Networks and Systems

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Lecture-45
Sampling Theorem and Exercise on Fourier Transforms

As an illustration of the application of the Fourier transform method, let us consider 1 application which is referred to as the sampling theorem.

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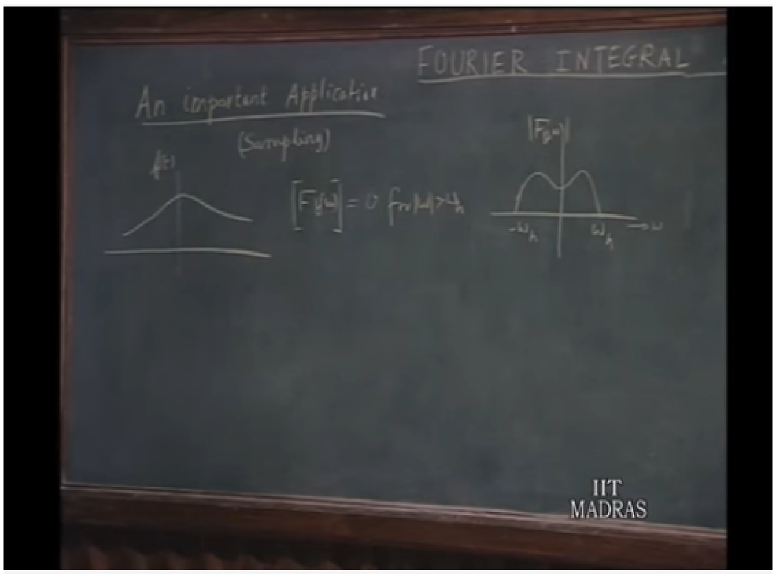

Okay, let us consider the function of time like this which is band limited in that the Fourier transform for that is confined to may be like this. So, the Fourier transform is 0 for values of omega beyond omega h or we can say that the highest frequency component that is present in f of t is omega h. No component of frequency beyond omega h and the positive side minus omega on the negative side exists in this particular f of t.

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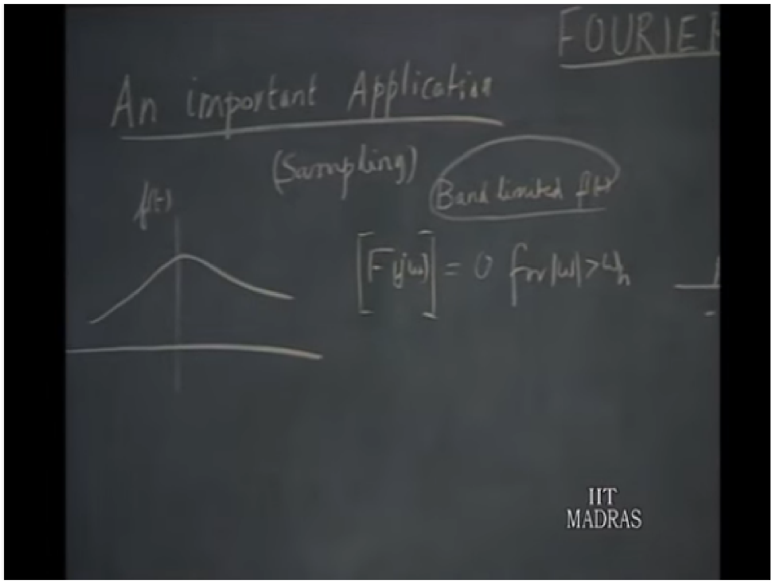

Such functions are called band limited functions band limited f of t that means, the frequency band of the component that exists in f of t are limited to this band nothing beyond that.

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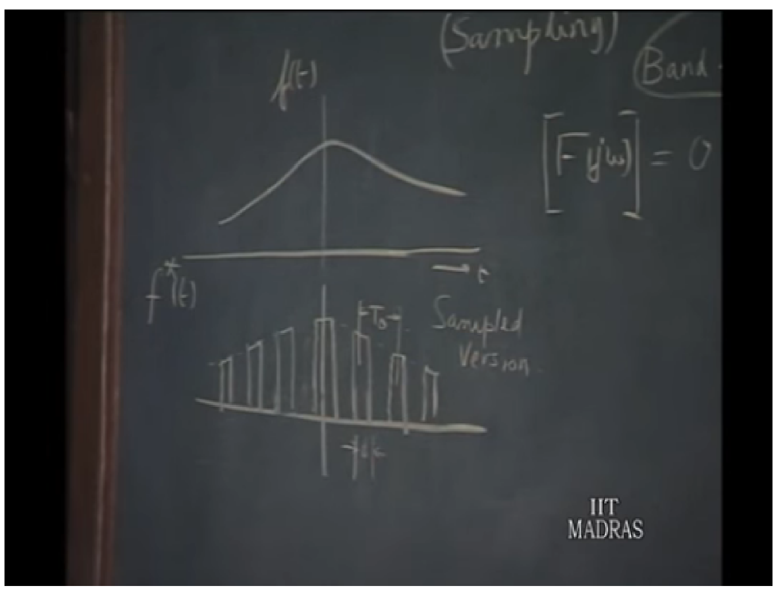

Suppose, we do not consider f of t for the entire time duration, but generate from f of t a sample version of that. That means, this is f of t we have variation; you do not consider the entire variation by take samples from that tiny interval like this. So, we consider f of t for small intervals of duration d and the period between 2 successive samples is t not. This we may call sampled version of f of t we will call that f star of t; this is sampled version.

