Networks and Systems
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Lecture-31
Examples and Some Properties of Fourier Transform

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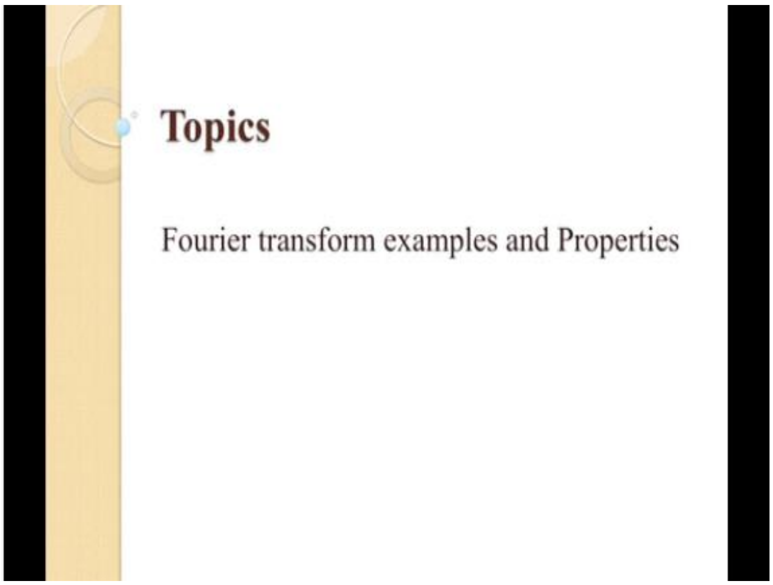

You recall that when, you had periodic impulse train in the case of Fourier series we found that all Fourier coefficients have the same magnitude.

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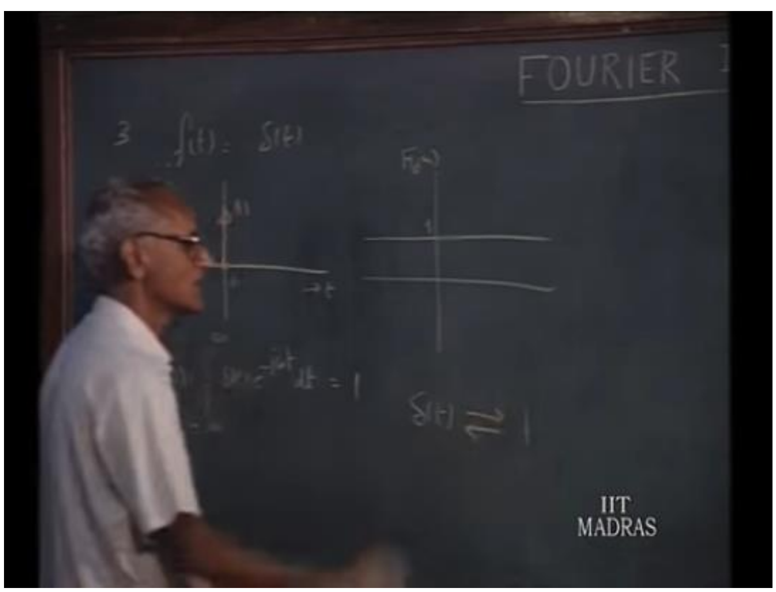

So let us, now take an impulse and see just 1 single impulse this is of course,, a periodic and see what its Fourier transform would like. So, f of t is taken to be a delta function. So, you have a unit delta sitting at t equal to 0 that is a time function. The Fourier transform for that f of j omega minus infinity to plus infinity the function of time is delta t e to the power of minus omega t dt.

And since, we have the delta function in the integrand naturally the value will be the value of e to the power of minus j omega t t equals to 0 that is equal to 1. So, we have very nice and compact result F of j omega equal to 1.Which means the Fourier spectrum so, you have impulse you have Fourier spectrum which is flat.

