Radiation Heat Transfer
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Lecture - 13
Integral methods for enclosures

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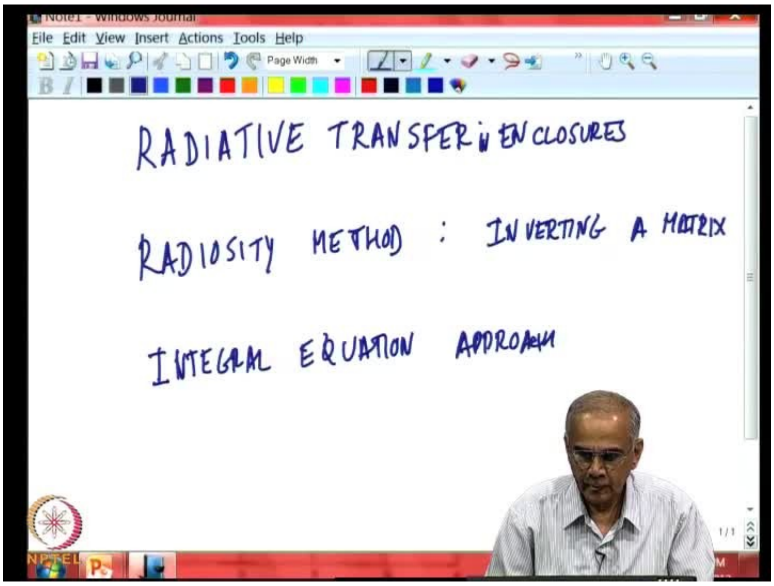

In the last we lecture, we looked at radiative transfer in enclosures and our primary focus was on the radiosity method and this involved essentially, inverting a matrix. Today with the availability high speed computers, inverting matrix is very easy job and done quite easily. So, most of the problems involving radiative transfer enclosures, is solved by this method, but today we will talk about a method somewhat older one, which has been used and is called the integral equation approach.

This method is not so popular now because it is somewhat more tedious than the matrix method, but we would like to introduce this method because when we later go on to study gas radiation, we have to solve integral equations. It is good to get an exposure to the way of solving integral equation through a simpler example. In this example, we will first set up the problem.

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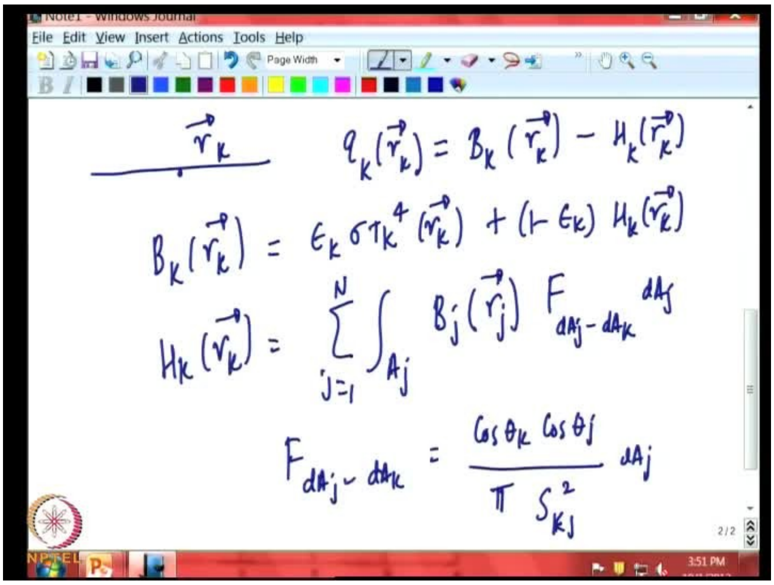

The idea is that if you have a surface on that surface, you have a location and that we will call as r vector k everything is varying continuously here. The heat flux at that location, we will use the symbol q_k, it is nothing but radiosity at that location minus the radiation. This formulation is not different from what we did earlier, except that everything is continuous variable not a discreet number.

We can also write the radiosity at that location as the function of the emission for a gray surface, everything function of the location r_k and of course, reflection. So, as far as the problem is concerned it is identical to what we did earlier except, everything is varying continuously. Though the important difference is the irradiation, the radiation is coming into this location.

Now, it will be integral over surface j of the radiosity of surface j. The shape factor F d A j and d A k times d A j. This is the radiation link surface A j, this fraction of the surface k We sum over also surface j equals 1 to n. This integration within that single surface. And all of us know by now that this differential shape factor is the basic quantity, which we know as cos theta k, cos theta j by pi S A j whole square d A j. This also is known.

This is the formal statement of the problem and as in the previous examples we have done, the aim is to know how the radiosity is varying with distance at the each of the surfaces. Previously, we solve this problem by dividing the surface into discreet surfaces. Then we use Matrix version and as pointed out, that is the preferred approach today because matrix inversion is very fast and efficient with the use of computers, but imagine that we were solving this problem more than 50 year, 60 year, or more than 100 years ago.

When we had no access to computers then you would have look for some analytical way of solving this problem, and people managed to solve it without using a computer. We just take a walk through history and look at how people solve this problem, in the pre-computer era. This will give us an insight into a method of solving integral equations analytically. Now, this is best done if we take simple example.