So, notice f of t is the continuous function of time we are considering a sampled version, we are taking samples each time for duration d and taking the intervals between the successive sample is t not.

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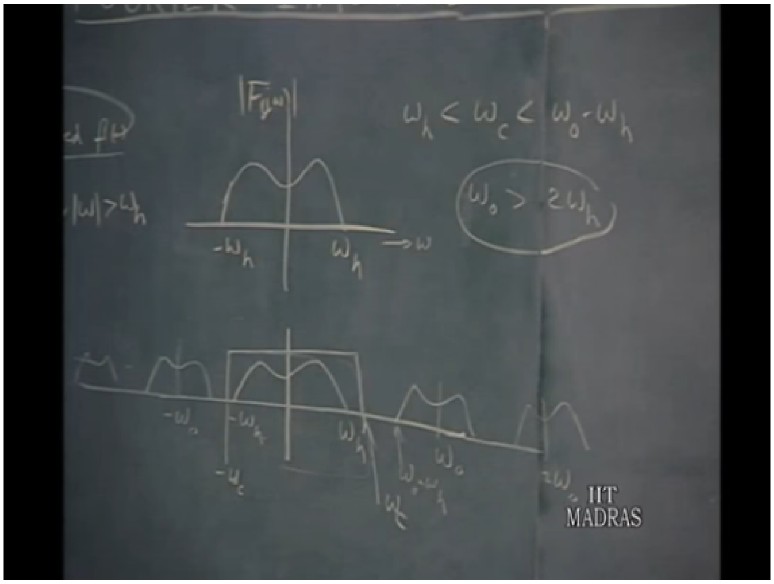

Now, it turns out that the Fourier spectrum of this sampled version will be something like this, if this is the original Fourier spectrum. The f star of t the fourier transform for that will be the same spectrum is repeated endlessly in both directions with a reduced scale factor. If this is omega h this is minus omega h and the sampling period is t not and let omega not be the corresponding angular frequency. Then the next section is centered around omega not, this is omega not and this is the other section is repeated at 2 omega not and so on.

So, endlessly it goes in both directions; that means, the sample version incorporates in it the Fourier transform of the original signal to a reduced scale. Besides, it contains several other reproduction of the same original spectrum at different scale. Now, imagine that the sample version is put through the low pass filter so that; we select only this portion of the spectrum. That means, we lift this spectrum out of assembly of the spectra section of the spectra and consider only this portion and avoid all others by putting this samples versions through a low pass filter.

Then the spectrum of this which is the resulting the output of the low pass filter is the same thing as the spectrum except for a scale factor. In other words the low pass filter output would be a replica of f of t reduced by a certain scale factor. So, it tells us that, if we have band limited function f of t if you sample it in this manner. This sample, from this sample you can extract the original function f of t. Even though we have missed out on the variation in these intervals in between, does not really matter as longs as this is band limited function from this sampled version you can recover the original signal.

And this is very important in communications particularly because, if you take these samples wide apart. Then in the blank periods you can send the other messages in the same communication channel. So, communication can be employed to send several messages in succession. So, you sample 1 particular message and then send it at regular intervals. And in t he blank periods you can send another messages, a second message, a third message so on and so forth.

This interleaving of these messages over the same communication channel is referred to as the time division time multiplexicy. And this is very common occurrence to affect utilize the communication channel to send several messages which are inherently band in character. Now, you can see in order for us to able to lift this section of the spectrum and remove the others you must have the low pass filter characteristic something like this. So, if the cutoff of the low pass filter is omega c.

Then omega c must be larger than omega h, but at the same time you do not want to pick up any portion of this spectrum. So, if this is omega not you have omega this must be omega not minus omega h. So, you must be able to choose a omega c which lies in between omega h and omega not minus omega h. Which in other words means omega is the cut off frequency must be less than omega not omega h and must be greater than omega h. And this is possible only if omega not is greater than 2 omega h.

So, only if omega not is greater than 2 omega h you will be able to switch in on a suitable cut off frequency of the low pass filter. Then omega not minus omega h must be larger than omega h. In other words what it says is your sampling frequency omega not, must be more than twice the highest frequency component that is present in your original signal; omega not must be larger than omega h. If it so, then you can choose a low pass filter and allow this sample version to go through it and out comes a version a continuous function of time which is a faithful replica of your original f of t.

I have not gone to the mathematics; just i am trying to explain how it works out to complete the mathematics.

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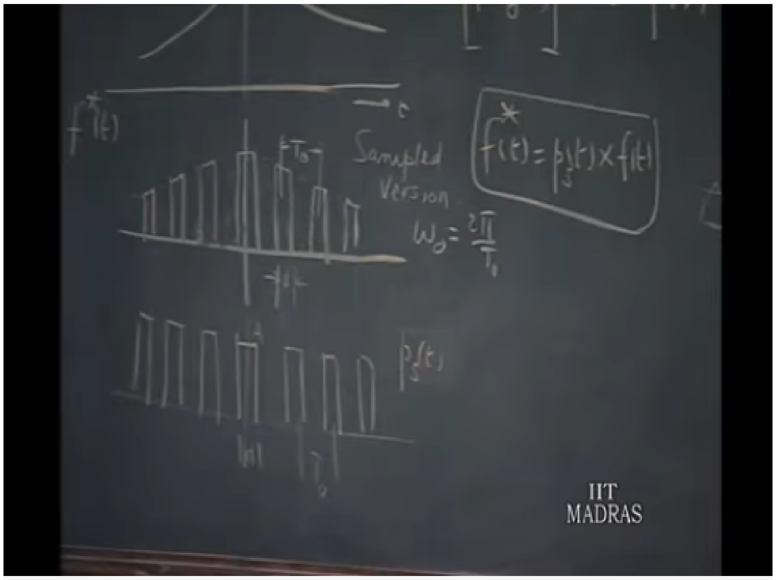

