Digital Circuits and Systems
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Module – 05
Algebraic Minimization Examples

Hello all welcome to the 5th module of this course, Digital Circuits and Systems. So, I want to give you a quick summary of this notion of Min terms and Max terms.

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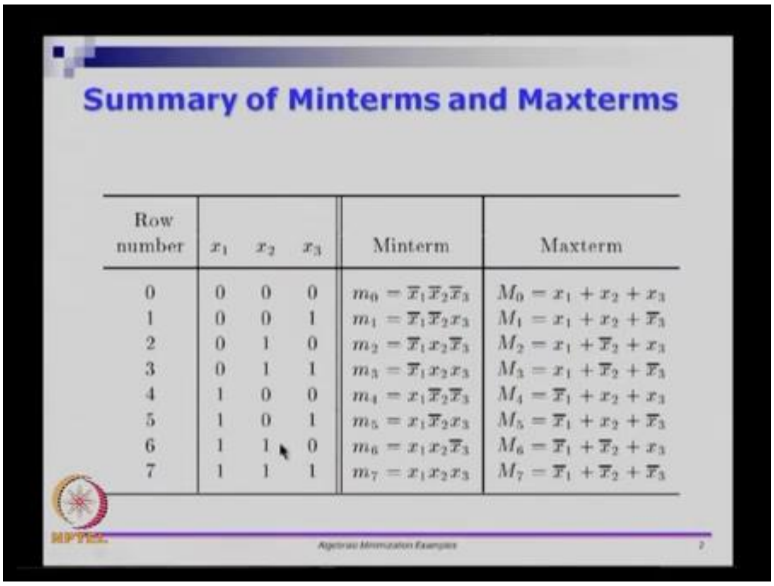

So, we have, you know the familiar truth table representation, we have 0 0 0 to 1 1 1 and they correspond to row numbers 0 to 7. So, I mentioned that the small case m is used for min terms and upper case M is used for max terms and the subscript is an indicative of what is the term that you are talking about. So, for a min term, if you say 0, it is actually 3 bits 0 0 0 and what it means is x 1 is 0, x 2 is 0, x 3 is 0 and so on. For max term the interpretation is, if you see a 0 0 0, then x 1 is in the uncomplemented form or x 2 in the uncomplemented form or x 3 in the uncomplemented form.

So, one thing that we can notice is, if you look at this small case m naught and the capital case M naught, you will see that the term here is actually the logical complement of the term here. So, use Demorgan’s theorem, you have x 1 or x 2 or x 3, so the complement of the sums is product of the complements. So, you can see that repeat, so I want to give a quick summary of min terms and max terms that we read in the last module. What we have is various row numbers here and their binary representation here.

I have mentioned that min terms will use this small case m and for max terms, we use upper case M. When we say m 0, we are talking about m x 1 in the zeroth row, m in the zeroth row, where everything is 0 and the interpretation is x 1 is complemented, x 2 is complemented and x 3 is complemented, so that is what you see here. The max term interpretation of that is x 1 is uncomplemented or x 2 uncomplemented or x 3 uncomplemented.

So, one thing you can notice is that this m and this M are actually logical complements of each other. So, use Demorgan's theorem, you see that the, you take this and you complement it, you will get these. So, complement of the sum here would be the product of the complements here. So, this is something that was interesting, it makes simplification easy, we will see how.

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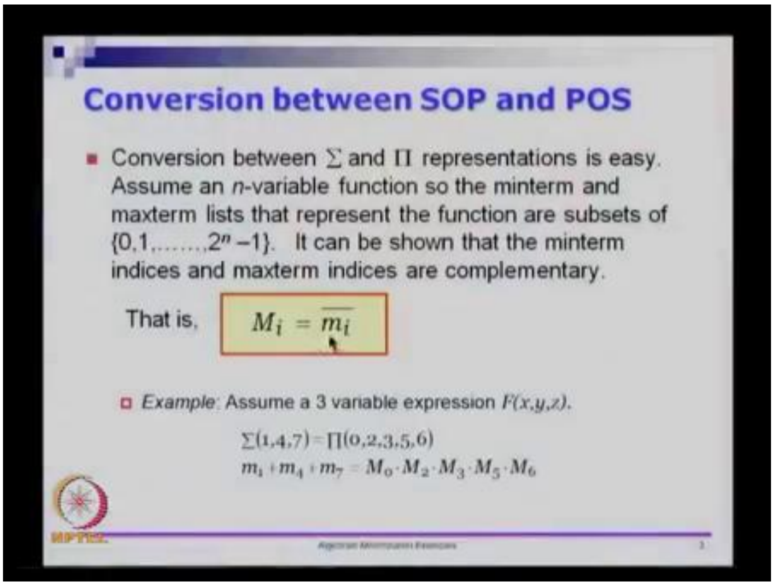

So, one thing you can notice is, if you generalize this set up, the capital M subscript i is the same as small m subscript i complement. So, the term that you have min term, the ith min term, you take the complement of that, that gives you the ith max term. And similarly, if you take the complement of the ith max term, that gives you the ith min term.

