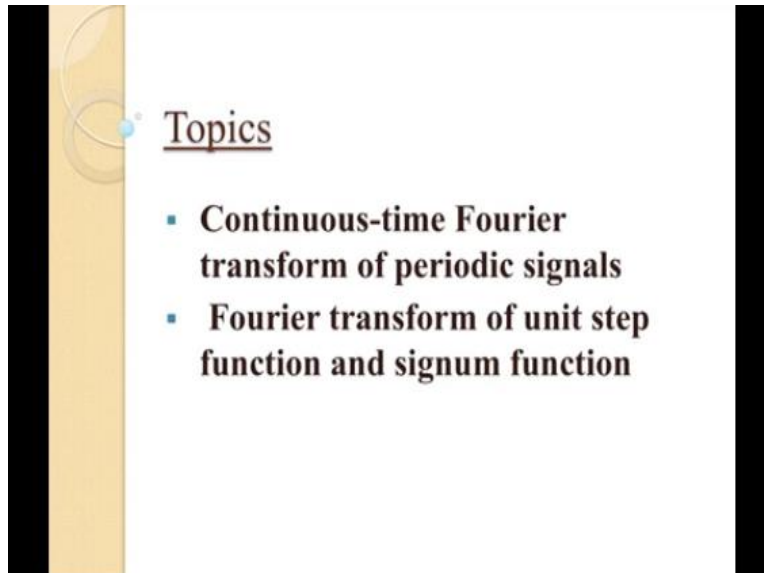


**Networks and Systems**  
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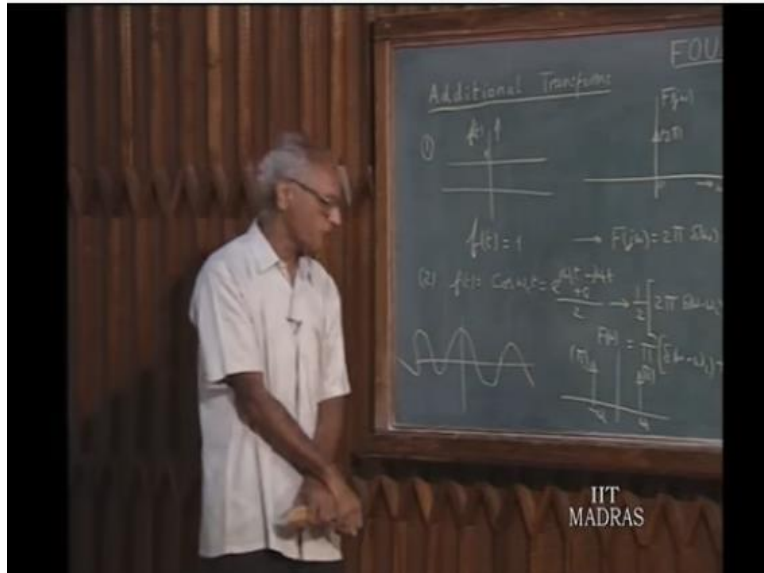
**Lecture-40**

**Fourier Transform of Periodic Signals, Unit Step and Signum Function**  
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In the last lecture, we were considering the Fourier Transforms of functions of time  $f(t)$ , which are not absolutely integrable over the infinite range. To do this we lay down certain formulas, which we can assume. These formulas can be justified on the base of distribution theory. And we will just assume these formulas in deriving these Fourier Transforms of the special functions. Special in the sense; that  $f(t)$  is not absolutely integrable over the infinite range.

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To start with, we took the case of a pure dc. So, if  $f(t)$  equals 1 then, we said the Fourier Transform for this  $F$  of  $j$  omega is  $2\pi$  delta omega. The spectrum: Fourier Spectrum then, will comprise just a single impulse at the origin on the omega axis. Since, this time function now is pure dc we cannot expect to have frequencies at any value other than, omega equal to 0 and get that point you have a definite non 0 coefficients, in the Fourier Series expansion for  $f(t)$  which is equal to 1.

And consequently you have a density Fourier coefficient density, which is infinitely large and since the Fourier coefficient density is defined in terms of the coefficient per cycles per second. Since, we are plotting on the basis of omega, it turns out to be  $2\pi$  delta omega, you concentration is  $2\pi$  times delta omega.

Now, let us extend this to a sinusoid. Suppose, I have  $f(t)$  as  $\cos \omega_c t$ , a sinusoid of frequency is angular frequency omega c. This of course, can be written as  $e$  to the power of  $j$  omega c t plus  $e$  to the power of minus  $j$  omega c t divided by 2. So, in other words  $f(t)$  consists of 2 exponential functions; half multiplied by  $e$  to the power of  $j$  omega c t and another half multiplied by  $e$  to the power of minus  $j$  omega c t.