Let us see, what it means is to get the sample version from f of t all you have to do is multiply f of t by a pulse strain. So, if you have a pulse strain of duration d we will call this p s of t. Let us say for the sake of generality an amplitude a duration d and the period p not. So, if you have such a pulse strain you multiply f of t by this pulse strain you get f star of t. So, f star of t is p s of t the pulse strain times f of t. Then we can find out the Fourier spectrum of the f star of t we will go to the mathematics now and show that this is indeed the type of spectrum that you get for f star of t.

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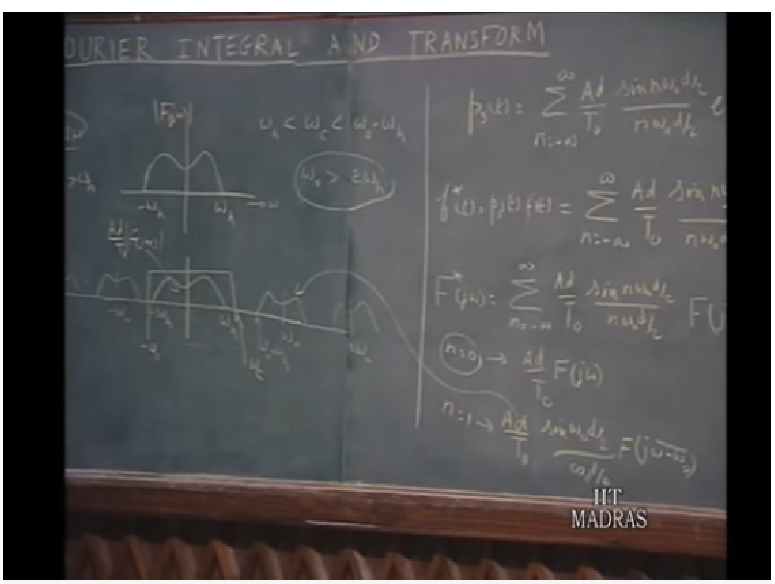

Let us see. P s of t is the periodic pulse strain therefore, it can be extended by means of Fourier series and we have done this type of expansions several times in the past. So, let me straight away put down Ad by t not sin n omega not d by 2 divided by omega not d divided by 2 is the cn coefficient multiplied by e to the power of jn omega not t. That is the time variation n ranging from minus infinity to plus infinity is your Fourier series expansion for p s of t the periodic pulse strain. Now, f star of t is p s of t multiplied by f of t. So, which means that it can be written in this fashion as the sum of several terms sin n omega d by 2 by n o mega not d up on 2 e to the power of jn omega not t times f of t.

So, f star of t can be thought of as a infinite sum of several times and this is the portion which depends on the time the rest of it is purely a coefficient. Now, to find out the Fourier transform of this we can find out the Fourier transform of each individual term and them up. And since, this is a purely a constant as far as time is concerned. So, we can write this as n from minus infinity to plus infinity of ad up on t sin n omega not d up on 2 by n omega not d up on 2. If the Fourier transform of f t is is f of j omega we have seen that the Fourier transform of f of t multiplied e to the power of jn omega not t is simply f of j omega minus n omega not.

That is multiplication by exponential in the time domain; is equivalent to translation in the frequency domain this is something which we have done. Now, let us see what we have got f star of j omega the fourier spectrum of the sample version consists of several terms like this: f of j omega minus not when n equal to 0 it is f of j omega; when n equals to 1 if f of j omega minus omega not. And for each value of n you have a particular coefficient.

So, if you take n equal to 0 the portion of this corresponding to n equal to 0 will be ad up on t n equal to 0 this is equal to 1 sin theta by theta when, theta equal to 0 is 1 f of j omega. If n equals 1, this becomes ad up on d this is t not all the ways this is t not ad up on t not sin omega not d up on 2 by omega not d up on 2 f of j omega minus omega not. So, this portion which corresponds to n equals to 0 is ad up on t not times f of j omega.

And this is something else ad up on t not times this quantity corresponds to this. So, we have seen now corresponding the central portion of the spectrum is ad up on t not times f of j omega that is it is reduced version of this.

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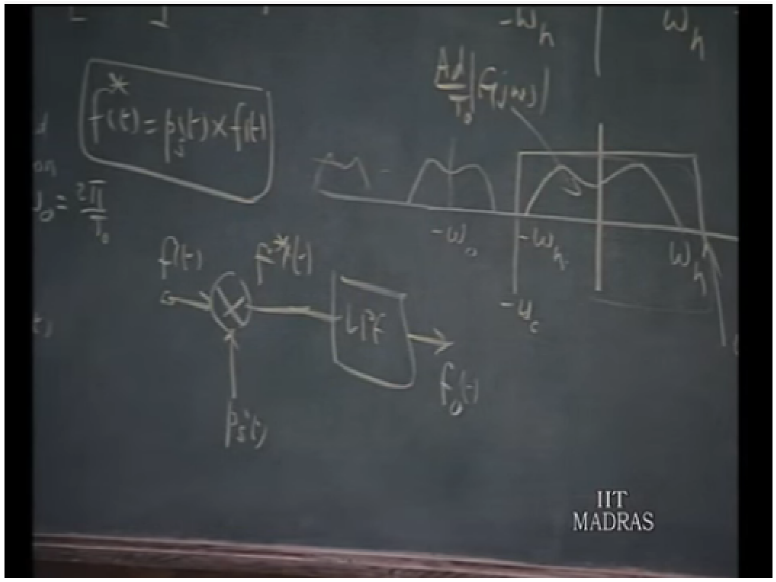