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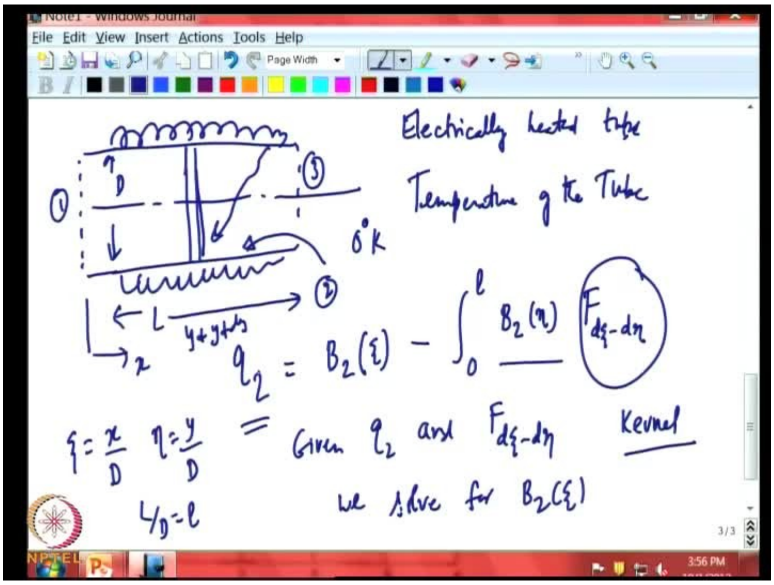

We will take a very simple example of a tube, which is electrically heated from outside. We will come across this problem in many situations, in which you have to heat a tube. So, certain amount of electrical energy is a supplied here. There is a constant heat flux here applied by the electrical heating, We want to know the temperature variation in the tube. As a functional distance and more specifically, you would like to know where a maximum temperature of the tube is and what upper limit of the temperature so that we can choose suitable material for this tube.

This is the simple problem we are going to do now, we will treat this ambient as 0 degree Kelvin this really for simplicity. Later, we can easily extend this problem when the ambient is not at 0 degrees temperature. Now, let us say that tube diameter is D, length is L and for convenience we will call this surface as 1, the other opening as 3 and the inner surface of the tube as 2. It is a 3 surface enclosure containing the inside the tube and the 2 side openings.

We can say that the heat flux to be supplied to the tube, has to be equal to the radiosity of the tube.

We have measured all the distances as x from the end of tube, we take a element here that element is between y and y plus d y. For convenience, we will non-dimensionalize everything. Psi φ is x by diameter of the tube and non dimensional y is y by again D. These are non dimensional quantities, which makes our life easier. The length by diameter ratio of the tube is small non dimensional number. So, with that you can see that the radiosity of the, this tube will vary with psi and minus irradiation will be radiation leaving other parts of the tube, and arriving here times the shape factor between d psi and d eta.

Once we know this shape factor F d psi d eta and q 2 is given specified electrical heating. So, given q 2 and F d psi d eta, we solve for B 2 of psi that is the proper statement, we can see that this is in integral equation because the unknown function is inside the integral. This is not commonly taught in most colleges. You are taught how to solve differential equations, you are taught to solve algebraic equation, very rarely you are taught how to solve integral equation, where the unknown function is inside the integral.

That is the purpose of this particular example, to tell you how you can solve the integral equation. Now, before we go to that we need to look at this what is called kernel. In integral equations, the unknown function here multiplied by kernel. The function also appears outside and that is a non homogenous term which is the specified the heating. What is this kernel, we kernel can easily obtained from standard table for shape factors.

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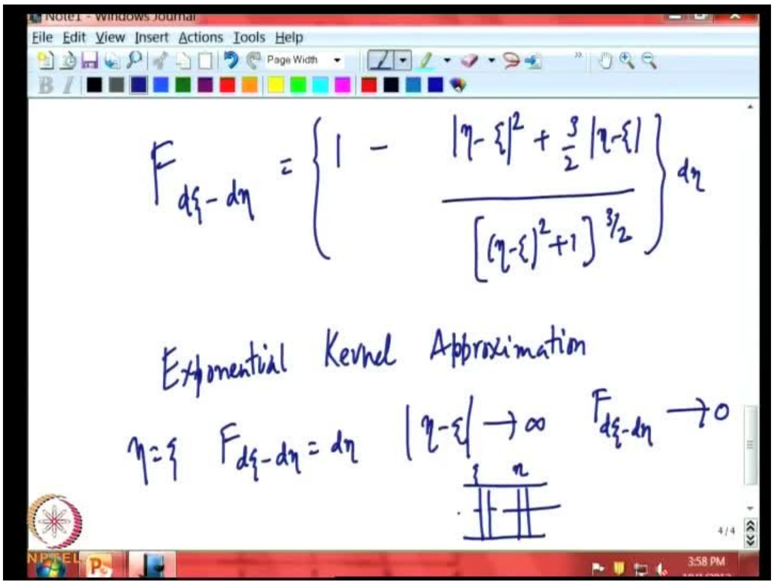

So, F d psi d eta will be equal to 1 minus modulus of square plus 3 by 2 modulus of difference in distances and at the bottom we have that acting as course multiplied d eta. This is the typical nature of the function. We can imagine that this will not be easy once, this is plugged into the equation, we wrote down it is not easy to solve this equation analytically. It can be solved numerically, there are lots of techniques available and that is not the aim of the lecture. The aim of the lecture is to find a way to solve this analytically because that gives more insight into the solution. We go for what is known as exponential kernel approximation.

This kernel we want to approximate as an exponential. Now, where you can see when eta equals psi, you can see that F d psi d eta equal to essentially 1 into d eta. Now, as you go further away as eta 1 as psi increases F d eta d psi tends to 0. This function has a largest value when eta equals psi, and as eta and psi if you recall eta and psi those at we have not forgotten, what we wrote in the last period. There is one element here another element here this element, which is psi and psi this is eta. So, when these two elements are far apart, than the shape factor between these two will be very low, when they are next each other that is when the shape factor is maximum. We want to replace this function by an exponential and for this case.