Now, the difference between half  $e$  to the power of  $j$  omega c t and  $f(t)$  equals to 1 is that, it is no doubt scale down by the factor 2. But, the time function is multiplied by the

exponent  $e$  to the power of  $\text{minus } j \omega c t$ . We had earlier discussed the property of the Fourier Transform.

If  $f(t)$  and  $F$  of  $j \omega$  form a transform pair, if  $f(t)$  is multiplied  $e$  to the power of  $j \omega c t$  it means: in the transform domain the frequency spectrum is shifted.  $e$  to the power of  $j \omega c t$  times  $f(t)$  has a Fourier Transform which is  $F$  of  $j \omega$  minus  $\omega c$ ; a property which we have studied earlier.

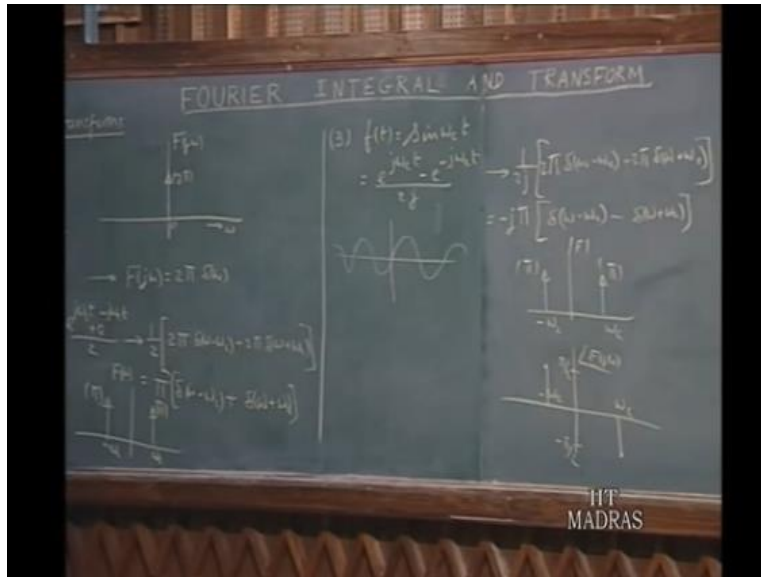
So, we make use the property and therefore, the Fourier Transform for this will turn out to be, first of all scale factor half is there. So, if  $f(t)$  has the transform  $1$ ,  $e$  to the power of  $j \omega c t$  has the transform  $2 \pi \delta(\omega - \omega c)$ . Likewise for the other term we have  $2 \pi \delta(\omega + \omega c)$  which of course, is  $\pi$  times  $\delta(\omega - \omega c)$  plus  $\delta(\omega + \omega c)$ .

So, what do we have? A pure sinusoid here a cosine function goes like this, this is  $f(t)$ . As far the transform is concerned you have  $F$  of  $j \omega$ . We have  $1$  impulse here, another impulse here at  $\omega c$  of the positive side and  $\text{minus } \omega c$ , on the negative side each of strength  $\pi$ .

So, what we have observe here is, once again this is a pure frequency terms that is, only  $1$  single frequency is present, the pure sinusoid  $\cos(\omega c t)$ . And therefore, the energy is concentrated only at  $2$  discrete frequencies as far as in the exponential presentation.  $e$  to the power of  $j \omega c t$  and  $e$  to the power of  $\text{minus } j \omega c t$ .

There cannot be energy at any other frequency. And these  $2$  terms together constitute  $1$  sinusoid frequency  $\omega c$ . So, this is a quite revealing that, even a pure sinusoid or a pure dc can be represented by means of Fourier integrals in this, Fourier transform in this manner

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Let us extend this idea further. It can be shown that, if you have  $\sin \omega_c t$  which of course, can be written as  $e^{j\omega_c t} - e^{-j\omega_c t}$  by  $2j$ . Obviously, the Fourier Transform by the same argument as before will be;  $\pi \times 2\pi \times \delta(\omega - \omega_c) - \pi \times 2\pi \times \delta(\omega + \omega_c)$  corresponding to second term, you have  $2\pi \times \delta(\omega + \omega_c)$ .

