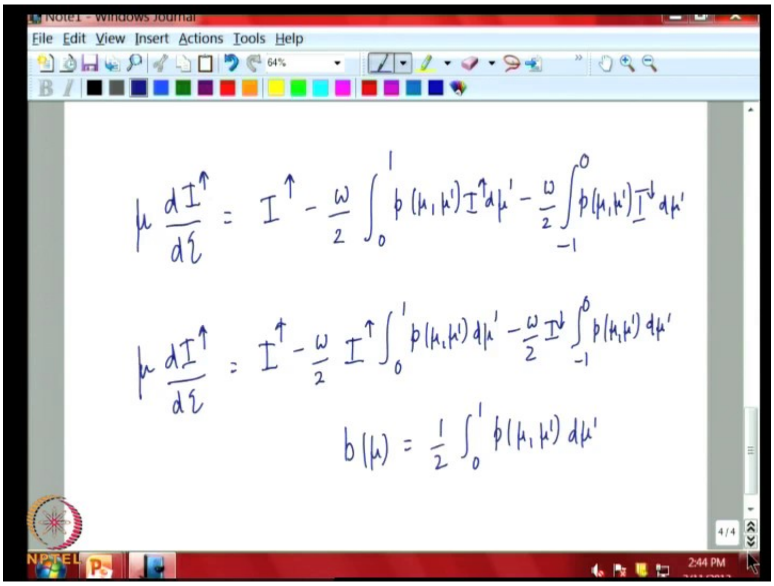

What we have done here is that, the intensity in the upper direction depends on what is already the upper direction, and times the radiation scattered in the upper direction by particles from radiation travelling upwards, and also radiation which is scattered upwards from the downward going radiation. Both these are accounted for in this calculation. The advantage is, since we are assuming that I up and I down are independent of angle, we are going to take it out of the integration; so, then, we are left with a simplified equation.

We have taken the I out of the integration, because it is a constant by our assumption. We get a simple integration of the phase function. Now, since we are going to encounter these integrals quite often, it is useful to define a b of mu as half, 0 to 1, p of mu, mu prime d mu prime.

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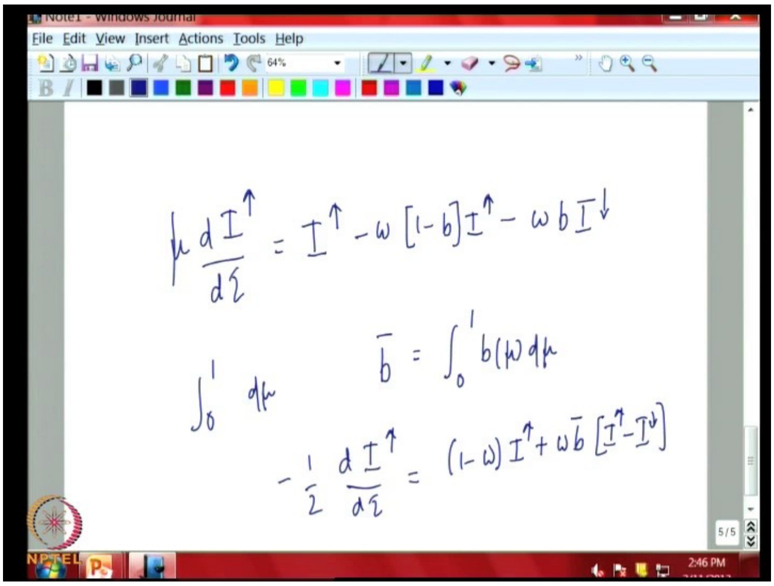

This represents the fraction of radiation that is scattered into the opposite hemisphere. It shows, a ray traveling in direction mu prime, how much of it is going in the opposite hemisphere due to scattering. Once we make this simplification, the final equation now for intensity upwards as a function of tau is written as, I upwards, minus single scattering into 1 minus b into I upwards minus omega into b into I downwards. We have made the equation symbolically very simple. Now, we can integrate the equation over mu, above equation, and the function will become much simpler. When we do that, and we define b bar, average value b as 0 to 1 b of mu d mu. In each hemisphere, this quantity will be defined. If we do all that, finally, we have visible very simple equation, minus half d I upwards by d tau, equals 1 minus omega I up plus omega into b bar into I up minus I down.

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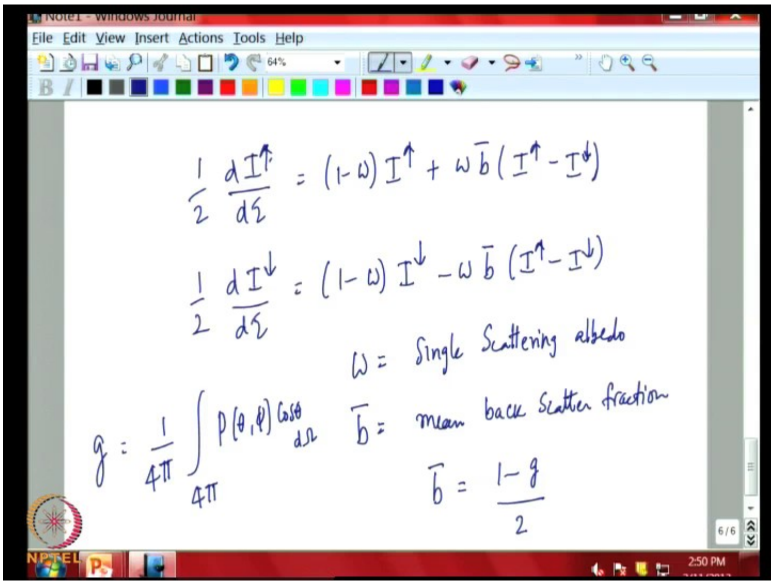