So, if you have something which is complement of each other, then the direction, so it goes in both directions. Then, what that tells us essentially is if you have this, then if you given a, let us say the sum of product expression like this. So, you have sum of 1, 4 and 7 and 1, 4 and 7 are min terms, because we said it sum of products. So, it is m 1, m 4 and m 7, the logical r of those, what this gives us as a powerful tool is that, if you are given the sum of products, you can write the product of sums really quickly.

So, the way to do that is, you look at all the terms that are possible. So, you have terms, min terms starting from 0, 1, 2, 3, 4, 5, 6, 7 for the three variable function, but you are given only three of them here, so 1, 4 and 7. So, the way to get the product of some expression is find out all the terms that are missing here and you put them down. So, you have 1, 4 and 7; you are missing 0, 1, 2, 3, 5 and 6 in the range of value 0 to 7.

So, you take all those terms and you take the max terms and add them together, so the sum of 1, 4 and 7 is the product of max term 0, 2, 3, 5 and 6. So, this is a very handy thing, it is a good thing to remember, because sometimes getting the product of sum is easier than the sum of products or the other way round. If this is a complicated expression you try and do this and convert back or if you think that this is easier to do, you get this and convert to this.

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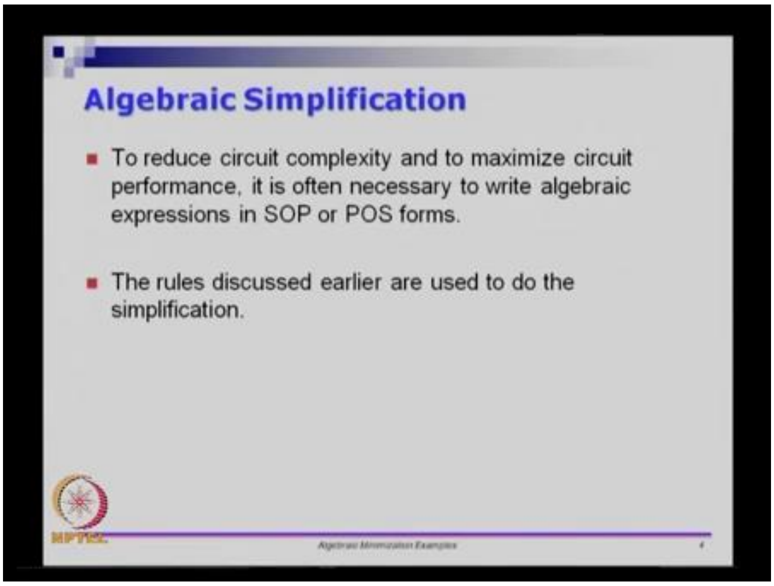

So, let us I am going to give you a few examples of how to do an algebra simplifications. So, we are going to use several rules that I discussed so far, you will also try and write things in SOP form and POS form and so on. So, let us start with a fairly simple one.

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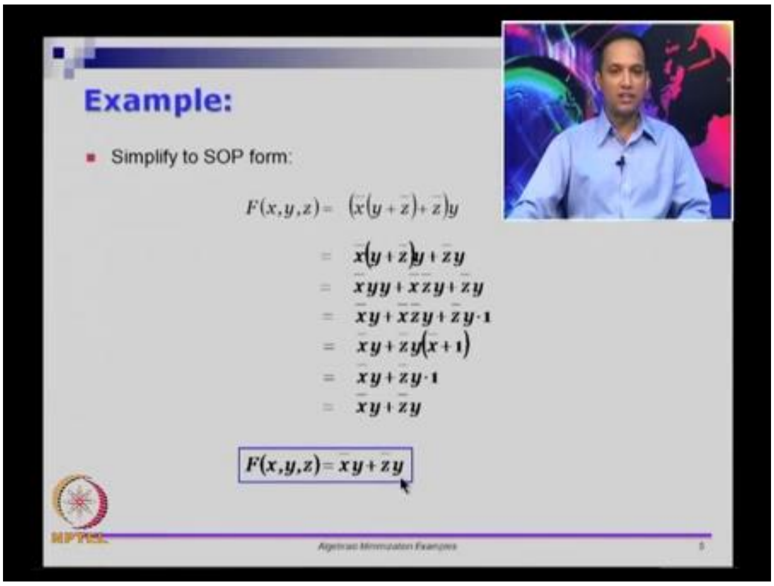

