VLSI Data Conversion Circuits
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Lecture - 21
Linearized Analysis

This is VLSI Data Conversion Circuits lecture 21.

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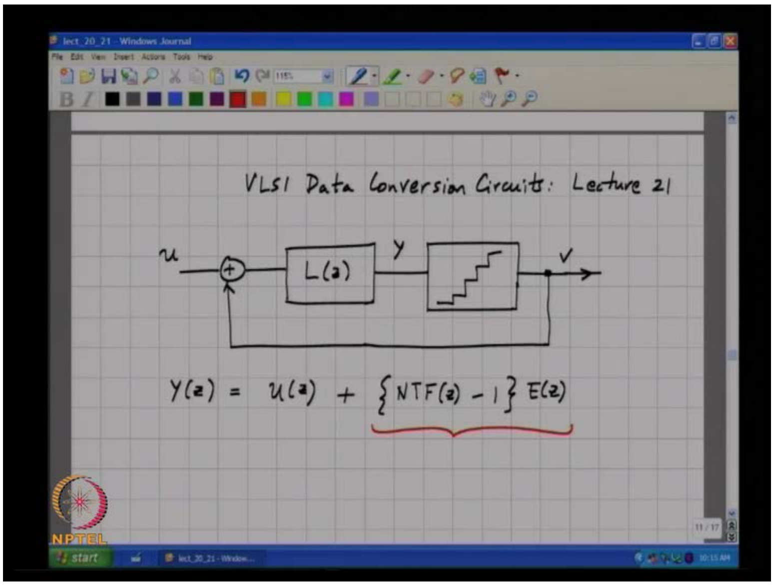

In the last class, we were wondering about the distribution of the error and the input to the quantizer. And we said that, if we assume that the quantizer is an additive noise source and the input to the quantizer can be written as the sum of the input to the modulator plus NTF of z minus 1 times E of z. And we were wondering, how this animal behaves and by assumption, this corresponds to uniformly distributed noise sequence, which is white which means, it is uncorrelated from sample to sample and so on.

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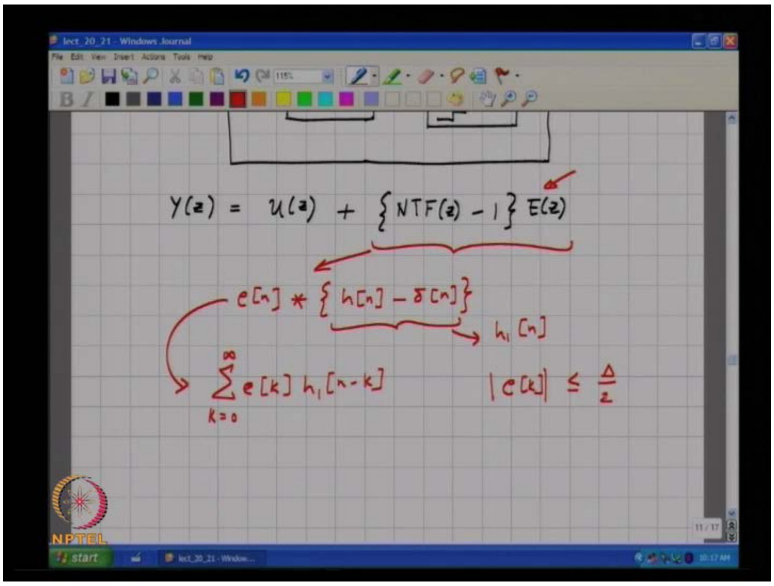

In the time domain, this corresponds to the convolution of e of n with a filter, whose impulse response is h of n minus delta of n. Let us call this h 1 of n, just not to keep carrying h minus delta all over the place. So, the output is, since this is a convolution, is e of k h 1 of n minus k, k equals 0 to infinity at 0 to n, I suppose so. So, this is the expression for the noise process, which is riding over the input, at the input to the quantizer.

Now, to find a bound on this what do you think you can do, please recall that e of k is less than or equal to delta by 2.

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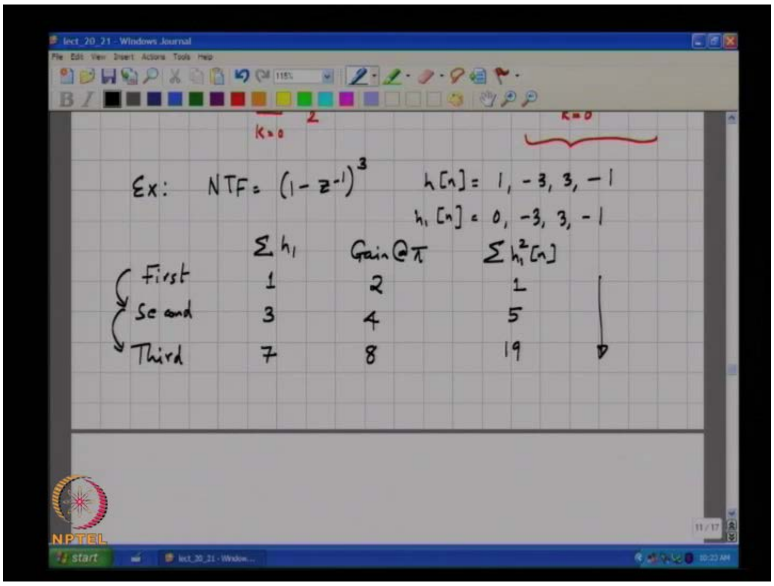

So, if this is the noise process riding over the input, it must follow that, this must be less than or equal to sigma k equal to 0 to infinity delta by 2 times modulus of n minus k. I mean, a b is less than mod a mod b and mod a is less than delta by 2, so this bound is equal to delta by 2 plus minus k equal to 0 to infinity mod h 1 of n. So, what is this telling you, I mean let us say you compare a first order and the second order modulators. What is mod h 1 of n in the first order of modulator, it is 1, in the second order modulator, it is 3.