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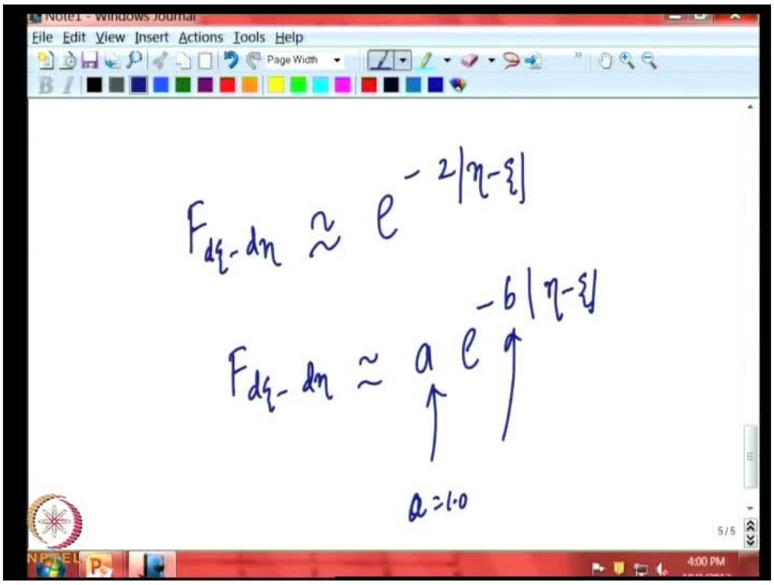

The exponential approximation is that F d psi d eta is approximately, equal to e to the power of minus two times eta minus psi. We might wonder how we arrived at this approximation. Now, there are actually two things involved. We assumed that F d psi d eta is approximately equal to a times e to the power of minus b mod of eta minus psi We need to find these two quantities. From the original function, we know that eta equals psi this function is 1, so we want to keep a equal to 1, if possible after that we need to find b.

The b that is chosen depends on what kind of approximation you want for this function. So, b is chosen such that the integral of this quantity over the length, if we recall the previous expression. We want this integral to be same both for the actual kernel and the approximate kernel. So, a and b are chosen so that at the origin it tends to the value 1. This quantity goes to 1 and as the distance between the two elements goes up it tends to 0. We will now show picture that illustrates that nicely.

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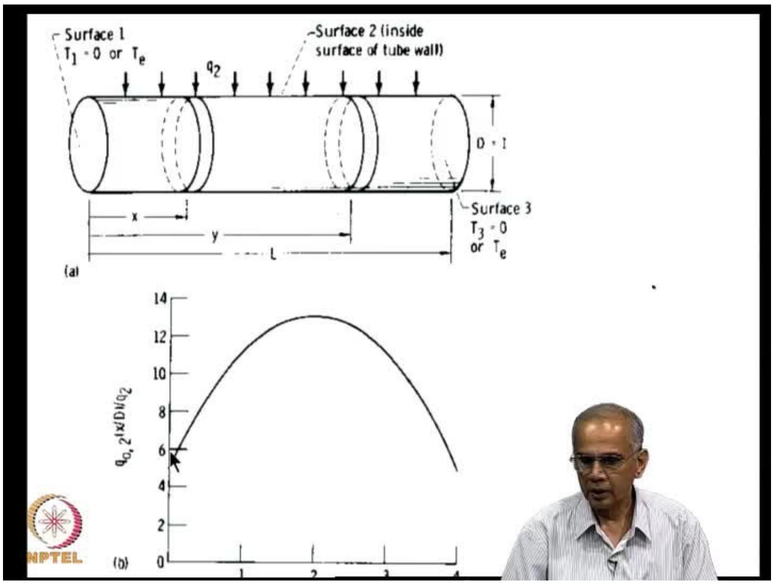

This is a tube where you looked at this is the problem example, x distance from one end and y is distance from the same end. These are two elements starts d psi and eta plus d eta. We can imagine the flux is going to vary somewhat like this, that is going to be realistic because , we are not seeing much of the ambient. It will be the hottest and the flux will be the highest radiosity and as we go to edges, where we are seeing the cold background here the radiosity will come down. These two are understood. Now, we go to the next picture.

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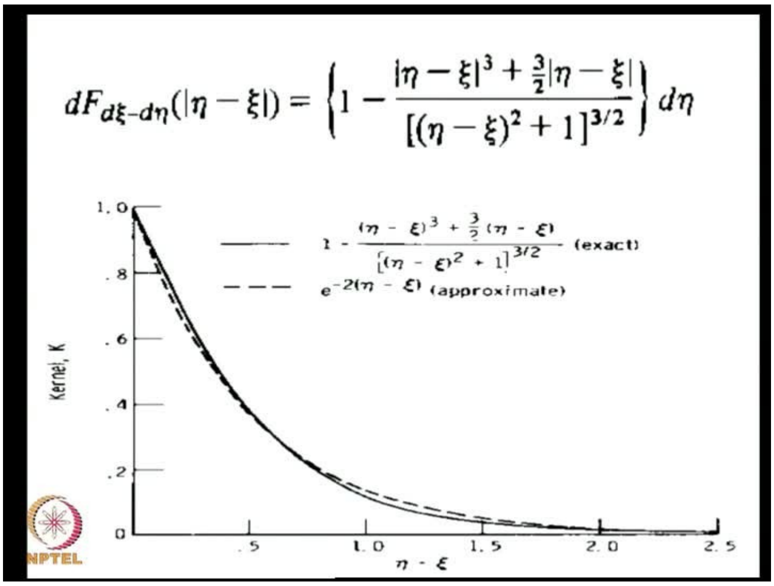

Here we show how we have made the approximation. The approximation that we have made the solid is actual function, the dotted line is the approximation. We can see that by choosing a equals 1 and b equals 2, we have ensured that this function is slightly lower than the actual function between 1 and between 0 and 0.5 and beyond 0.75 its somewhat higher, but the area under curve is almost same. That is our main criteria we want the function to correctly reproduce the behavior of the kernel at the origin.

In addition the only other constant we have which is b. So, chosen that area under the curve is actually more complicated function and exponential approximation should be same. We can see that it is a quite good, the fit between the approximate function and the actual function and we except that this will give us some useful result. Now, we go back to our solution methodology.