So, the implementation of this is straight forward you have f of t you can have multiplier p s of t. And this is your f star of t you put this a low pass filter and you get f not which is a replica of your original f of t. This is the general what is called the sampling theorem which referred to a literature of sampling theorem. That is, band limited signal can be recovered from it is sampled version, when the samples are taken at regular intervals with a frequency f not which should be larger than, 2 times the highest frequency component present here.

In practice, because of the limitations of the low pass filter that we have the design of the low pass filters. In practice, you may need to have a sampling rate employ sampling rate which is higher than 2 omega h. Even though in theory any sampling rate greater than 2 omega h will suffice. But, in practice we have to take 3 or 4 times the highest frequency component because the limitation of your design of the low pass filters. To conclude our discussion of the Fourier transform methods let me offer a few remarks at this stage.

Ideally, the Fourier transform is obtained by integrating the f of t over a infinite interval of time from minus infinity to plus infinity. In practice, when you have a signal, it is normally of finite duration and therefore, the integration is to be carried over for the duration of the signal. And this poses no problem. But, on the other hand if you a type of signal which is continuous infinitely long for infinitely long signal and you like to find out its Fourier transform in practice. You may not be able to have a analytical description of the signal.

Then such cases what we can do is we can statistically process this information that is, if you take a finite length of the signal like for example, a peak signal or so. Take a finite length and find out it frequency characteristic and where the energy rise on which portion of the spectrum and so on. And it turns out that the statistical processing of the signal quite useful 1; that means, even if you take other samples of the signals the essential character of the spectrum does not change.

Therefore, this way it is 1 can handle signal even though they are need not be known for an infinitely length of time. A second point which i would like to emphasize once again is, that the Fourier transform method enables us to find out the transient behavior of network and system, essentially, using steady state method which we have emphasized once again earlier when we are working out the problems.

That is what we are trying to do is we are considering the transient excitation as the sum of infinite number of tiny sinusoids and we find the response to each sinusoid using essentially steady state methods frequency response methods. Then add all the responses together to find out the total response. When i say summation it is actually the integration that is being done because, these are all a continuous band of frequency that you are having.

But, the principle is the same summation is essentially a integration; integration is essentially the summation. Now, it is this fact that transient behavior of networks and systems can be analyzed on the basis of steady state sinusoidal theory. That places the Fourier transform method and Fourier theory and important gives it important place in communication engineering. In communication engineering essentially, we have to do deal with transient signals and transient processing of transient signals.

But, the communication network can be characterized in terms of their steady state behavior under sinusoid excitation condition. And using that information we are able to process the behavior of the network under transient signals. And some people assert that this is 1 of the most important idea in communication engineering is the way, in which we tackle this transient signals by the Fourier transform method. Now, we also seen that the Fourier transform methods we have taken networks and found out the transient in the network using the Fourier transform methods.

And we mentioned that we have certain restrictions earlier; not only, on the type of excitation signals. Which can be handled by the Fourier transform method and we also said we have placed some restriction on the network also the natural response of the network should die down with time for the Fourier transform method to work in a convenient fashion. Now, there are several such restrictions when we apply the Fourier transform of method to the transient solution network and systems.

A competitor to this is the Laplace method which we will take up later. For the Laplace transform method offers greater conveniences for the analytical solutions of the transient in network and system. Mainly because, it enlarge the class of time functions it can be handled; that means, certain types of functions like e to the power of 2 t. For example, cannot be handled by the Fourier transform method, but Laplace transform method can handle this.

Secondly, the several delta functions which may get in the Fourier transforms are avoided when use the Laplace transform method because of these reasons Laplace transform is a more convenient tool for analytical evaluation of transient and linear systems. The same time where the networks are characterized in terms of their frequency behavior; in terms of their frequency response conditions. Like communication network, like filters for example, the Fourier transform method is very convenient tool.

Because, these frequency response characterize of systems and networks can experimentally determine h of j omega can be experimentally determine. H of s may not be so easy experimentally determine. So, where the networks are characterized by the frequency response behavior then Fourier transform method trans out to be very convenient tool to employ. For example, the behavior of the low pass filter that we have discussed little earlier. Also, in terms in certain branches like a control engineering and so on it is the frequency response of a system that we are interested in.

To assess the stability of the control system sometime we like to know the frequency response characteristic of this. So, in such cases of course, the Fourier transform method and handling of the various components to the frequency response approach turns out to be very convenient. They are also well defined methods where once you know the frequency response you can calculate the transient response graphically. So, because of all these reasons Fourier transform method plays the vital role in analysis of network and systems.

But as i mentioned for the transient analysis of network perhaps the Laplace transform methods is the more useful tool and that will take up later. But nevertheless as i mentioned wherever the networks or systems are characterized by the frequency response Fourier transform method turns out to be more convenient tool. And this stage i will give you set of problems as an exercise for you to work out the Fourier analysis transform methods. Now, let me write down a set of problems for you as an exercise under the topic Fourier transform and the Fourier integral.