By all these approximations, we have reduced the calculation of the intensity going upwards, in terms of just two parameters, the single scattering albedo and the mean back scatter fraction. So, remember that, since there is a downward intensity coming here, you need 2 equations; let us write down the 2 equations now, which we have to solve for; one is what already we wrote, and I will repeat that.

Now, since this is the unknown term here, we have to write the equation for the downward intensity, This will come out as, 1 minus omega I downwards minus omega b bar into I up minus I down. We have two coupled equations, two coupled, ordinary differential equation for the intensity upwards and downwards. There are only two parameters of interest here; one is single scattering.

Another one is b bar, the mean back scatter fraction. Now, first quantity, which is the quantity, which we call the mean back scatter fraction is a quantity that is related to a quantity, which we call as the asymmetry parameter which we defined earlier, lectures if you recall; when we talked about Rayleigh scattering, we clearly identified a quantity called g.

The quantity g was nothing, but 1 over 4 pi, 0 to 4 pi, p, the phase function, function theta and phi, cos theta d omega. This was the asymmetry parameter, and one must recognize that, the asymmetry parameter and the back scatter fraction must have a connection. Now, the way we have defined the back scatter fraction, it is connected to the asymmetry parameter that we defined earlier as follows.

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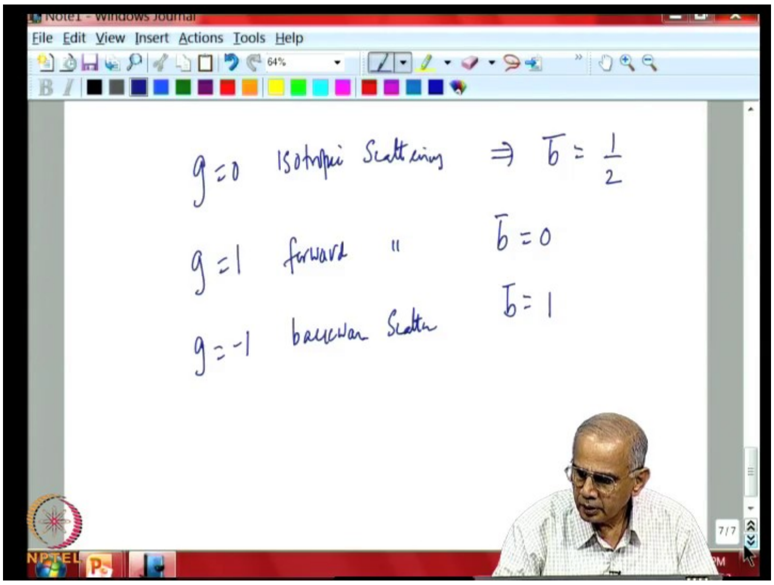

If we recall, in our earlier discussion, when g was 0, we had a isotropic scattering, which is equivalent to mean back scatter fraction of half; because isotropic scattering than the amount scattered upwards and downwards will be same. Similarly, when g equals 1, there is only forward scattering. When g is 0, we have isotropic scattering and hence, b is equal to half.

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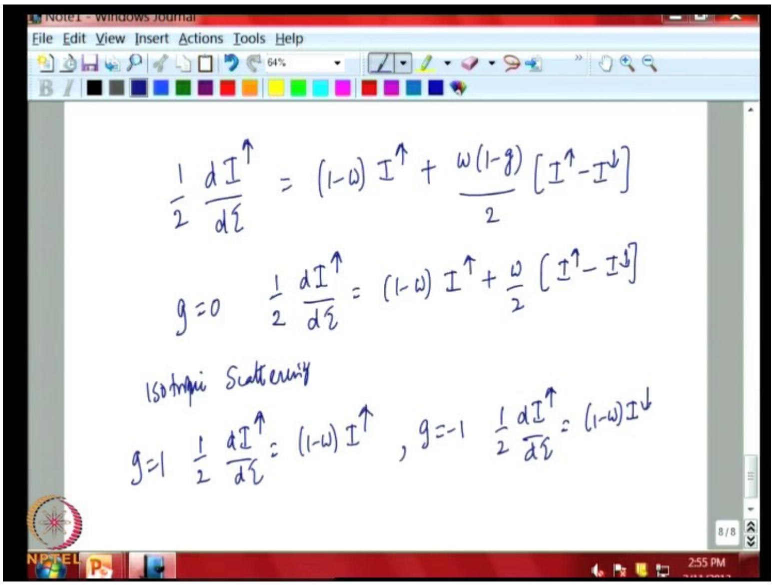

On the other hand, when g equals 1, you have only forward scattering; then, b delta has to be 0, because this is a back scatter fraction; if everything is scattered forward, the back scatter fraction is 0. Similarly, when everything is backward scattered, b delta has to be equal to 1.