So, this expression F of x, y, z it is given there and clearly it is not in any form. So, it is not product of sums, it is not sum of products either and the example is asking you to simplify that you into the sum of products form. So, what we need is, we need terms which are all product terms and we want a logical OR between those. So, this is not sum of product, because you have a product, x bar into and y plus z bar and that kind of thing is not allowed in the sum of product form.

So, let us see, how to simplify it. So, the first thing you can do is, you can see that x bar here is anded with y or z bar and there is also y, that is anding from the other side. So, if I take all these terms and expand it, so this is x bar here, y and z bar here and this y here, this z bar is associated only with y, so you expand that. So, it is fairly straight forward algebra here. Then now you bring x and y inside, so this y gets x bar and y, so it is x bar y, y. So, this z bar gets x bar and y, so it is x bar z bar y and this is z bar y.

And if you notice, there is y and y here, that is the same as y, so x bar y, y is same as x bar y, then x bar z bar y, we write it as it is. And let us say z bar y, we write it as z bar y 1, because we see that x bar y is occurring in this term also, you write it as a z bar y 1. If you simplify it further, this term remains as it is and the z bar y, which is common between these two which means we can eliminate x bar.

So, essentially, we get x bar y plus z bar y, so this is a valid sum of product form. So, x bar y plus z bar y is a valid sum of product form. The same thing, if you want to write it in product of sum form, in this expression at least it is fairly easy, all you have to do is, you can see that y is common between these two. So, it is y into 2 x bar plus z bar that, is actually a valid product of some expression also. You can take this expression here that you got from SOP and you can quickly convert to POS for this example. So, it is just x bar plus z bar into y, x bar plus z bar is a valid sum term and y is a valid sum term, so the product of those two is a valid POS.

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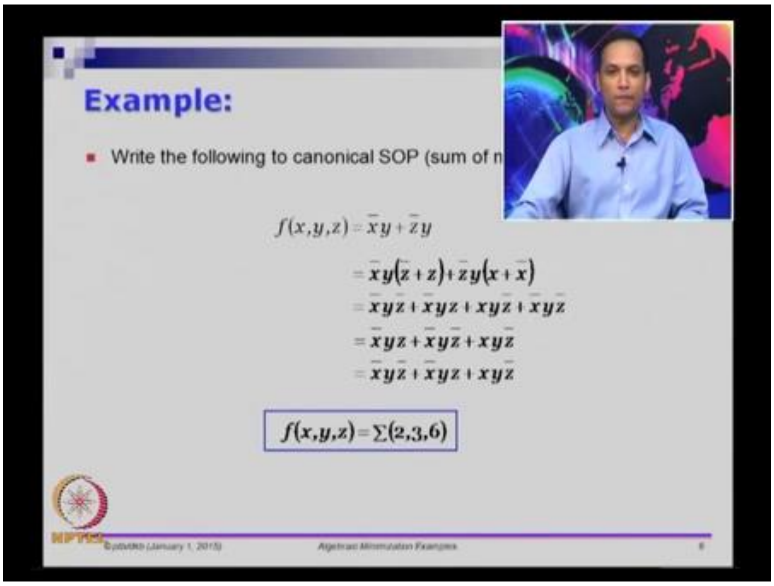

Let us move on to another example, we want to write something now in the canonical SOP form. So, let us look at this one here, x bar y and z bar y is what we had in the previous example. So, the first question is I am asking to do this in canonical SOP and why is this not canonical SOP, so let us look at that. The left side we have three variables x, y and z and in a canonical SOP, we need all the variables to be in all the terms.

So, remember this should be normal terms which means, we you should have min terms that capture the same logic functionality on the right side. So, we have F of x, y, z; this one does not have z term and this one does not have x term in it. So, this is a valid SOP, it is not a canonical SOP however. So, if you want to convert something like this to a canonical SOP form, what you have to do is this. So, this is missing a z, so I am going to introduce z bar plus z for this term, this term is missing x. So, I am going to introduce x plus x bar.