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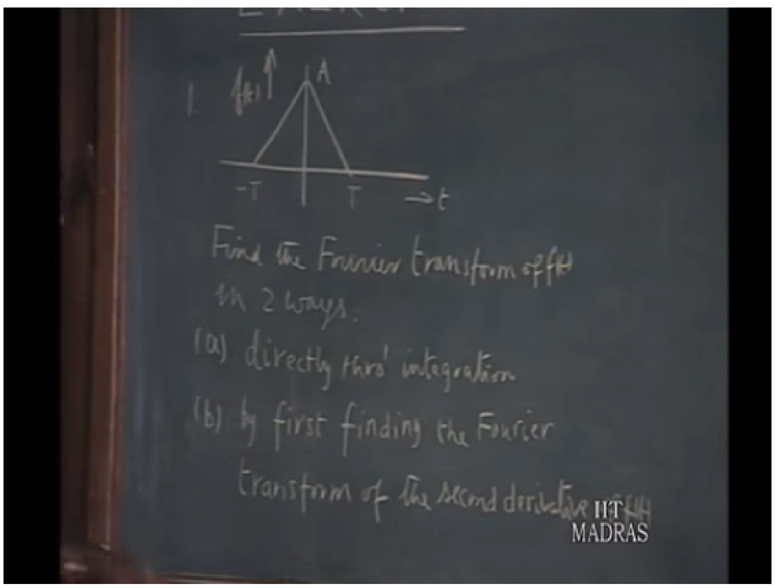

Let us consider, f of t which has got this type of variation t minus t this is a this is f of t. Find the Fourier transform of f of t in 2 ways; a: directly through integration applying the standard formula for the Fourier transform. That is, minus infinity to plus infinity f of t e to the power of minus j omega t dt that type of integration.

b: By first finding the Fourier transform of the second derivative of f of t. The motivation for this is if you take the derivative of this you get a pulse here and another pulse here and if you take the second derivative you have impulses 3 impulses. To find out the Fourier transform of the impulses quite easy. So, you find the Fourier transform of the second derivative of f of t and from that deduce the Fourier transform of the original function f of t.

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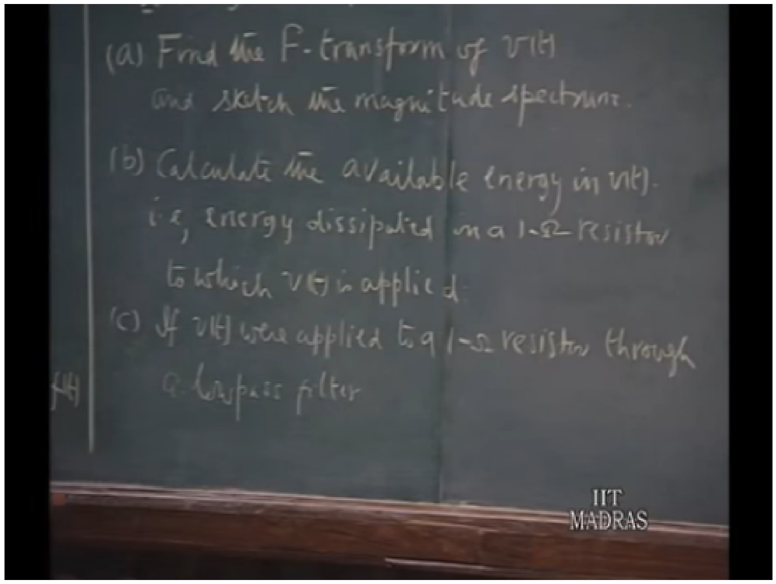

Problem number 2: consider the voltage signal e to the power of minus a modulus t a is being a real positive constant; a: find the Fourier transform of v of t this is voltage signal. Let me say, this is v of t find the Fourier transform of v of t and sketch the magnitude spectrum. b: calculate the available energy of v of t the signal v of t. What you mean by the available energy?

If this v of t was applied to a 1 ohm resister what is the energy dissipated in the resister v of t in the resister r. That is what is meant by available energy v of t that is, energy dissipated in 1 ohm resister to which v of t is applied.

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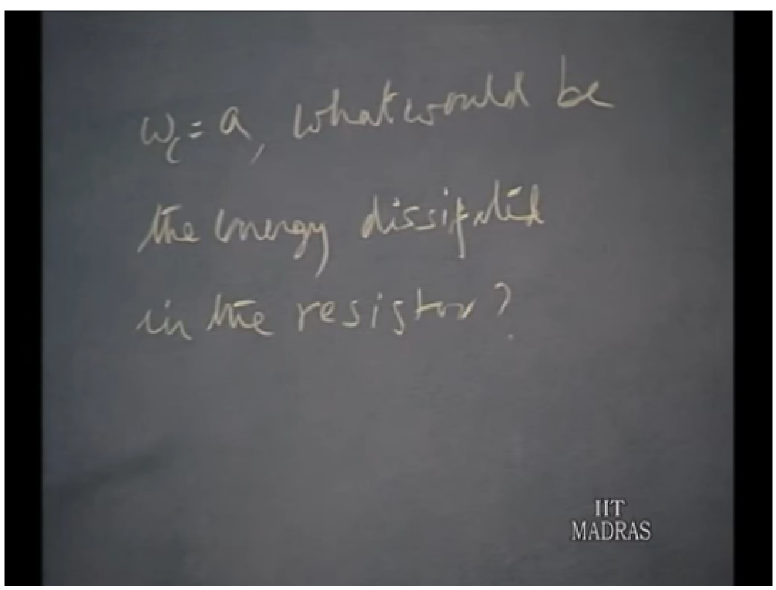

c: if v of t were applied to a 1 ohm resister through a low pass filter, with cut off frequency omega c is being equal to a. What would be the energy dissipated in the resister? That is you have a low pass filter which cuts off all frequencies beyond omega c omega greater than a then, what is the energy dissipated in the resister.