And as we will go along, we will see how we came from the first order to the second order modulator can be repeated again and you will get a third order modulator, where the noise transfer function becomes 1 minus z inverse whole to the 3 and you are expecting to do much better in band and worse out of band. Because, so for example, an NTF which is 1 minus z inverse the whole cube h of n is what, 1, minus 3, 3, minus 1, the sum must be 0.

So, h 1 of n therefore, will be 0, minus 3, 3 and minus 1, so what is the upper bound on the noise process riding over the modulator input at the input to the quantizer 7 delta by 2. So, first order, second order, third order, sigma h 1 is 1 for the first order, 3 for the second order and 7 for the third order and what is the out of band gain, rather the gain at pi. For the first order it is 2, for the second order it is 4, for the third order it is 8 and what can you say about sigma h square of n or let me say, sigma h 1 square of n.

In this case, it is 1 and in the second order example, it is 5, in 19, please note that, the square root of sigma h 1 square of n, this gives you this times delta square by 12 is the variance of the noise to the input to the quantizer. And this will and square root of this must be significantly smaller than the absolute upper bound; so I mean, what do you notice as far as the trend here is concerned.

As you go from first order to second order to third order, what is happening to the inband noise, inband noise is going down, what is happening to the out of band noise that is, the noise at high frequencies, the high frequency noise is going up. And what can you say about the variance of the noise of the input to the quantizer inside the loop, it is also going up. So, as I said, since we see improvement from first to second and second to third order modulators, the obvious question that arises is, why cannot I continue this add infinite term.

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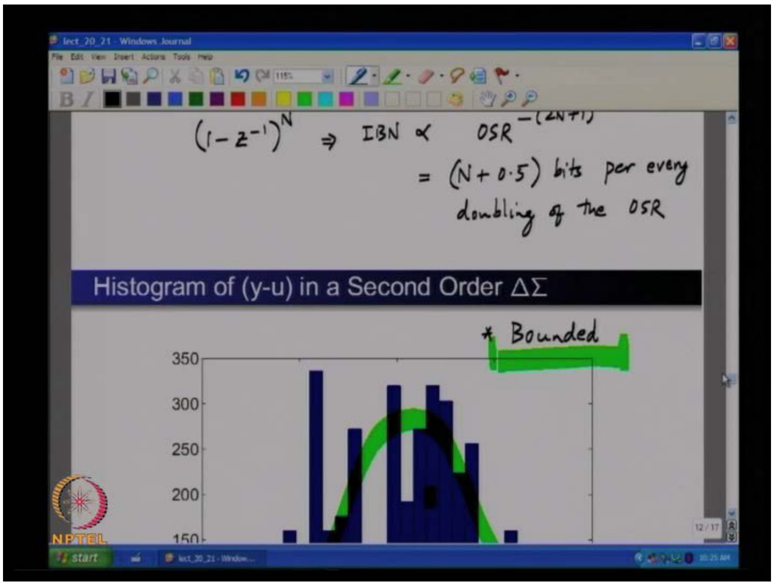

And in general, I will therefore end up with a noise transfer function, which is of the form 1 minus z inverse whole to the N. So, the inband noise will be of the form, what will be the integrated inband noise, will be proportional to OSR. See, 1 minus z inverse to the N means that, the noise transfer function at low frequency is goes as omega to the N. So, when we square it, you will get omega to the 2 N, when you integrate it, you get 2 N plus 1.

So, this must go as OSR to the power minus 2 n plus 1, so which is therefore, how many bits per every doubling of the OSR, N plus half bits for every of the OSR. And as we saw during the last class, the input to the quantizer is noisy with the transfer function from the quantizer noise process, the input of the quantizer being NTF of z minus 1. And in the second order case, if we kind of said, why do not we assume that Gaussian, because it seems to be high in the middle and tapering off towards the ends.

We of course realize it, cannot be truly Gaussian, because this is bounded, whereas the true Gaussian is not bounded. Yet, a Gaussian is a good approximation and that is, it is not just fancy, but you must understand that, e of n is assumed at least to be a white sequence, in other words uncorrelated from sample to sample. The shaped quantization noise is formed by taking this white sequence and passing it through a filter which means, that at any one instant of time or at any one sample, is a linear combination of I mean, several samples of the quantization noise.

Because, the output as I mean, the input to a quantizer is a convolution of e of k, which is the quantization noise sequence with some filter transfer function. And e itself is uncorrelated from sample to sample and the input to the quantizer is simply a sum of several of these uncorrelated random variables. So, if you take many uncorrelated random variables with the same distribution and you add them together, what can you say about their probability distribution function.