So, if we already know that expression x plus x bar is 1 and z bar plus z is 1, so introducing that is not a mistake. So, this line is taking something from here, whatever is missing we have return them out, now this line is just expanding it. So, if you take x bar and y into z bar, so that is here x bar y and z is that is here and z bar y, x the terms are rearranged, because we know that logical AND is commutative. So, this is x, y, z bar and finally, x bar y, z bar that is here.

So, now, it is look like, we have four terms and each of these terms have x, y and z in them. So, the question is, is this canonical? So, the quick answer to that is, it is not canonical, because there are terms which are repeated in that. So, if you notice, there is x bar y z bar and there is x bar y z bar even here. So, this term is repeated twice and a canonical SOP will have each product term appearing only once.

So, the next line actually changes that, we have x bar y z plus x bar y z bar plus x, y, z bar. So, we have this here and this expression, the final step is, we are arranging them in some order. So, this order, so it is same, this last line is same as this line, except that it is rearranged in some other order, the product terms are shuffled around. So, if you look at this, this corresponds the min term 0, 1, 1 or 3. This corresponds to the min term 0, 1, 0 or 2 and this corresponds to the min term 1, 1, 0 or 6. So, there is 3, 2 and 6, we are rewriting that as 2, 3 and 6.

So, that is what we have here, you can write F of x, y, z as a canonical expression, you can write it as sigma of 2, 3 and 6. So, at this point, I want you to think about this. If I gave you this expression and if I ask for you the canonical POS, you might have done this already, if I want a canonical POS at this form, all we have to do is, take sigma of 2, 3, 6. We know that all the terms that are here, we remove those and put the rest of the terms in. So, sigma of 2, 3, 6 in the canonical POS form is the same as phi of 0, 1, 4, 5 and 7. So, I want you to go and think about it and if possible even do that as an exercise.

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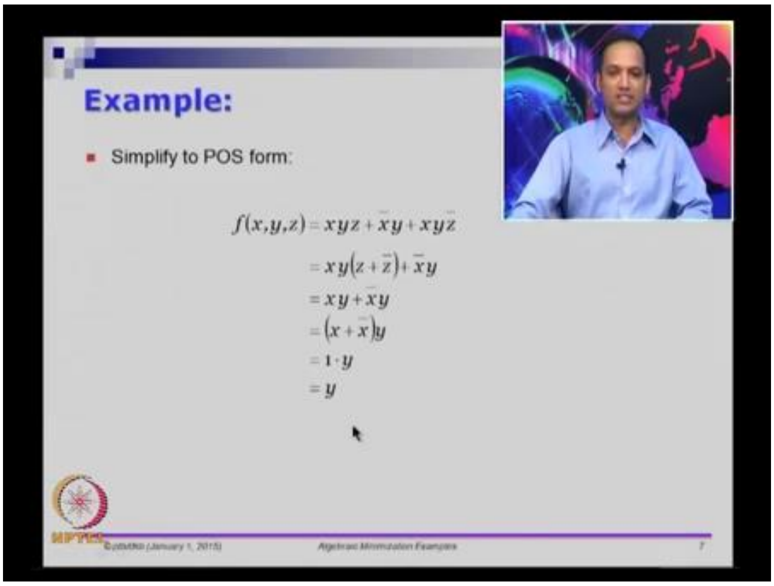

Let us look at another simple statement. We want to simplify something to the POS form. So, in this case, we are given something in the sum of product form and we want to convert that into a product of sum form. So, we could go and take this route of taking the sum of product form and let us say, write it in the canonical SOP and convert that back into the POS form. However, in this example specifically, it is the product of the some forms terms, turns out to be very simple.

So, we have x, y, z plus x bar y z bar, so x, y is common between this term and this term. So, you take that out that, you get z plus z bar and this is x bar y. So, we know that z plus z bar is 1, so that gets killed. So, we have x plus x bar y and again, y is common between these two terms, so we get y and 1 which is the same as y. So, in this case, it turns out that the POS form, it is actually fairly simple, but this is only the POS form, this is not the canonical POS form. So, if you want the canonical POS form, you have to do something else.

