Basic Electrical Circuits
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Lecture – 95

In this lesson, we will go about formally proving Reciprocity Theorem for resistive networks.

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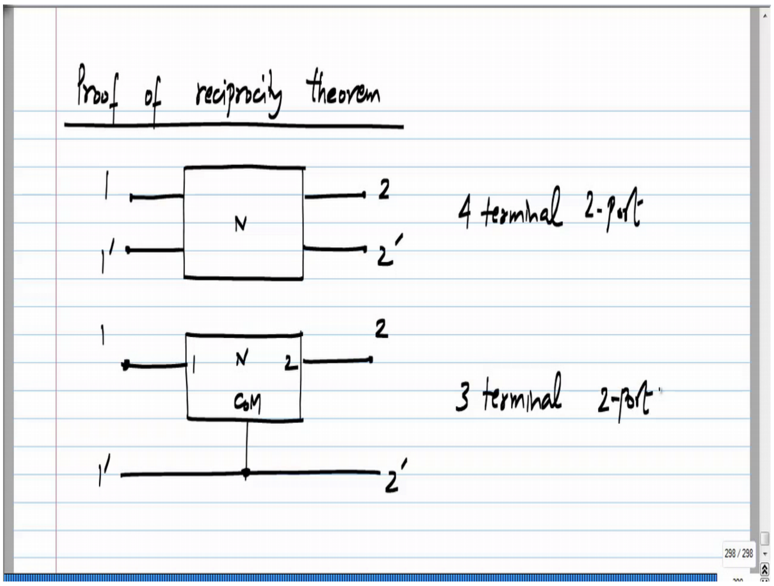

Doing so, we will recall some other properties that we have seen for resistive networks earlier. Now, first let me make a distinction between two kinds of two ports, this is the picture I have been drawing so far, 1 1 prime and 2 2 prime. Now, a sub set of two ports can be of this form, 1 2 and this 1 prime and 2 prime are really the same terminal, so this is the common terminal and port 1 port 2 and a common terminal. Now, this is a more general representation, but a number of two port networks fall into this category.

So, first I will take this type of two port network, proved reciprocity and then, extend it to that one. Once we do it for this, it turns out to be trivial for the other case; it turns out to be lot easier for this given the kind of circuit analysis we have already studied. Now, this is known as a 4 terminal two port and this is known as a 3 terminal two port.

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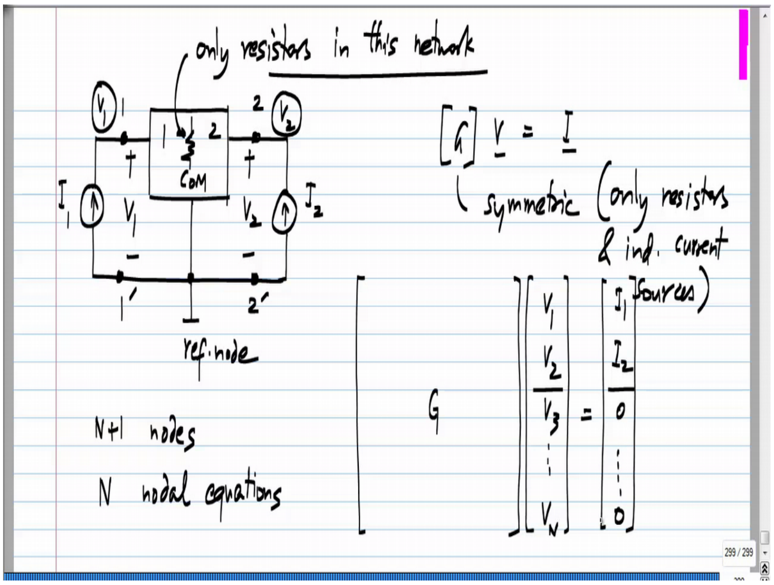

Now, to go about proving reciprocity I will use this picture. So, 1, 2 and the common node, this is the first port, this is the second port and I think of both ports being excited by current sources I 1 and I 2. So, this corresponds to the z parameter picture of the two port. Now, like I earlier showed if you proved reciprocity with z parameters, then; that means that all the other parameter sets also have the corresponding reciprocal relationships.