So, we can say now this particular pair of transforms delta t as a Fourier transform which is equal to 1. That means, the coefficient density at all frequencies ranging from minus infinity to plus infinity equals 1 very nice and useful result.

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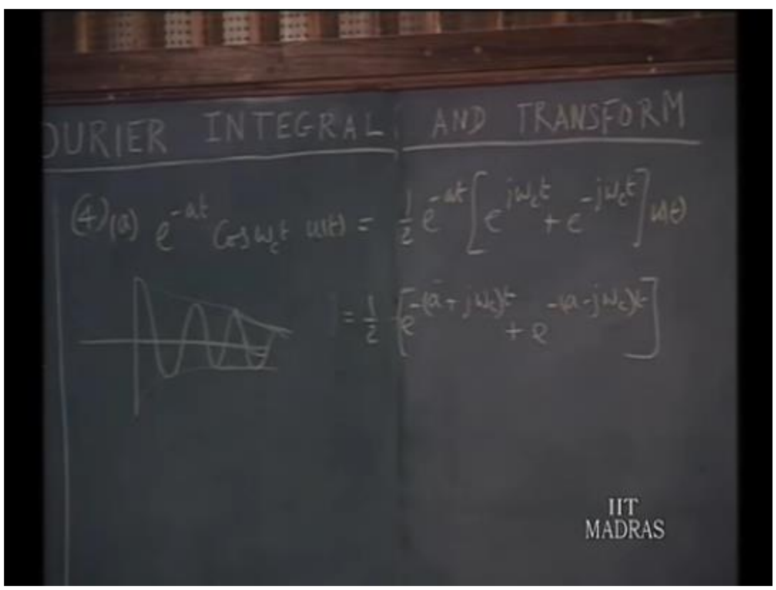

Let us, take now a more complicated example e to the power of at cos omega ct u t that means, the time function would be something like this which is oscillating sinusoid, but with decaying amplitude. Now, this can be written as 1 half of e to the power of minus at e to the power of minus j omega c t plus e to the power of minus j omega c t which is 1 half of e to the power of minus a plus j omega c t plus e to the power of minus a minus j omega c t.

So, this decaying exponentially decaying sinusoid can be a thought of as 2 exponential functions of this type. And very first example we observed that e to the power of minus a plus j bt can also, have the same type of Fourier transform as when you have e to the power minus j 1 over plus a we have 1 over j omega plus a plus j omega in this case.

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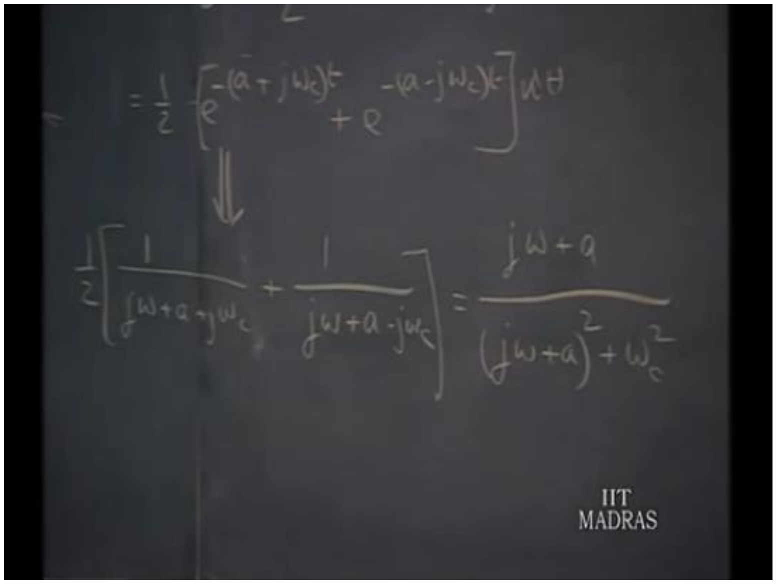

So, the Fourier transform for this will be half of 1 over Fourier transform for this particular function for this i also i have ut that continuous. J omega plus a plus j omega c that is for the first function, second function we have 1 over j omega plus a minus j omega c. And these 2 can be combined and together we have an expression which looks like this j omega plus a over j omega plus a whole square plus omega c square.