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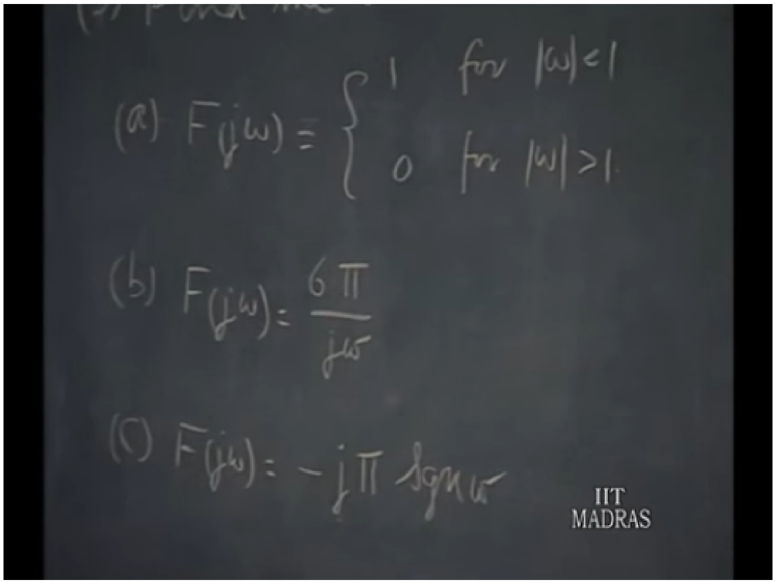

Find the inverse Fourier transforms of the following functions; a: f of j omega equals 1 for omega magnitude less than 1 0 magnitude greater than 1, b: f of j omega equals 6 pi up on j omega, c: f of j omega equals minus j pi signum function of omega.

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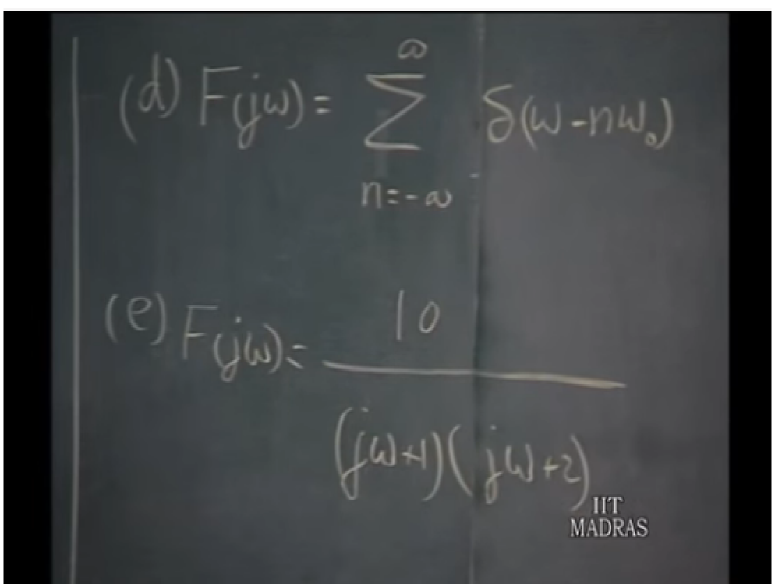

d: f of j omega equals a set of impulse functions periodically occurring intervals of omega not, e: f of j omega equals ten up on j omega plus 1 times j omega plus 2. That is the third problem.

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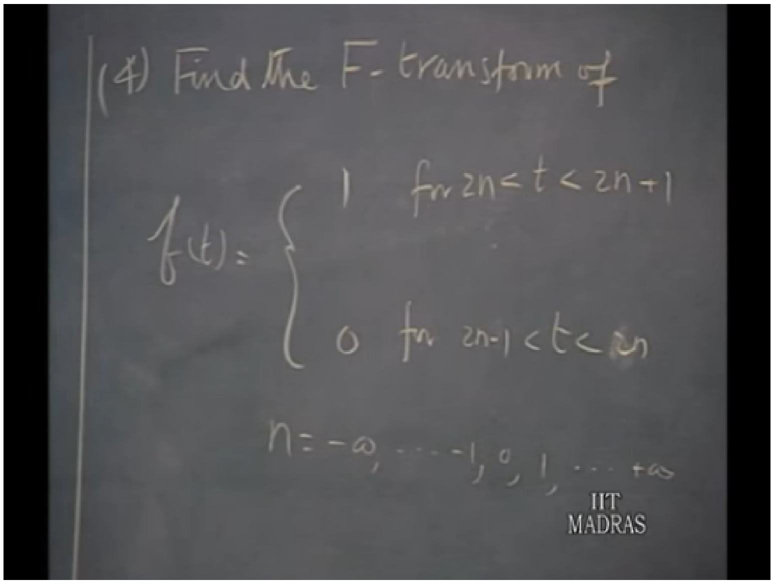

Fourth problem find the Fourier transform of a function f of t defined as 1 for t between 2 n and 2 n plus 1 and 0 for t between 2 n minus 1 and 2 n. Where, n ranges from minus infinity minus 1 0 1 etcetera to plus infinity for all integrals values of n positive and negative. That is the fourth problem.

