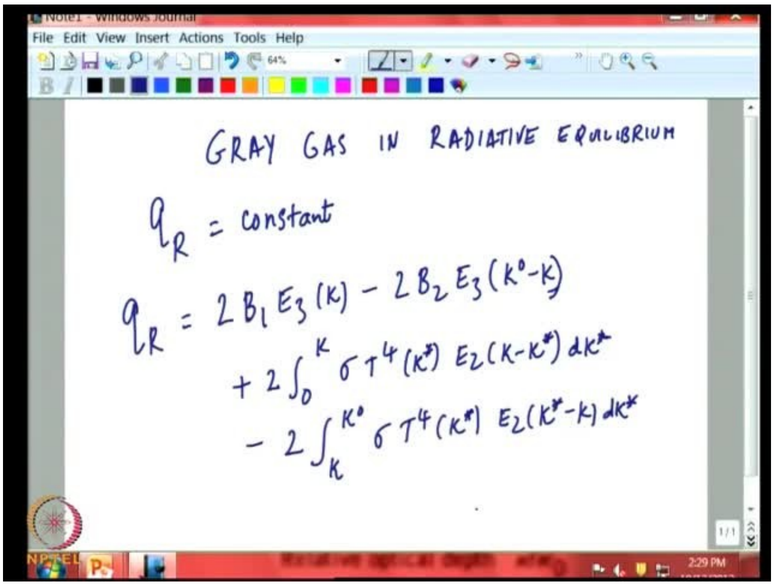
Radiation Heat Transfer
Prof. J. Srinivasan
Center for Atmospheric and Oceanic Sciences
Indian Institute of Science, Bangalore

Lecture - 17
Radiative equilibrium

(Refer Slide Time: 00:26)

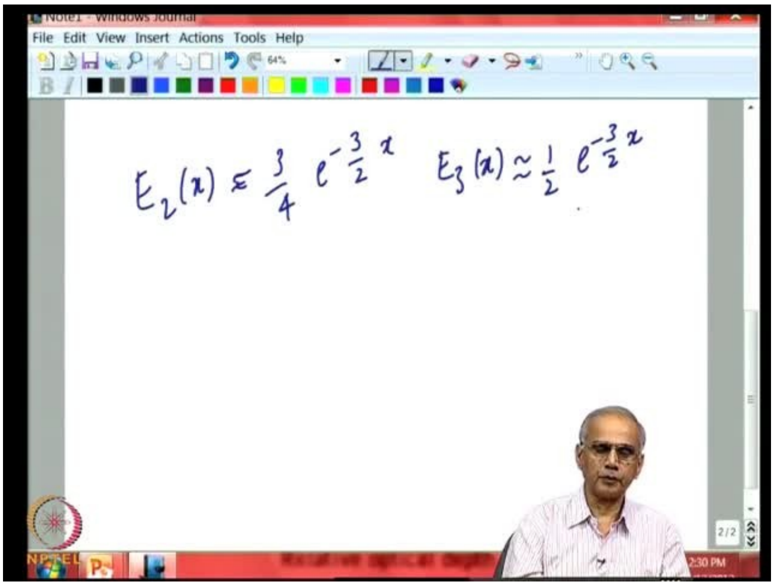


In the last lecture, we looked at gray gas in radiative equilibrium. In one dimensional problem in radiative equilibrium, the radiative flux is a constant. We need to find the temperature distribution in the gas under this condition. We started with the general radiative flux equation, which in this case was the following which we had already discussed. The challenge we faced in this kind of problem was the unknown gas temperature was inside the integral, and hence pose this special difficulty in the solution. The third and the fourth terms in this equation are terms, in which the unknown occurs inside the integral.

Today, we can solve this problem numerically, We can start with some initial gas of the gas temperature variation, and then integrate this, get the value of q r if the value of q r is not what is not a constant, in radiative equilibrium then you alter the assumed profile and We can do this iteratively, very easily on the computer until you get the answer you want. Now, the purpose of these lectures is not to teach you how to solve the numerically, but to come up with some simple solutions to these equations which provide some insight in to the nature of the solution.

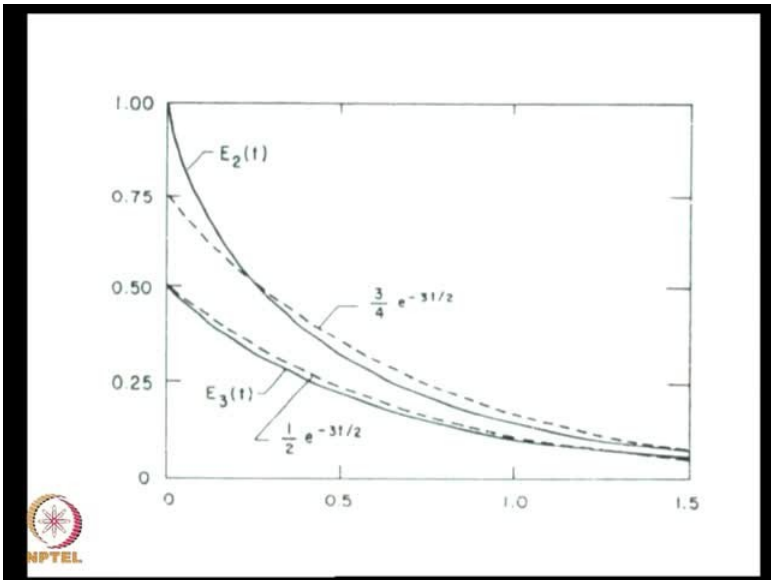
What we did was, we look at this equation and since our great difficulty was these two integrals here, we went about the task of finding a way to eliminate these terms We realize that if you replace this exponential integral functions by exponentials, then we know then exponential on differentiation repeats itself. If you differentiate this twice, the exponential will repeat itself and then you subtract the equation that is obtained on differentiation from this equation, then we can eliminate these two terms.

(Refer Slide Time: 03:55)



So, briefly if you recall we replaced E 2 of x as 3 by 4 e to minus 3 by 2 x and E 3 of x by half e to minus 3 by 2 x. We pointed out that purpose of this kind of approximation is primarily to ensure that the terms, which like E 3, which are outside the interval they should have a correct value at x equal to 0 at x equals 0 E 3 is half. This is correct and this slope there is chosen such that, the area under curve is approximately right and this can be shown.