If you take a large number of these samples and then, go on adding stuff together, eventually by central limit, they will I mean, the sum will have a cumulative distribution function, which is the integral of the Gaussian, which in this case turns out to be equivalent to saying that, the PDF will tend to a Gaussian. I mean, there is a big difference between tending to Gaussian and being Gaussian and you know, this is the difference.

One of course, we know for sure that, it s bounded, so it can never really get Gaussian unless you have a large number of terms and so on, infinite number of terms in fact. But, as I said, I mean everything that is probably I mean, needs to be known about a Gaussian is known. So, it is a very useful approximation to make so that, we can now exploit all the knowledge that has been developed over the generations on Gaussian processes and what happens to Gaussian processes as they processed, either linearly or nonlinearly, you understand.

So, that is the intuition at least for assuming that, the shaped quantization noise, not quantization noise, quantization noise is still uniform. But, once you pass this through a filter, where I mean with a reasonably large number of coefficients, you start adding up where is uncorrelated random variables and result starts to look like random variable, whose PDF is Gaussian. So, as we just saw, increasing the order of the loop seems to be giving us better and better and better results.

So, the question is, what really I mean, is this sounds too simple, so you just go on adding integrators and you can make a very very bad quantizer look like a quantizer, which is extremely good. So, there must be a, if I were an engineer, I would say this sounds too good to be true. So, it must break at some point and the question is, where and why of course. So now, we will take a look at, why this assumption is not I mean, you cannot go on putting integrators in the loop, go on increasing the order of the noise transfer functions and make.

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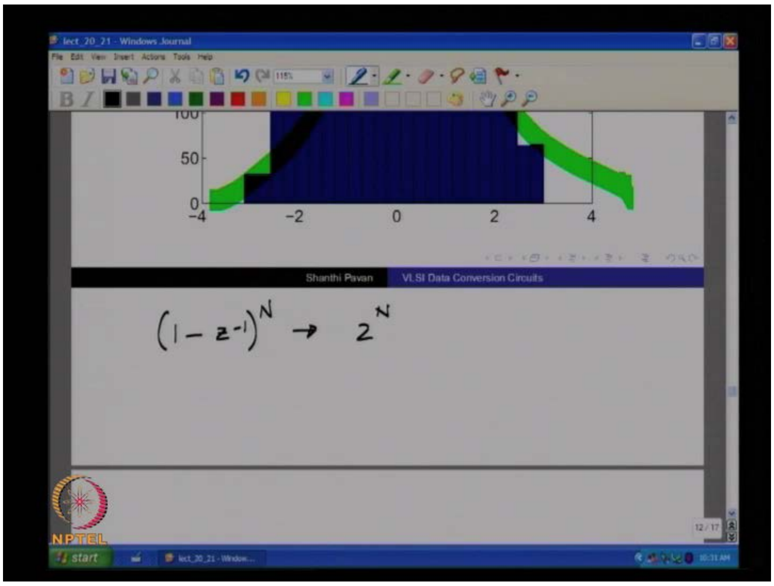

In other words, 1 minus z inverse to N, where N is a large number, with the a motivation being that, you want to reduce the inband noise. Inband noise is not quite a viable thing in practice and we will now go ahead and discuss this. Before I proceed, I had like to remind you that, the out of band gain of this noise transfer function is 2 to the N and as we go on increasing the out of band gain, what can you say about the variance of the noise at the input to the quantizer, it goes on increasing.

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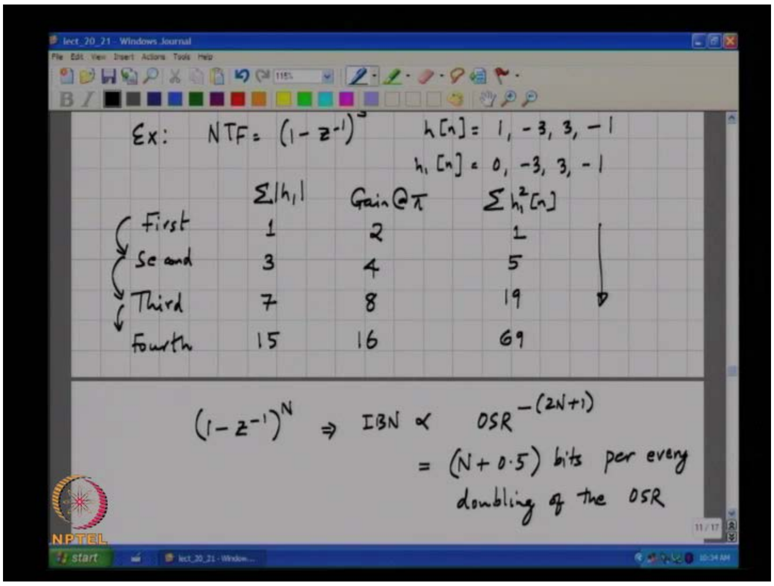

We saw going from first order to second order to third order, that the variance of the noise at the input to the quantizer went up from delta square by 12 to...

Student: 5 delta square by 12

To 5 delta square by 12 to 19 delta square by 12, so you can only imagine what happens when you make a fourth order modulator. What is sigma h 1 for the fourth order case?