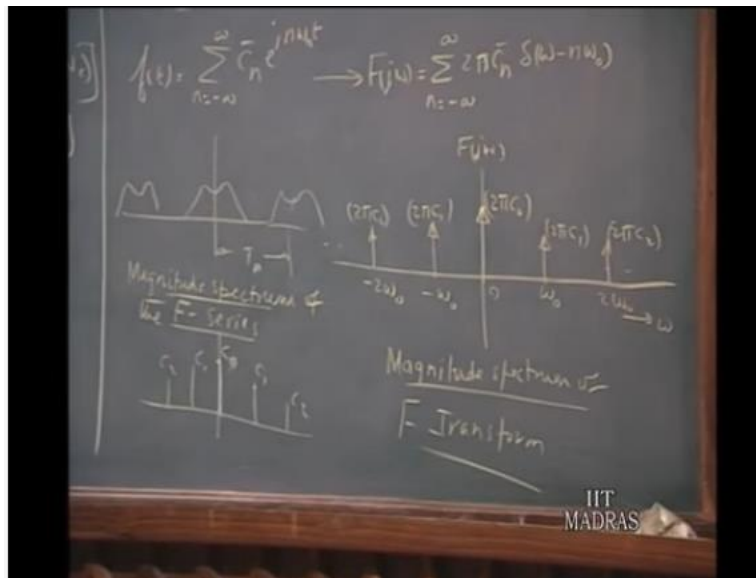
So, this will be  $\pi$  times  $j$ ,  $\pi$  times  $j$  can take out as a common factor  $\delta(\omega - \omega_c)$  and  $\pi$ . So, it is pure sinusoid therefore, we have sine wave extending from minus infinity to plus infinity and for the Fourier spectrum you have magnitude spectrum  $f$  magnitude, we have as before 2 impulses at  $\omega_c$  and  $-\omega_c$ .

Each of strength  $\pi$  and the angle of  $F$  of  $j\omega_c$  will be for the frequency  $\omega_c$  and  $-\omega_c$ , these are the components are present only at these 2 frequencies when the impulse at  $\omega_c$  is  $\pi$ . Therefore it is  $90^\circ$  just plus  $90^\circ$ . Therefore, this is  $\pi$  up on 2 and this is  $\pi$  down on 2.

So, this is the angle spectrum phase spectrum and this is the magnitude spectrum. Now, why do we stop at this point, when after all now you are able to find out the Fourier

Transform of d c a pure sinusoid? I can as well go to the extreme case, of a pure of non sinusoid periodic functions. And indeed it can be shown that, the non sinusoidal periodic function which admits the Fourier Series can also, have a corresponding Fourier Transform. Let us see how it goes.

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A general periodic function: Suppose,  $f(t)$  is a periodic function which admits the Fourier Series representation in this manner;  $n$  from minus infinity to plus infinity. So, this periodic function is something which repeats itself at regular intervals, with a time period  $T_n$  and  $\omega_0$  be is  $2\pi$  over  $T_n$ .

So, that is the fundamental frequency. Now, we have seen that a constant will have a Fourier Transform which is  $2\pi$  times delta  $\omega$  times constants. This is 1 therefore; this is  $2\pi$  times delta  $\omega$ . Now, each of 1 these terms in the Fourier Series expansion will therefore, have a corresponding Fourier Transform.

And the summation of all those transforms of individual components will, be the Fourier Transform of the entire function. So, it is from the discussion, it is quite obvious  $F(j\omega)$  will be equal to. For this term  $C_n$  of course, you have  $2\pi$  times  $C_n$ . If, it is  $C_n$  alone, it would have been  $2\pi$  times  $C_n$  times delta  $\omega$ , but you have  $e^{jn\omega_0 t}$ . Therefore, this becomes delta  $\omega - n\omega_0$ .

And you have all such terms starting from,  $n$  from minus infinity to plus infinity. So, that is its Fourier Transform. So, a periodic function of time, which has the Fourier Series, also has the corresponding Fourier Transform. Now, the Fourier Transform of this would have a magnitude spectrum like this;  $F$  of  $j\omega$ .

Let us see, a dc it will have  $2\pi$  times  $C_0$ , at  $\omega = 0$  it has  $2\pi$  times  $C_1$  and so on and so forth. At  $-\omega$ , it will have  $2\pi$  times magnitude spectrum is even spectrum  $2\pi C_1$  and you may have a  $2\pi C_2$ . These are the strength of the impulses etcetera.

Now, if you plot the Fourier spectrum of the Fourier Series in the convention of sense, what you would have done is the spectrum of the Fourier Series. What we talked about the spectrum when we were discussing the Fourier Series, we would have plotted a line spectrum like this  $C_1, C_0, C_1, C_2, C_1, C_2$  etc.

This is the magnitude spectrum of the Fourier Series, magnitude spectrum of the Fourier Series. Now, what we have in the case of Fourier Transform is; instead of this being the coefficients we have impulses. So, instead of finite constants here, we have impulses. The strength of each impulse is  $2\pi$  times the corresponding coefficients here.