So, it is the network that is reciprocal, so you can prove reciprocity by proving the reciprocal relationship for any set of parameter; that is you can prove z 1 2 equals z 2 1 or y 1 2 equals y 2 1 or h 1 2 equals minus h 2 1 or g 1 2 equals minus g 2 1, any of them implies the others. So, we will start with this, by the way the most important thing this network now consists only of resistors, so that is important. Now, let me you try and use nodal analysis for my proof and I will choose this common node as the reference node.

So, then the port 1 voltage V 1 is nothing but, the voltage at this node and port 2 voltage V 2 is nothing but, the voltage at this other node with respect to the references node. So, now, if I set up the nodal analysis equations for this, let say that inside there are a number of nodes, that say that there are total of n plus 1 nodes. Now, I choose it to be like this, so that I have n equations, I exclude the reference nodes and I have n nodal equations.

Now, the two nodes corresponding to the two ports, I choose to label them node 1 and node 2 and the remaining nodes in the circuit are labeled 2 to N and there are only two independent sources in the circuit I 1 and I 2, because inside we have only resistors. So, now, clearly the nodal analysis equations for this can be set up as some G matrix times the unknown voltage vector equaling the independent source vector, which consists only of current sources in this case. Now, recall that when you have only resistors and current sources, nodal analysis was easiest and also the conduction matrix had some nice properties.

So, this conduction matrix was symmetric when we had only resistors and independent current sources, so we have a symmetric conductance matrix. Now, let me sort of write this in an expanded form, I have my G matrix here and I have my voltage vector V 1 V 2, which corresponds to the two port voltages and after that I have V 3 all the way to V N. Remember, this is N plus 1 node circuit, so there are N node voltages and N independent KCL equations.

Now, this equals the source vector and what we have in the source vector is basically the independent current source value being injected into a particular node. So, clearly into node 1 we are injecting I 1 and into node 2 we are injecting I 2 and the rest of the entries here will be 0, because we have no other sources in the circuit. Now, what do we need to prove reciprocity? Let me copy this over.

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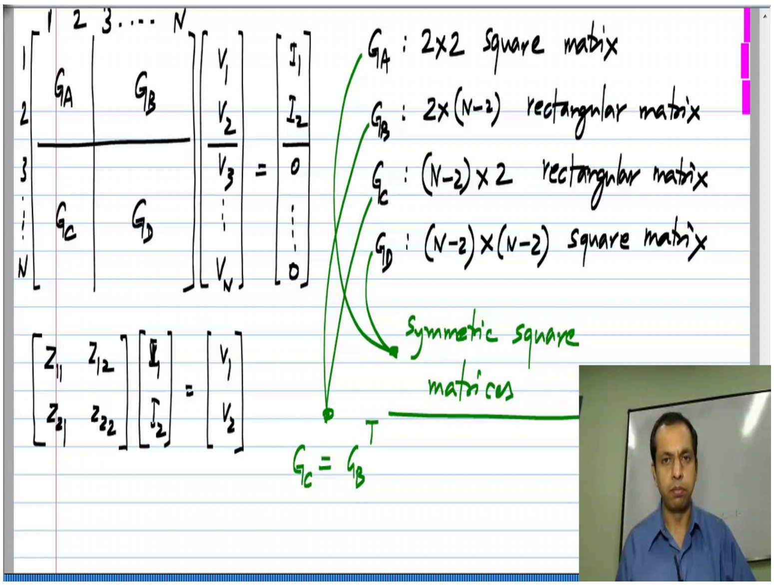

The z parameter description of this network would be z 1 1, z 1 2, z 2 1, z 2 2 times I 1 I 2 equals V 1 V 2. So, we can see parts of this here, we have this I 1 and I 2 over there, V 1 and V 2 over there, but we have all these extra variables, which we have to eliminate and come up with the relationship of this type to see if the network is reciprocal. To do that, what I will do is the following. This G matrix, I will sub divide into four parts.

So, that is here I have taken 2 rows and 2 columns, so let me write down the numbers this is 1 and 2 and then, I have 3 all the way to N. And similarly, I have taken 2 columns 1 and 2 and then, I have 3 all the way to N. Now, I will have basically four sub matrices in each of this, so let me call this G A, whatever goes into the first two rows and first two columns. So, clearly G A is a 2 by 2 square matrix. Now, here I will have G B, G B is in general broader than it is tall, it is a rectangular matrix.