Student: 15

1 4 6 4 1, you remove the 1, so it is 15, what is the gain at pi, it is 16 and what is the sigma h 1 square, it is 16 plus 36 plus 16 plus 1, which is 69. So, you can see that, the variance of the noise in the input of the quantizer is increasing very rapidly indeed. And that makes sense, because the out of band gain is of the gain at pi or at high frequencies is dramatically higher and the variance of the noise is proportional to the square of the gain.

So, it makes sense that, the variance noises has increased by a large factor, in fact I mean, if you kind think of this, see most of the energy in the NTF minus 1 is coming from which frequencies?

Student: high frequencies

Coming from high frequencies, what has happened to high frequency gain?

Student: Very large

Has gone up by a factor of...

Student: 2 power N

From 8 to 16, is gone up by factor of...

Student: 2

2, so if most of the energy is coming from around pi, what must you expect for the variance of the noise.

Student: ((Refer Time: 19:26))

Should go up by factor of 4 and what are you see...

Student: ((Refer Time: 19:34))

One is 19 delta square by 12 and the other one is 69 delta square by 12, indeed it is very close to 4 and from 5 to 19, again it is close to 4, 1 to 5 is again close to 4, you understand. The intuition behind the stuff increasing by a factor of 4 every time is that, most of this energy is coming from high frequencies, the high frequency gain is doubling every time, so it makes sense that, it keeps increasing by a factor of 4 every time.

So, increasing the order of the noise transfer function to 1 minus z inverse to the N, very quickly cause the input to the quantizer to have a noise, whose variation goes on increasing very rapidly with N. Now, the question is, what does it do to the modulator loop as such, we must remember that, the quantizer has a finite range beyond a certain range, the output of the quantizer is...

Student: Saturating

Is saturating, the range of the staircase is not infinite.

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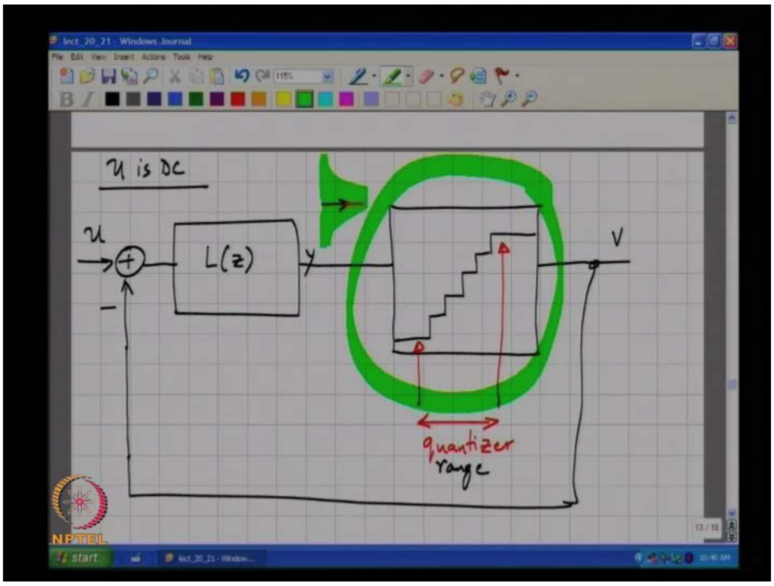

So, let me draw home that point, so this is the range of the quantizer, beyond which the output of the quantizer simply saturates to some value. Now, let us understand what happens when such a quantizer is embedded inside a feedback loop. So, again I am going to assume some generic loop filter say, L of z. For a given noise transfer function, you can calculate L of z by using, how is the L of z related to the noise transfer function.

Student: ((Refer Time: 22:29))

1 by.

Student: 1 plus L of z.

1 by 1 plus L of z is the noise transfer function and this is v, this is y, this is u, now let us assume that, u is purely dc. So, y therefore will be, how will the waveform at y look like, it is a constant sequence. So, how do you think y will look like.

Student: ((Refer Time: 23:46))

It will look like some dc level on top of which, there will be...

Student: Some noise

Noise, some noise will be riding on top of this dc quantity and we understand that, the variance of this noise will keep increasing as...

Student: ((Refer Time: 24:24))

As the noise transfer functions gain at high frequencies goes on increasing, so this is the actual input into the quantizer. And the output of this quantizer is a quantized version of this input, which is dc plus some noise. So, it is a non linear function of dc plus some noise and see, dealing with linear systems is hard enough. So, you can imagine that, dealing with non linear systems are lot harder, especially when they are enclosed in feedback loops.

So, a natural way of solving a problem is to, what about you think you do I mean, imagine you were a researcher in the beginning of 20 th century, where a lot of linear stuff was figured out. But then, you started encountering systems with non linearity, so what do you think you might do.

Student: ((Refer Time: 25:46))

So, reasonable thing to say is, let me try and see if I can use my knowledge of whatever I have learnt in a linear control systems class, which is all the various stability criteria and all that, which are only applicable to...

Student: Linear systems

Linear systems, let me see if I can somehow hack a non linear system and in a manner so that, I will still be able to apply my theory, which was designed for...

Student: Linear systems

Linear systems, so in this system, which is a linear loop filter followed by a non linearity, what should I do. I know, I should try and eliminate nonlinearity, so what would I do...