Likewise, you can plot this spectrum then, you will find that around omega c you have some kind of peaks, but we do not do that.

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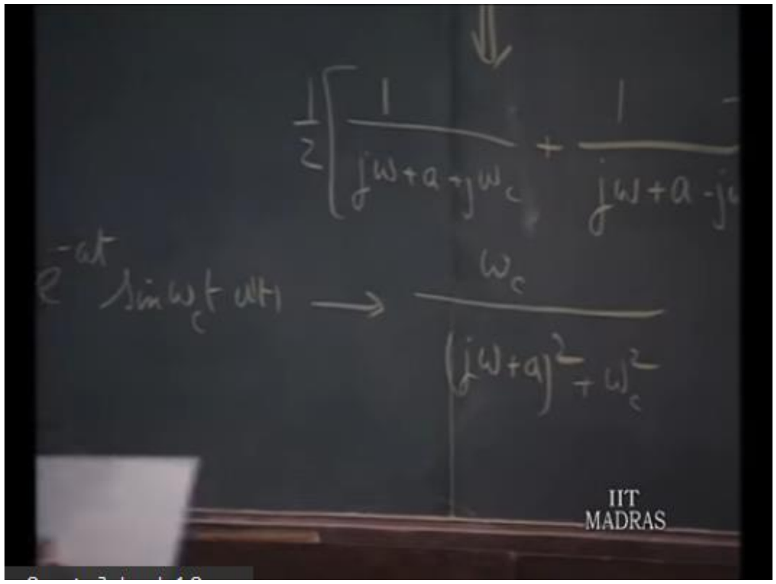

Similarly, e to the power of minus at sin omega c t ut can be shown to have a Fourier transform which is equal to omega c over j omega plus c whole squared plus omega c square right.

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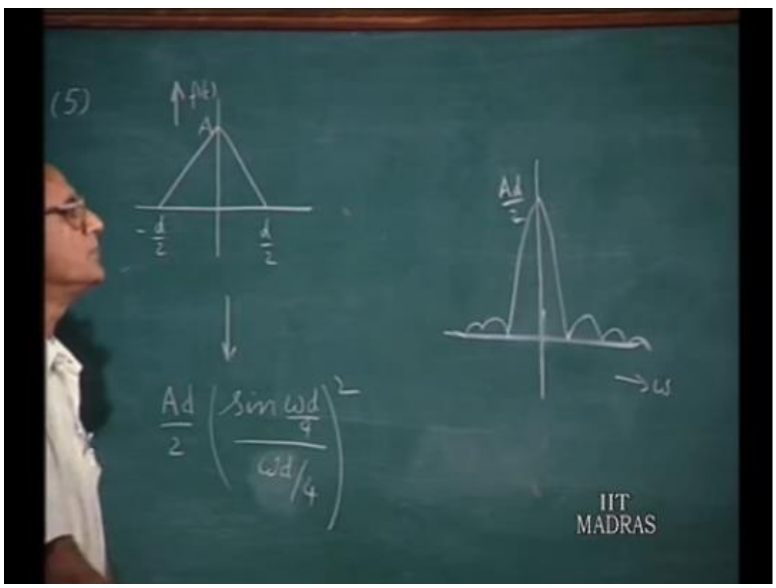

And the last example if i have a function like this, A d upon 2 minus d upon 2 this is f of t. That is a pulse the form of a triangle lasting from minus d upon 2 to plus d upon 2 it can be shown that this will have a Fourier transform which is equal to A d upon 2 sin omega d upon 4 by d omega d upon 4. So, this is something like sin theta by theta type of arrangement, but squared that means, the decay is faster. The spectrum for this would be like this and this i leave to you as an exercise to show.