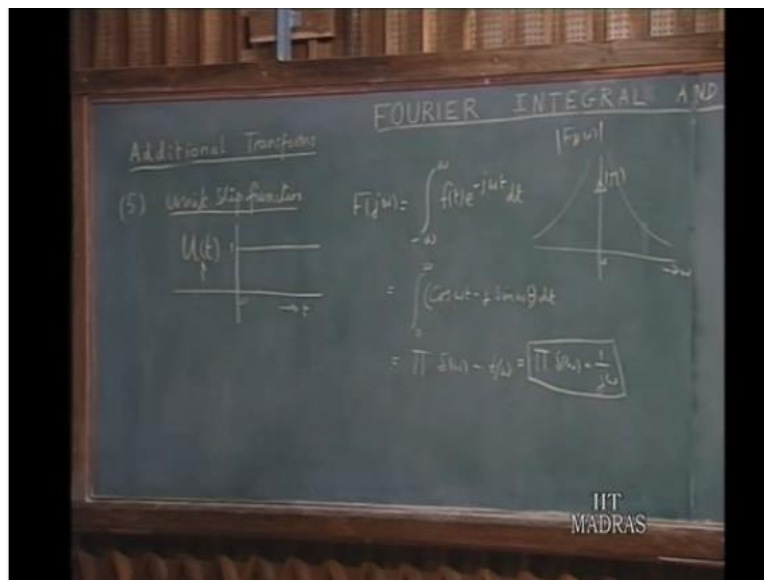
Therefore, this is the Fourier magnitude spectrum of the Fourier integral, in the sense; this is the magnitude spectrum of the Fourier Transform. So, we have now seen that a Fourier integral gives more generalized concepts of, finding out the various harmonic components of a time functional function of time.

It can be used not only for a periodic functions, but a periodic functions as well. Because, once we have the Fourier Series for a periodic functions in the Fourier Transform domain all it means the same, line spectrum is carried out carried over into the transform domain as well. But, instead of coefficient finite values of coefficients, we have impulses each of which has strength which is proportional to the Fourier coefficients.

Since, this particular wave form has frequency component adjusting at  $0$   $\omega$   $0$   $2$   $\omega$   $0$  and nowhere else, here also the coefficient density, we have concentration of energy at these particular frequencies and nowhere else. So, it is not a continuous spectrum because,  $f(t)$  is periodic not only here, here here. And even in the case of dc can be considered to be a periodic function.

So, the conclusion that we get is when we have a periodic function of time then, the spectrum will turn out to be a line spectrum. Let us now continue with our discussion and derive some additional Fourier Transforms of functions of time, which are not once again absolutely integrable over the infinite range. But at the same time they are not periodic as well, like the cases we just now considered.

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To start with, let us take the unit step function  $u$  of  $t$ . This of course, has a value 1 for positive  $t$  and 0 for negative  $t$   $u$  of  $t$  is the unit step function in the sense. Now, we like to find out the Fourier Transform for this. So,  $F$  of  $j$   $\omega$   $1$  over  $2$   $\pi$  integral from minus infinity to plus infinity of  $f(t)$   $e$  to the power of minus  $j$   $\omega$   $t$ .

And since,  $f(t)$  is 0 for negative values of time, you can start integration from 0 to infinity of  $f(t)$  is 1 in that range. So, you have  $\cos \omega t$  minus  $j$   $\sin \omega t$  minus  $j$   $\sin \omega$

$t dt$ . And from the formulas which we had developed earlier, this is seen to be  $\pi \delta(\omega)$ , the formula which we noted down in the last lecture.

And this will be  $\frac{1}{j\omega}$  plus  $\pi \delta(\omega)$ . This is the Fourier Transform of unit step function. So, if you plot this spectrum, there will be an impulse of magnitude  $\pi$  sitting at the origin and you have a continuous spectrum of decreasing amplitude as  $\omega$  increases corresponding to  $\frac{1}{\omega}$ .

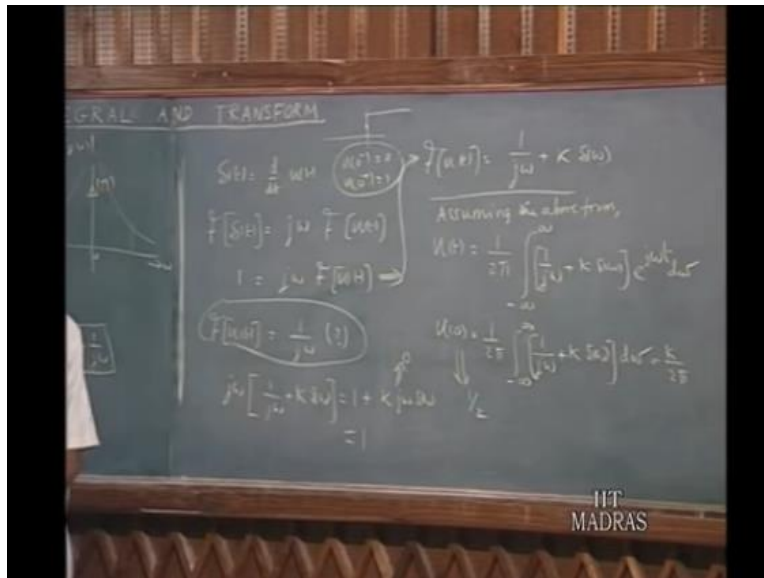
So, that will be the magnitude spectrum that we have. Now, look at these, this particular function of time  $u(t)$  is either even or odd. Therefore, we cannot expect the Fourier either real purely real or purely imaginary. Indeed it has got a real component and purely imaginary component.