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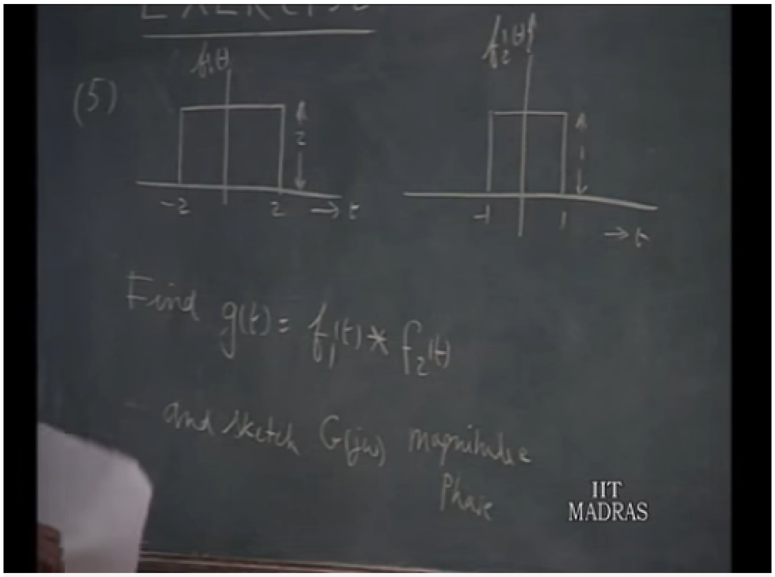

Let us consider 2 functions f 1 t a pulse of amplitude 2 lasting from minus 2 seconds to plus 2 seconds, and another f 1 t f 2 t lasting from minus 1 to 1 second of height 1 unit. Find g of t which is obtained by the convolution of f 1 t and f 2 t and the sketch g of j omega which is the Fourier transform of g of t sketch g of t; that means, magnitude and phase.

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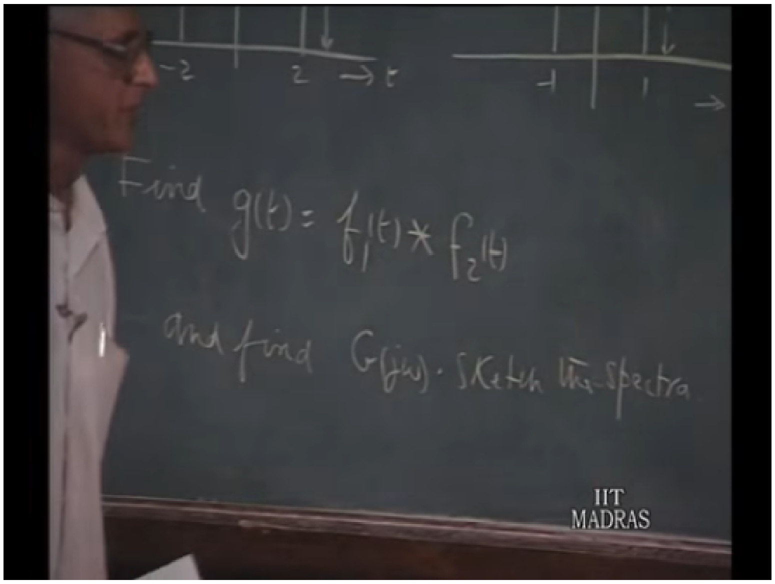

We will change this g of j omega and sketch the spectra. Find g of j omega the spectra of g of omega.

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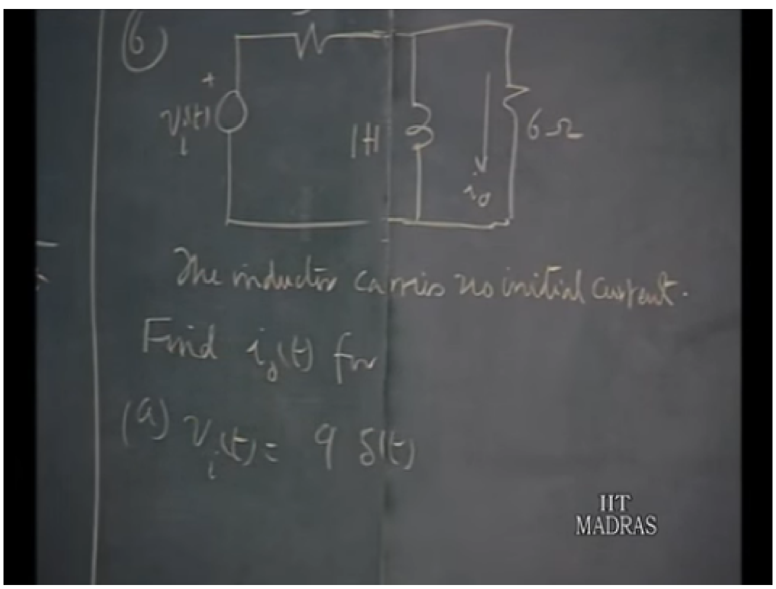

Sixth we have circuit in which there is a voltage source v i of t impressed at cross a circuit like this. This 1 henry inductor 6 ohm resister and this current is i not. The inductor carries 0 initial current no initial current that is before the excitation is applied the inductor is carrying no current.