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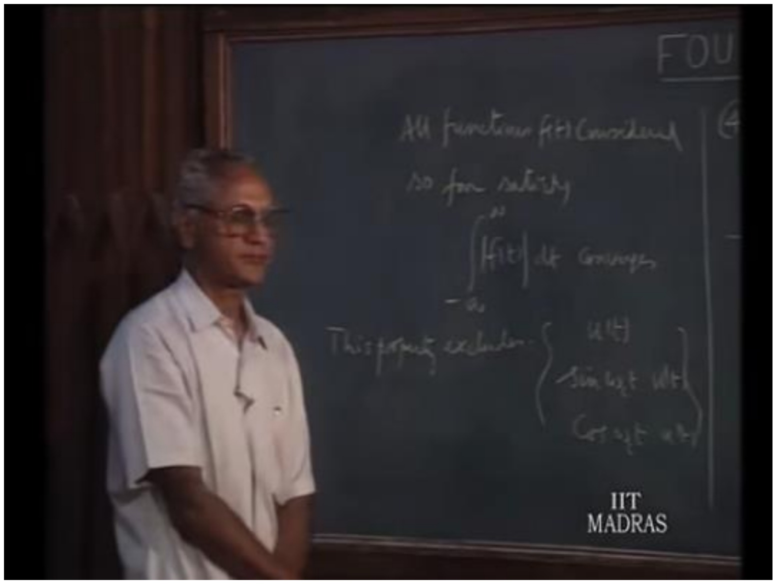

Now, all the function that we have taken so far are nice well behaved functions which have the property that, all functions considered so far considered so far satisfying the relation from minus infinity to plus infinity f of t dt converge. That means, the magnitude of the function is integrable from minus infinity to plus infinity that is why, we have exponentially decaying function, exponentially decaying function sinusoid here.

A pulse with last for a finite amount of time either in a rectangular pulse or a triangular pulse and impulse can be integrated that is equal to 1. So, all such functions have we have considered nice properties like this, this excludes this property excludes some desirable function which we should like to find Fourier. For example: u of t last from 0 to infinity sin omega ct u t a sinusoid cos omega ct ut functions like this are important.

And we come across them quite commonly, but in the classical sense these functions will not satisfy this property therefore, we do not find the Fourier transform for this not yet anywhere. In the classical Fourier transform theory these functions were excluded from this scope. But that impulse functions have become a matter common usage once, we have impulse functions we should be able to find out the Fourier transform of these functions as well which we do a little later.

The mathematical justification for such usage comes from what is called the distribution theory which has scientified the use of impulse functions and related mathematics we will not however, go into that. But nevertheless, we assume certain relations which are sanctioned and use those results to find out the Fourier transforms of such functions as well a little later. But before that, we like to have a closer look at the some of the properties of the Fourier transforms in order for us to get familiarity with their characteristic before we move on to finding out the Fourier transforms of special functions of this type.

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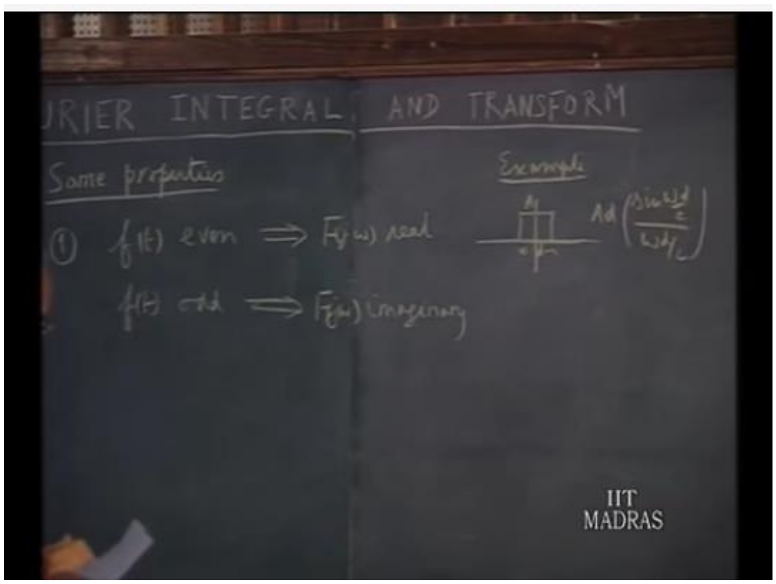