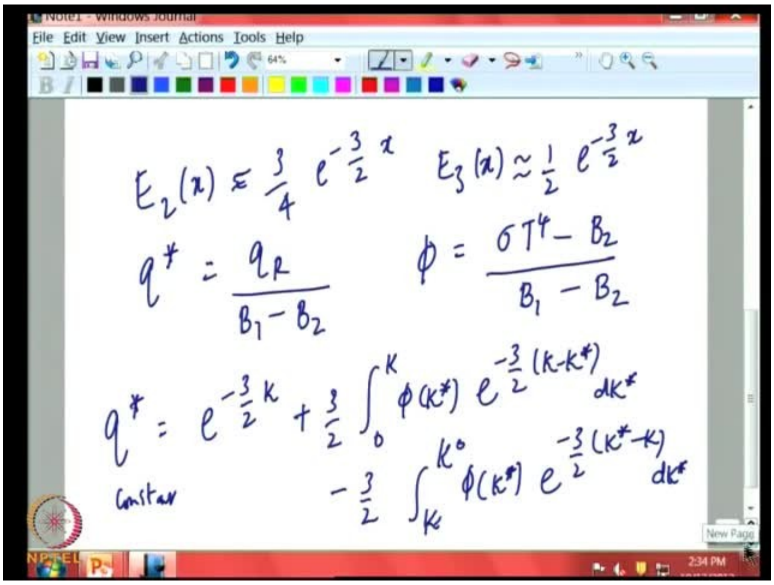
(Refer Slide Time: 04:44)



This is the example showing, how good the approximation of the exponential integral function, when you replace it exponentially is. We can see that in the case of E 3, we are able to fit the complex function quite well, for values of this case t here 0 to 1.5. In the case of E 2 function we are not doing so well at the origin and not so well in these regions. But remember E 2 appears inside the integral. What is important is not the getting of the E 2 correctly, but the integral of E 2 correctly and notice here that E 2 has been so approximated that from 0 to around 0.25, E 2 is under estimated and from 0.25 to around 1 is over estimated.

We take the area under curve between 0 and 1.5 it will be same of both these functions. These function have been chosen very carefully after many trial and errors such that the function E 3, which appears outside the integration is represented quite well over the entire region. In the case of function E 2 your only ensuring that the integral of the function is accurate. So, with those approximations, that is given here we got this expression for q star.

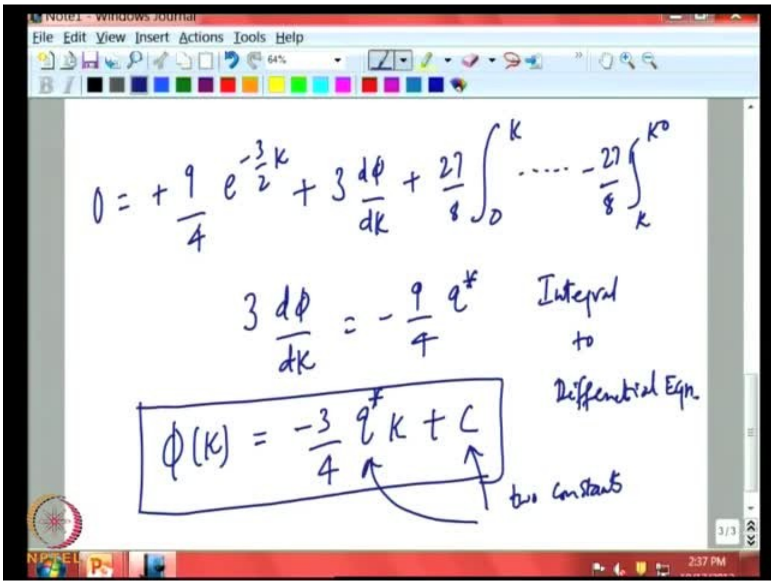
(Refer Slide Time: 06:19)



If we recall q star is of the non dimensional radiative flux. We also had a non dimensional temperature for the gas, which was the temperature of the gas minus radiosity surface two by 1 minus 2. So, with these non number representation and with these approximation, our expressions for q star was e to the power of minus 3 by 2 kappa plus 3 by 2, 0 to kappa phi of kappa star e to the power minus 3 by 2, kappa minus kappa star d kappa, star minus 3 by 2 kappa to kappa 0, non dimensional temperature kappa minus kappa. Notice that this arguments here are always positive so that this always represents the d k term.

This is the equation now we are going to solve the approximate equation after replacing the, exponential integral function by exponentials. This expression was differentiated twice and since we are dealing with radiative equilibrium, quantity q star is a constant so on differentiation goes to 0. So after two differentiation you will get the following expression.

(Refer Slide Time: 08:18)



We will get 0 equal to minus plus 9 by 4 e to the power minus 3 by 2 kappa, this is after two differentiation. This is from Leibnitz's rule because the limits of the integral are not a constant, but functions of kappa, then we have 27 by 8, 0 to kappa term and minus 27 by 8 kappa to kappa 0 term. These two integrals are there, but they are identical to these two integrals except for the constants in front.

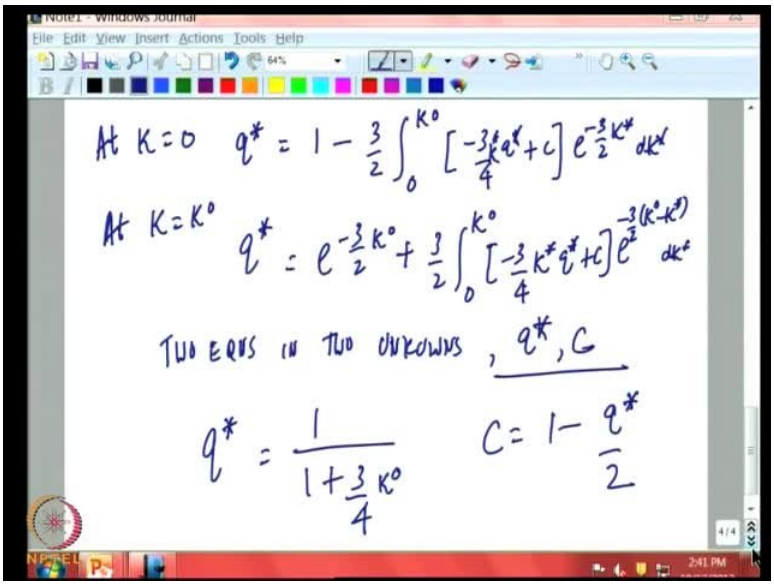
We take this equation multiplied by 9 by 4 then these two will be identical to these two. We subtract this equation that equation so when you do that you get a very simple expression, which is minus 9 by 4 q star. Essentially we have managed to convert integral equation to a differential equation. This is a major achievement because all of us have learnt techniques to solve the different equation, in our earlier courses. This is easily solved.

The conversion of the integral differential equation is very convenient because this can be solved very easily. It is first order ordinary equation. This becomes is equal to minus 3 by 4 q star is of constant because we have radiative equilibrium, where that is into kappa plus a constant. We have got the final result for the temperature distribution the gas, which is linearly proportional to kappa they will length coordinate, and there are two constants q star and c.

Now, in a differential equation if we want to obtain the value of the constant you have to use boundary condition. But recall that we started with the integral equation. The integral equation does not require boundary conditions, boundary conditions are built into the equation. So, when we convert a integral equation to a differential equation, all we can do is to ensure that satisfies this differential equation at two locations, in the region of interest to you.

For example, we can ensure that you satisfy the equation at kappa equals 0 or kappa equal to kappa 0. This equation now we can write down at kappa equal 0, this term drops out. So, 0 to kappa 0 that is one expression.