The relation between g and b is simple. Now, if you convert everything in terms of g, which is our new parameter, we can write a new set of equations in terms of g, instead of b, which is only a transformation of symbols here.

This is a result which clearly shows that, if there is only forward scattering and no backward scattering, and g equals 1, the second term will drop out; because then, there will be no connection with the downward intensity into the upward intensity. On the other hand, if there is only backward scattering, and g is minus 1, this would become 1. This term will cancel out, and we would have 1 minus omega I downwards.

What we would be saying that, let me write down, so that we appreciate this. When g equals 0, you have a simple equation, which we have dealt with earlier; this problem has been solved earlier; this is isotropic scattering. Now, if g equals to 1, this term drops out; we have half d upwards d tau, depending only on this parameter. If g is minus 1, minus 1 and minus 1 is plus 2, by 2, 1; This   will cancel with that and we will have 1 over half d upwards d tau, will be equal to 1 minus omega I downwards. So, g is minus 1, minus 1 minus is 1; 2; this omega gets cancelled with this omega; this will cancel with this; We are left with 1 minus omega term here; This omega by 2 here will add up to this; then we will get the following answer.

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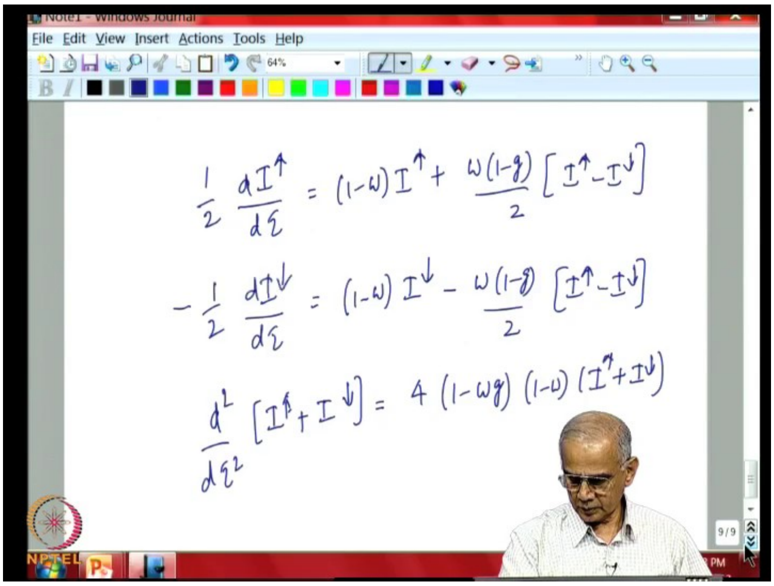

This result clearly shows in the second case, that the upward scattering here depends only on downward scattering, because everything is back scattered. Here, the upper scattering depends only on the upper intensity, because there is no backward scattering. Now, let us write down the two equations that govern this problem in terms of g, before we solve it. The first equation is what I wrote earlier, and the second equation for the downward intensity which is, can be written down by inspection because of the symmetry of the problem. We have two equations, in two unknowns, that is I up and I down, in terms of two parameters, the single scattering albedo and the asymmetry parameter. We can make this problem little more interesting and we can combine these two equations and with the slight manipulation, we will get this equation.

We can see that, we have an equation for the, essentially the mean of the upward and downward intensities, in terms of the asymmetry parameter and the single scattering albedo, and one can derive a similar equation for the difference between the two they look quite similar.

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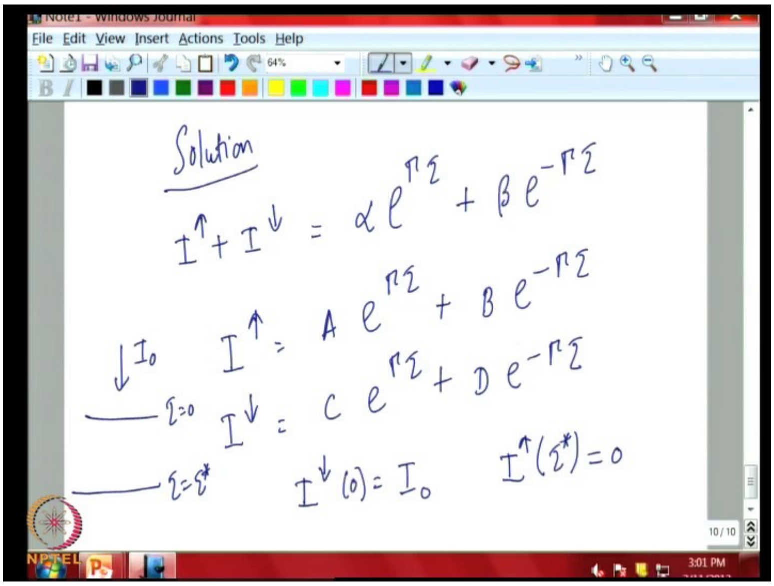

We can solve these equations fairly easily. The solution of the above equation, a part of it we have done earlier; when we dealt with isotropic scattering. Now, we are dealing with scattering which is not isotropic, because there is an asymmetry parameter which is not equal to 0. So, solution to this problem is, I up plus I down, which is equal to some constant alpha, e to the power of gamma tau plus another constant beta, e to the power of minus gamma tau. There is a similar expression for I up minus I down. In general, we can write, I up is equal to a e to the power of gamma tau plus b e to the power of minus gamma tau; while I down can be written as c e to the power of gamma tau plus d e to the power of minus gamma tau. We can connect the two of these constants here, because they are not completely independent. Finally, we will connect everything to the total reflective layer called total albedo, called r infinity.