So, what we now like to do is discuss the properties of Fourier transforms 1 this property will run somewhat parallel to the properties we discussed when, we talked about the Fourier series after all Fourier integral Fourier series are related to each other. Therefore, it is natural for us to talk about properties which are similar to what we discussed in the case of the Fourier series.

So, if f of t is even then it means that F of j omega is real. So, when you substitute j omega in the Fourier transform it trans out to be real. Example what we have seen this pulse this is an even function and then we found that F of j omega is real there is no j term in F of j omega. If f of t is odd then, it turns out that F of j omega will be imaginary there is a j term sitting outside will take just give an example which i will not however, derive that.

Let me, write this this is A, this is d we know this is Ad sin omega d upon 2 divided by omega d upon 2. Because, i am repeating this again and again because, this come so frequently it is better to remember that.

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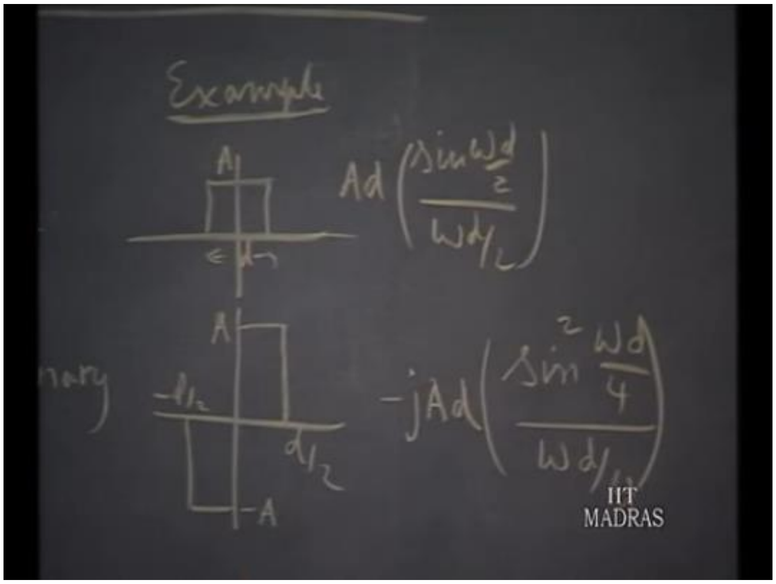

Now, suppose i have a function like this A d upon 2 minus d u on 2 this is certain a not function. We have pulse of value A is lasting from 0 to d upon 2 and minus A is lasting from minus d upon to A. Its Fourier transform Trans out to be minus j Ad sin squared omega d upon 4 divided by omega d upon 4. So, you observe that this is immediately j times the j sign attached to it is purely imaginary.

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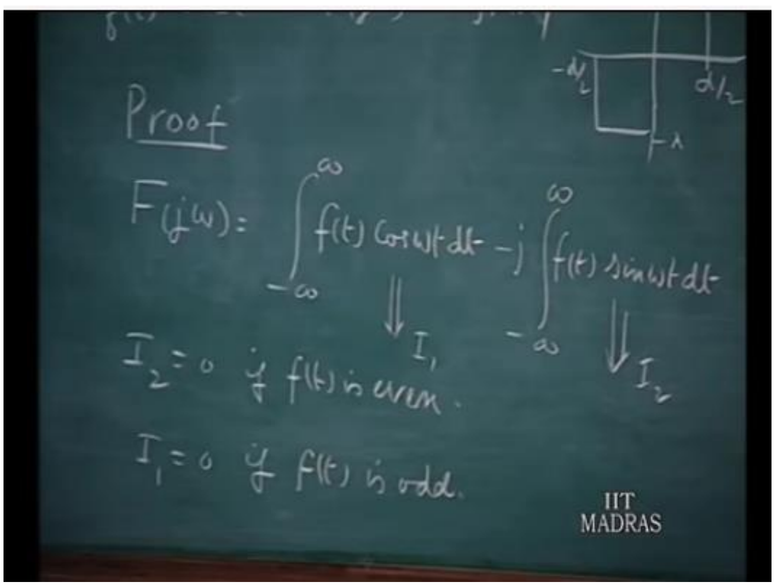