(Refer Slide Time: 11:55)

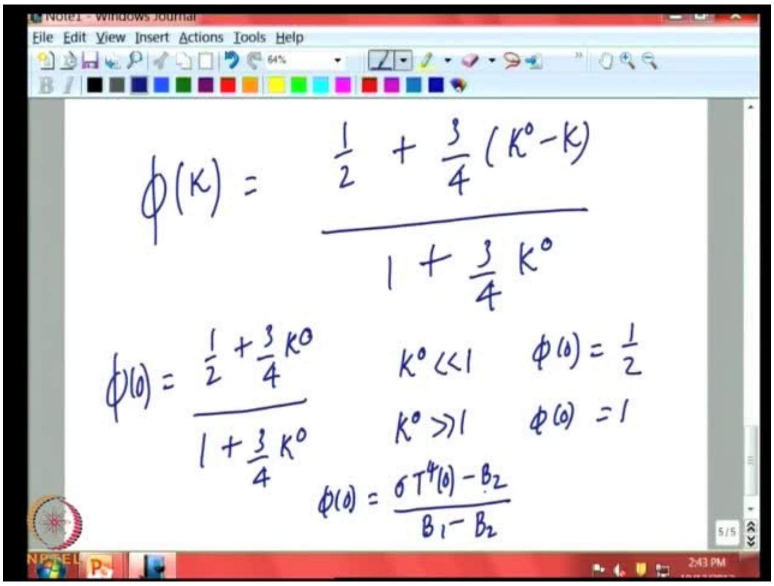


We say when kappa equals 0 expression of q star becomes 1 kappa at 0, 1 plus 3 by 2 minus 3 by 2 first term will go away kappa is 0. Second term is 0 to kappa 0 and phi of kappa over then minus 3, 4 q star plus c into e to the power of minus 3 by 2 into kappa star minus kappa, d kappa star. Kappa is 0 here so that term will also drop out minus 3 by 2 kappa star. This is what we have to now integrate now integrating this is not very difficult; we have to integrate by parts.

This equation as to be integrated by parts and you get one equation, we can say at kappa equals kappa 0 then next one, will give you q star equal to minus 3 by 2 kappa 0 plus 3 by 2, 0 2 kappa 0 minus 3, 4 kappa star q star plus c e to the power of minus 3 by 2 kappa 0 minus kappa star d kappa star. We have two equations here two equations in two unknowns that is q star and c, we can solve for it that is how values. We can integrate by parts and finish this integration.

We have two equations between q star and c you solve for it and if you do it, we will get the following result. Q star will be equal to 1 by 1 plus 3/ 4th of kappa 0. This is a number and c will be 1 minus q star by 2. We have got the complete solution to the equation, we know what q star is We know what the c is so the final expression for the temperature distribution of the gas.

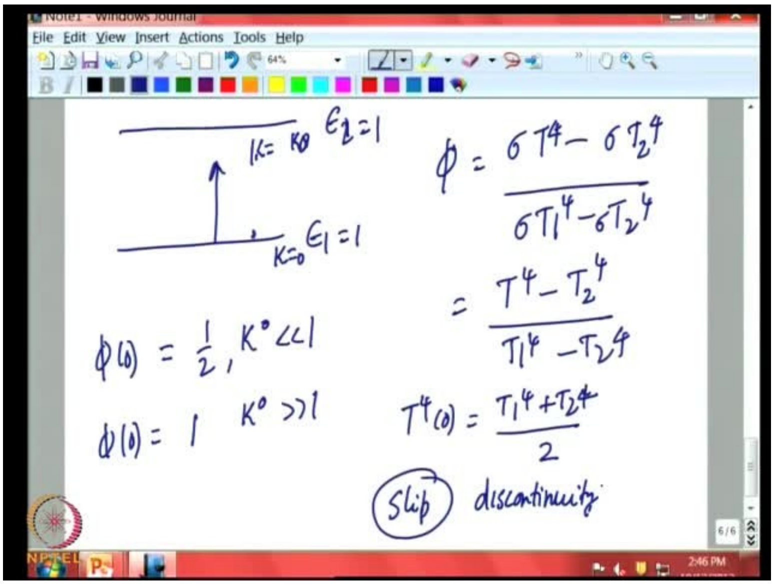
(Refer Slide Time: 15:22)



The temperature distribution the gas is nothing but half plus 3, 4th kappa 0 minus kappa divided by 1 plus 3, 4th kappa 0. This is a final expression for this temperature distribution and we notice that there are several interesting features. One get advantage of getting such an analytical solution is we can prohibit and understand what it means. For example, temperature at the lower wall phi equals 0 and kappa equals 0 gives you half plus 3/ 4th kappa 0 by 1 plus 3, 4th kappa 0.

Now, notice two features here that kappa 0 much less than 1 thin limit, phi of 0 is equal to half and for kappa 0 much, much greater than one the thick limit phi 0 equals 1 and what is phi 0, if we recall the definition of phi that we can came up originally. So, phi of 0 will be sigma T to the power of 4 minus at kappa is equal to 0 minus B 2 by B 1 minus B 2. Now, if you look at the expression we may not see much use of it, but imagine the two plates are black.

(Refer Slide Time: 17:26)

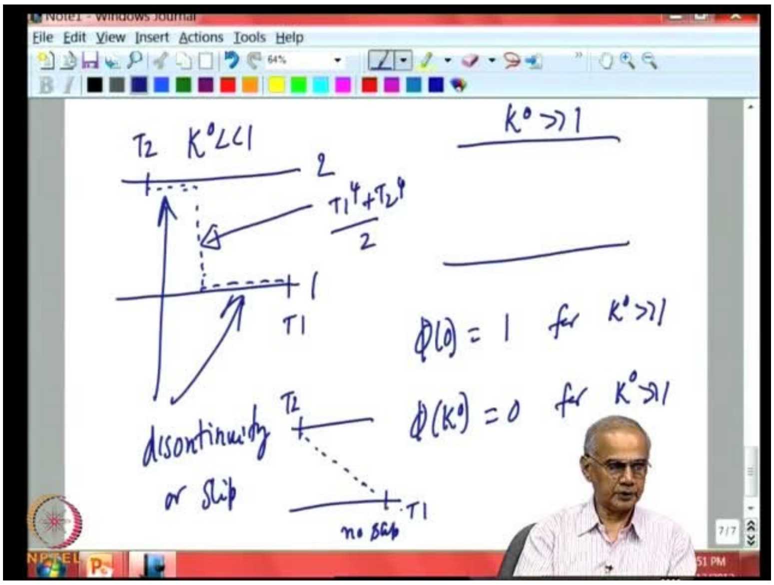


The two plates at the top and bottom both have emissivity equal to 1 then phi will become sigma T to the power of 4 minus sigma T to the power of 4. This is will enable us to interpret the results much more easily. We will take this case for the present purpose two parallel black plates with the gas in between and the question, we want to ask is what is the value of phi of 0. Phi of 0 was found to be equal to half for kappa 0, much, much less than 1. First we must assume kappa was half when phi 0 is much greater than 1 we will get is equal to 1.

This result we got this now, what is the meaning of that because now phi represents the temperatures. This is how kappa is measured so this is kappa equals 0 and this is kappa equals kappa 0. Let us now write this experiment more clearly here the kappa is measured from the lower plate here. This is kappa 0 this is kappa this is kappa of 0 here and kappa equals kappa 0 there and so when kappa equals 0 very small value of the optical depth, you find that T 4 of 0 this is equal to half.

It was T 1 to the power of 4 plus T to the power of 4 by 2. The temperature of the gas at the wall here is not equal to the wall temperature. That is what is known as slip. There is a temperature discontinuity. This we had briefly mentioned last time let us again point out.