We will define that quantity r infinity now, for convenience. To solve this equation, we need a boundary condition, and the boundary condition that we will say that, downward intensity, that is coming in, let us say, solar radiation, that is I 0; and at the bottom, the total optical depth. We have, a layer with tau equals 0, tau equals to tau star this is the incoming radiation, This surface is perfectly non-reflecting. The upward intensity of the bottom surface is 0.

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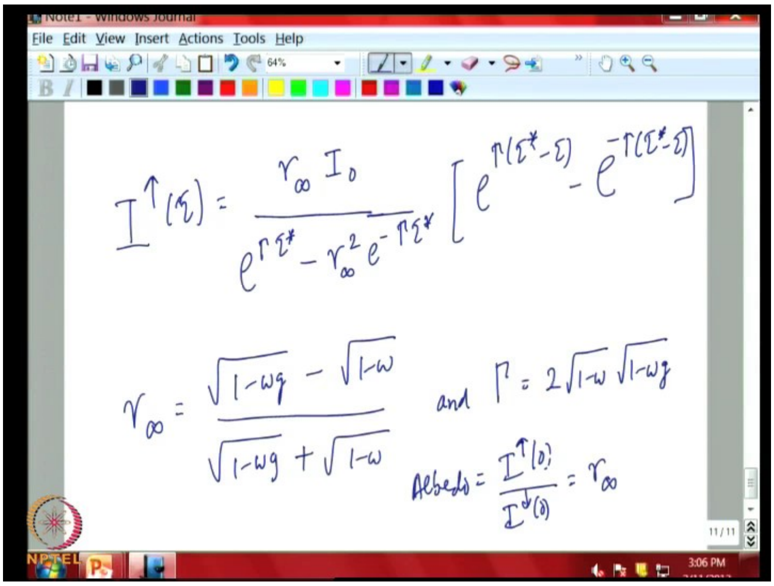

This is the simplest case that we can visualize, If you solve this equation with these boundary conditions, we will get the following expression for upward and downward intensity. First, I will write the upward intensity, as the function of the optical depth and we will define the parameter r infinity, which we will define soon. So, one is exponentially increasing and another one is decreasing term. Here, now, we have introduced two parameters. We should define what these parameters are. One is, r infinity, and we will understand the meaning of this symbol very soon.

Now, we can see that, there is some kind of total reflectivity of a thick medium because, when omega 1 is 1, this quantity becomes 1. So, when the single scattering albedo is 1, everything is reflected. When omega is 0, this goes to 0. This is a, essentially a parameter related to the total reflectivity of the medium, which we will highlight and gamma, the parameter that comes in the solution, is related to the single scattering albedo and asymmetry parameter. We have been able to obtain the upward intensity for a given I 0, the downward intensity, in terms of this parameter r infinity, this parameter gamma. All of them are connected only to two factors which control the reflectivity in a scattering medium, which is the single scattering albedo, which tells you what fraction of the radiation is scattered as compared to the radiation, which is both scattered and absorbed. If omega tends to 1, we have a very highly scattering medium. If omega tends to 0, we have a highly absorbing medium g, the asymmetry parameter, tells us something about the direction in which the scattering occurs. If the scattering occurs equally in all directions, then, g is 0 and we have isotropic scattering, which we have dealt with earlier.

On the other hand, if the scattering is predominantly in the forward direction, then g tends to plus 1; while if scattering is predominantly in the backward direction, the g tends to minus 1. This is the main quantity of interest and if we define, and we have defined albedo before also; it is the ratio of upward intensity by downward intensity, This can be shown in this case, to be anything, but r infinity.

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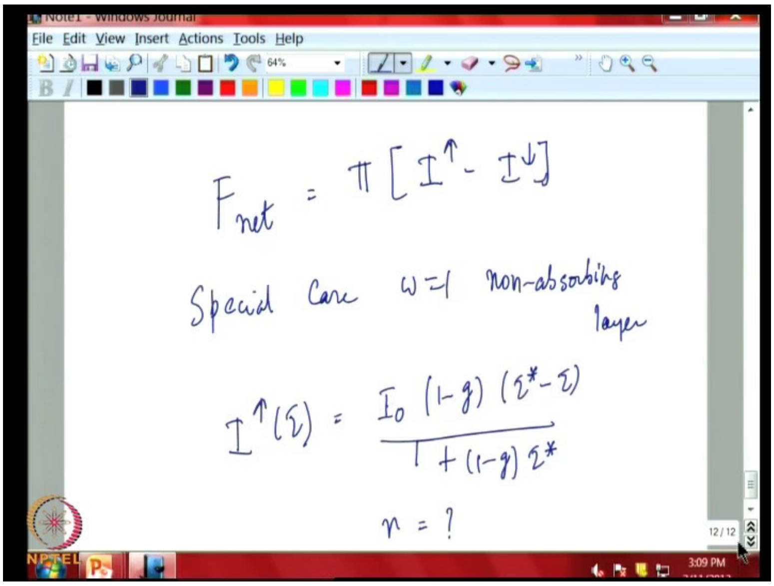