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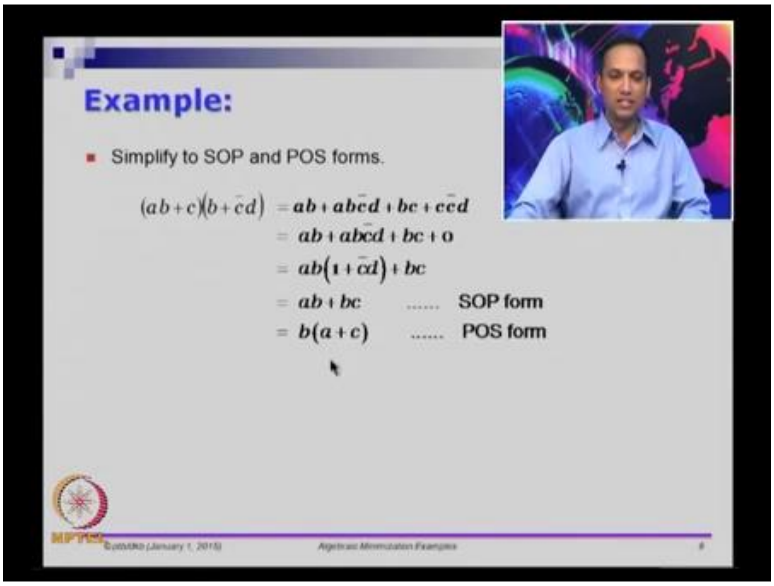

So, let us look at this example, simplify to SOP and POS forms. So, let us say, I have a b plus c into b plus c bar d. So, this is clearly not in the SOP form, nor in the POS form. So, it is neither of those, we want to convert that both SOP and POS form. So, if you are asked a question like this convert to both SOP and POS form. So, first thing you have to do is, get it in one form, so let us try and get it in the SOP form first.

So, you expand the product out, so a b into b is a b b, which is a b itself, then you have a b into c bar d, that is a b c bar d, then you have c into b, that is b c, then you have c into c bar d, that is c c bar d, so we have these four terms. So, this is the sum of product, but we can still simplify it, because there terms that are repeating. So, if you want this in a slightly better way, then the first thing you can recognize is c c bar d is 0. So, we can remove that, so a b plus a b c bar plus d bar plus b c.

So, many times, when you look at this, when the statement say simplify, we are asking for the most simplest form in which we can write it. So, clearly if we leave it here, that is also a sum of product, but it is not the most simplified form. So, if we leave it in this line a b plus a b c bar plus b c, again it is a valid sum of product. But, we want to see, that is the simplest sum of product form in which we can write, it turns out it is not on the case, so let us look at this one, a b is common between these two. So, you can write it as a b in to 1 plus c bar d plus b c, so at this point, it is just a b plus b c.

So, a b plus b c, you cannot simplify it any further and still have a SOP form. So, this is the simplest SOP form, it turns out getting the fewest form from this is straight forward, b is common between these two, so b into a plus c. So, remember the question is not asking you to do canonical SOP and canonical POS, we are just asking write the simplified expressions for these sides, this case it turns out to be a straight forward process.

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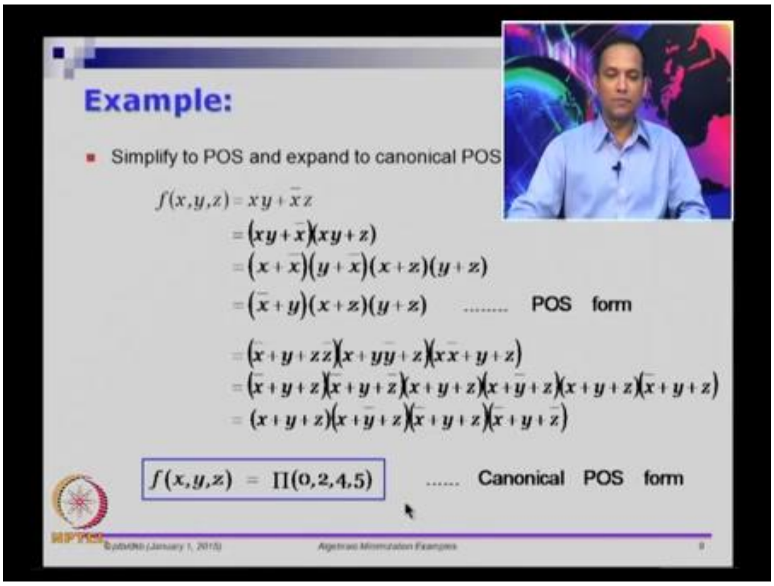

So, let us look at another example, in this example, we want to simplify to the POS form and expanded the canonical POS form. So, what that means is, first of all have to take this sum of product expression and see, how to write it in the product of sum form. So, this is not product of sum, we have two products and they are summed up. So, we want to write it as instead as product of sums.