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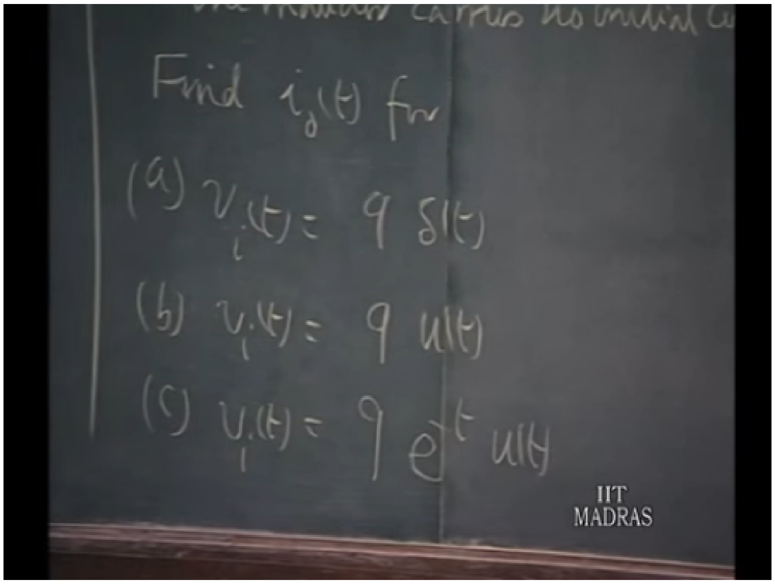

We are asked to find the current i not of t for the following excitation v i of t equals 9 delta t, b: v i of t equals 9 u t, c: v i of t equals 9 e to the power of minus t ut. So, this is the 6 problem this is the transient analysis of this problem.

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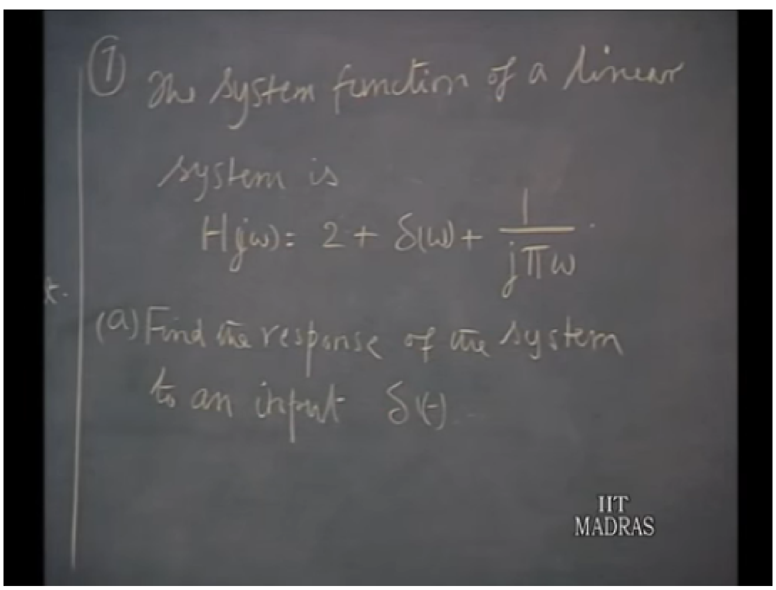

7 the system function of a linear system is h of j omega equals 2 plus delta plus 1 over j pi omega. Find the response of the system to an input delta t that is impulse response of the system is asked to found out,

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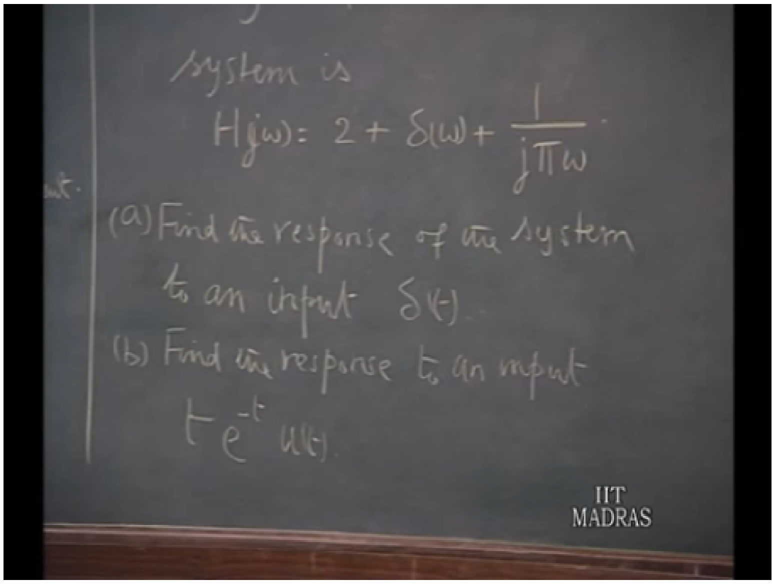

b: find the response to a input which is equal to t multiplied by e to the power of minus t ut. So, in both these cases use make use the system function find out the input Fourier transform multiply them out you get the output Fourier transform. Find the inverse Fourier transform you get the response we have not found out we have not discuss the Fourier transform of functions this type.

But, we should able to derive them using the from fundamentals t e to the power of minus t u t you have a Fourier transform which is, 1 over j omega plus 1 whole squared that we should able to derive.