The albedo of the semi-infinite cloud is r infinity; that is, a quantity of great relevance, for example, in understanding cloud physics and quantity like that. Now, given the information, we are, can also calculate the net heating rate. The net heating rate will be nothing, but pi times I up by I down; this is of course, a quantity of great interest in calculating the heating of the layer, or atmosphere. Now, let us take a special case of a non-absorbing cloud; that is, we set omega equal to 1; special case. It is essentially a non–absorption or purely scattering layer. In which case, our equation for upward intensity becomes nothing, but I 0 into 1 minus g into tau star minus tau as shown above.

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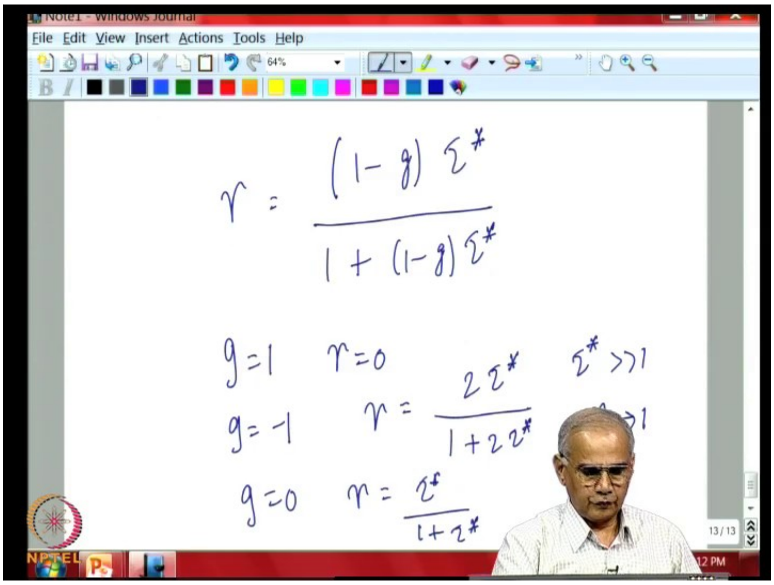

In this simple situation where there is no absorption, the upward intensity depends only on the asymmetry parameter and the optical thickness. We can see that, in this case, the albedo which is ratio of the upward downward intensity, r will be nothing, but, this can be calculated now. So, r will come out as, the reflectivity albedo as 1 minus g into tau star, 1 plus 1 minus g into tau star. This is a very simple result and we can see again, that if g equals 1, everything is scattered in the forward direction, or has to be 0. This is understood from the definition of g. If g equals to minus 1, everything is back scattered; that becomes 2; this becomes 2. So, r becomes 2 tau star by 1 plus 2 tau star; and of course, in the limit, as the tau star tends to infinity, or becomes very large rather, r has to tend to 1.

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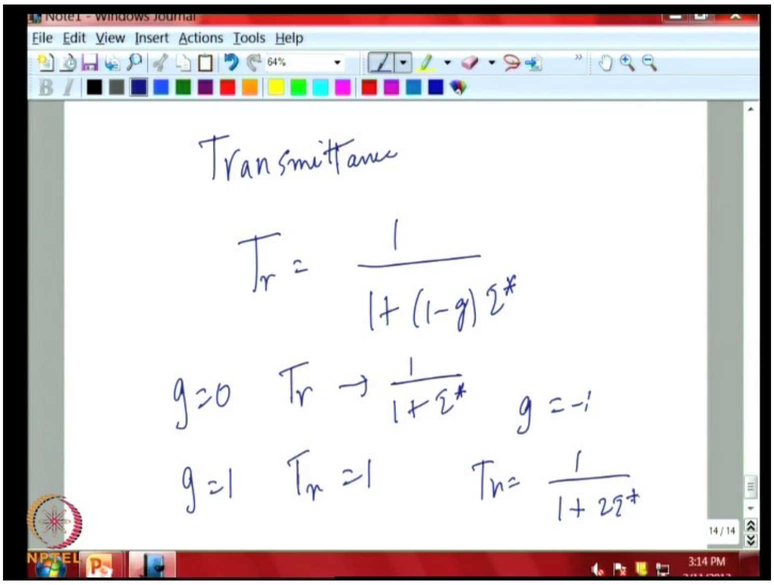

That is quite simple. The other case where we have isotropic scattering, r becomes tau star by 1 plus tau star. We clearly got a simple idea of what happens when there was forward scattering, what happens when there is backward scattering and what happens when there is isotropic scattering. We can see typically that, in the case of backward scattering, very often the reflectivity can be twice that of the isotropic scattering, especially when the optical depth is small.