So, this is a SOP form, let us see how to do that. So, let us look at x, y here and let us look at x bar z here. So, clearly, this is not in the form that we had right now. So, the first step is to do is, so if you have x y plus x bar z. One way to rewrite that is, take this term here x bar z, logically OR it with the complete term, product term here. So, x y plus x bar into x y plus z. So, what we have done is, we have taken this term here and split it into two product terms. So, one can quickly argue why this should be correct. So, you have x y plus x bar in to x y plus z.

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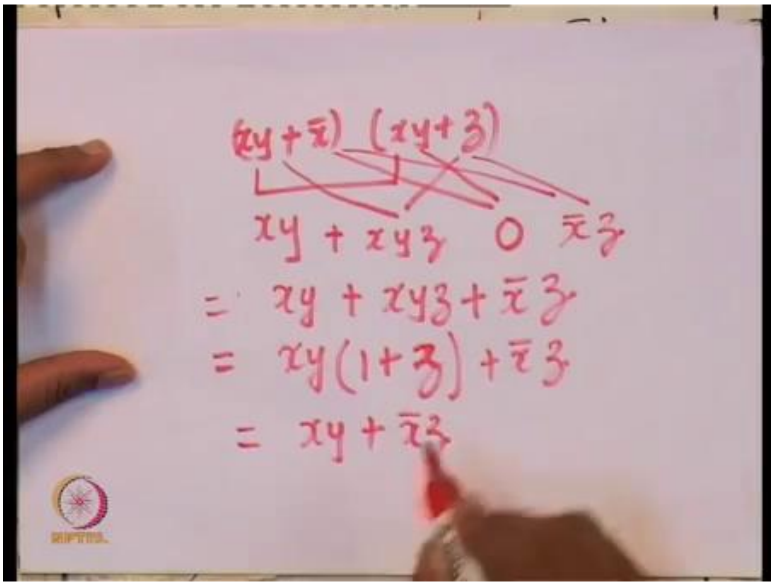

So, if you go back and look at it, so let us see, if we can try and show if these two are the same, x y plus x bar into x y plus z. So, if take these two terms that gives me x y, if I take these two terms that gives me x, y, z. If I take these two terms, that is x bar into x y, it is actually is 0, because x bar is cancel with x and the last two gives me x bar z. So, this is the same as x y plus x, y, z plus x bar z and if you do this. So, x y is common between these two, so I can rewrite it as x y plus x bar z, so it is fairly easy here.

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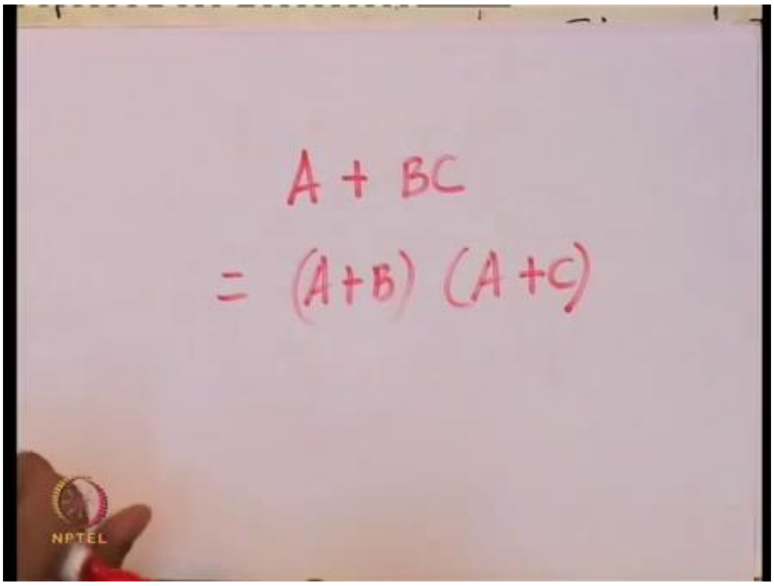

So, what the thing that I am using is, I am using this theorem which is essentially a plus b c is the same as a plus b into a plus c. So, this is the theorem that I have used to expand to the first line.