(Refer Slide Time: 15:22)



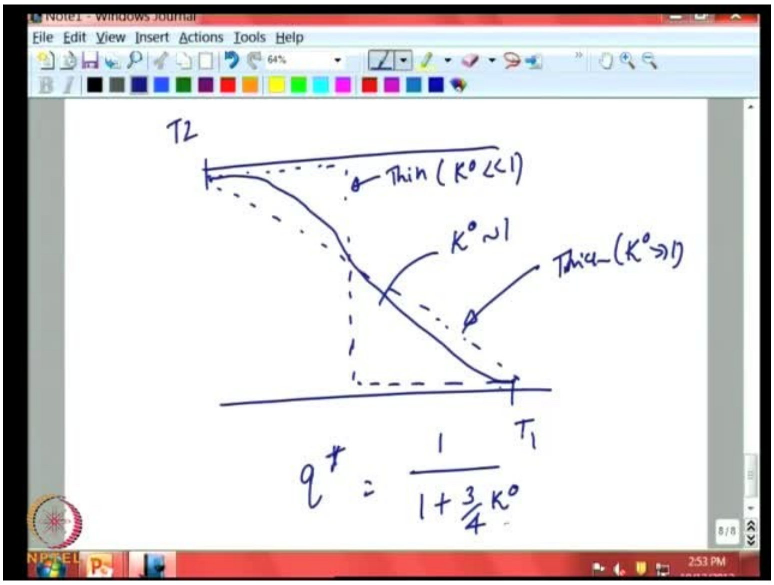
So, kappa 0 is much, much less than 1 the temperature distribution of the gas between the two plates here this is T 2 this is T 1 let us say so it is more like this. This value is T 1 to the power of 4 plus T 2 to the power of 4 by 2. The gas is isothermal and the temperature of the gas is not equal to the wall temperature at either on the top, or at the bottom. There are large temperature discontinuity in both plates and this discontinuity or slip is a very important feature of radiative transferring gasses. This is not seen that frequently, in the case of conduction heat transfer in gasses, because we are there dealing with situations, where the molecular mean free path is small compared to length scale. That this kind of slip is not seen, but as we as we recall in the thermal flask problem, we know thermal flask between the two wall the air is almost evacuated.

The mean free path of the molecules between the two walls is quite large. The molecules do not collide to each other they more often collide with the two walls. In that case also we will find a slip and the two walls. But in a normal situation encountered when the air is not evacuated, when we are dealing with atmospheric conditions in this room where the mean free path is very small.

In that situation where the optical depth of the photon is very large, the photon mean free path is very small compared to the length scale of interest and in such a case if you recall phi of 0, for large kappa 0 is equal to 1 and phi of kappa 0 is equal to 0 for kappa 0 much greater than 1. What is happening in this case is if we now draw temperature distribution, in this limit here is temperature T 2 here is temperature T 1, it will be varying linearly and there will be no slip, no discontinuity because in this case the photon mean free path is so small, compared to the length scale that the phenomenon is similar to conduction heat transfer, that you encounter it normally in engineering.

We see a continuous temperature profile. What we saw was that the temperature profile due to radiation heat transfer in the radiative equilibrium, does very linearly with the distance there if you already look at T to the power of 4, and not T and in the third limit, there is large discontinuities on either side those are slip and in the thick limit there is no discontinuity. So, to make this point even more clear we will draw it once more.

(Refer Slide Time: 25:05)

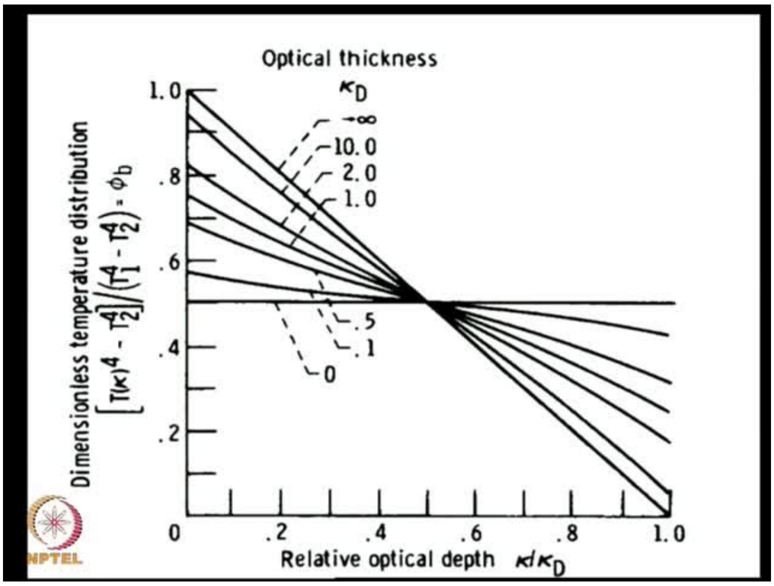


So, here is the top wall and the bottom wall here it is T 2 here it is T 1. We want to draw 3 sketches, one is in the optically thin limit, other is in the optically think limit and the next one is in the intermediate conditions like this. This is thin, this is thick for very small kappa 0, this is for very large kappa 0 and this is kappa 0 of the order 1.

We clearly see that the shape of the functions depends very much upon the thickness, the absorption coefficient of the gas and whether, the mean free path of the photon is very small a very large compared to the distance between the plates, but they key point to remember is that there is always a slip accept at the thick limit. We have to redraw the, this sketch for the intermediate case it will be always a slip here and it will go like this and there will be a slip here. There will be slip in these two cases only times no slip is when we had the thick limit. In normal situations in the radiative heat transfers slip is always there.

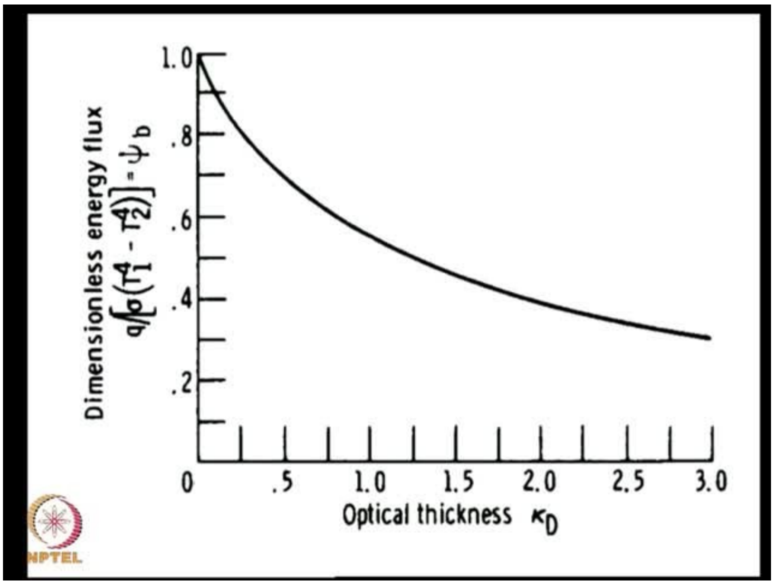
Now, the slip may vary with the optic thickness of the medium and only in the thick limit that the slip tends to 0, but does not reach till very large optical thickness. The second important result which we got from our analytical solution is a fact that the non dimensional heat flux, goes as 1 plus 1, 3 by 4 kappa 0. This result tells you that the heat flux goes to 0 at very large optical depth that is not surprising. The gas is highly absorbing hardly any heat is transferred from plate 1 to plate 2, and in the limit of optical thin this term is very small. All the radiation leaving wall one reaches wall two. We have two limits that come very clearly. Now, let us go back and look at these results in the light.