But if the optical depth is very large, then it does not really matter; if the optical depth is very large, in both cases, surface tends to 1. So, our understanding is that, if the layer has a large optical depth, we are talking about 10, or above 10, then it does not really matter, whether the scattering is isotropic, or non-isotropic, the reflectivity is around the same value, 1. This insight we are getting fairly easily; we can also define a transmittance, which is also useful for many calculation. We will write down the transmittance. So, both reflectivity, or albedo and transmittance are quantities of great relevance in practical application. The amount of radiation transmitted is 1 by 1 plus 1 minus g into tau star for a non-absorbing layer. Again, you see that, if g is equal to 0, you have isotropic scattering; tau is nothing, but I will call the transmittance with a different notation, because we do not want to confuse with optical depth. The transmittance is Tr, so, transmittance tends to 1 plus 1 by tau star, when g equal to 1, that is forward scattering, then transmittance is equal to 1. Not surprising, all the radiation is transmitted in the forward direction; there is no absorption. We must have 1. And, when g is equal to minus 1, we have transmittance is equal to 1 by 1 plus 2 tau star. So quite clearly, when there is backward scattering, the amount of radiation that is coming in the forward direction is much smaller, than when there is forward scattering. And, as a matter of fact, in the limit of very high optical depth, high tau star, and the transmittance can go to 0, if there is backward scattering. This simple analysis of a layer of non-absorbing particles has provided a general insight about the nature of albedo and transmittance in a very simple context.

What has to be kept in mind in the analyses is that, if the only quantity of interest in a given application is the transmittance or albedo, and there is not much interest in the angular dependence of these quantities, then the kind of analyses that we have done here is quite adequate. But if we need further information on the dependence of albedo or transmittance, on angle, then this analysis is not useful; analysis, azimuthally averaged, and finally, also averaged over all angles.

Our focus only was on up and down. This is useful for many applications, which are concerned with fluxes and overall hemispherically averaged quantities, but if you are interested in quantities varying with an angle, then the analysis done so far is not adequate. So, one needs to go to a more sophisticated model, which accounts for the variation of intensity with angle, which was not done so far. So, let us see the kind of approximations that we need to make, when the intensity is a function of angle.

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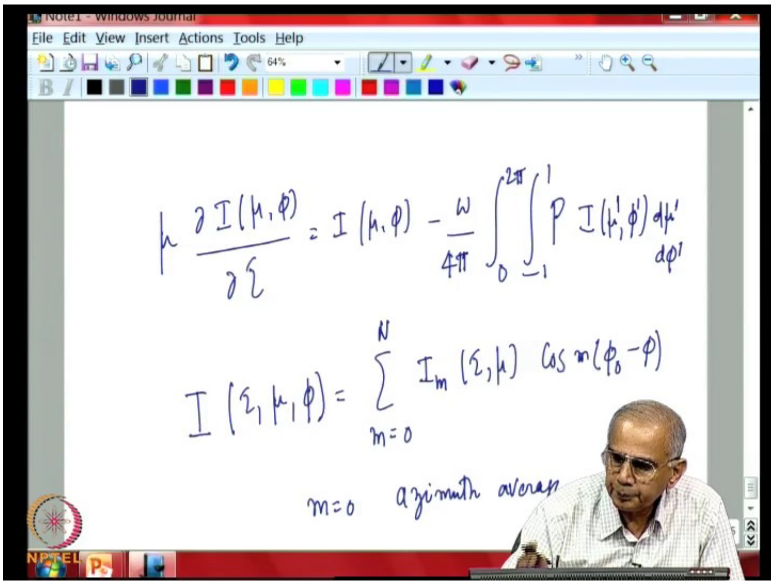

Once more, we will ignore the dependence on the azimuth, but first, we will write down the equation with the azimuth angle explicitly indicated, so that, we understand. The general form now is, both angular dependence explicitly there, and the pure scattering problem we are dealing with; this is scattering phase function into the intensity. This is the general problem of scattering; they neglected the emission term in this analysis. Now, the most rigorous way of doing this problem is to expand I, which depends on the optical depth, the zenith angle and the azimuth angle, in terms of spherical harmonics where I of m, i depends now on, tau and mu, cos of m, minus phi.

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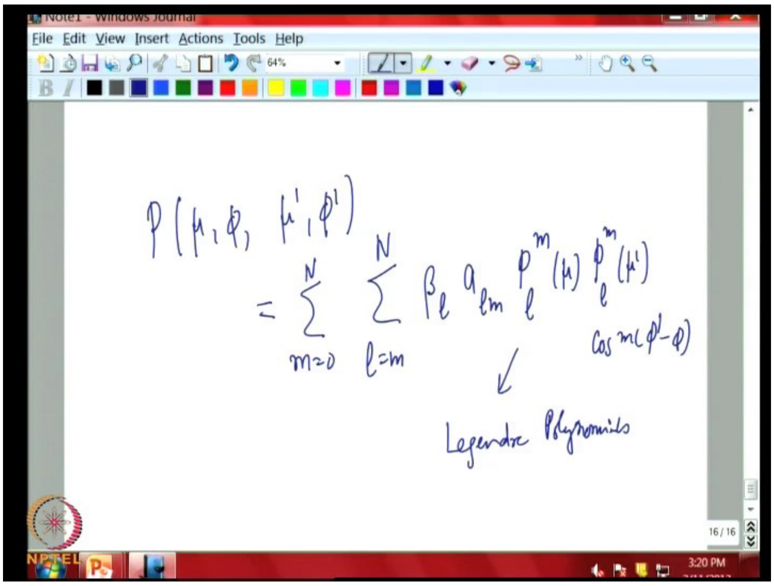

So, essentially, instead of assuming things are independent of the azimuth angle, which was the early approach we took, now, we explicitly want to account for the azimuthal dependence of intensity in a cosine expansion and Legendre polynomial expansion, and then, in this case, m is equal to 0 will represent the earlier result, the azimuth average. If we want to go beyond azimuth average, then this kind of expansion is called for. Then, this function p, which is our phase function, depends on the angle mu pi and also angle mu prime and pi prime.