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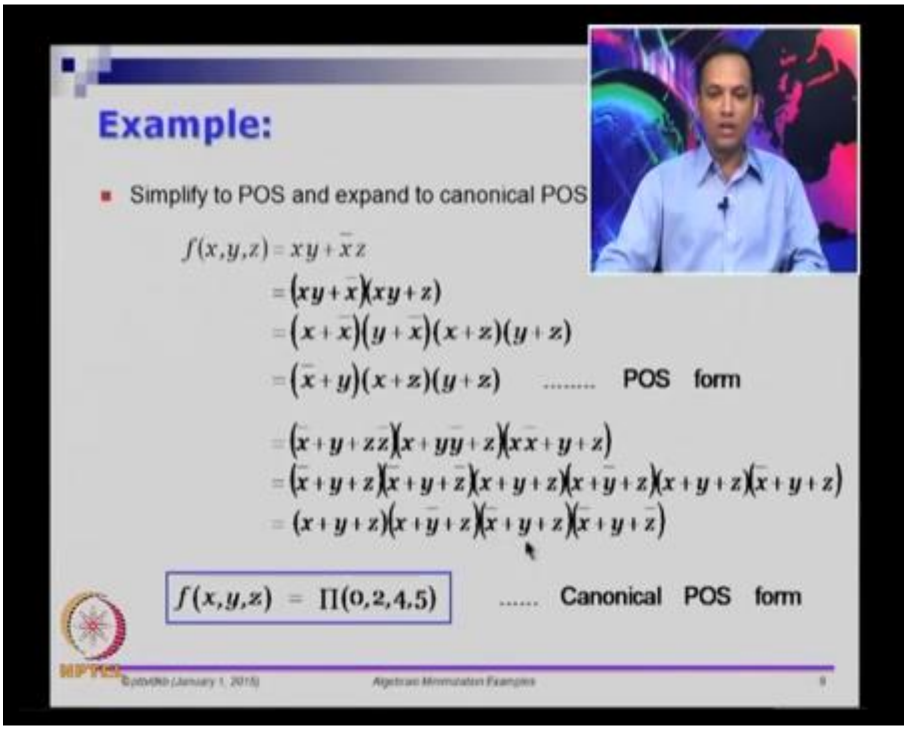

So, let us move on, let us see how to get this further. So, we have x y plus x bar, again this is a product term and that is a product term but, you have sum of products. We want to convert that to product of sums, we will use the same theorem as before, you have one term here and you have a product here, you want to remove it.

So, I use the same thing a plus b c is a plus b into a plus c, so in this case, it is x bar plus x and x bar plus y. So, you can see this x bar appears in these two terms and this x and y is split up as two different sum here and this x y plus z, we can write it as x plus z into y plus z. So, it is possible to do that, at this point, we have four product terms and four sum terms which are added with each other.

And if you notice, so we have x bar plus y here, so we have rearrange and written it here, then we have x plus z, we have it here, y plus z, we have it here, x plus x bar is just 1, we can ignore it, so this is the POS form. So, if you go back and actually expand this product, we will see that the same as this one, if you take this expand it and simplify it, you will eventually end up with this expression.

So, this is the POS form, the question is this a canonical POS form, so quick answer is, it is not, because again if you look at these terms here. So, we have three sum terms, this term does not have a z, this term does not have a y and this term does not have a x. So, we need to introduce those terms in each one of those and the way, we do that is just like what we did in the first step.

So, we did this earlier, x bar plus y, this term is missing z and z bar. So, we introduce a z z bar here, this one is introduce is missing a y to make it a valid canonical POS, we need y in this x plus z also. So, we write y y bar and for y plus z, x term is missing, so we write it as x x bar. So, in this case, x x bar is 0, y y bar is 0, z z bar is 0, so it is okay to logically OR is 0, because it is not going to change the value of this expression.

Now, will use the same trick, so we have something of the form a. So, we have something of the form a plus b and c. So, let us say this is my a, this is my b and this is my c, we have something of the form a plus b c, you are going to write it as a plus b into a plus c, so that is what in this line. So, x bar plus y plus z z bar, we write it as z here and z bar there, then y y bar get split up as y here and y bar are there and x x bar split up as x here and x bar there.

So, it is looks like, we have six terms here and however these six terms there are some reputations here. So, this is not the canonical SOP, because it is not canonical POS, because we have some terms, which are repeating, if you remove the terms that are repeating, it turns out that, this is the expression here. Now, like I did earlier for sum of product, we may want to go and write this as the product of sum form using the pi notation.