So, this unit step function therefore, has a Fourier Transform which is  $\frac{1}{j\omega} + \pi \delta(\omega)$ . And let us see how we can sort or justify this or we know that, unit step function is obtained by integrating a delta function or in other words, a unit impulse function is the derivative of the unit step function.

Since, we know the rule for derivative of  $f(t)$  how it gets transform in the transform domain. Let us see how both these relationships are related to each other.

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We know that delta t is d by dt of u t. Therefore, we would expect that the Fourier Transform of delta t will be obtained by multiplying the Fourier Transform of u of t by j omega j omega Fourier Transform. So, you take the Fourier Transform of u of t multiply by j omega, you must get the Fourier Transform of delta t.

On this basis we conclude, the Fourier Transform of unit step function delta function is 1. And this is j omega times Fourier Transform of u of t. If this is so, the Fourier Transform of u of t can we conclude from this; divide everything by j omega can we say this is j omega. If so, then Fourier Transform of u of t is 1 over j omega that is not in accordance what now just derived.

So; obviously, there is something inconsistent here. The steps here also seem to be harder, but the catch here is whenever you are dividing by j omega, you must also take possibility of a delta function is being presented. In fact, if you have j omega times 1 over j omega plus some K times delta omega.

Suppose, I have an impulse of magnitude K sitting at the origin, if you multiply right through you have 1 plus K j omega time delta omega. And whenever a function f of omega is multiplied by delta omega, that f of omega can be replaced by a f of 0. Because,

after all this delta function here has value at omega equal to 0 only and nowhere else and therefore, the value of this function at omega equals 0 is only 1 that matters.

And in this case  $K j \omega$  equals 0. Therefore, this is equal to 1 this is equal to 0. So, in other words  $j \omega$  multiplied by this is equal to 1. Therefore, if you take 1 over  $j \omega$  it could be  $1/j \omega$  plus  $K \delta \omega$  as well. So, that is where, we have a discrepancy here.

This is of course, 2, but it is not a general expression. The general expression for this would be  $F(j \omega)$  from this it follows that, the general expression for this, you want to solve for this  $f(u, t)$  must be Fourier Transform of  $u$  of  $t$ . The most general way in which this equation can be solved is if you write  $1/j \omega$  plus  $K \delta \omega$ .

So, this is whenever you are dealing with impulses 1 has to be careful in this manner, in the algebraic solutions of equations, you divide by  $j \omega$  you must also take into account the possibilities that it is the delta function is there. Because, if you multiply by this  $j \omega$  it is equal to 1, the second term dropped out.

Having agreed to this, now what is the value of  $K$ ? How do we get the value of  $K$ ? In the earlier derivation we saw that  $K$  equals to  $\pi$ . So, how do we justify this  $\pi$  from these reasoning? Now, let us see assuming that Fourier Transform of assuming this relation assuming the above relation let us, calculate  $u$  of  $t$   $u$  of  $t$  therefore, must be.

So, assuming the above, assuming the above form. We can obtain  $u$  of  $t$  through the Inverse Fourier Transform formula which is  $1/2\pi$  minus infinity to plus infinity of  $F(j \omega)$  which is  $1/j \omega$  plus  $K \delta \omega$   $e$  to the power of  $j \omega t$   $d \omega$ . That is the Inverse Fourier Transform formula.

Now, in this formula let me substitute the value after all, this is the function of  $t$ , this is the function of  $t$ . Let me substitute the value of  $t$  equal to 0. Now, what is the value of  $u$  of 0? I ask this question. And on the other side, I substitute  $t$  equals to 0. Therefore, 1

over  $2\pi$  minus infinity to plus infinity of  $\frac{1}{j\omega + K\delta(\omega)} e^{-j\omega t}$  now, becomes  $\int_{-\infty}^{\infty} \frac{1}{\omega} d\omega$ .