It has 2 rows and N minus 2 columns and in this location, I will have G C and G C will be taller than it is wide, it will have N minus 2 rows and 2 columns, it is also a rectangular matrix and finally, G D which is a square matrix of N minus 2 rows and N minus 2 columns. Now, the important part if we have a circuit with only resistors and independent current sources, we know that the G matrix is symmetric.

Now, what does it mean? If the G matrix is symmetric; that is above this diagonal we have symmetry, so; that means, the G A and G D are also symmetric square matrices, because the G matrix is symmetric, G and G D are also symmetric square matrices. So, G A and G D are also symmetrical and also, because this G matrix is symmetric this G B and G C are not independent of each other. So, it is very easy to see that these two are related by G C by being equal to G B transpose.

So, the symmetric of this matrix means that G C will be the transpose of G B. So, with this we can go ahead and try and come up with the relationship of this type involving only two variables V 1 and V 2.

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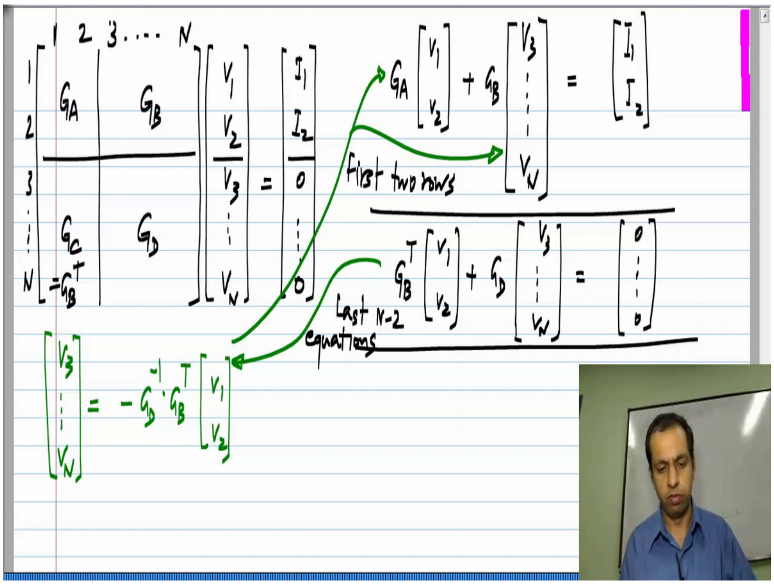

What I will do now? Is to split up this one with two different variable vectors you see, what I mean first I will write G A times V 1 V 2 plus G B times V 3 all the way down to V n will be equal to I 1 I 2. So, essentially what I have written are the first two equations, so the first two equations are the first two rows will correspond to this equation, because we have only this part of matrix coming into picture. Now, if you are little confused about this you can consider some matrix with individual entries and write it down and see.

So, G A will multiply this part of the unknown vector and G B will multiply that part of the unknown vector. So, these are first two rows or the first two equations, now I will write it for the remaining ones and I will also use the fact that G C is G b transpose. So, G C will multiply this vector V 1 V 2 and G D will multiply to the rest of this unknown vector. So, will have G B transpose times V 1 V 2 plus G D times V 3 up to V n being equal to a vector of all 0's and these are the last n minus 2 nodal equations you have the first two nodal equations over here and the last n minus 2 equations over here.

Because of the way we choose the nodes numbers all the independent sources appear in this part and nothing is here and what we wanted to eliminate was this extra unknown variables vector V 3 to V n and that we can do by using the second equation. From this one, we have essentially I solve for this unknown vector. So, I will get the extra unknown vector V 3 to V n to be minus G D inverse times G B transpose times V 1 V 2; that is we have got this unknown vector in terms of the desired vector V 1 V 2.

Now, what I am going to do is substitute this into the first two equations is this clear. So, then, essentially I have to substitute this over here this is, where I have the extra unknown vector.