Student: ((Refer Time: 26:36))

I will try and I mean, approximate the only non linear element in the picture, which is...

Student: The quantizer

The quantizer by...

Student: A linearity back up

By a linear block.

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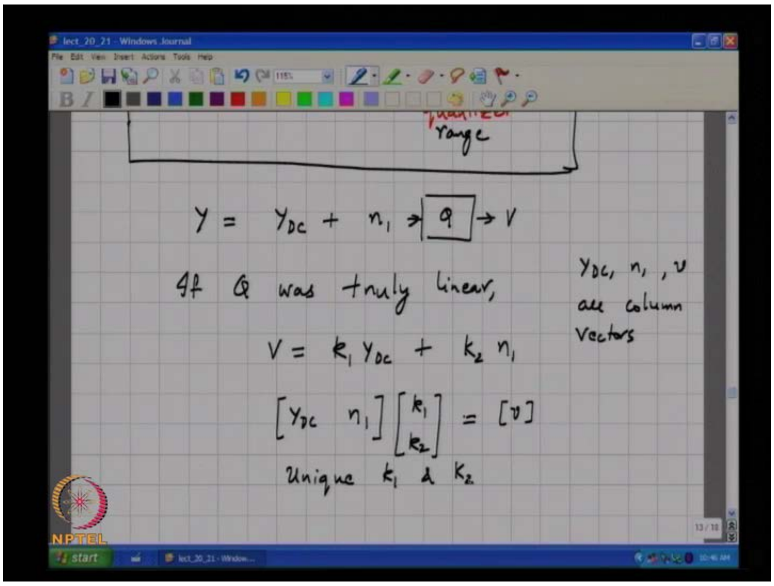

So, in other words I say that, I understand that Y is composed of two components, one is Y D C and why is Y D C there?

Student: Input

Because, the input is dc, Y D C plus let me call this n 1, where n 1 corresponds to the...

Student: Shape quantizer

A shape quantization noise which is floating around in the loop, so the input to the quantizer is Y, which is Y D C plus n 1. The output of the quantizer is V, so this goes in through a quantizer and the output is V. So, what do you think, I can do to approximate the quantizer as a linear block.

Student: Linear combination of Y D C and n 1.

And...

Student: And plus some noise component

So, what you will do is say, I have this box, I saw output V, I do not know what is inside the box. I have two inputs Y D C and n 1 and one output V, I want to find the linear approximation to this box. If this was truly a linear box then, if Q was linear then, what does it mean?

Student: ((Refer Time: 28:53))

V must be...

Student: ((Refer Time: 28:56))

Some k 1 times Y D C plus k 2 tines n 1 I mean, let me rephrase or repeat the argument again. I have a box with two inputs Y D C and n 1 and one output V, if this box was truly linear then, the output would be represented as a linear combination of...

Student: Inputs

Of the two inputs weighted here with k 1 and k 2 and how would I find k 1 and k 2.

Student: ((Refer Time: 29:46))

So, if I was given the sequence Y D C and if I was given sequence n 1 and given the sequence V, how will I find k 1 and k 2?

Student: ((Refer Time: 30:00))

No I mean, you have no choice, let us say these are the recordings from some plant, you have Y 1, you have this sequence Y D C is given, n 1 is given, V is given as data and I want you to compute k 1 and k 2, what will you do?

Student: We have to separate two points.

So, if Y D C and n 1 was given then, I will form a column vector Y D C, another column vector n 1, this multiplied by k 1 and k 2 must be equal to V, where all these are Y D C, n 1 and V are column vectors. So, in other words, if had a sequence which was 10000 points long and if Q was truly linear then, basically I need to determine k 1 and k 2, all I need are two equations. So, I will take at two instances of time, I will find, I will look at Y D C, n 1 and V, two equations, two variables and I can solve.

Even though there are in principle 10000 equations and all these will be linearly dependent. So, when you have more equations than unknowns, the problem is under constrain and you can find a, you can solve this equation, you will get a unique k 1 and k 2. Now, what do you think will happen, so this is let me put this down here, unique k 1 and k 2. Now, what do you think will happen when, if Q is not linear and what do you think will happen?

Student: ((Refer Time: 32:59))

So, you cannot get...

Student: Solutions

I mean, we will not be identically equal to k 1 times Y D C plus k 2 times n 1, there will also be some...

Student: ((Refer Time: 33:12))

There will be some...

Student: ((Refer Time: 33:15))

There will be some error, now the question is, what is my best approximation to I mean, of this system to a linear one.

Student: If choose k 1 and k 2 (( )) tends to I mean, mean square error of e goes to 0.

So, reasonable approach is to say, I cannot find, given that the system is non linear, I cannot find a k 1 and k 2 such that, V exactly equals k 1 times Y D C plus k 2 times n 1, you understand. Then, the next question is, I mean that is fair enough, because this is a...

Student: Non linear

It is a non linear block, so the next question is, I know it is non linear, so I must expect error, but can I be smarter about choosing k 1 and k 2. And as he said, a reasonable thing to do is to choose k 1 and k 2 such that, the mean square value of this error is as...

Student: Small as possible.

Small as possible, in other words, we still have this set of equations, but what we need to do, is to find the least square solution to this set of equations.