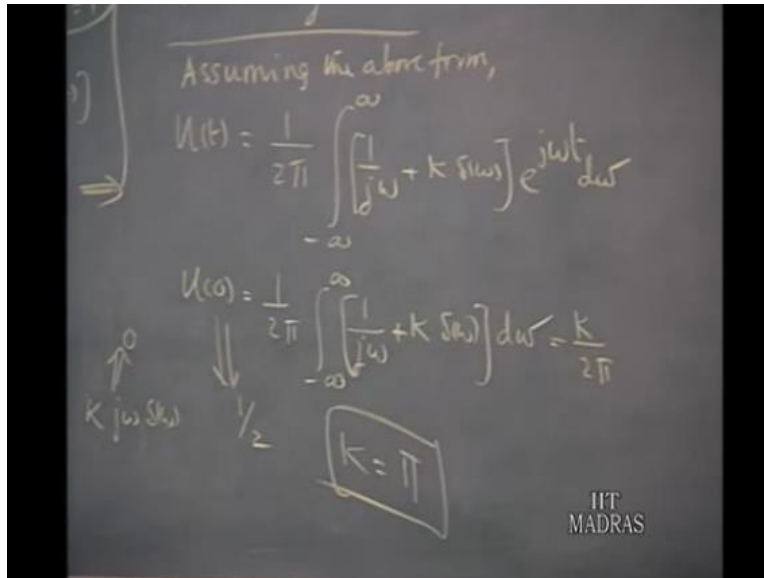
Now, we observe carefully the expression on the right hand side  $\frac{1}{\omega}$  is an odd function of  $\omega$ ,  $\frac{1}{\omega}$  is an odd function of  $\omega$  and integrating between the symmetrical limits minus infinity to plus infinity. So, the positive and negative area cancels themselves out and therefore, the contribution of this term  $\frac{1}{\omega} d\omega$  whatever it might be will, be 0 whatever it might be on the positive side and whatever might be on the negative side it is going to be 0.

So, the only contribution that comes from this is from the second term. And when you are integrating from  $K\delta(\omega)$  over the infinite range, that infinite range covers the where the impulse function stands. And therefore, this will be  $\frac{K}{2\pi}$  because, integral from minus infinity to plus infinity of  $\delta(\omega) d\omega$  equals 1.

Therefore,  $\frac{K}{2\pi}$ . But, what is the value of the step function at the origin? You recall from the Fourier Series theory, we said whenever there is a discontinuity of any functions, the Fourier Series converges to the average value of the limit you get from the starting from the right side and the left side.

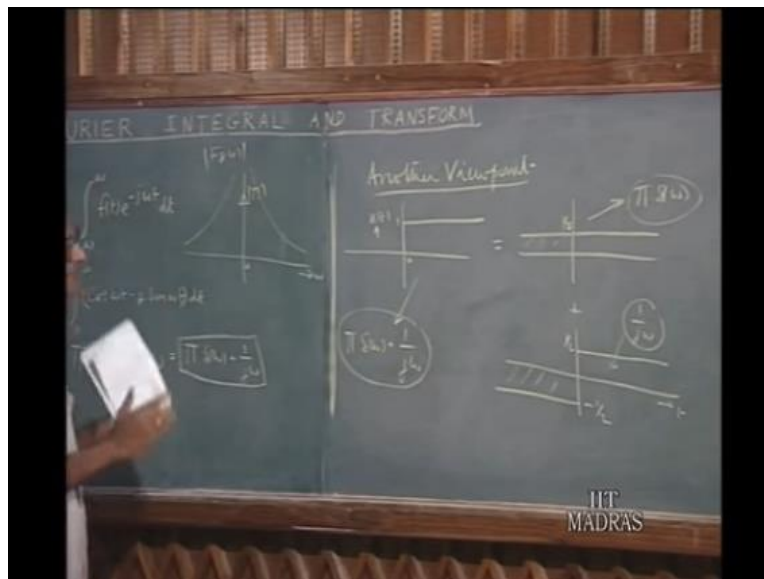
So, in other words here also  $u(0^-)$  is 0  $u(0^+)$  is 1. Therefore,  $u(0)$  and the point  $t$  equals to 0, this will turn out to be half. This is how the Fourier Transform and Fourier Series both of them converge. And since  $u(0)$  is half and that must be equal to  $\frac{K}{2\pi}$

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K equals pi and that indeed is the values that we obtained here. So, this is how tied up the formula that we obtained, starting from the Fourier Transform of the impulse that we know already.

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Now, let us look at some from another angle. Let us say another view point. We can always think of u of t to be the sum of; suppose, there is a pure dc term half, what else should be added to the pure dc term to get this? I must add to this a function which has the value half for positive t and minus half for negative t.