The proof of this is straight forward if you write the expression for F of j omega as minus infinity to plus infinity of f of t cos omega dt minus j times minus infinity to plus infinity of f of t sin omega t dt. We recognize that e to the power of j omega t is cos omega t minus j sin omega t. So, we split the defining integral for Fourier transform into 2 parts like this now if you call this integral i 1 and call this integral i 2.

Then, we notice that f of t is even the product f t cos omega t is also even. Therefore, we are integrating as far i1 is considered between 2 symmetrical limits. So, whatever sequence of values this integrand takes from minus infinity to 0 it takes from minus infinity to 0 as well. Therefore, the integral i 1 will be twice the value of the integral from 0 to infinity.

On the other hand, i2 involves ft times sin omega t therefore, that is the odd function of time and we are integrating between the symmetrical limits. Therefore, the value of the integral from minus infinity to plus 0 will be exactly the negative of the integral from 0 to infinity. So, as a consequence i2 is 0 if f of t is even similar arguments we show that if f of t is odd f of t sin omega t is even.

Therefore, the second integral will be non-zero, but the first integral which involves f of t cos omega t that turns out to be the integrand is odd. Therefore, if f of t is odd, f of t cos omega t is odd and you are integrating between symmetrical limits. Therefore, i1 will be 0 if f of t is odd. So, we have the situation that if f of t is even then, the second integral vanishes and the first integral i1 is real.

Therefore, f of j omega itself is going to be real. On the other hand, if f of t is odd the first integral vanishes and i2 remains, but i 2 proceeded by j therefore f of j omega will be purely imaginary.

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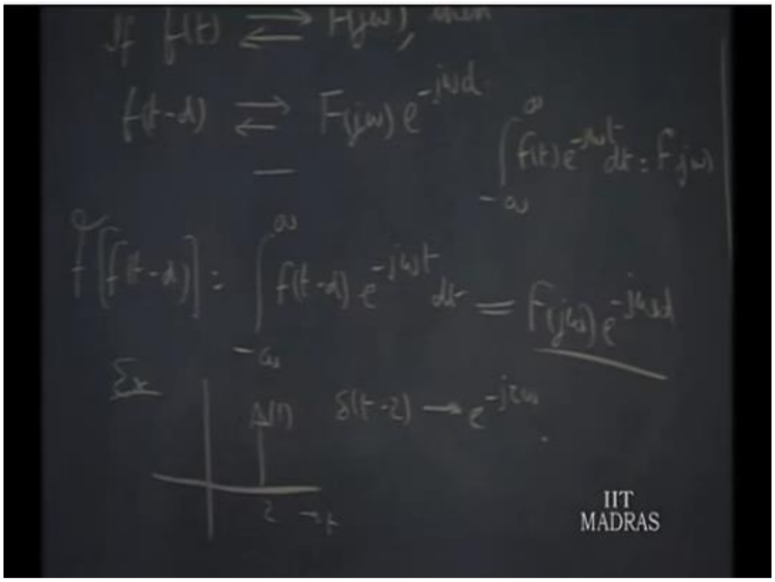

Let us, now discuss another property of the Fourier transform translation in time if f t has the Fourier transform F j omega then, F of t minus d that means, you delay the signal to by some amount then this turns out to be F j omega e to the power of minus j omega. This is the useful result if you delay this time signal by d units, the Fourier transform gets multiplied with by e to the power of minus j omega d.