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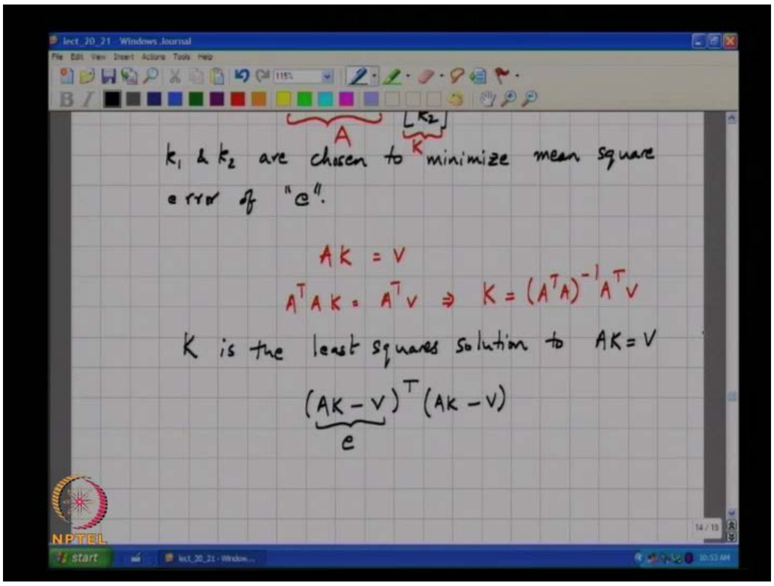

So, find k 1 and k 2 are chosen to minimize mean square error of e, you understand and how would you find the mean square error. You know least square, so if I call this matrix A...

Student: ((Refer Time: 35:52))

So, A times K and if I call this K is V, so least square solution corresponds to A transpose times A times K is A transpose times V which means, K the least square solution will be A transpose A inverse times A transpose times V. This matrix A will it be a thin tall mate I mean, tall skinny matrix or short fat matrix.

Student: ((Refer Time: 36:30))

It will be more rows and columns, which will be a tall skinny matrix, so K is the least square solution to A K equal to V. And how will you find the mean square error, you actually find A K which is your best approximation to V you subtract V. And then, find it is, so this is basically e, e transpose times e gives you the sum of the squares of V and that divided by the number of samples will give you the mean square e, is this clear.

If you do not know least squares solution to a set of equations, you can read it up by at this point all that I want you to do, the aware of is that you should appreciate that, it is possible to choose k 1 and k 2 such that, the error is minimized. So, this quantizer is 2 inputs and one output, so in other words and its non-linear, so if you plot this on a, if you want to show this in a picture, the input versus output is a surface. It is a very a non-linear, because it got steps all over the place, however what we are trying to do in this linear approximation is to fit a.

Student: Straight line

Fit a plane to that surface, and if you have some surface and if you want to fit a plane to it, there is a way of choosing the plane such that, the error is minimum, you understand. And I mean the least square solution is just a way of computing those two coefficients k 1 and k 2.

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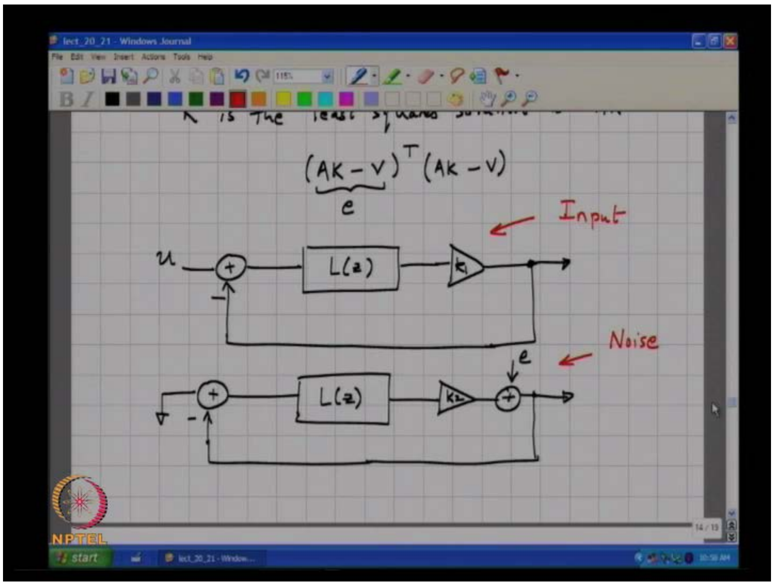

Now, that we know you can divide it by the number of right samples, so now that you have L of z, and you replaced the quantizer by a block, which is please note that in general k 1 need not be equal to k 2. So, k 1 is the gain for the dc part k 2 is the gain for the noise part and e is the error, so this error will what is the main of this error.

Student: ((Refer Time: 40:10))

It will be 0 mean, because if the error is not 0 mean, I can always now what do you call, you know choose the coefficients such that, the error is 0 mean. Now, I can represent the delta signal loop given that, I have a linear approximation to the quantizer by two loops, one which only deals with a dc path and one which only deals with this part. The motivation for breaking it up into these two systems is simply to I mean, you represent the quantizer by a block, which is giving one gain for dc and another gain for the noise plus adding an error.