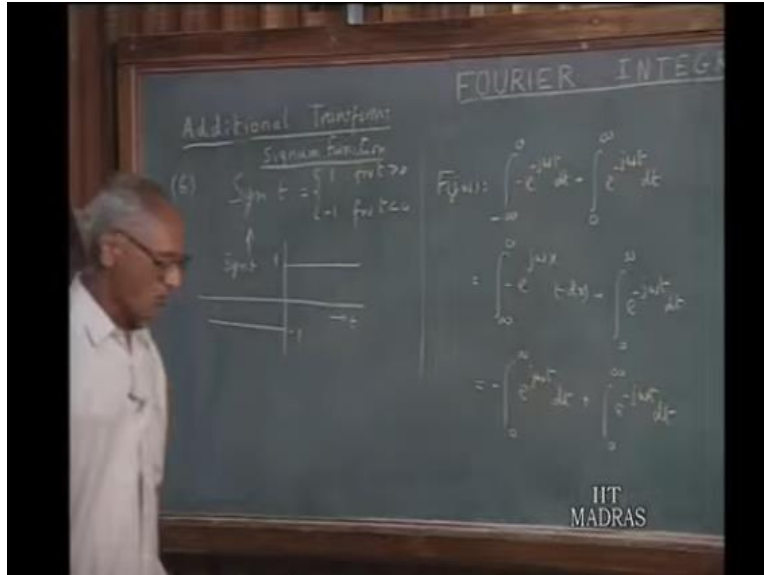
If, I add these 2 functions of time, I would indeed to get unit step functions because, on the negative side this area is cancelled by this area and the positive side both of them both these values add. So, it will be plus 1 for positive  $t$  and 0 for negative  $t$ . Now, we know that the Fourier Transform for this a pure dc if, is 1 it would have been  $2\pi\delta(\omega)$ , but this is half therefore, this  $\pi\delta(\omega)$ .

And we know the Fourier Transform for this, we just now derived is  $\pi\delta(\omega) + \frac{1}{j\omega}$ . So, it would appear that the Fourier Transform of this should have been  $\frac{1}{j\omega}$ . So, we have a unit step function can be resolved into 2 functions 1 is pure dc and another which has the odd function of time, which has the value half for positive  $t$  and minus half for negative  $t$ .

And this is an odd function and therefore, its Fourier Transform is purely imaginary and everything ties nicely and therefore, it appears as if, that a function like this which has the negative value for negative  $t$ . And equal and opposite value, which is the positive value over positive  $t$  has the Fourier Transform of this type  $\frac{1}{j\omega}$ . And this is the type important function that appears again and again.

So, we will try to derive this, a Fourier Transform for such functions independently that is, the what we will do next, but now we can see that a unit step function consists partly of a pure dc and partly of the function like this. We just now saw that, an interesting function which has the constant for positive  $t$  and constant for negative  $t$ , but the 2 constants are negative of each other.

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A function like this is defined as a signum function and usually symbol like this. This is signum function of  $t$  denoted as  $\text{sgn } t$  has the value 1 for  $t$  greater than 0 and minus 1 for  $t$  less than 0. And such a function is will have a time variation like this. This is called signum function so; obviously, its Fourier Transform can be derived in the usual way; from minus infinity to 0 its value equal to minus 1.

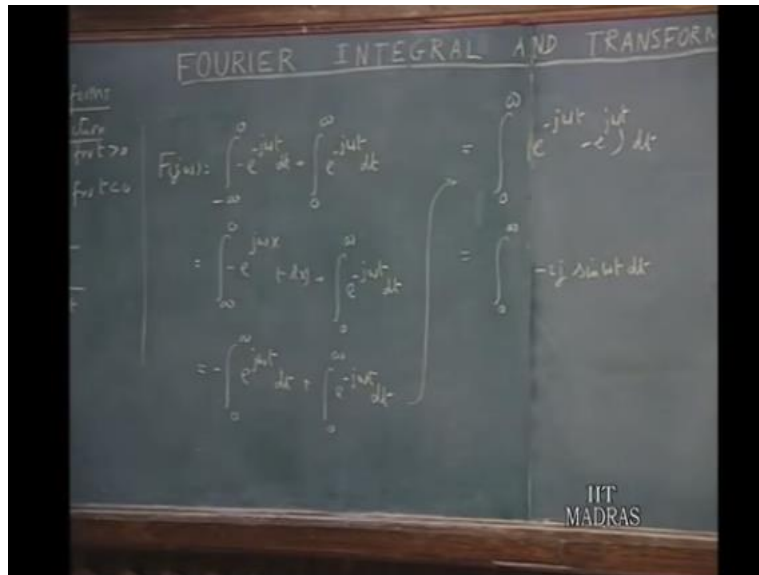
Therefore, I will write this as minus  $e$  to the power of minus  $j$  omega  $t$   $dt$  and I split it up this integration into 2 parts: minus infinity to 0 and 0 to infinity 0 to infinity its value equal to 1. Therefore, 1 times  $e$  to the power of minus  $j$  omega  $t$ . Now, I can put  $t$  equals minus  $x$  if I do that, it will be minus of  $e$  to the power of minus  $j$  omega  $t$  equals minus  $x$  therefore, I can write this as plus  $x$   $j$  omega  $x$ .

And  $dt$  equals minus  $dx$  and the integrals limits will become infinity to 0. My idea is to combine these 2 terms in some fashion and put them between the same limits of integration 0 to infinity of  $e$  to the power of minus  $j$  omega  $dt$ . Now, in this integration  $x$  is dummy variable of integration.