(Refer Slide Time: 28:13)



This is the result now showing the between two black walls, the temperature distribution for various optical depth kappa by kappa 0 here, and various values of kappa 0 and this is called kappa D in this paper, and this is a numerical solution then on the computer, not analytical solution. We can see clearly that in the optically thin limit the temperature hardly varies. In the optically thick limit it varies linearly in T to the power of 4 and in the thin limit there is a discontinuity here, in the thick limit there is no discontinuity. So, all that is visible very nicely in this picture or the remaining solution.

(Refer Slide Time: 29:11)



The next picture shows the non dimensional heat flux, written to parallel plates which are black as for the optical thickness kappa 0. If we look at the optical thickness up to 3, we can see that the heat flux goes on deceasing, but does not decrease rapidly it goes 1 over 1 plus kappa. Now, let us see how both the results shown here on the slide, are exact numerical solutions obtained on the computer and today, these can be done very easily with great speed.

(Refer Slide Time: 30:08)

Optical Depth	q*(approx)	q*(exact)
0.2	0.8696	0.8491
0.4	0.7692	0.7458
0.6	0.6897	0.6672
1	0.5714	0.5532
1.5	0.4706	0.4572
2	0.4000	0.3900
3	0.3077	0.3016

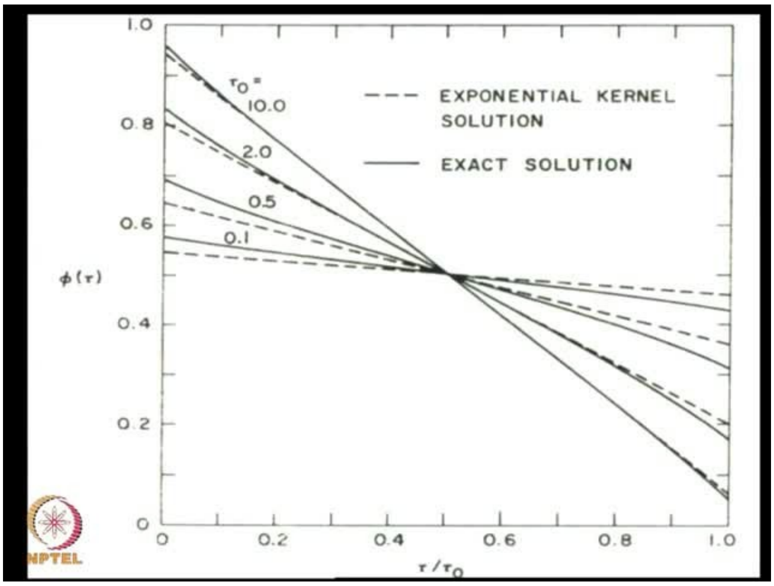
$$q^*(approx) = 1 / \{ 3\kappa_o/4 + 1 \}$$

Now, let us compare the result that we obtain which is given below. Approximate result we got from the exponential kernel approximation, with the exact result obtained by numerical integration. It is approximation result given below exact result for various optical depth. We notice that these two results are very close. Now, at optical depth of point 2, we can see that the change difference is about point this is 0.85 this is 0.87, 0.02 and here the difference is 0.006.

One can say with great confidence that the approximate solution is pretty accurate and the largest error is of order of somewhere 3 percent. In many engineering applications the technique, gives us the accuracy of 3 percent. We are satisfied with errors of 3 percent because one of the challenges, we face in solving radiation problem is the fact that the actual data required to calculate optical depth. For example, the absorption coefficient or the wall emissivity are not known very accurately.

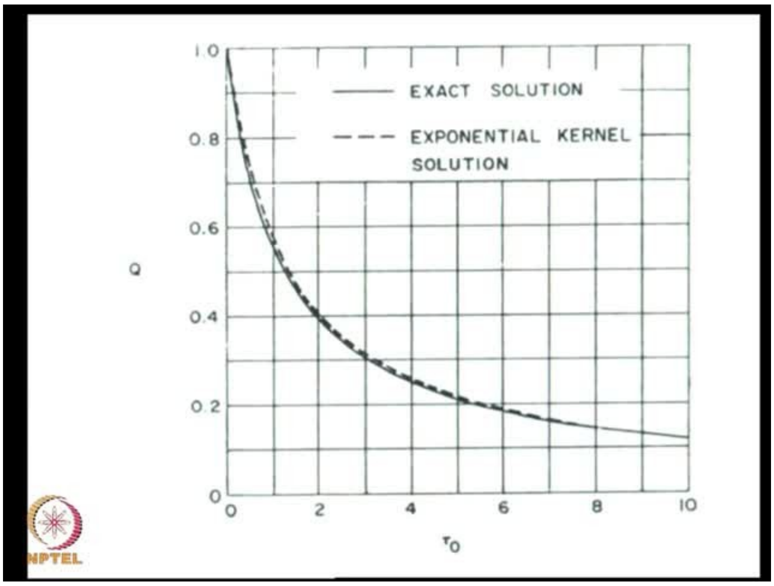
One of the biggest challenges in solving real world problems in engineering, in radiation heat transfer is the fact that the data that is available to you is not that accurate. That is why very often one is happy if an approximate method gives us the result close to the exact within an accuracy of less than 5 percent. This example is going go example where we are quite happy with the accuracy of the solution and this is the kernel approximation, which we discussed little early.

(Refer Slide Time: 32:21)



This is a comparison between the exact solution in continuous lines and the kernel approximation, which is dotted line and once more we can clearly see that the approximation using kernel approximation is very close to the numerical solution at high optical depth. But as the optical depth decreases we can see some error is creeping in, in the case of temperature profile.

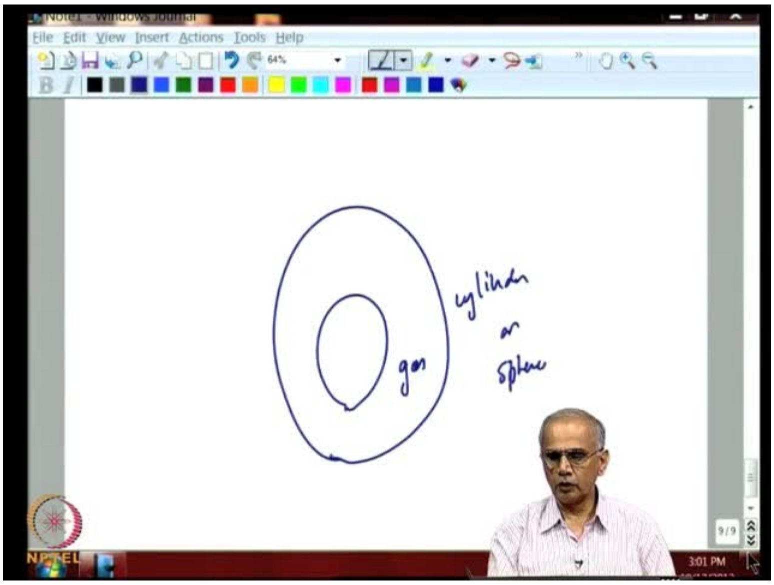
(Refer Slide Time: 32:55)



If we go further and look at the non temperature flux, we can see that it is pretty accurate. This is the pictorial depiction of the table we just now saw and indicating that at most optical depth, the kernel approximation slightly over estimates the exact solution, but if you go to extreme high optical depth the results are very close to being exact. Now, let us go back to the solution we have here. We are solving this problem by kernel approximation and in this example of the radiation between two parallel plates with gas, this is a one dimension problem.