What is responsible for the noise I mean finally, what is responsible for this n 1.

Student: ((Refer Time: 42:03))

This e itself you understand, please recall I mean, this is a little suttle in it, it takes some time, you must go back and think about this carefully and realize, why this way of breaking up the modulator makes sense. You must understand that, the noise n 1 is a consequence of e. one way to appreciate that is that, if the quantizer was perfectly linear, there would be no n 1 and there would be...

Student: No e

No e either, this whole jumping around at the input of the quantizer is coming because of Quantization, which is error pro. So, it makes sense that, this n 1 is a consequence of e, which is going around the loop and so, you can break this system up into...

Student: Two parts

Two parts, please note that that is, I mean this is something that you have been always used to doing. If you have multiple inputs in a linear system, you can break it up, you can think of it as many systems, identical systems there, each with one input. And there, there is no confusion at all, because it is a linear system, the basic system remains the same, only the location of the input keeps...

Student: Changing

Changing, here there are two inputs and the gain for each input is different, so the two systems will be different. So, this is the linear system corresponding to the input and this is the linear system, which models the way in a quantization noise, you know flows in the loop.

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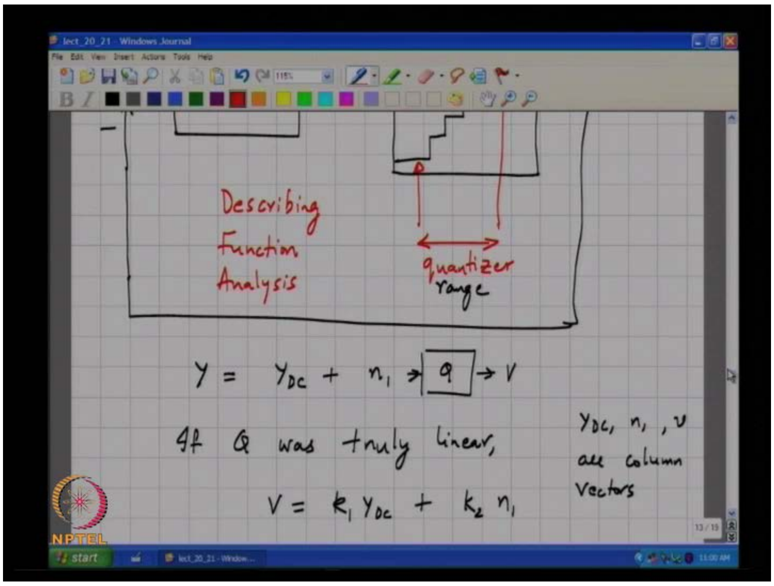

So, this kind of analysis, where you take a non linear system and approximate it by it is linear part and some error, is what is called describing functional analysis. And please note that, this is only I mean, only an approximation, where we try and use the background that we have developed in linear control systems to a problem, which is...

Student: Nonlinear

Nonlinear and therefore, presumably very difficult to tackle analytically, so I mean you should not be surprised that, you know I mean, if you assume this and then, you conclude that, I mean you find after your analysis, that something is fishy or weird or does not make sense, you should not be too surprised. Because, the underlined assumption is not fundamentally correct, it is like assuming 1 equal to 2 and then, you can prove that, the sun goes around the earth.

I mean, the argument may be perfectly true from the step that you assume 1 equal to 2 and all the way up to proving that, sun moves around the earth. Then, you can say, you know my arguments are all right, you know how is that I am coming to totally wrong conclusion and I mean no surprise, because you assumed 1 equal to 2, you understand. So, this is only a weir I mean, this is only an attempt to understand a horribly non linear system, where it is most likely not possible to get analytic solutions by using the stuff that we know from linear systems, you understand.

And this is a very routine thing that was I mean, that is used in non linear control systems and it works quite well in many systems, though one should not be surprised that, it does not work in several classes of systems. Now, given that, we are armed with this new knowledge, that is a kind of look at, what happens in a real quantizer.

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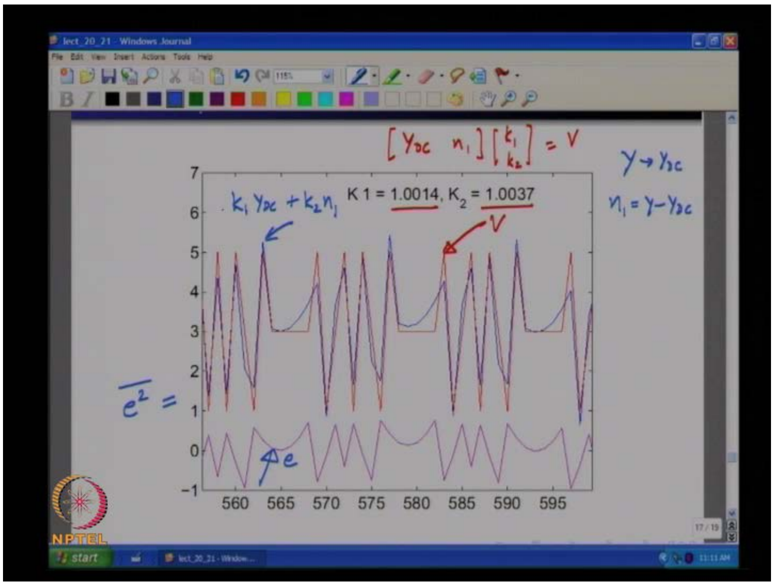

I have taken a second order delta sigma modulator and I put in a dc input, just for fixed ((Refer Time: 47:23)) 3.1415, there is nothing wholly about this and I plotted, so which is the input you think, the red one or the blue one.