It means, the magnitude of this being equal to 1 the Fourier transform magnitude will remain the same, but the phase will be disturbed by the amount minus omega d. Very similar, to what we had in the case of cn coefficients in the Fourier series the proof is quite straight forward to find, the Fourier transform of f of t minus d all you have to do is from minus infinity to plus infinity f of t minus d e to the power of minus j omega t dt this defining relationship for the Fourier transform of f of t minus d.

Now, the what we have to do further is quite evident for you all you have to do is bring it we know that minus infinity to plus infinity of f of t e to the power of minus j omega t dt is F of j omega F j omega from this. Therefore, we have to bring it into that form so, all you have to do is put t minus d as x.

Then, f of x e to the power of minus j omega t plus instead of t t minus d is x therefore, t equals d plus x and then, dx we get and then you have e to the power of minus j omega t term is coming outside. Then, you have minus infinity to plus infinity of f of x e to the power of minus j omega dx which is indeed the F of j omega. So, it can be shown that this is equal to F of j omega very useful result extremely useful result.

As an example simple example I will work out suppose, I have delta function here of unit magnitude sitting the time t is equal to 2. That means, this delta t minus 2 and we know the Fourier transform of delta t is 1 the Fourier transform of this is e to the power of minus j 2 omega. Because, you have to multiply the Fourier transform of delta t is 1 you are multiplying this by j e to the power of minus j omega d the delay is 2 second therefore e to the power of minus j 2 omega.

We will work out, some more examples in the next lectures, but before that let us now summarize what we have done today. So, in this lecture we started with review of the Fourier transform defining integral and the Inverse Fourier transform. Then, we constructed the Fourier transforms of a few typical type functions.

Specifically, these are the impulse functions, the rectangular pulse functions, the exponentially decaying sinusoidal functions, a triangular pulse functions and we observed that, the if the Fourier transform turns out to be either is real turns out to be real. Then, there is no necessity for us to plot the magnitude spectrum and the phase spectrum separately.

Because, the entire function is real we can exhibit this by means of single spectrum which represents the value of F of j omega. We also, introduce ourselves to a new function Sinc x which is defined as sin pi x over pi x: Sinc x has the value 1 at x equal to 0 and vanishes when x takes the integral value like 1 2 3 minus 1, minus 2, minus 3 etc.

So, this new function is helpful in finding debiting the Fourier transforms of pulse functions. Then, we took up for steady some important properties of the Fourier transforms: the first property we studied is that, if f of t is even then F of j omega is real and if f of f is odd F j omega is purely imaginary. So, we also saw that we also mentioned earlier that F of j omega is real.

Then, the magnitude spectrum and the phase spectrum need not be exhibited separately they can be exhibited by 1 spectrum which use both this magnitude and phase of course, either will be 0 or 180. Same thing can be done if the spectrum is purely imaginary, in that case we will plot of F of j omega by j that means, the imaginary part of F of j omega is can be plotted by means of 1 spectrum and because, the real part is 0 we can also do that extend this principle of having only 1 spectrum even F of j omega is imaginary by keeping at the back of mind.

Whatever, values which we plot of the spectrum are purely imaginary quantities. The second important property that, we studied was that if the function f of t is delayed by tau seconds then, the Fourier spectrum gets modified F of j omega multiplied by e to the power of minus j omega tau. That means, the new function will have the same magnitude spectrum as the earlier function f of t, but the phase will be modified.

This is not surprising because, we saw a similar result in the case of the Fourier series where the cn coefficients the magnitude will remain undisturbed if the periodic pulse periodic function is delayed by certain amount of time per tau seconds only the case will get modified. After all f of j omega is closely related the cn coefficients therefore, we have the similar result here also. We will stop at this point and continue our discussion of the properties of the Fourier transforms in our next lecture.