Now, this has to be expanded in terms of m equals 0 to n as earlier; and then, Legendre polynomials will come in now; beta and a are constants, which we have to evaluate. These are the Legendre polynomials, which we will show what they are. These are the Legendre polynomials, are the appropriate basis functions for radiation transfer.

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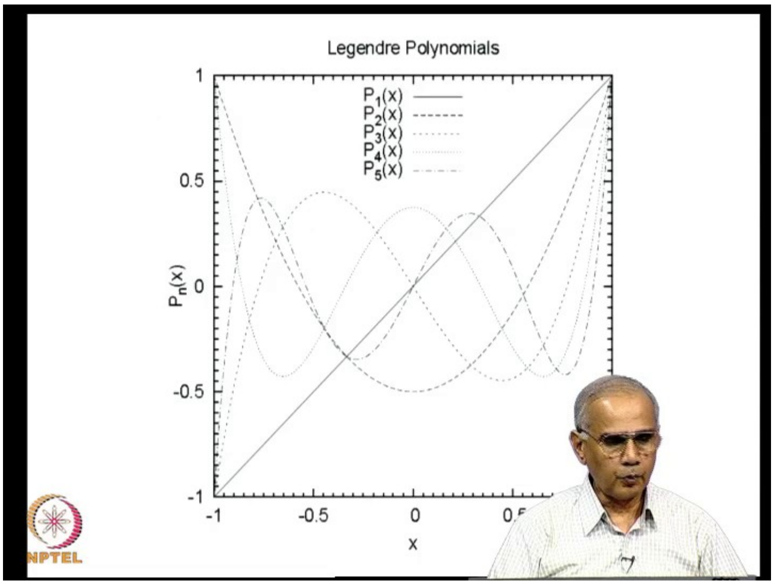

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The first few Legendre polynomials are given by:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

One can show that, they are the most optimal choice. You could make other choices, but they will not converge, or be as accurate as these functions. Let us look at how these functions look like. These functions were discussed earlier. This is the example of the Legendre polynomials of order 1, 2, 3, 4 and we can see that, the first polynomial is a linear function; the second is a quadratic and the third is a higher order and so on. These are simple algebraic functions, which are optimal for this problem; If we want to further know what these look like, these are the axis one beyond axis x and so on. These polynomials are the ones which are used in many of the problems of this kind.

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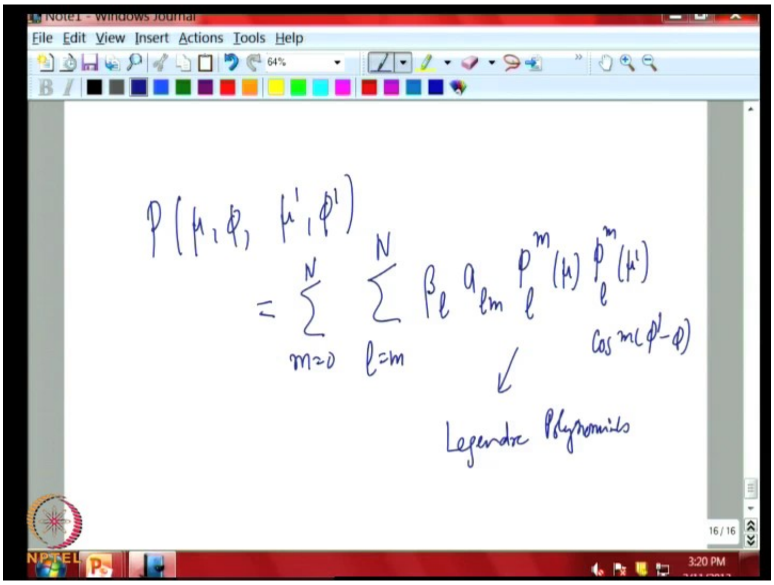

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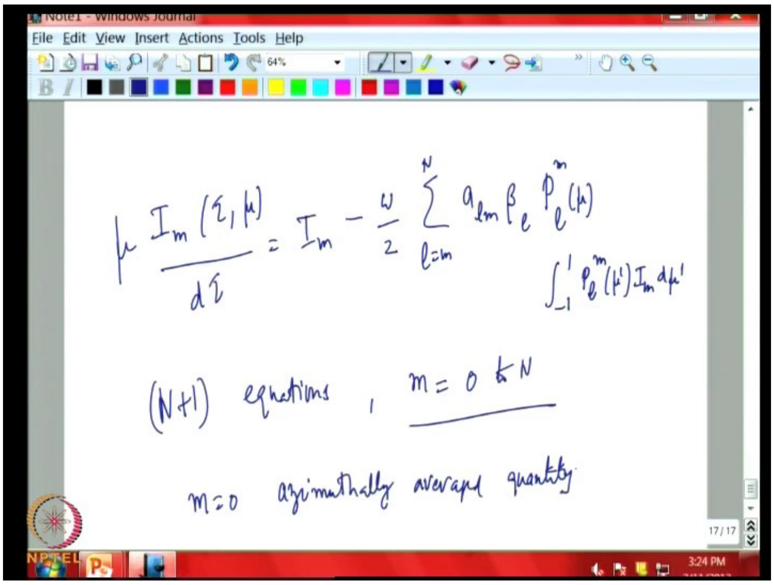