So, let us take a minute to think about, how to write in this POS form, we have x plus y plus z. So, this is in the uncomplimented and this in the uncomplimented and this is in the uncomplimented form. So, this is the same as m, capital M subscript 0. So, this is the max term and this is the zeroth max term, then you have this is 1 and 0 and 1. So, we have that as 2, then you have 1, 0, 0; that is 4 and you have 1, 0, 1 that is 5.

So, x plus y plus z is the zeroth max term, this is max term in the second row, this is the max term in the 4th row and this is the max term in the fifth row. So, F of x, y, z is phi of 0, 2, 4, 5. So, at this point if you want to canonical SOP is just sigma of 1, 3, 6 and 7, because those all the terms that are missing here.

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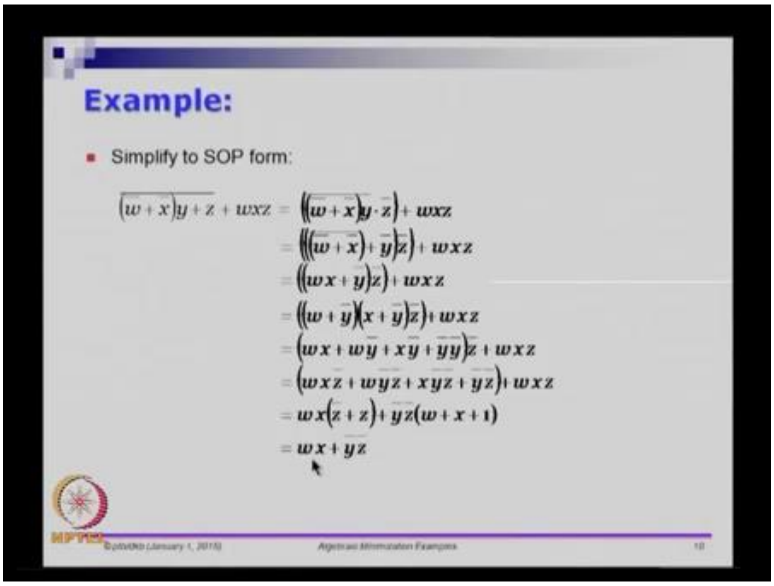

So, let us do final example, before we wrap up, so we want to simplify this expression to the sum of product form. So, usually, we use some of products, because it is slightly more intuitive to work with some of products than with the product of some forms. So, we will see, how to convert this to the sum of product form, this is slightly more complicated expression.

So, it has complements on some terms which are mix of sums and products and so on, this is not just sum of products. So, this is a product term, however, this is not appearing as the product term, we want to be able to simplify it. So, the first thing, we can do this, do here is, you look at this expression, this is of the form a plus b compliment. So, we can use DeMorgan's theorem and say, there is the same as a complement and b complement, so a is this term up to y and b is just z.

So, this is a complement into b complement, so this is slightly better, however, we still have a complement that is at the top here. So, this w bar plus x bar and y, the whole thing is complemented here, again we can de Morgan's theorem and say, this is a b complement is a complement or b complement. So, this is a, a is w bar plus x bar let us say and b is y, then this is a complement or d complement.

And again, there is still the complement on the two terms, we need to be able to simplify that and that is w bar and x bar is complement as same as w x. So, now, it is slightly better, we do not have these complements running over multiple terms, the complements are actually contained on each. So, you have only literals, which are complemented not whole expressions which are complemented.

Once, you get that there is a fairly straight forward process, so we have w x plus y bar into z bar plus w x z, we can rewrite that as w plus y bar. So, w x plus y bar, we can write it as w plus y bar into x plus y bar into z bar, you already have that, plus w x z. So, at this point from here, you can branch off and do something completely different end up with the same result, but there trying to re enforce of few ideas here.

So, if I have the something of the form a plus b c, we have a plus b into a plus c, the z bar is actually common to both of them. So, that just appears as the term here, the product plus w x z. Now, you expand everything out, so this is w x, w y bar, x y bar, y y bar and so on. And you have the z bar which comes here plus w x z and if you simplify it, it turns out it t is w x plus y bar z bar.

So, from this point you may be able to jump here directly, slightly more involve though. So, however this one, the process that we have is slightly more complicated from here to here is actually slightly simple, I suggest that you go and do that. So, I have one form of derivation here, you try and take this expression and get here directly if it is possible. So, you can take that as one of your exercise problems. So, this brings me to the end of this module, I will see you in module 6.