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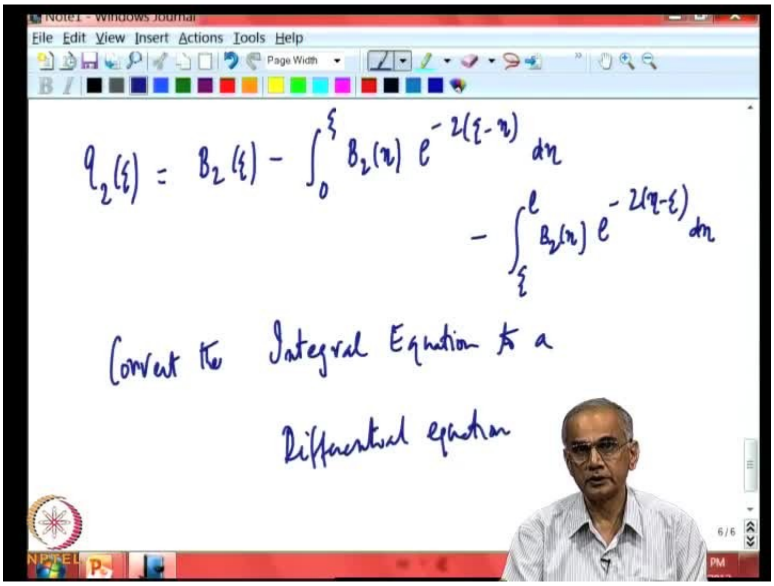

Once we make this approximation we can write down the heat flux on the left hand side is equal to the unknown radiosity on the right. Now, since there is a modulus function involved it is better to split the integral 0 to l into two parts, 0 to psi then you have the unknown radiosity due to the minus. Now, module involved we are dealing with psi less than eta. This will come at what as psi minus eta into d eta, the second term will be minus going from psi to l again b 2 of eta, e to the power of minus 2 eta minus psi d eta.

What we have done is to avoid the modulus term, we split the integral 0, l into two parts 0 to psi and psi to l. In a both cases the exponential is decaying, which is to be believed and now we have the full integral equation in front of us. Now, one of the great advantages of choosing an exponential kernel approximation is that exponential function, repeats itself on differentiation. That advantage we want to exploit, our aim is to convert the integral equation to differential equation. That is because we know well known techniques to solve differential equations, all of you have studied it.

On the other hand, very few have a mastery of methods of solving integral equations. If you convert the integral equation to the differential equation then we can use standard techniques, which we learnt as well as a easily available in the literature. So, how do we get differential, we have to eliminate these two integrals. We can see very easily that when you differentiate this function twice, exponential will repeat itself. Then we subtract the equation we have obtained after two differentiations from the original equation all the integral terms will disappear. We will be left with a differential equation. This before you go for this you need to know little bit about, what is known as Leibnitz rule.

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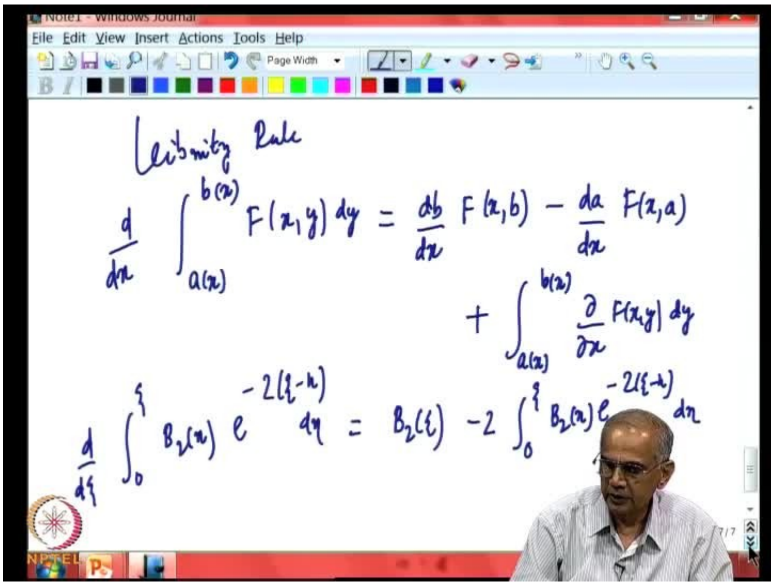

Now, Leibnitz rule is concerned with the following issue. Suppose if we want to differentiate an integral, where the limits of the integral are functions of the x and inside that is another function of x, y and d y. The Leibnitz rule states in order to differentiate a function in which the limit of the integral or itself is not a constant, but functions of this quantity x. We have to first differentiate this function and evaluate this function at y equals b minus differentiate this function, the lower limit evaluate the function the lower limit, then you can take the differentiation inside.

So, Leibnitz rule lays down the condition that you cannot just mearly take the differential from outside to inside the differential equation, if the two limits in the integral are not constant. Many of you may have done it because you may have done it with limit, which are independent of x, in which case these two terms drop out and we have left with this, this all of you are familiar with, but many of you may not familiar with the situation where the limits of the integral are not constants, but functions of x.

Now, let us apply that to one of the example. Suppose, if we want to know what is d by d psi of 0 to psi B 2 of eta, e to the power of minus 2 psi minus eta d eta. Here we see that there upper limit psi is a function of this quantity. In applying Leibnitz rule you have to say d by d psi of psi which is 1, and calculate this function at eta equals psi that is 1. We are left with B 2 of psi then this of course, doesn't continue with 0, doesn't continue with psi and not a function of psi. Then you take the integral in psi than this differentiation comes in and gives you minus 2. It is minus 2 into 0 to psi B 2 of eta e to the power of minus 2 psi minus eta d eta. We are done the differentiation similarly, we have to do the differentiation for the other function that is there where, the lower limit is a function psi.