I can reintroduce  $t$  and then, I notice as the negative sign here and then negative sign here therefore, they cancel each other out. And interchange the limits of integration therefore,

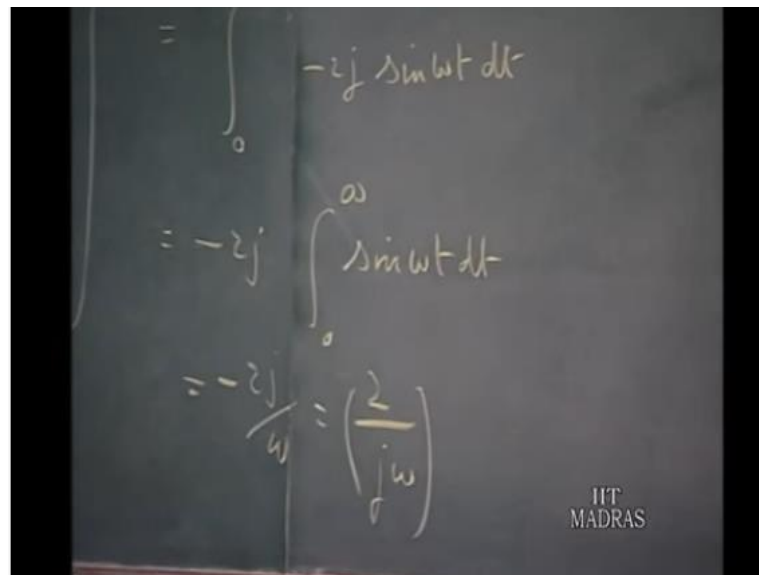
there is a another negative sign is coming. Therefore, this becomes 0 to infinity minus of e to the power of j omega t dt plus 0 to infinity of e to the power of minus j omega t dt.

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We can combine these 2 as 0 to infinity of e to the power of minus j omega t minus e to the power of j omega t dt. e to the power of minus j omega t minus e to the power of j omega t dt 0 to infinity. This will be minus 2 j sin omega t minus 2 sin omega t dt.

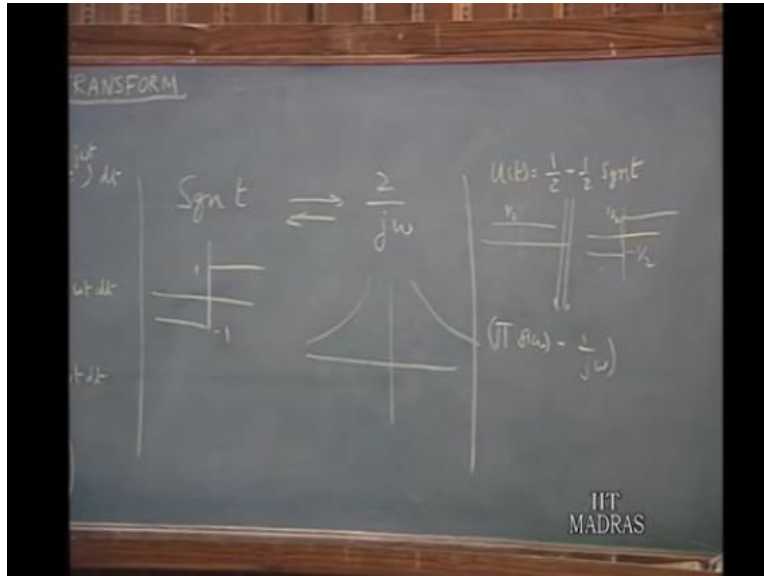
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And minus 2 j can be pulled out the integral and what is the 0 to infinity of sin omega t dt. And this integral has a special value, which we integrated in the last lecture that will

be  $1/\omega$ . Therefore, this will become  $-2j/\omega$  or I can write this as  $2/j\omega$ .

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So, the upshot of this now, to say that the signum function of  $t$  minus 1 plus 1 that is, signum function of  $t$  has its Fourier Transform  $2/j\omega$ . And therefore, when we had in the derivation of the unit step function, you recall that once again that let me redo what we done there.

A Unit step function can always be thought of; as 1 half plus 1 half the signum function of  $t$  because, if you add a pure dc of magnitude half and the signum function which has minus half and plus half for negative and positive sign then, the sum of these will indeed result in  $u(t)$ .

And therefore, the Fourier Transform for this, from the dc part you get  $\pi\delta(\omega)$ . It is the even function therefore, this is real and for the signum of  $t$  you get  $2/j\omega$  but, we have a coefficient half here this is  $1/j\omega$  that is the Fourier Transform. So, a signum function can be represented in the fashion and it has a Fourier Transform which is  $2/j\omega$ .



So, what we earlier said,  $1/j\omega$  corresponds to a function of this type, we wondered if that is so then, we justified now based on this particular analysis. Now, you also observe that is the purely odd function of time therefore, the Fourier Transform turns out to be purely imaginary. Let us take next, a truncated sinusoid.