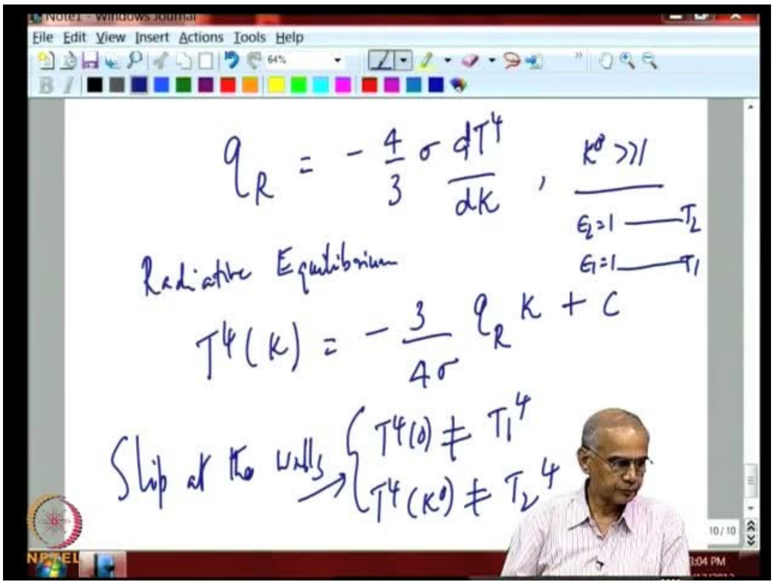
We manage to get the solution very easily. Now, the question we ask if there are many examples in practice in which, we cannot invoke kernel approximation and get a right answer. For example, is that of 2 parallel plates suppose, we had 2 cylinders or 2 spheres then of course.

(Refer Slide Time: 34:19)



The 2 cylinders with gas in between this can be a cylinder or a sphere. In such cases the kernel approximation is not going to simplify the problem substantially. We would like to know whether there are other ways of solving the problem, which avoid solving the full integral equation that we encounter in radiation transfer. One approach that can be attempted is why not solve the equation in the thick limit.

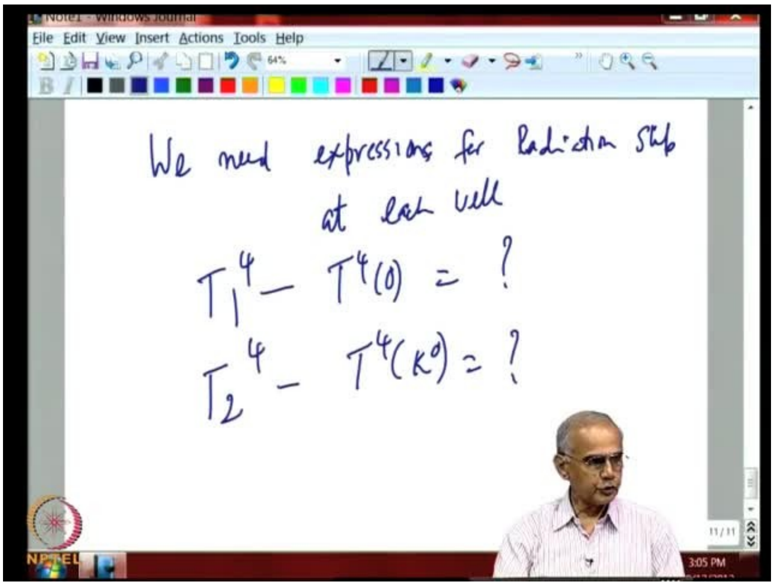
(Refer Slide Time: 35:07)



In the thick limit we saw that q r minus 4 by 3 sigma d to the power of 4 by d kappa. The question now is whether we can solve this problem in radiative equilibrium, and get result, which are similar to what we obtain by the kernel approximation. We integrate this equation under radiative equilibrium. This equation is very simple and we get T to the power of 4 a function of kappa is equal to minus 3 by 4 sigma q r the constant in radiative equilibrium, linearly because of this kappa because the constant.

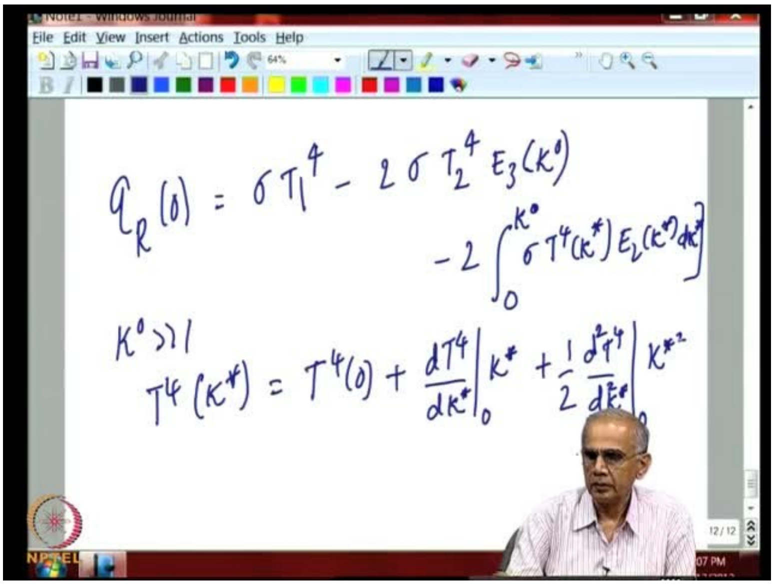
This is a result we get by indicating that equation in thick limit. We apply the boundary conditions, but applying boundary condition is a challenge here because we know that there is a slip. Because there is a slip at the walls the top wall is at T 2 bottom wall is at T 1 black plates. In this case the questions is that we cannot say that this is equal to T 1, we cannot say at the top these two are not possible because of slip. Since, we do not know the temperature of the gas near the top wall or the temperature of the gas at the bottom wall, we cannot apply the boundary conditions.

(Refer Slide Time: 37:28)



Therefore, we need expressions for radiation slip at each wall. Somebody has to give an expression of how each of these varies. These 2 slip values have to be obtained from some other source. We will show that it is possible because what you will do is, we will look at the radiative flux at the bottom wall, which from the basic equation of radiation heat transfer.