Now, we do this twice, when we do this twice and we subtract it from the original integral equation. These terms will cancel out because when we do two differentiation as a exponential, than in the first one if we have minus sign, you differentiate again you get a plus sign. So, finally you are left with the function same as the function that you have seen in the first equation. When we subtract, the twice differentiated function from the original function, all the integral term will cancel out. We will see how to eliminate the integral. And when eliminate the integral finally, you are left with a very simple equation.

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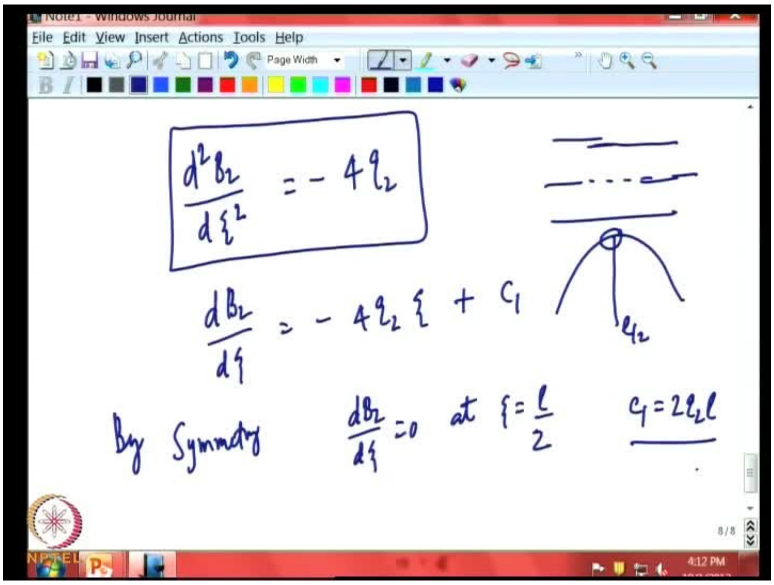

It is d square B 2 d psi square is equal to minus 4 q 2. The integral equation is simplifies to a differential equation because we have used the exponential kernel approximation, where the kernel is repeating itself. Now, what are the bonding conditions Of course, this is differential equation, all of you know how to solve it. And after solving it we will have two constants that is we differentiate once d B 2 by d psi will go minus 4 q 2 psi plus C 1. Now, the problem that we have stated is symmetric.

So, by symmetry we know that the radiosity has to reach a maxima at the middle of the tube. We can see that the tube is there, we plot the radiosity and right at the center of the tube at l by 2, it will be a maxima. This slope will be 0. We take that and psi goes l by 2 is 0, you can get c 1, c 1 becomes 2 q 2 l, put l by 2, here you get that this is where your c 1 then you integrate once more.

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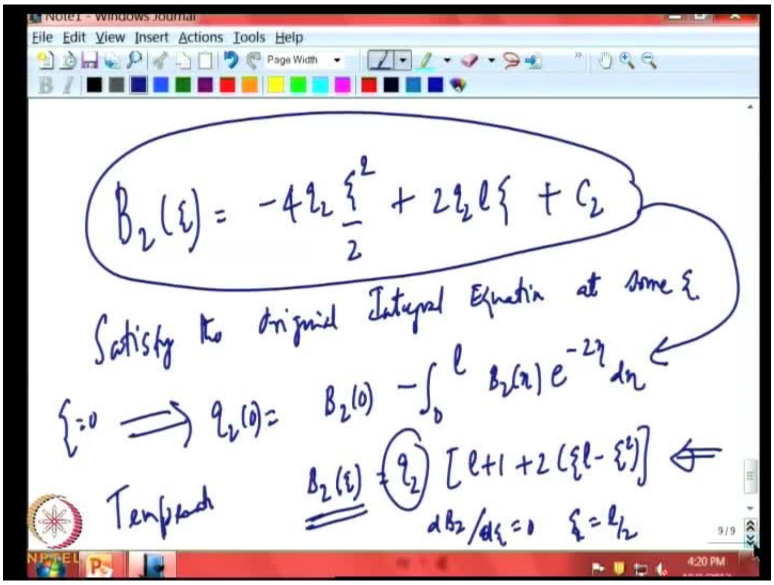

Now, integrate once more you get B 2 of psi is equal to minus 4 q 2 psi square by 2 plus 2 q 2 l psi plus c 2, one more constant. We do not see any other boundary condition, which is easily available. We exploited symmetry, but we don't have it, but although we do not have boundary condition, original equation that we wrote down, has to be satisfied at all points.

So, one way to do it is to satisfy original integral equation at some psi. Here take psi to 0 for convenience. We know that q 2 of 0 has be equal to B 2 of 0 minus 0 to l B 2 of eta, e to the power of minus 2 eta d eta, we know that at psi equals 0. We plug this expression into this integral, integrate and when you do that you will get a expression for B 2 of 0. That is c 2, we will get the expression for c 2 and that concludes the problem that we need to solve.

We saw that they were two constants we applied because we have second order differential equation, one constant came for symmetry and another constant came by satisfying this equation, integral equation at some point it can be psi equal to 0, psi equals l are in the equals l between anywhere does not matter. Satisfied at one point, and put this function back in and integrate it we will get c 2. We will get the final solution for the radiosity. We can then even estimate, the actual temperature that is obtained in this result.

Now, having solved the equation we can now ask if we solved it correctly so that we can check the result that we got will B 2 of psi is equal to q 2 into l plus 1 plus 2 into psi l minus psi square. So, once we get c 2 the answer should get. We can verify that d B 2, d psi is indeed 0 at psi equals l by 2 that you can see easily because d B 2 by d psi is nothing but 2 l minus 2 psi Then you put it as psi equals l by 2. We can see that l minus 2 psi l by 2 l minus l is 0 and this becomes 0. This is the final result for the variation of radiosity with length and once, you know radiosity and we know the imposed heat flux, you can also calculate the temperature.