So, once we have used these polynomials in our analysis, then the equation of ray transfer now becomes much simpler; and let me write down what equation we will get. We will get equation for the I of m now, because everything is expanded in terms of spherical harmonics. We removed the phi dependence through m. Now, it only depends on tau and mu as before; but now, we are accounting for azimuth variation, which we have not done earlier.

We are inside now the integral over all azimuth angles. This integration over the azimuth angle is now, has been done. We have now, n plus 1 equations; that is, I am going from 0 to n.

We have n plus 1 equations in n plus 1 unknowns; the m equals 0 case is what we had done earlier; that is the azimuthally averaged quantity.

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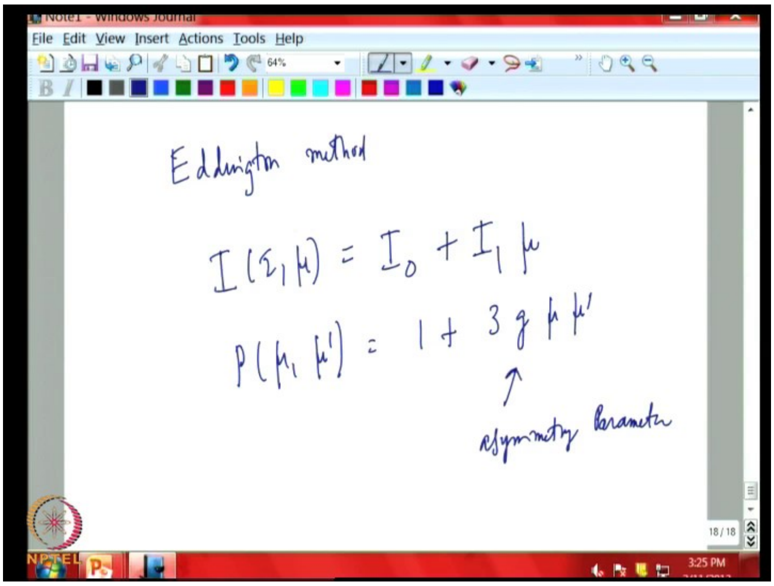

That was a simple problem which we performed earlier in this lecture. So, m equal to 0 is the azimuthally averaged quantity, which we have dealt with in the early part of the lecture. Now, we want to do something more and we adopt the so called, the Eddington method. In the Eddington method, we do not assume intensity is uniform in the two hemispheres; we allow for the intensity to vary and the simple assumption, that is, that was proposed by Eddington is that, I is equal to I zero plus I 1 mu.

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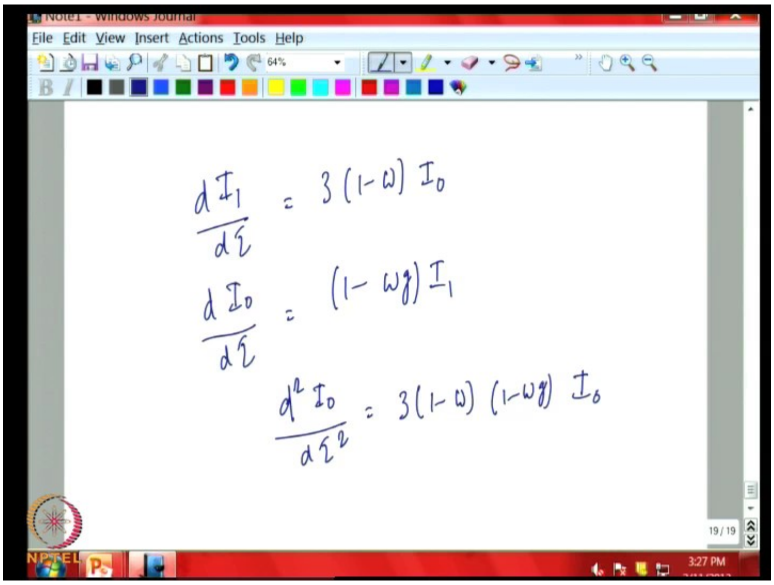

Essentially, he is varying the distance vary in the upward and the downward directions, and on account of these assumptions, the phase function also is simplified now. It becomes 1 plus, in terms of the asymmetry parameter. We have now, a simple equation for I 0 and I 1. We substitute all these in the equation that is given earlier. Now, we will get two coupled equations for I 1, which is 3 into 1 minus omega I 0, This will be 1 minus omega g into I 1. We can eliminate I 1 from these equations and you get a single equation for I 0, which is very similar to what we had derived earlier for the azimuthally averaged case. This analysis shows that both the, analysis done early in this lecture and the Eddington method, which is little more, generalized intensity, which is dependent on angle, lead to very similar equations for I 0. This can be solved and the solutions we will get will be similar to what you got earlier this will be continued in the next lecture.