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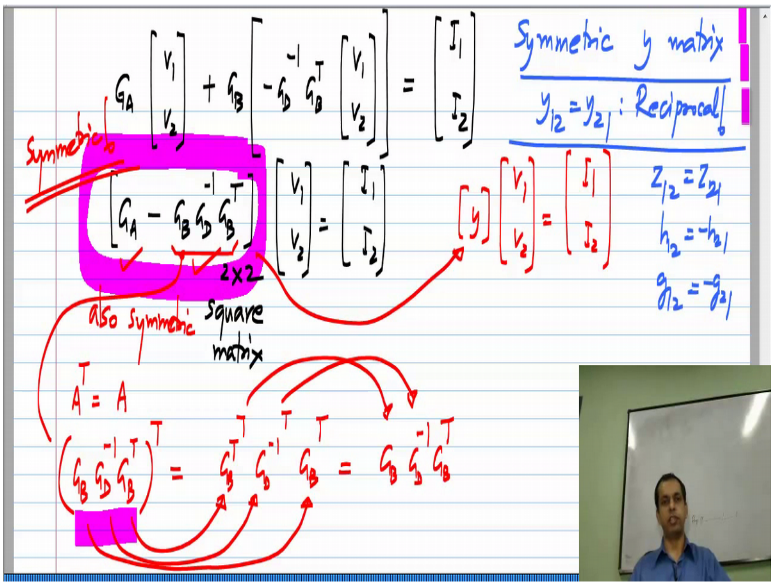

So, if I do that I will get g a times V 1 V 1 V 2 plus G B times the extra unknown vector which we saw was minus G D inverse G B transpose times V 1 V 2 and the whole thing equals I 1 I 2. Now, I have this relationship G A minus G B G D inverse G B transpose this whole thing is a 2 by 2 square matrix G A is 2 by 2 G B has 2 rows and n minus 2 columns G D has n minus 2 rows and n minus 2 columns and G B transpose has n minus 2 rows and 2 columns.

So, finally, you have end up with 2 by 2 this whole thing times this vector V 1 V 2 equals I 1 I 2. Now, this is the relationship for this particular circuit and you see that already this is the y matrix y parameters matrix of the circuit, because in terms of y parameters we would have the y parameters matrix times V 1 V 2 equals I 1 I 2 and if you want z parameter matrix we have to invert this.

But of course, most importantly, what we want to see is this matrix symmetric the y parameter matrix is symmetric; obviously, the inverse z parameter matrix will also be symmetric and either way we already proved reciprocity. So, to prove reciprocity we have to prove that y 1 to equal y 2 1 or z 1 to equal z 2 1 that is y parameter is a symmetric or the z parameter matrix is symmetric. Now, let us look at the this G A it is symmetric we know that comes right from the symmetry of this G matrix, now what about this part how do you prove symmetric a matrix is a symmetric if its transpose equals the matrix itself, so that will test.

So, let transpose this G B G D inverse G B transpose is or matrix say and we transpose that and you know that when we transpose a product of matrices you will get the product of transpose of a individual matrices, but in opposite order. So, first we get this 1 G B transpose and the transpose of that next will get that, which is G D inverse and the transpose of that and finally, will have G B transpose and clearly this is g b transpose and transposed again.

So, that is equal to G B itself now this G D itself is a symmetric square matrix. So, it inverse will also be symmetric, so when I transpose G D inverse I will get the same thing, because G D inverse is also symmetric. And finally, I have G B transpose. So, the transpose of this is exactly the matrix itself, because that is the matrix I started off with by G B times G D inverse times G D transpose. So, this is also symmetric, now we have the sum or difference between two symmetric matrices, so that also a symmetric.

So, this whole thing here we have proved the symmetric. So, that is symmetric and that is nothing, but the y matrix of our circuit. So, that is symmetric which means that the two port network is reciprocal. So, the conclusion is that the y matrix is symmetric and, which mean y one two equals y 2 1 and hence it is reciprocal and y 1 2 being equal to y 2 1ne automatically means z 1 2 being equal to z 2 1 or h 1 2 being equal to minus h 2 1 or G 1 2 being equal to minus g 2 1.

So, all of this automatically comes from that 1, so using our knowledge of properties of conductance matrix that appears in nodal analysis we are able to prove that a resistive two port network is reciprocal of course, we have restricted the kind of two port networks we have taken only a three terminal two port network, but it can be generalized to arbitrary three port network that is four terminal two port networks in a very simple way.