This is the method that we have shown for solving enclosure problem, where the variation radiosity is continues and where you are able to use the kernel approximation than you are getting complete analytic solution for radiosity, for temperature and any other property that what you want to derive. This is an interesting result, a very special case and these results are very useful to us because tomorrow, when you solve this problem numerically using radiosity method and matrix inversion.

You would need to verify your solution against some other solution. At that time this kind of problem will act as your test case. You may divide this tube into hundred parts, and have hundred radiosity, inverse the matrix and ultimately plot the function, and see how close it comes to this expression here. These are very useful results that we have obtained analytically, which can be used to tune a new code because when you solve the problem numerically, you have to make judgments about accuracy.

We have to decide if you want to divide the tube to hundred parts or thousand parts, before you do matrix inversion. Now, that really depends upon what kind of accuracy that you want to attain. The accuracy goal has to be stated clearly and once, you state your accuracy goal than we can decide suppose, we want either radiosity or temperature accurate to one percent then that will determine, how fine you want to divide tube into many, many parts.

This analytic solution will help you come up with the appropriate number of discrete elements, you need to have in that solution. We give the example although these examples are somewhat simple, but it illustrate the usefulness of a few analytical solutions, which will enable you to compare the standard numerical results with analytical solutions. With that we conclude for the time being our discussion on the enclosures. Now, we move on to a very important topic so, far we looked at radiative transfer and surfaces, radiative transfer between surfaces using the concept of shape factor and enclosures. We have given a fairly a good survey of what is going on this area, but the most important problem that we have to deal with now is gas radiation.

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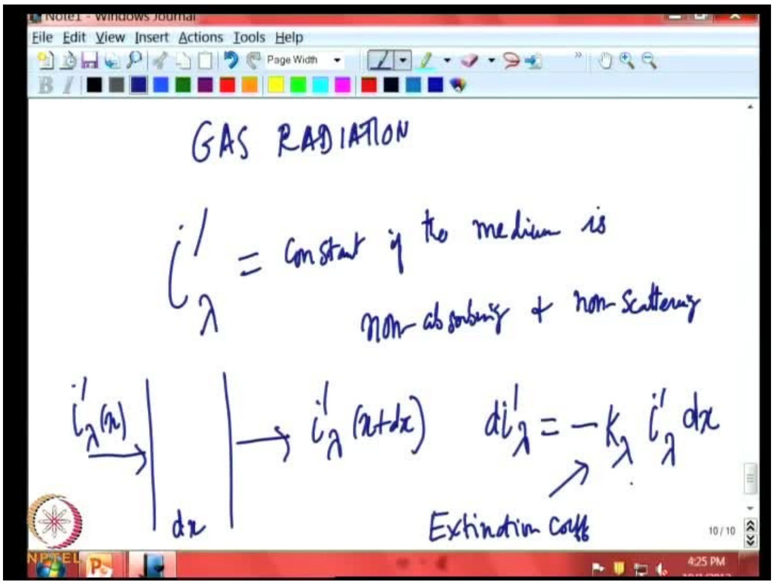

So, far we assumed that radiation leaving a surface, and when it reaches another surface in between there is no medium, which absorbs or scatters radiation. Now, we want to tackle the problem, where we want to look at an absorbing scattering medium between two surfaces. We want to know how it attenuates radiations, and how to calculate that attenuation. If we recall the basic concept of intensity in a given direction given wave length was so defined that when this radiation is going through a medium, where there is no absorption or scattering this remains constant. If the medium through which it is going is non absorbing and non scattering.

Now, we do ask our self what will happen to this quantity, if the medium is absorbing or scattering. If the medium is absorbing or scattering, immediately, you can see that this quantity cannot be constant because we think this is a stream of photon, some of the photons will be absorbed, some will be scattered out and some of the photon will be scattered in. We had account of all those features to do that we need some basic laws, and definition to proceed further.

So, let us think of a very simple example of medium of thickness d x. The radiation intensity arriving at d x is that and what is going out at x plus d x is something else. The two are not equal. The change in the intensity between these two is obviously d i prime lambda. Now, the question is what does it depend upon, what does the change in intensity depend on. One thing we can easily see is that the intensity decreases, this quantity has to be negative that is quite clear. Secondly, it adds proportional to the thickness of the element that is quite obvious, the thicker the element there is the more absorption and more scattering.

We exploit the change in this that has to be proportional to the thickness. It also proportional to the incoming intensity because this intensity is a measure of the photon flux. If there are more photons, there will be more decrease of the intensity. There are 100 photons here and ten photon can be absorbed as a it goes through the medium here. Now, there are 400 photons, we except 40 photons to absorb. That is expressed from the fact that d i prime lambda is proportional to i prime lambda and the thickness of the element.

There is something in the front of this which is an unknown, and which is what we call the extinction coefficient. This statement you are saying is that change in intensity in a given direction and wave length is a proportional to the incoming intensity, proportional to the thickness of the element, it is negative because decreasing and the proportionality constant is called the extinction coefficient.