Student: Red one

The red one is the input and what is the blue wave form.

Student: Input without quantization.

This is y, this is u and as we can see, the input to the quantizer is, you know you can think of it is some noise process riding over.

Student: Dc

Dc input and the quantized output ((Refer Time: 48:02)) of course, will have only discrete levels. In this particular example, I have chosen a quantizer with delta equal to 2 and the levels going from 7, 5, 3, 1, minus 1, minus 3, minus 5 and minus 7. How many levels are there in a quantizer, it is an 8 level quantizer.

Student: ((Refer Time: 48:37))

With a delta of 2, so when I put in 3.1415, the output, you know the average of output must be...

Student: 3.1415

3.1415, so it makes sense that, the output is hovering between 3, 5, 1 and occasionally 7 and on the average I mean, it seems likely that, the output voltages of the quantized sequences.

Student: 3.14

3.14, now as I said, this is y and I found Y D C, it is simply the mean value of Y, what is the mean value of the output V, it must be the same as u, 3.1415. So, what I did was, found the least square solution to, so Y D C n 1 times k 1 k 2 equal to V. So, I found k 1 and k 2, so k 1 turns out to be 1.0014, k 2 happens to be 1.0037. and why do you think this makes sense or does this makes sense at all?

Student: ((Refer Time: 50:34))

Why?

Student: ((Refer Time: 50:40))

See, please note that, when the input is 3.1415, are we saturating the quantizer at all.

Student: ((Refer Time: 50:54))

The quantizer steps start from minus 7 through 7 with deltas of 2, we are well within the linear range of the quantizer and this is a second order delta sigma. So, the variance of the noise is not too high, so we are not presumably saturating the...

Student: Quantizer

Quantizer, so in other words, since we are not saturating the input to the quantizer and the input to the quantizer is busy in the sense that, there is some activity. The output should be approximated as input plus.

Student: Some noise

Quantization noise, where this quantization noise is uniform and white and all these other stuff. As you have seen in the assignment, if the input is busy, you will have this linear additive noise assumption is close to being valid. So, I mean, the average gain for the signal for dc is the same as the average gain for noise, because you are not saturating the quantizer. So, it makes sense that, the two gains are very close to 1, this here represents what do you think.

Student: ((Refer Time: 52:19))

What sequence is that?

Student: ((Refer Time: 52:28))

No, look at the red sequence., what levels does it take on.

Student: ((Refer Time: 52:44))

Yes, so what might that be?

Student: Output of quantizer

It is the output of the quantizer V, what is u I mean, what is the sequence in blue.

Student: ((Refer Time: 53:03))

The blue sequence happens to be...

Student: The input ((Refer Time: 53:12))

K 1 times Y D C plus k 2 times n 1, so this is the best fit to the quantizer output and how do you think I got that. How do you think I mean, can you suggest a way, in which I can get this.

Student: ((Refer Time: 54:00))

How do I get n 1?

Student: ((Refer Time: 54:04))

How do I get n 1?

Student: ((Refer Time: 54:16))

So, during the operation, you observe the input to the quantizer y, from this you can get Y D C, that is simply the average value of y, how will I get n 1.

Student: ((Refer Time: 54:37))

Yes

Student: ((Refer Time: 54:46))

So, n 1 is simply y minus Y D C and I know the output V, so it is quite straight forward to...

Student: Find k 1 and k 2

Find k 1 and k 2, it is simply the least square solution, which makes the fit the best, so k 1 times Y D C plus k 2 times n 1 is the blue curve from there, clearly there is some error and this is the error, e. I went and computed the mean square value of e, what do you think the mean square value of e is.

Student: ((Refer Time: 55:54))

What do you think the mean square value of the error will be I mean, if the quantizer is not overloaded, the output of the quantizer can be represented as, how can I relate the output of the quantizer to the input.

Student: ((Refer Time: 56:30))

Pardon

Student: Input plus some noise.

Input plus what noise?

Student: Quantization noise.

Quantization noise, so what is the mean square value of that quantization noise?

Student: Delta square by 12

Delta square by 12, so what do you expect the mean square value of this noise to be?

Student: ((Refer Time: 56:45))

No, what is the difficulty I do not understand, you just said it right now, why cannot.

Student: ((Refer Time: 56:58))

See, the output is nothing but, input plus I mean, just now think of the quantizer, we have some input which is busy, the output is a quantized version of this. The quantizer is not overloaded, so the output of the quantizer is nothing but, input plus quantization noise. And of course, the output is I mean, so the output we are saying is, we have fit it to k 1 times Y D C plus k 2 times n 1 plus e, and k 1 and k 2 are both close to 1. The input to the quantizer is Y D C plus n 1.

So, what must the mean square value of e be?

Student: ((Refer Time: 57:46))

Delta square by 12 and what is delta here?

Student: 2

2, so what is delta square by 12, one third, you understand, so we will continue with this in the next class.