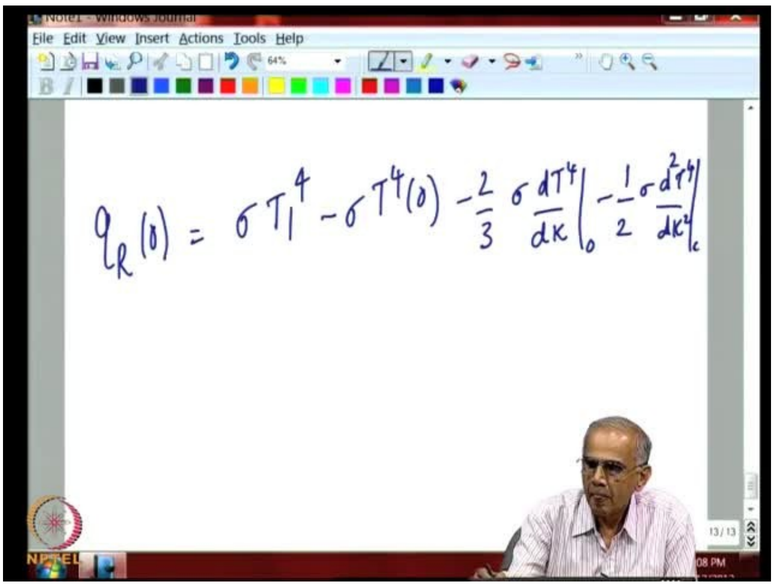
(Refer Slide Time: 38:24)



This is the expression which is similar to what we started out, when solving the kernel approximation, but here we are not making any kernel approximity we are solving the exact equation. Now, let us invoke the fact that you are in the thick limit We expand the temperature inside the integral, in a tailor series around the bottom wall.

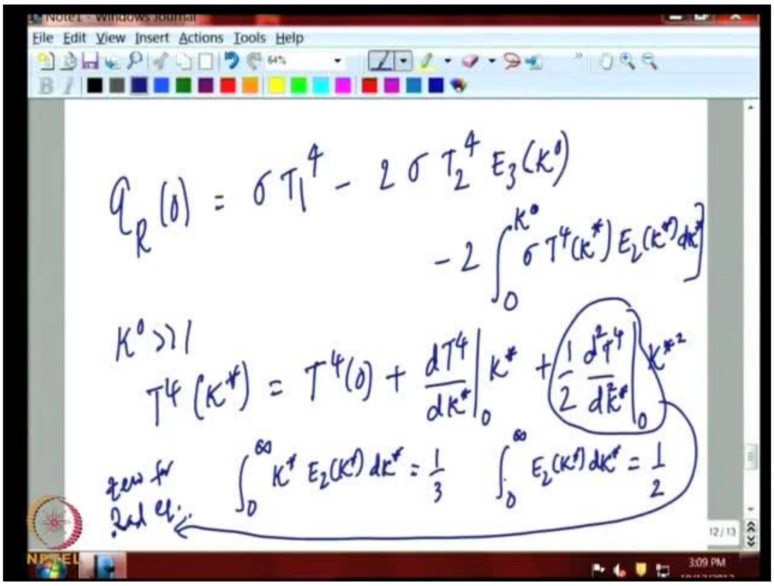
This will be nothing but derivative of this at 0 plus second derivative at 0 and kappa star square. This can be plugged into this integral and remember these 3 quantities are not varying only this varying. We can integrate this equation fairly easily by putting inside. So, on integration you get the following expression for radiative flux to the bottom wall.

(Refer Slide Time: 40:34)



This the first term, which is then changed the second term and this two-third comes from the envelope half.

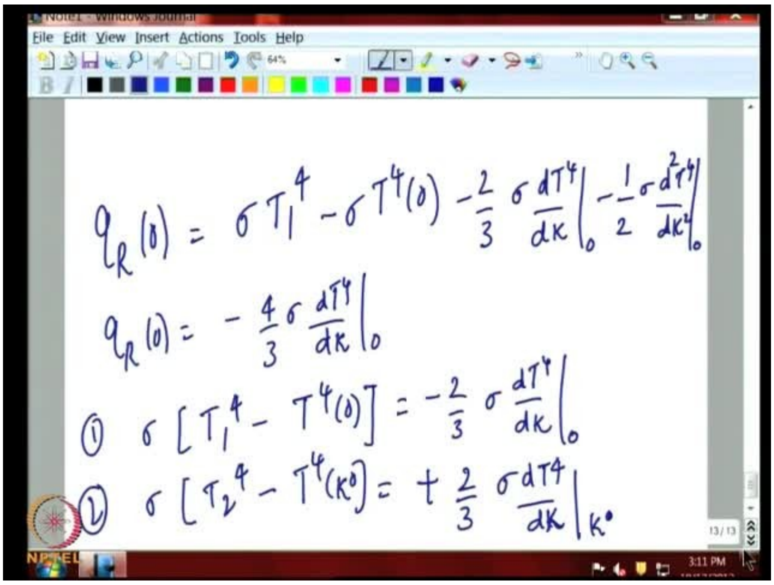
(Refer Slide Time: 41:10)



We are letting this limit go to infinity and so because this cover is very large you got infinity, then this thing comes out. This kappa star one quantity of intensive is 0 to infinity kappa star E 2 of kappa star, E kappa star this can be shown to be equal to one-third and when we substitute this here we will get only 0 to infinity E 2 of kappa star d kappa star, this equals 3 and this will give us R.

These are the two expressions that we will be utilizing and remember that in this case of radiative equilibrium, this term anyway is 0, this is 0 for radiative equilibrium. That term is not important in this example, but there are other examples where you will have to worry about that. If we put all this in all the numbers in there we are getting what is given here as expression for q r of 0. We also know that from the basic equation for optically thick limit.

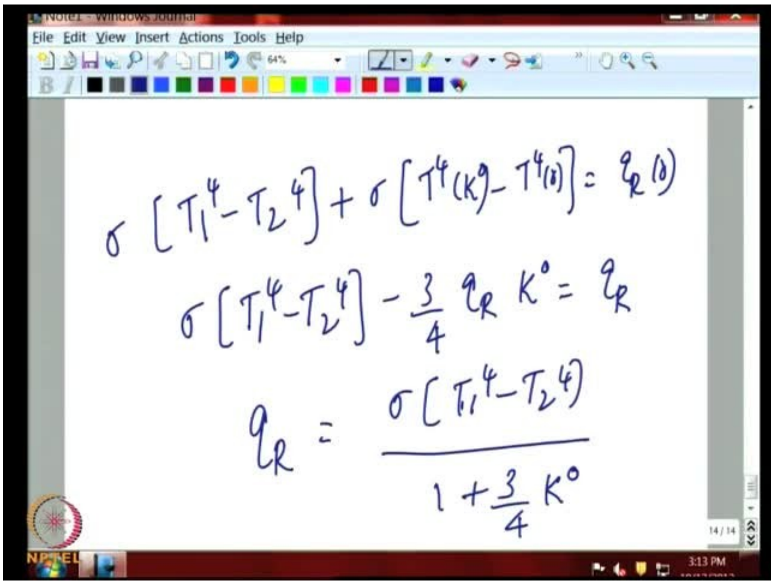
(Refer Slide Time: 43:01)



We also know that q r of 0 is also minus 4 by 3 sigma d to the power of 4 by d kappa at kappa equal, this you already know and so we combine out this is that so finally, we get expression which is saying that the temperatures slip at the bottom wall is equal to minus two-third, this is 0 because of radiative equilibrium. So, essentially we take this two-thirds here on this side and that is my four-third, this is my two-third.

Similarly, apply the same thing to the top wall, we get the following expression for the slip, but this time it will be plus. Now, what we do is we subtract let us call this equation one and equation two subtract 1 minus 2, and we will get minus four-third slope and this slope your constant and minus four-third sigma is nothing but your radiative flux.