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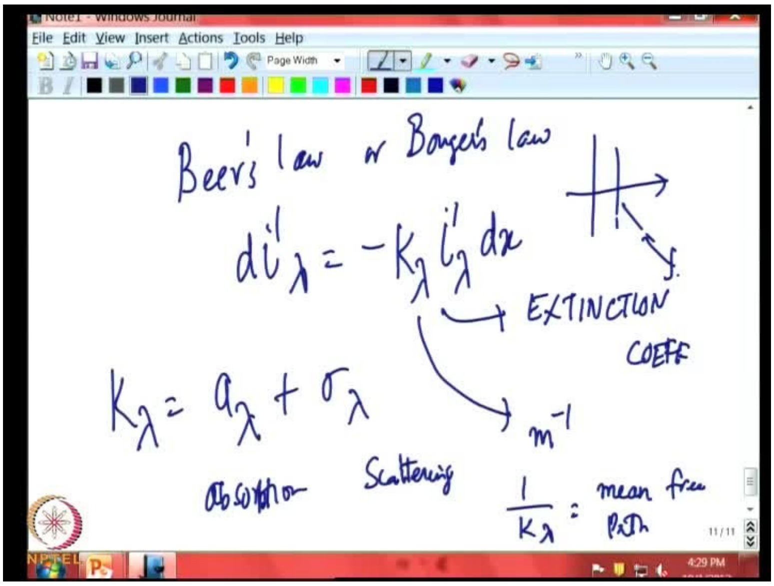

Now, this is something called as the Beer's law and Bouger's law that is the matter of which text book, you will look up. Now, this law was discovery by many people in different branches of science, this has been used by people dealing with illumination and optics. It is used by people who do astrophysics. It is also use by people how do radiation in furnaces. So, and of course, it is also used in nuclear radiation laws.

This law has been discovered in a wide variety of contexts and some of them call Beer's law or Bouger's law. The name is not important except that, this law follows logically. We can verify this law in the laboratory by doing experiment, but what we have stated so far is it follows essentially, from common sense because we are expecting the decrease intensity depending on thickness of the layer and the incoming radiation, and the proportionality constant k lambda is called the extinction coefficient.

This extinction coefficient has two parts, one is absorption coefficient and another is scattering coefficient. And note that as per this equation, this quantity has to have a units of eta minus 1 so that this product is non dimensional and this same as this. So, extinction constant is very important thing because 1 over extinction co-efficient can be thought of as mean free path on the photon, that is the mean distance that a photon travels before it is absorbed or scattered.

This quantity is very useful quantity because if k lambda is shown to be 1 meter minus 1 then we say that photons travel approximately 1 meter, before they are absorbed or scattered. That is our physical understanding of k lambda that 1 of k lambda is measure of the approximate distance on an average, but a photon travels before it is absorbed or scattered. It is sum of two quantities, the absorption co-efficient where in the photon is absorbed by the material and disappears from your calculation, or photon travelling in given direction, suddenly changes direction and goes some other direction.

As for as this direction concerned you have lost that photon. So, once the photon is scattered out is not following this part that we have initiated, this part is going some other part from the direction. As far as this direction is concerned, we have lost the photon because it is scattered out of this direction. Now, in this first part of the course we will focus primarily on the absorption co-efficient, and take up scattering much later in the course because it is much more complicated. So, absorption is our main focus.

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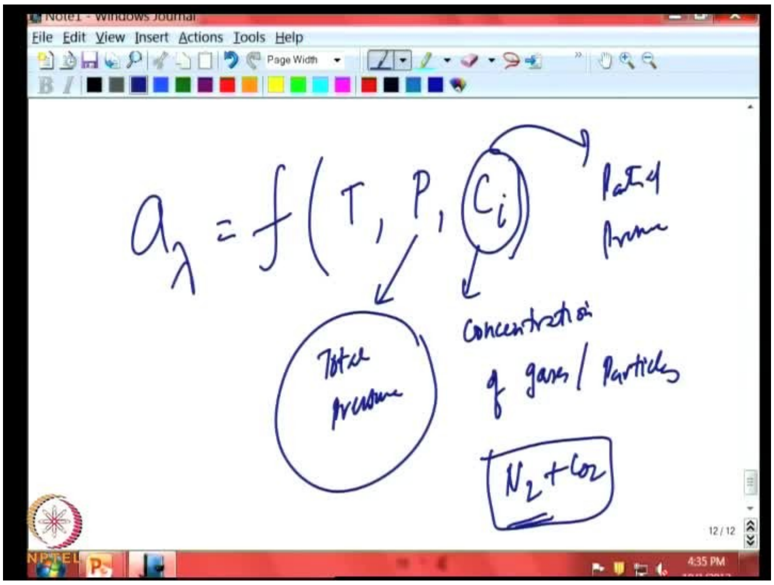

The absorption coefficient of a gas or medium can be a function of many things of course, function of a wave length. Function of the temperature of the medium obviously, depends on the total pressure, depends on the concentration of the individual gas. We can provide this function, how the absorption coefficient of gas or any other medium depends on temperature, total pressure and the concentration of gases or particle.

Once that is given to you, this function then of course will be quite straight forward, but we must mention that this is most full quantity that we deal in radiation because a lambda is an extremely, strong function of wave length. It varies very rapidly from let's say from one wavelength to other. It is a very extremely complex function, and hence a lot of time has been devoted by hundreds of people over a last period, to find out how to solve this problem, where in the basic function we have got a lambda is an extremely strong function wave length.

We got a very high value then within few 0.5 or 0.2 microns it will come down to almost transparent. Essentially a lambda fluctuates very rapidly and that as all of you must now realize, is not something computer likes. Computer likes function which are monatomic in nature, which very slowly and which can be easily represented by a simple function. But unfortunately radiation transfers in gases a lambda, as you will see later is extremely complex function of lambda. It is not easy to specify a function in the class.

We will spend some time later in this course asking, how this function varies with lambda. How we ultimately handle this complexity which is inherent in gas radiation heat transfer. Now, the only thing that is important about this result we have written is that it depends on total pressure, and this of course depends on the partial pressure of the gas. The absorption co-efficient of a gas depends not only on the temperature, but both on total pressure and partial pressure.

This is the dependence which confuses most students. For example, suppose you have mixture of nitrogen and carbon dioxide in a container. Nitrogen is transparent to almost all radiation, but carbon dioxide is strong absorption in the infrared. All though N 2 does not directly absorb any radiation because it is a, linear diatomic molecule, we have to figure out what role nitrogen is playing in this mixture.

Nitrogen itself will not be able absorb any radiation because it is a linear diatomic molecule, but it can assist carbon dioxide in trying to absorb some radiation. The way nitrogen indirectly affects the absorption of the infrared molecules in this situation, one need to understand that what nitrogen does when it collides with carbon dioxide and broadens, the absorption curve of that mixture.

The higher temperature and higher pressure is always preferred we want to calculate, the absorption coefficient of mixture of gases. Now, this concept will be dealt with great detail later in the course, when we look at how to get this function a lambda.

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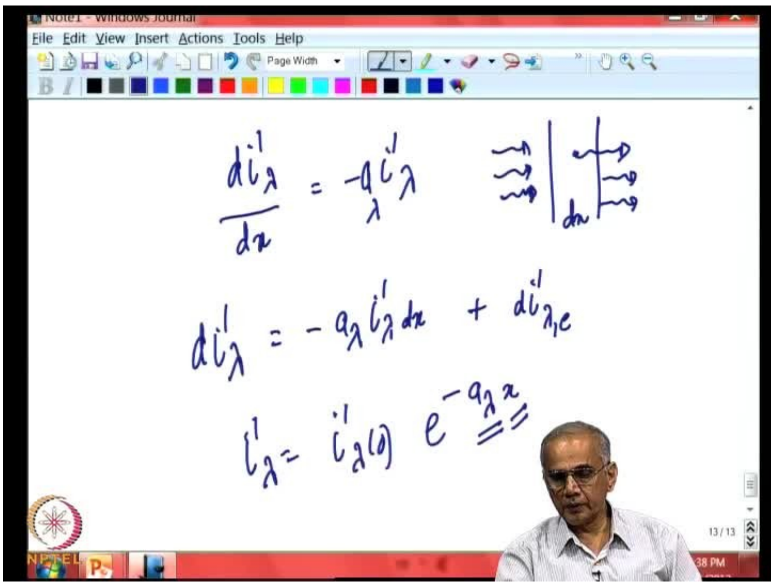

Right now, we will look at an absorbing medium and neglecting scattering function, and go back to original equation which we wrote down which is d i prime lambda d x is equal to minus i prime lambda. Now, this quantity deals only with absorption, and so we will write into a prime lambda no scattering is looked at right now. Now, all of you know by now that any medium liquid, solid or gas, if it absorbs radiation, it also emit radiation. We have to put an emission term here which we have so far not mentioned. The way to look at this the change in intensity while going to certain path is equal to loss of intensity due to absorption, and some gain in intensity, due to emission.

The real rate of change of intensity will depend both on the loss of photon due to absorption, as well as the gain that may occur due to emission by a photon. Essential by looking on this is there are photons coming here partly absorbed, and two of them come here, but can always by emission from inside which can add to that, that is a second term.

The second question is how to estimate the intensity emission by the gas. For that we have go and appeal to Kirchhoff's law, all of you know that if we neglect the term and solve for the i prime lambda, it will be solved i prime lambda of 0, e to the power of minus a lambda x assuming a lambda is a constant. In the absence of the emission intensity declines exponentially, and how rapidly this exponential decay occurs depends upon the absorption coefficient of gas, and the length traveled by the radiation peak. This is a very important result and is used in this field widely. But notice that we can also define absorptivity of this gas absorption coefficient which is a dimensional quantity.

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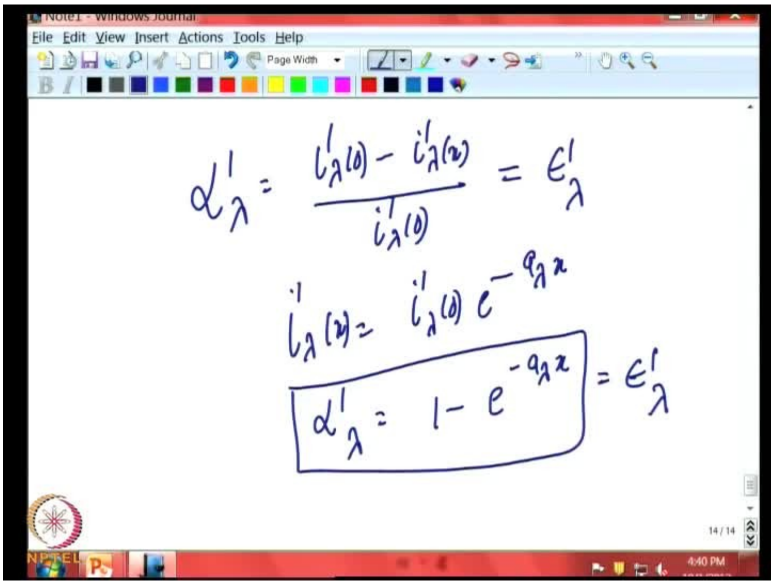

Now, we talk about absorptivity, which is fraction of radiation that is absorbed that is what was coming in and what went out divided by what is coming in. This is absorptivity. Now, if we remember Kirchhoff's law that directional spectral absorptivity has to be is equal to directional spectral emissivity. This is following from the basic statements Kirchhoff's law that states that the directional spectral absorptivity, and directional spectral emissivity are equal. If they are equal then you can write emission. We have 1 minus i prime lambda x minus i prime lambda 0.

This we already solved if you recall. We will get in the absence of emission this kind of simple expression for intensity. The final expression for absorptivity if we substitute it here, it is i prime lambda is 0 you can write then alpha prime lambda is equal to 1 minus a lambda x. This is important expression and so this we know is equal to emissivity. This is an important use of Kirchhoff's law for gases. We will continue this lecture next time where, we will use this Kirchhoff's law to estimate the emission intensity, which we did not include in the first derivation.