(Refer Slide Time: 44:55)



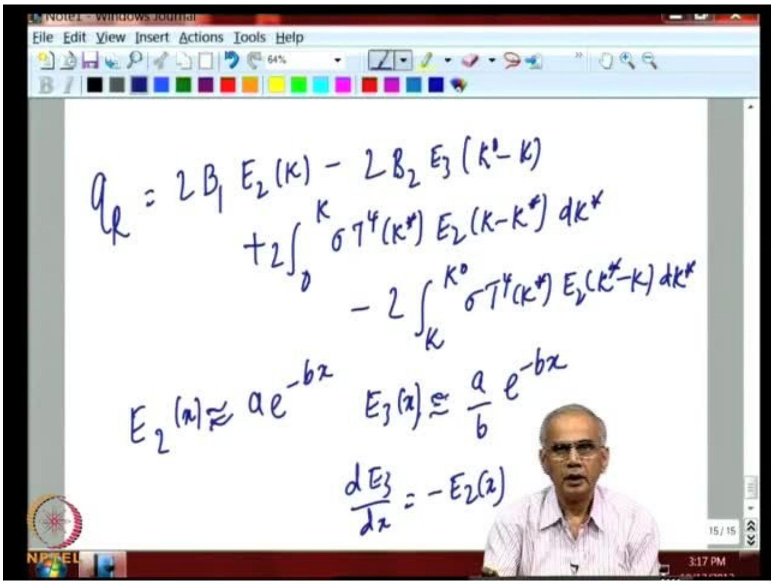
The final expression you are going to get for the radiative slip condition is this and from the basic integration of the optical thick limit is also known, it is minus three-fourth q r into kappa 0 is equal to q r. We can take this q r to other side so we will get q r radiative flux is equal to sigma T 1 to the power of 4 minus T 2 the power of 4 by 1 plus three-fourth kappa 0. What is amazing is that this result we are which we have obtained only for the optically thick limit, but by using this slip conditions this result is identical to the result obtained from the kernel approximation, which is valid both in the thick and the thin limit and also intermediate cases.

Between two parallel plates you do a derivation for the radiative flux, using either the kernel approximation or using the thick limit and the radiative slip conditions, answer is same. Now, that is partly an accident because this number three-fourth derivation here in this derivation you may not always get in the case of the kernel approximation because that very much depends, upon what is the approximation you are going to use for the exponential integral functions.

We have to use e to the power of minus three-fourth x and that is how we got three-fourth, if you had used root 3 which is preferred by some other people, we will get root 3 here. We can see that there are some variations here, and those variations really depends upon which approximation we use for the exponential integral functions, but the results that is really of relevance is that in this specific problem whether, we use the exponent kernel approximation or take the thick limit and apply the boundary conditions, using slip conditions then we get the expression for q r between 2 flat plates, which is identical to what you would get to the kernel approximations.

We are getting results in two different ways, one is already from thick limit applying the slip condition the other case plotting kernel approximation, the answer is same and that is very, very useful. But it is important to recognize that this result is identical only because of these certain approximation, we made in the case of the kernel approximation. To ensure that we realize that results are sensitive to the approximation used we want to show, how in general we can convert the integral equation that you obtained, in the radiative that is this equation in general.

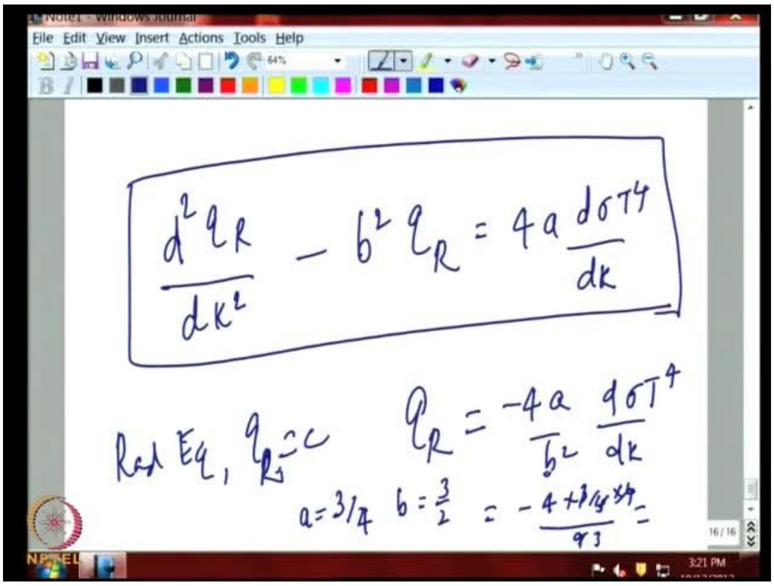
(Refer Slide Time: 48:49)



This is one which we started out today. We want to do is convert make a general conversion of this integral equation to differential equation, and if you have followed the kernel approximation that we adopted, we can see this can be done in general. What we do here is we assume that E 2 of x is a e to the minus b x and to be consistent, we assume that E 3 of x is a by b of minus d x.

We are doing that because we want to ensure that d E 3 d x is equal to minus E 2 of x that we must at least try the mathematical identity, the relation between E 3 and E 2 must satisfy. If we assume E 3 as a by b is minus a b x and we differentiate this x we will get this quantity. If we use this approximation replacing the exponential integral function by exponentials, if we do that and substitute it into the equation and differentiate twice, like before and subtract we will get the following expressions.

(Refer Slide Time: 50:43)



This is a very useful result, which shows that the original integral equation for q r can be converted to a differential equation for q r. In many situations this equation is much more easy to solve, then the original integral equation. The only point we want to make is that we look at radiative equilibrium, in which case q r is the constant this term drops out. We will get a expression for q r in this case, this term drops out as minus 4 a by b square d by d kappa sigma T to the power of 4.

This is the Rosseland diffusion limit, but notice that depending on what a and b use. For example, we had used previously a as three-fourths and b as 3 by 2. So we substitute that there we will see here we get minus 4 into 3 by 4 by 9 by 4. We get 4 by 3. So, 4 by 3 that you got earlier is because of the choice of a and b. If we choose different a and b then we won't get 4 by 3 but we get some other number.

So, 4 by 3 is the exact limit of the thick limit, but we start from a differential approximation and depending on your choice of a and b, we may get somewhat different result. This equation for the radiative heat transfer is quite useful because notice that there is a differential equation, we can do this in various coordinate systems other than plane parallel, we can do it in cylindrical spherical or other coordinates. Today with the powerful tools that are available for solving differential equations numerically, this is a very useful tool to have.

What we have seen is that there are two different ways solving the basic radiative heat transfer problem in gasses, one is taking the full integral equation and solving numerically. The second is taking the differential equation and making an exponential kernel approximation and converting into a differential equation and then solving it. Then we saw there is the method in which we assume thick limit. Apply the slip conditions and get an expression, which can be valid for a fairly wide range of optically thicknesses.

All these we get for a gray gas because it was a simpler to do, and because most real gasses are not gray, we need to look at ways we have to extent this to non gray situation and that is about will be the main topic for discussion. In the next lecture, how to extend this result for a gray gas to a non gray gas and how the results change, while going from a gray medium to a non gray medium. That is the what we discuss in the next lecture and because all real gasses are non gray and so it s important to understand how non gray limit behaves.